Parrondo's paradox - a new paradox in probability theory

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> The rules for *game 2* are more complicated: if the capital is divisible by 3, then the winning chances are only $1/10$, but otherwise (if the capital is *not* divisible by 3) they equal $3/4$.

Both games are fair in the following sense: if many games are played, then wins and losses balance. (This will be made more precise below.)

But strange things happen if one switches between the two games. For example, if one decides by throwing a fair coin whether to play with game 1 or game 2 in the next round it might happen that a winning game results: There is a positive number g such that the expected gain after m rounds of the game is close to $m \cdot g$.

But strange things happen if one switches between the two games. For example, if one decides by throwing a fair coin whether to play with game 1 or game 2 in the next round it might happen that a winning game results: There is a positive number g such that the expected gain after m rounds of the game is close to $m \cdot g$.

This is Parrondo's paradox.

Demo

Survey:

- The mathematical background and an analysis of the pararadox
- Optimal strategies and fractals
- „Greedy“ strategies for collective Parrondo games and stochastical dynamical systems

The mathematical background and an analysis of the pararadox

In both games only the position modulo 3 is relevant, and for an analysis of the possible gain it is sufficient to know the expectation of the gain for the various positions modulo 3.

The expected gain is zero for game 1 at all positions. At game 2 it is

$$0.1 \cdot 1 + 0.9 \cdot (-1) = -0.8$$

if the capital is zero modulo 3 and

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And after any round the position modulo 3 has changed. For game 1 one passes with equal probability to one of the other positions, and also for game 2 one can easily evaluate the probabilities to reach one of the next states. For example one passes from position 0 (divisible by 3) to position 2 with probability 0.9.

The appropriate abstract frame is the following:

Definition: A *random walk with reward* is given by a number s, the state space $S := \{0, 1, \ldots, s-1\}$, a stochastic matrix $\mathbf{P} = (p_{ij})_{i,j=0,\ldots,s-1}$ and a „gain vector" $\mathbf{w} = (w_0, \ldots, w_{s-1})^\top$ (which might have negative components).

The interpretation: one starts at 0 and follows a random walk on S which is governed by P; in any step one gets a reward which corresponds to the value of $\mathbf{w}$ at this position.

Here are the matrices and the gain vectors for the original Parrondo games:
For game 1 all w_i^1 vanish and the stochastic matrix is

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

.
For game 2 $\mathbf{w}$ is given by

$$w_0^2 = -0.8, \; w_1^2 = 0.5, \; w_3^2 = 0.5,$$

and

$$\mathbf{P}_2 = \begin{pmatrix} 0 & 0.1 & 0.9 \\ 0.25 & 0 & 0.75 \\ 0.75 & 0.25 & 0 \end{pmatrix}.$$

The analysis of a random walk with reward is not difficult if one uses wellknown results from the theory of Markov chains: If **P** is an ergodic stochastic matrix, then there exists a probability vector π with $\pi^\top \mathbf{P} = \pi^\top$.

This probability vector has as its components the probabilities to find the random walk (after a sufficiently large number of steps) at the various positions of the state space.

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It is precisely this interpretation by which the two Parrondo games are fair. (In the second game the reward vector is $(-0.8, 0.5, 0.5)^\top$ and the equilibrium equals $(5/13, 2/13, 6/13)^\top$.)

Now suppose that we are given r fair random walks with rewards, we will change between the various walks stochastically. This means that one has numbers $\lambda_1, \ldots, \lambda_r$ in $[0, 1]$ with $\sum \lambda_\rho = 1$ so that with probability λ_ρ the next step is done using the rules of the ρ's walk.

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One can easily show that this new game corresponds to a random walk with stochastic matrix

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And this implies: The vaue of the „stochastically mixed" game is

$$\langle \mathbf{w}_\lambda, \pi_\lambda \rangle,$$

where π_λ is the probability distribution of $\mathbf{P}_\lambda$.

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But typically this function (which here is sketched for the case of two games) looks as follows: here a $\lambda \in [0, 1]$ is mapped to the equilibrium distribution associated with the mixture $(\lambda, 1 - \lambda)$.

This means that then *all* mixures are fair or losing games or winning games.

But there are also situations where the curve attains both positive and negative values. Then for certain mixtures one gets fair games, and for others winning or losing games result.

> Computer experiments show that this phenomenon only occurs if the games under consideration are „very poorly“ mixing. It is open until now how this could be made precise.

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A qualitative explanation of the phenomenon (and thus a qualitative explanation of the Parrondo paradox) could be given by an analysis of the map $\lambda \mapsto \pi_\lambda$: π_λ is generated as an eigenvector of a matrix the components of which are linear in λ. Therefore - by Cramer's rule - the components of π_λ will be rational functions in λ, and the possible powers increase with the dimension.

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(Nonlinearity is also the basis of some other contemporary paradoxes. At *Astumian's paradox*, for example, the nonlinearity of the absorption probabilities as a funciton of the mixtures plays a crucial role.)

Optimal strategies and fractals

We assume again that there are given r fair random walks on $S = \{0, \ldots, s-1\}$ with reward. The walk starts at 0, and there are to be played m rounds.

In the preceding section we have chosen once and for all a probability distribution on $\{1, \ldots, r\}$ which was used in every round. But one could also determine at the very beginning different probability distributions $\lambda^{(\mu)}$ for all rounds μ.

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The problem: What is the optimal choice of the $\lambda^{(\mu)}$?

To find a solution one will combine the following facts:

- If K denotes the set of probability measures on $\{1, \ldots, r\}$, then K is a compact convex set. Consequently also K^m is compact an convex.

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It follows that the optimal strategy will be a deterministic one: Start in round 1 with game ρ_1, continue with game ρ_2, etc. until ρ_m.

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From this point on it is appropriate to consider the problem in a more abstract framework. The starting point is the observation that an application of the $\mathbf{P}_\rho$ only changes the probability distribution for the position of the walk and that this suffices to calculate the expected gain.

Here is the setting which has successfully used to treat Parrondo games mathematically.

One needs

- a complete metric space M.

  In our case this is the space of probability measures ν on $\{0, \ldots, s-1\}$, provided with the l^1 norm.
- Contractions $\Gamma_1, \ldots, \Gamma_r : M \to M$.

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- And finally there are continuous „reward funcions“ $\phi_\rho : M \to \mathbb{R}$ for $\rho = 1, ..., r$.

  In the present situation these are the functions $\nu \mapsto \langle \nu, \mathbf{w}_\rho \rangle$.

Then one can define the following „game“: Start at any x_0 and choose a ρ_1. This gives the gain $\phi_{\rho_1}(x_0)$, and x_0 is transformed to $x_1 := \Gamma_{\rho_1}(x_0)$. Now choose a ρ_2, the immediate gain is $\phi_{\rho_2}(x_1)$, and the next position is $x_2 := \Gamma_{\rho_2}(x_1)$.
And so on. How should on choose the ρ_μ in order to have a maximal total gain after m rounds of the game?

Now fractals come into play. If F denotes the smallest nontrivial subset of M which is invariant with respect to all Γ_ρ, then this F usually has a fractal structure. Here are some examples which resulted from certain Parrondo games (probability measures on $\{0, 1, 2\}$ are represented with the help of barycentric coordinates as points of a triangle):

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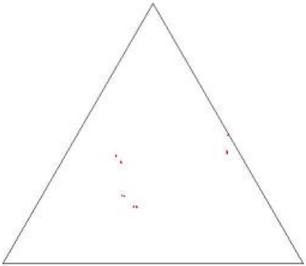

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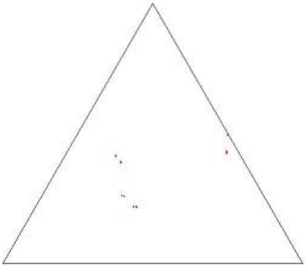

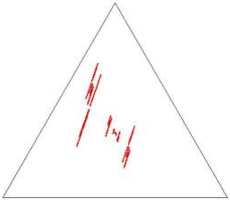

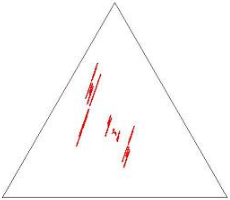

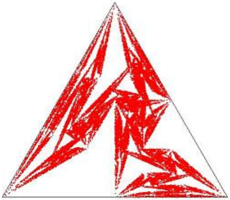

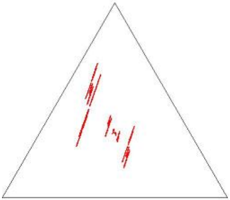

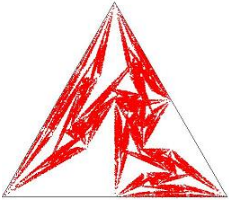

F is important since after some moves only points of M which are close to F are of interest. This is caused by the contraction property of the Γ_ρ.

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What does this help? Suppose that it would be sufficient to find the $\rho_1, \ldots, \rho_m$ such that the optimal gain is assumed up to an error of ε. Since the ϕ_ρ are uniformly continuous and the Γ_ρ are contractions one may replace the x_μ by points which are „sufficiently close“.

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This implies that one may pass to a walk on a finite δ-net on the fractal F.

The starting problem (the optimal choice of the ρ) is transformed in this way to the problem to find an optimal path (maximal total weight) of length m in a directed graph with weights

But for this problem there exists a theory: With an amount of work which is quadratic in the size of the graph one can find an optimal solution, and this consists in a sequence of closed cycles.

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First it is now clear how the fractal dimension of F comes into play: One has to note that small resp. large fractal dimension means that for given $\delta > 0$ one finds δ-nets with few or only with many elements.

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And secondly one now understands why in the first „experimental“ studies of the Parrondo paradox one always has considered periodical changes of games as optimal.

In the original Parrondo game, for example, one has recommended the choice 122122122122. . . .

One can show that with $M, \Gamma_\rho, \phi_\rho$ there is associated a constant γ which can be thought of as „the value of the game“: When playing optimal one can gain $m \cdot \gamma$ in m rounds of the game.

It is, however, very hard to determine γ exactly.

„Greedy“ strategies for collective Parrondo games and stochastical dynamical systems

We consider the same situation as in the last section: M, the Γ_ρ and the ϕ_ρ.

This time we define a random walk as follows:

- We start at a fixed x_0. Choose a ρ_1 such that the „reward“ $\phi_{\rho_1}(x_0)$ is as large as possible. If there are several candidates choose one stochastically.
- Take this ρ_1 for the first step: one moves from x_0 to $x_1 := \Gamma_{\rho_1}(x_0)$.
- Now choose a ρ_2 with maximal $\phi_{\rho_2}(x_1)$. And so on.

In the classical Parrondo situation this „game“ occurs if a large number of players is involved. They decide before each move which game should be taken next: game 1 or game 2? And they choose that possibility which gives rise to a maximal gain in this round. (This is a typical „greedy“ strategy).

M corresponds again to the probability measures on $\{0, \ldots, s-1\}$, the components of an $x \in M$ stand for the fraction of players which are in the correponding states.

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Computer simulations show a rather strange phenomenon: The development of the gain during the game is „somehow periodic“. More precisely: If X_m is the position in round m and G_m the expected gain in the m'th round (i.e., the expectation of $(\max_\rho \phi_\rho)(X_m)$), then the sequence (G_m) is „usually“ periodic in the following sense:

For every $\varepsilon > 0$ there are a number l and an m_0 such that

$$|G_{m+k\cdot l} - G_m| < \varepsilon$$

(all k, all $m \geq m_0$).

To be able to make the restriction „usually“ more precise it is appropriate to introduce a new definition. Let A_ρ denote that subset of M where ϕ_ρ is the largest of the ϕ-functions. These are closed subsets for which the union is M. And then one can prove the following two results:

- If all A_ρ are clopen one has the „periodic“ behaviour which has been experimentally observed.
- There is a counterexample where the G_m vary chaotically: not even the Cesáro limit exists.

The proof uses the fact that in the case of open A_ρ the random walk can be modeled by a Markov chain with finite state space.

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The „key lemma“ is the following assertion:

> If M is a compact metric space such that $M = \bigcup_\rho A_\rho$ with clopen A_ρ ($\rho = 1, \ldots, r$), then there is a $\delta > 0$ with the following property: If $x, y \in M$ are given with $d(x, y) < \delta$, then
>
> $$\{\rho \mid x \in A_\rho\} = \{\rho \mid y \in A_\rho\}.$$

In this way M can be replaced by an η-net, where $\eta \leq \delta$ is so small that $d(x, y) < \eta$ always implies that $|\phi_\rho(x) - \phi_\rho(y)| < \varepsilon$ (for all ρ).

And then the theory of finite Markov chains can be applied: The state space (this is the η-net) splits into a transient part and subsets where the chain operates periodically.

The counterexample is rather technical. M, however, is simple: the unit interval. The fractal which occurs here is the Cantor diskontinuum.

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Thank you for your attention.