

Direct connections on jet groupoids

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This paper is dedicated to the memory of the late Kirill Mackenzie (1951-2020)

Abstract

Direct connections on Lie groupoids are smooth local right inverses of the anchor map which preserve the units. They were introduced by N. Teleman and J. Kubarski as generalized parallel transports on the frame groupoid of a vector bundle. We show that a Lie groupoid over a connected base admitting a direct connection is necessarily a gauge groupoid, and that direct connections generalize principal connections. Viceversa, we show that they induce infinitesimal connections on the associated Lie algebroid, hence linear connections on associated vector bundles, and that the transitive ones are exactly the parallel transports of their induced linear connection. We study the curvature of a direct connection, which measures its failure to be a groupoid morphism, and show that it refines the curvature of the induced infinitesimal connection, thus completing a result announced by J. Kubarski. Finally we characterize the Lie groupoids admitting a flat direct connection as the flat ones.

We then investigate the jet prolongations of direct connections. Using the new notion of jet primitive bisection, we prove that the jet prolongation of a direct connection is entirely determined by that of the trivial direct connection on the pair groupoid (the identity), and we compute the associated infinitesimal connection. We exhibit three natural jet prolongations of the identity on the pair groupoid, built from Jacobi fields of geodesic variations, one of which recovers the exponential jet connection of I. Kolář and W. M. Mikulski. Finally, inspired by the theory of regularity structures, we construct a higher-order direct connection on the frame groupoid of a jet bundle which is not a jet prolongation. For this, we introduce a covariant Taylor expansion of sections of a vector bundle induced by a direct connection. Besides the usual local terms given by covariant derivatives, this expansion features new non-local integral corrections governed by the derivatives of the curvature along geodesics, which vanish exactly when the direct connection is a genuine parallel transport. We conclude with a local model for jet regularity structures on vector bundles based on this covariant Taylor expansion.

Contents

1	Introduction	3
2	Lie groupoids and Lie algebroids	6
2.1	Lie and gauge groupoids	6
2.2	Local maps and local morphisms	9
2.3	Bisections and gauge transformations	10
2.4	Groupoid actions	12
2.5	Lie algebroids	13
3	Direct connections on Lie groupoids	20
3.1	Direct connections and generalized parallel transports	20
3.2	Direct connections on frame groupoids	22
3.3	Infinitesimal connections induced by direct connections	24
3.4	Curvature of direct connections	29
3.5	Flat groupoids	31
4	Direct connections on jet groupoids	34
4.1	Jet prolongation of groupoids	34
4.2	Jet prolongation of Lie algebroids	37
4.3	Jet prolongation of direct connections	39
4.4	Jet prolongation of direct connections on the pair groupoid	42
4.4.1	Parallel jet direct connection	43
4.4.2	Jacobi jet direct connection	45
4.4.3	Exponential jet direct connection	47
4.5	Flat jet groupoids	48
4.6	Taylor expansion with respect to direct connections	49
4.7	Higher-order direct connections and jet regularity structures	59
A	Appendix: Jet bundles and jet prolongations	63
A.1	Jets and Taylor polynomials of smooth local functions	63
A.2	Higher jet groups and higher frame bundles	64
A.3	Higher tangent bundle	65
A.4	Jet prolongation of fibre bundles and vector bundles	66
A.5	Jet prolongation of structure groups	66
A.6	Jet prolongation of principal bundles	67
A.7	Associated jet bundles and reductions	69
A.8	Jets and covariant derivatives	69
	References	70

1 Introduction

The symmetries of a vector bundle are of two different natures. On the one hand, the structure group of the bundle acts along the fibres by relating its different frames, and rules the so-called *internal* (or *gauge*) symmetries of the bundle. On the other hand, many geometric situations require to compare fibres sitting over *distinct* points of the base manifold, and involve the so-called *global* symmetries of the bundle. Such symmetries can no longer be encoded by a group, since they can be composed only when the arrival point of one matches the departure point of the next, and they are naturally described by a *Lie groupoid*. A Lie groupoid $\mathcal{G} \rightrightarrows M$ consists of arrows, each attached to a source and a target in the base manifold M , with the exact properties we need: the arrows with equal source and target form a group above each base point, and code the internal symmetries, the others can be composed whenever the target of the first coincides with the source of the second and they are invertible, so they let the base manifold carry its own symmetries and finally code the global symmetries of the bundle. Groupoids were introduced by H. Brandt in 1926 [Bra26] and endowed with topological and differentiable structures by C. Ehresmann [Ehr52]. They are nowadays pervasive in mathematics, physics and computer science, see for instance [Wei96, Alt21].

A connection is the tool that turns such symmetries into an actual comparison of nearby fibres, by prescribing a parallel transport between them, that is, a specific arrow of the symmetry groupoid with given nearby source and target points. In the classical setting of a principal bundle, a principal connection selects, among the gauge symmetries, those that realize the parallel transport along the paths of the base, and infinitesimally it is described by a Lie-algebra-valued differential form. In the groupoid setting, the infinitesimal symmetries form a *Lie algebroid*, a fibred generalisation of a Lie algebra whose sections generalise the vector fields on the base, and the corresponding infinitesimal notion of connection, due to C. Ehresmann in 1952 [Ehr52] and enhanced by S. Kobayashi in [Kob57], is a section of its anchor map. Integrating these infinitesimal data to a finite parallel transport on the groupoid has traditionally required a path lifting, as in the *path connections* introduced by J. Virsik in 1971 [Vir71, Appendix] and thoroughly studied by K. Mackenzie [Mac05, Chapter 6]. Such a transport therefore depends on the choice of a path on the base manifold.

An alternative notion of connection, defined more generally for a groupoid map $\mathcal{G} \rightarrow \mathcal{H}$ over M which is a surjective submersion, is the *multiplicative Ehresmann connection* recently introduced by R. Loja Fernandes and I. Mărcuț in [LM23] (and in previous works by other authors cited therein), which compares to the path connection when the map is the anchor to the pair groupoid. This is a horizontal distribution on $\mathcal{G}$ complementary to the anchor fibres, devised to study local models of Poisson submanifolds, and for this it is assumed to be compatible with the groupoid composition. It is ruled by specific connection forms and therefore it is essentially an infinitesimal theory. Like the path connections, then, a multiplicative Ehresmann connection produces its parallel transport by the horizontal lifting of paths in the base, and is therefore path-dependent. Unlike a general path connection, however, it is encoded by an intrinsic geometric datum on $\mathcal{G}$ and is by construction compatible with the composition.

Path-independent notions of connection on a Lie groupoid were first developed by A. Kock in the framework of Synthetic Differential Geometry [Koc89, Koc07, Koc17], where they are required to preserve the units and the inversion, and a *curvature* measures the obstruction to preserving the composition. More recently, N. Teleman and J. Kubarski considered a relaxed version of such connections on the frame groupoid of a vector bundle $E \rightarrow M$, no longer preserving the inversion, suitable to capture the geometric content of the Chern classes of E , and called them *linear direct connections* [Tel04, Tel07, KT06]. The generalisation to arbitrary Lie groupoids was announced by Kubarski in [Kub08]. As Teleman observed, and in contrast with path connections, a direct connection need not be smooth, so that it can be defined also on singular or non-smooth base

manifolds. In this paper we take up this point of view and develop the theory of *smooth* direct connections up to the level of jet groupoids and of a new covariant Taylor expansion of sections of a vector bundle.

A *direct connection* on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth local right inverse $\Gamma : \text{Pair}(M) \rightarrow \mathcal{G}$ of the anchor map which preserves the units (Definition 3.1). It is *flat* if it preserves the composition, and *transitive* if it preserves the composition along the geodesics. Its guiding geometric model is the parallel transport of a linear connection on a vector bundle E along the small geodesics of a base manifold M equipped with a torsion-free linear connection: to a pair of nearby points (y, x) it assigns the arrow of $\mathcal{G}$ from x to y coding the transport along the unique geodesic joining them. This is a transitive direct connection which is not necessarily flat. Compared with path connections, a general direct connection dispenses with the choice of a path and need not be compatible with the groupoid composition, hence it encodes geometric informations that are invisible at the infinitesimal level.

In Section 3 we develop the general theory of direct connections on Lie groupoids. We show that a Lie groupoid over a connected base which admits a direct connection is necessarily a gauge groupoid (Proposition 3.4), that every direct connection induces an infinitesimal connection on the associated Lie algebroid (Proposition 3.13) and hence a linear connection on the associated vector bundles (Corollary 3.14), and we characterize the usual parallel transports as the transitive direct connections (Proposition 3.5 and Theorem 3.18). We then introduce the curvature of a direct connection, which measures how far it is from being a groupoid morphism, and show that its antisymmetric part reproduces the usual curvature tensor of the induced infinitesimal connection (Theorem 3.24), while its full expression carries finer, non-infinitesimal invariants. We close the section by characterizing the Lie groupoids admitting a flat direct connection as precisely the flat ones (Proposition 3.26), recovering in our language a class of groupoids largely studied by K. Mackenzie.

In Section 4 we study the jet prolongations of direct connections, building on the jet prolongation of groupoids and Lie algebroids after C. Ehresmann [Ehr55] and I. Kolář [Kol07]. Attaching to each direct connection on a jet groupoid its jet primitive bisection, we prove that the jet prolongation of a direct connection is entirely determined by that of the trivial direct connection Id on the pair groupoid, and compute its associated infinitesimal connection (Theorem 4.11). We then exhibit three explicit jet prolongations of Id on $\text{Pair}(M)$, arising from three Jacobi fields of geodesic variations that differ by the initial value of the field and of its covariant derivative (Theorems 4.15, 4.17 and 4.20). We also compute their infinitesimal connections and show that the first two are, in a sense, orthogonal, while the third reproduces the exponential jet connection on $J^n TM$ of I. Kolář [Kol07] and W. M. Mikulski [Mik07].

The last part of Section 4 is motivated by the theory of regularity structures for stochastic PDEs, introduced by M. Hairer [Hai14b, Hai14a] and further developed on manifolds by A. Dahlqvist, J. Diehl and B. Driver [DDD19] and on vector bundles by M. Hairer and H. Singh [HS23]. Starting from a direct connection Γ on the frame groupoid of a vector bundle E , we introduce a covariant Taylor expansion of local sections of E whose displacement along the fibres is ruled by Γ (Theorem 4.29 and Definition 4.24). Besides the usual local terms given by higher covariant derivatives, this generalized Taylor expansion contains new *non-local* integral corrections based on the derivatives of the curvature of Γ along geodesics. These corrections vanish precisely when Γ is the parallel transport of a linear connection, so they are a genuinely new feature of general direct connections. Using this expansion we build a higher-order direct connection on the frame groupoid of a jet bundle which is not a jet prolongation of a direct connection on E (Theorem 4.34), and we sketch a covariant local model for E -valued jet regularity structures.

The paper is organized as follows. Section 2 recalls the necessary background on Lie groupoids, bisections, groupoid actions and Lie algebroids, with an emphasis on gauge and frame groupoids

and a systematic treatment of local maps and local bisections. Section 3 develops the theory of direct connections on Lie groupoids, and Section 4 their jet prolongations and the associated Taylor expansion. An appendix (Section A) collects the background on jet bundles and on the jet prolongation of principal and vector bundles.

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2 Lie groupoids and Lie algebroids

In this section we recall basic facts and fix our notations on Lie groupoids, their bisections, their natural actions on fibre bundles, and on their infinitesimal counterpart (Lie algebroids). For the purpose of the present work, we only need to focus on gauge groupoids arising from principal bundles and on the frame groupoids of the associated vector bundles. The only original part of this section is the systematic treatment of local maps between groupoids, local bisections, and local actions on vector bundles. Our main references are the standard textbooks by K. Mackenzie [Mac05], I. Moerdijk and J. Mrčun [MM03] and E. Meinrenken [Mei17], and the pedagogical introduction by A. Kumpera [Kum15].

2.1 Lie and gauge groupoids

Let M be a smooth manifold. A smooth **groupoid** on M is a manifold $\mathcal{G}$ together with the following smooth structure maps:

- (i) a **source** $\mathbf{s} : \mathcal{G} \rightarrow M$ and a **target** $\mathbf{t} : \mathcal{G} \rightarrow M$,
- (ii) an associative **multiplication** (or **composition**) $\mathbf{m} : \mathcal{G} \times_M \mathcal{G} \rightarrow \mathcal{G}$, $(g_2, g_1) \mapsto g_2 g_1$, defined on the set of composable elements $\mathcal{G} \times_M \mathcal{G} = \{(g_2, g_1), \mathbf{s}(g_2) = \mathbf{t}(g_1)\}$,
- (iii) a **unit** $\mathbf{u} : M \rightarrow \mathcal{G}$, $x \mapsto \mathbf{u}(x) =: 1_x$ with $\mathbf{t}(1_x) = \mathbf{s}(1_x) = x$ such that $1_{\mathbf{t}(g)} g = g = g 1_{\mathbf{s}(g)}$ for any $g \in \mathcal{G}$,
- (iv) an **inversion** $\mathbf{i} : \mathcal{G} \rightarrow \mathcal{G}$, $g \mapsto g^{-1}$ with $\mathbf{s}(g^{-1}) = \mathbf{t}(g)$ and $\mathbf{t}(g^{-1}) = \mathbf{s}(g)$, such that $g g^{-1} = 1_{\mathbf{t}(g)}$ and $g^{-1} g = 1_{\mathbf{s}(g)}$.

The induced map $\mathbf{a} = (\mathbf{t}, \mathbf{s}) : \mathcal{G} \rightarrow M \times M$ is called the **anchor**. From the axioms it follows that the source and the target are surjective maps, the unit is injective and the inversion is bijective. The manifold M is called the **base** of the groupoid and can be identified with the **set of units** $\mathbf{u}(M) \subset \mathcal{G}$. A groupoid is compactly denoted by $\mathcal{G} \rightrightarrows M$ and the structure maps $(\mathbf{s}, \mathbf{t}, \mathbf{m}, \mathbf{u}, \mathbf{i})$ are tacitly understood.

A group G can be seen as a groupoid $G \rightrightarrows *$ over a single point, with trivial source and target maps. Hence groupoids generalise groups.

Given a groupoid $\mathcal{G} \rightrightarrows M$ and points x, y in M , we use the following notations:

- $\mathcal{G}^x := \mathbf{s}^{-1}(x)$ for the **s-fibre** of x , with restricted target map $\mathbf{t}^x := \mathbf{t}|_{\mathcal{G}^x} : \mathcal{G}^x \rightarrow M$,
- $\mathcal{G}_y := \mathbf{t}^{-1}(y)$ for the **t-fibre** of y , with restricted source map $\mathbf{s}_y := \mathbf{s}|_{\mathcal{G}_y} : \mathcal{G}_y \rightarrow M$,
- $\mathcal{G}_y^x := \mathcal{G}^x \cap \mathcal{G}_y$ for the **(anchor) fibre** of (y, x) , whose elements are sometimes denoted by g_y^x .

Similarly, for $U, V \subset M$, we set $\mathcal{G}^U := \mathbf{s}^{-1}(U)$, $\mathcal{G}_V := \mathbf{t}^{-1}(V)$ and $\mathcal{G}_V^U := \mathcal{G}^U \cap \mathcal{G}_V$.

For any $x \in M$, the set $\mathcal{G}_x^x$ is a (non-empty) group with the composition and unit 1_x , called the **vertex group** (or the **isotropy**) at x . The non emptiness of the set $\mathcal{G}_y^x$ above two distinct points x, y in M defines an equivalence relation ($x \sim y$ if and only if $\mathcal{G}_y^x$ is non empty), whose equivalence classes are called **orbits** of $\mathcal{G}$. The orbit of a point x in M is the set $\mathcal{O}_x = \mathbf{t}^x(\mathcal{G}^x) = \mathbf{s}_x(\mathcal{G}_x) \subset M$. The **orbit space** of $\mathcal{G}$, denoted $M/\mathcal{G}$, is the quotient of M by the relation and gives a foliation of M which is possibly singular. The groupoid $\mathcal{G}$ is **regular** if the orbits have all the same dimension, that is, if the foliation is regular. The groupoid is **transitive** if it has a single orbit M , which happens if and only if the anchor map is surjective.

A smooth groupoid $\mathcal{G} \rightrightarrows M$ is a **Lie groupoid** if the source and the target are surjective submersions. This guarantees the following nice properties [MM03, Theorem 5.4], [Mac05, §1.5]:

1. The units $\mathbf{u}(M) \subset \mathcal{G}$ form a submanifold of $\mathcal{G}$ and the composable elements $\mathcal{G} \times_M \mathcal{G}$ form a manifold.
2. The fibre $\mathcal{G}_y^x$ above each pair of points is a closed submanifold of $\mathcal{G}$ (possibly empty).
3. The vertex group $\mathcal{G}_x^x$ above each point is a Lie group¹.
4. The orbit $\mathcal{O}_x$ of each point is an immersed submanifold of M and the restricted maps $\mathbf{t}^x$ and $\mathbf{s}_x$ are both principal $\mathcal{G}_x^x$ -bundles on $\mathcal{O}_x$.
5. If $\mathcal{G}$ is transitive then the anchor map is a surjective submersion.

Thanks to these properties, Lie groupoids admit an infinitesimal calculus (via Lie algebroids) analogous to that defined on Lie groups (via Lie algebras), and are suitable to study smooth actions on fibre bundles and the related symmetries.

A groupoid can be seen as a small category whose objects are the points of the base manifold and all morphisms are invertible. A morphism between two groupoids is a functor between the (category-theoretic) groupoids. We focus on morphisms between two Lie groupoids $\mathcal{G}$ and $\mathcal{G}'$ over the same base manifold M , called **morphisms over M** . These are smooth maps $\phi : \mathcal{G} \rightarrow \mathcal{G}'$ such that

- (i) $\mathbf{s}' \circ \phi = \mathbf{s}$ and $\mathbf{t}' \circ \phi = \mathbf{t}$,
- (ii) $\phi \circ \mathbf{u} = \mathbf{u}'$,
- (iii) $\phi(gg') = \phi(g)\phi(g')$ for any composable g, g' in $\mathcal{G}$, and therefore $\phi \circ \mathbf{i} = \mathbf{i}' \circ \phi$.

A morphism ϕ is an **isomorphism** of Lie groupoids if the map ϕ is a diffeomorphism. A **subgroupoid** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a groupoid $\mathcal{G}' \rightrightarrows M$ together with an injective Lie groupoid morphism $\iota : \mathcal{G}' \hookrightarrow \mathcal{G}$ over M , giving the inclusion.

Example 2.1 (Trivial Groupoid). Given a Lie group G , the cartesian product $M \times G \times M$ defines a Lie groupoid on M , called **trivial groupoid with vertex group G** , with source $\mathbf{s}(y, g, x) = x$, target $\mathbf{t}(y, g, x) = y$ and composition $(z, h, y)(y, g, x) = (z, hg, x)$. Therefore the unit is $1_x = (x, 1, x)$ and the inverse is $(y, g, x)^{-1} = (x, g^{-1}, y)$. $\square$

A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called **trivial** if it is isomorphic to a trivial groupoid. Any trivial Lie groupoid is clearly transitive.

Example 2.2 (Pair groupoid). The **pair groupoid** of M is the trivial groupoid with trivial vertex group $G = \{1\}$, namely the cartesian product $\text{Pair}(M) = M \times M$, whose elements (y, x) have source x , target y , composition $(z, y)(y, x) = (z, x)$, unit $1_x = (x, x)$ and inverse $(y, x)^{-1} = (x, y)$. The set of units of the pair groupoid coincides with the **diagonal of M** , $\mathbf{u}(\text{Pair}(M)) = \text{Diag}(M) = \{(x, x), x \in M\}$. $\square$

For any Lie groupoid $\mathcal{G} \rightrightarrows M$, the anchor map $\mathbf{a} = (\mathbf{t}, \mathbf{s}) : \mathcal{G} \rightarrow \text{Pair}(M)$ is a Lie groupoid morphism over M , which maps the units of $\mathcal{G}$ onto the units $\text{Diag}(M)$ of $\text{Pair}(M)$, but is not globally surjective in general.

¹However, the vertex groups over distinct points x and y are not necessarily isomorphic, since the fibre $\mathcal{G}_y^x$ can be empty and the vertex groups can belong to separate connected components of $\mathcal{G}$, even if M is connected.

Example 2.3 (Gauge groupoid). Let $\pi : P \rightarrow M$ be a principal bundle with structure group G acting on the fibres from the right, transitively and without fixed points. The **gauge** or **Atiyah groupoid** of P is the quotient manifold

$$\mathcal{G}(P) = P \times_G P := (P \times P) / \sim$$

under the equivalence relation $(p, q) \sim (pg, qg)$ for all $p, q \in P$ and $g \in G$, whose elements are equivalence class of pairs (p, q) in $P \times P$ denoted $[p, q]$. The source and target maps are given by the bundle projection, namely $\mathbf{s}([p, q]) = \pi(q)$ and $\mathbf{t}([p, q]) = \pi(p)$ for all $p, q \in P$. The partial composition is defined by $[p, q][p', q'] = [p, q'g]$ only if $\pi(q) = \pi(p')$, where g is the unique element in G such that $q = p'g$. The units are $1_x = [p, p]$ for any $p \in P_x = \pi^{-1}(x)$, and the inverse of $[p, q]$ is $[q, p]$.

For instance, the trivial groupoid $M \times G \times M$ is the gauge groupoid of the trivial principal bundle $P = M \times G$. This holds in particular for the pair groupoid $\text{Pair}(M)$ whose structure group is trivial.

Gauge groupoids are characterized by a very simple local structure. In fact, if the base manifold M is connected, they are equivalently determined among Lie groupoids $\mathcal{G}$ by any of the following conditions [Kum15, Lemma 2] [Mac05, Propositions 1.3.3 and 1.3.5] [Mei17, Theorem 3.10]:

1. $\mathcal{G}$ is **transitive**, i.e. there is a single orbit M (hence the anchor is a surjective submersion).
2. $\mathcal{G}$ is **locally trivial**, i.e. for any x in M there exists an open neighborhood U of x in M such that $\mathcal{G}_U^U$ is isomorphic to the trivial groupoid $U \times \mathcal{G}_x^x \times U$.
3. $\mathcal{G}$ admits a **section atlas**, i.e. there exists a point $x_0 \in M$ and a collection of local sections of $\mathbf{t}^{x_0} : \mathcal{G}^{x_0} \rightarrow M$, that is, an open covering $\{U_\alpha\}$ of M and smooth maps $\sigma_\alpha : U_\alpha \rightarrow \mathcal{G}_{U_\alpha}^{x_0}$ such that $\mathbf{t}^{x_0} \circ \sigma_\alpha = \text{Id}_{U_\alpha}$, called **local decomposing maps**. This implies that the restriction $\mathbf{t}^{x_0}$ of $\mathbf{t}$ is a surjective submersion (which is not *a priori* ensured in a Lie groupoid), and one can define its **transition functions** $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathcal{G}_{x_0}^{x_0}$ by $g_{\alpha\beta}(x) = \sigma_\alpha(x)^{-1} \sigma_\beta(x)$.
4. The anchor $\mathbf{a} : \mathcal{G} \rightarrow M \times M$ is a locally trivial fibre bundle with fibre $\mathcal{G}_{x_0}^{x_0}$ for any choice of $x_0 \in M$, and structure group $\mathcal{G}_{x_0}^{x_0} \times \mathcal{G}_{x_0}^{x_0}$ acting on the left on the fibres by $(g', g) \cdot h = g'hg^{-1}$, for any $g', g, h \in \mathcal{G}_{x_0}^{x_0}$.
5. The source $\mathbf{s} : \mathcal{G} = \bigcup_{x \in M} \mathcal{G}^x \rightarrow M$ is a locally trivial fibration in principal G -bundles on M , with $G = \mathcal{G}_{x_0}^{x_0}$ for any choice of $x_0 \in M$. The principal $\mathcal{G}_x^x$ -bundle $\mathbf{t}^x : \mathcal{G}^x \rightarrow M$ is called the **vertex bundle** of $\mathcal{G}$ at the point x .

□

Example 2.4 (Frame groupoid). The **frame groupoid** of a vector bundle E is the set of linear isomorphisms $\varphi_y^x : E_x \rightarrow E_y$ between its fibres

$$\text{GL}(E) = \bigcup_{x, y \in M} \text{GL}(E_x, E_y) \rightrightarrows M$$

with source $\mathbf{s}(\varphi_y^x) = x$, target $\mathbf{t}(\varphi_y^x) = y$ and the usual composition $\eta_z^y \circ \varphi_y^x : E_x \rightarrow E_y \rightarrow E_z$ of linear maps. The frame groupoid is isomorphic to the gauge groupoid of the **frame bundle** $FE \rightarrow M$ of E . If E has fibre $\mathbb{R}^r$, this is the principal $GL_r(\mathbb{R})$ -bundle with fibre $F_x E = \text{GL}(\mathbb{R}^r, E_x)$ given by the set of linear isomorphisms $\varphi_x : \mathbb{R}^r \rightarrow E_x$ (the **frames** of E_x) and projection $\pi(\varphi_x) = x$. The isomorphism

$$\mathcal{G}(FE) \cong \text{GL}(E) \rightrightarrows M$$

is given by $[\varphi_y, \varphi_x] \mapsto \varphi_y^x = \varphi_y \circ (\varphi_x)^{-1} : E_x \rightarrow \mathbb{R}^r \rightarrow E_y$.

If $E = P \times_G \mathbb{R}^r$ is the vector bundle associated to a principal G -bundle P , through a linear representation of G on $\mathbb{R}^r$, then the structure group $GL_r(\mathbb{R})$ of E can be reduced to G and $P \rightarrow FE$ is a reduction of the frame bundle. This means that each element $p \in P_x$ can be seen as a frame $p : \mathbb{R}^r \rightarrow E_x$. Then, there is an associated groupoid map $\mathcal{G}(P) \rightarrow GL(E)$, which is injective if and only if the representation $G \rightarrow GL_r(\mathbb{R})$ is faithful. $\square$

2.2 Local maps and local morphisms

Being interested in jet prolongations of maps defined on groupoids, we will need to consider maps possibly defined only locally. In a fibre bundle, an object is *local* if it is defined in an open neighborhood of a base point. For a groupoid, this notion must be adapted to the fact that its very essence is to relate distinct base points: *locality* then means that the points to be related are sufficiently close to each other, wherever they are in the base manifold. This leads to the following definition, which extends that of a *local morphism* given in [Mac05, Definition 6.1.6].

Definition 2.5. A **local map** between two Lie groupoids $\mathcal{G}$ and $\mathcal{G}'$ over the same manifold M is a smooth map $\phi : \mathcal{U} \subset \mathcal{G} \rightarrow \mathcal{G}'$ defined on an open neighbourhood $\mathcal{U}$ of the units $\mathbf{u}(M) \subset \mathcal{G}$, which preserves the source, the target and the units, that is,

- (i) $\mathbf{s}' \circ \phi = \mathbf{s}$ and $\mathbf{t}' \circ \phi = \mathbf{t}$,
- (ii) $\phi \circ \mathbf{u} = \mathbf{u}'$.

If $\mathcal{U} = \mathcal{G}$, we call ϕ a **global map**. To distinguish local from global maps at a glance, we denote local maps by $\phi : \mathcal{G} \rightarrow \mathcal{G}'$. The local map ϕ is a **local morphism** of Lie groupoids if it also preserves compositions, i.e.

- (iii) $\phi(g'g) = \phi(g')\phi(g)$ for any composable g, g' in $\mathcal{U}$ whose product $g'g$ lies in $\mathcal{U}$.

In this case, it also preserve inversions, i.e. $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in \mathcal{U}$ such that g^{-1} lies in $\mathcal{U}$. We denote local morphisms by $\phi : \mathcal{G} \rightarrow \mathcal{G}'$, as in [Mac05, Definition 6.1.6]. A global morphism in this sense is the same as a groupoid morphism of Section 2.1.

Example 2.6 (Local maps on the pair groupoid). Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A local map $\phi : \text{Pair}(M) \rightarrow \mathcal{G}$ is defined on an open neighborhood $\mathcal{U}$ of the diagonal $\text{Diag}(M)$, that we shall call a **diagonal domain**² in $M \times M$. Since $\phi(\text{Diag}(M)) = \mathbf{u}(M) \subset \mathcal{G}$, the image $\phi(\mathcal{U}) \subset \mathcal{G}$ is a neighborhood of the units $\mathbf{u}(M) \subset \mathcal{G}$ (not necessarily open). Since ϕ is invertible on $\text{Diag}(M)$, by the local inverse theorem there exists a diagonal domain $\mathcal{V} \subset \mathcal{U}$ such that $\phi(\mathcal{V})$ is a homeomorphic image of $\mathcal{V}$ in $\mathcal{G}$ containing $\mathbf{u}(M)$. $\square$

A local map between two Lie groupoids cannot always be extended to a global one, even if it is a local groupoid morphism. Mackenzie proved in [Mac05, Theorem 6.1.10] that this is possible under rather restrictive conditions, namely when $\mathcal{G} \rightarrow \mathcal{G}'$ is a local groupoid morphism of locally trivial Lie groupoids over the same base manifold M , if the source-fibres of $\mathcal{G}$ are connected and simply connected (or alternatively for the target-fibres). In particular, if M is connected and simply connected, any local groupoid morphism $\phi : \text{Pair}(M) \rightarrow \mathcal{G}$ over M can be extended to the whole pair groupoid (but of course not any local map $\phi : \text{Pair}(M) \rightarrow \mathcal{G}$).

²This terminology is borrowed from [DDD19, Def. 83]. Diagonal domains are called first neighbourhoods of the diagonal in [Koc17, §1].

2.3 Bisections and gauge transformations

Bisections of Lie groupoids play the role of sections of principal bundles, with the essential feature that they reproduce not only groups of vertical transformations acting along the fibres (the gauge groups), but also groups of horizontal transformations of the base manifold. For the purpose of this work, local bisections are the key notion to define jet groupoids.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A **local bisection** of $\mathcal{G}$ is a smooth local section $\sigma : M \multimap \mathcal{G}$ of the source $\mathbf{s}$ defined on an open subset $U \subset M$, that is, $\mathbf{s} \circ \sigma = \text{Id}_U$, whose composition with the target $\mathbf{t}$ is a diffeomorphism $\varphi_\sigma = \mathbf{t} \circ \sigma$ between U and an open subset $\varphi_\sigma(U) \subset M$ (which might not intersect U). If $U = \varphi_\sigma(U) = M$, σ is a **(global) bisection**. The space $\mathcal{B}(\mathcal{G})$ of global bisections of $\mathcal{G}$ is a group with the following operations:

- (i) **multiplication** of σ and σ' given by $(\sigma' \bullet \sigma)(x) = \sigma'(\varphi_\sigma(x)) \sigma(x)$ for any $x \in M$, which corresponds to a semidirect product law on pairs (φ_σ, σ) ,
- (ii) **unit bisection** given by the unit map $\mathbf{u} : M \rightarrow \mathcal{G}$ of the groupoid $\mathcal{G}$,
- (iii) **inverse** σ^{-1} given by $\sigma^{-1}(x) = (\sigma(\varphi_\sigma^{-1}(x)))^{-1}$ for any $x \in M$, where φ_σ^{-1} denotes the inverse diffeomorphism of M while the external $()^{-1}$ denotes the inverse in the groupoid $\mathcal{G}$.

See [SW15] (and in particular Theorem 2.8) for the topological properties of the group $\mathcal{B}(\mathcal{G})$. The subspace $\mathcal{B}_M(\mathcal{G})$ of bisections $\sigma : M \rightarrow \mathcal{G}$ such that $\varphi_\sigma = \text{Id}_M$ is a subgroup of $\mathcal{B}(\mathcal{G})$. The map

$$\varphi : \mathcal{B}(\mathcal{G}) \rightarrow \text{Diff}(M), \quad \sigma \mapsto \varphi_\sigma = \mathbf{t} \circ \sigma \quad (1)$$

is a group homomorphism to the group of diffeomorphisms of M (not necessarily surjective), which sends $\mathcal{B}_M(\mathcal{G})$ to $\{\text{Id}_M\}$. In fact, $\sigma' \bullet \sigma = (\sigma' \circ \varphi_\sigma) \sigma$ (seen as a pointwise groupoid multiplication), and therefore

$$\varphi_{\sigma' \bullet \sigma} = \mathbf{t} \circ ((\sigma' \circ \varphi_\sigma) \sigma) = \mathbf{t} \circ (\sigma' \circ \varphi_\sigma) = (\mathbf{t} \circ \sigma') \circ \varphi_\sigma = \varphi_{\sigma'} \circ \varphi_\sigma.$$

The space of local bisections $\mathcal{B}_{\text{loc}}(\mathcal{G})$ is a pseudo-group with the above operations, because the product $\sigma' \bullet \sigma$ of two local sections defined respectively on two open sets U' and U is defined only if the target space $\varphi_\sigma(U)$ of σ has a non-empty intersection with the source space U' of σ' . The subspace $\mathcal{B}_{M,\text{loc}}(\mathcal{G})$ of local bisections $\sigma : U \rightarrow \mathcal{G}_U^U$ such that $\varphi_\sigma = \text{Id}_U$ is a sub-pseudo-group of $\mathcal{B}_{\text{loc}}(\mathcal{G})$. The map in Eq. (1) descends to a pseudo-group homomorphism

$$\varphi : \mathcal{B}_{\text{loc}}(\mathcal{G}) \rightarrow \text{Diff}_{\text{loc}}(M), \quad \sigma \mapsto \varphi_\sigma = \mathbf{t} \circ \sigma$$

from local bisections of $\mathcal{G}$ to the set of diffeomorphisms between open subsets of M , that we improperly call *local diffeomorphisms*, which sends $\mathcal{B}_{M,\text{loc}}(\mathcal{G})$ to the collection $\text{Id}_{\text{loc}}(M) = \{\text{Id}_U, U \subset M \text{ open}\}$ of local identity maps.

To summarize, we have the following group or pseudo-group maps between (local) bisections of $\mathcal{G}$ and (local) diffeomorphisms of M :

$$\begin{array}{ccc} \mathcal{B}_M(\mathcal{G}) & \hookrightarrow & \mathcal{B}(\mathcal{G}) \\ \downarrow & & \downarrow \\ \{\text{Id}_M\} & \hookrightarrow & \text{Diff}(M) \end{array} \qquad \begin{array}{ccc} \mathcal{B}_{M,\text{loc}}(\mathcal{G}) & \hookrightarrow & \mathcal{B}_{\text{loc}}(\mathcal{G}) \\ \downarrow & & \downarrow \\ \text{Id}_{\text{loc}}(M) & \hookrightarrow & \text{Diff}_{\text{loc}}(M) \end{array}$$

Example 2.7 (Bisections of gauge groupoids and gauge transformations). Given a principal G -bundle $P \rightarrow M$, a **principal automorphism** of P is a bundle isomorphism $\Phi : P \rightarrow P$ above a smooth diffeomorphism $\varphi : M \rightarrow M$ (that is, if $p \in P_x$ for some $x \in M$, then $\Phi(p) \in P_{\varphi(x)}$) such that $\Phi(pg) = \Phi(p)g$ for any $g \in G$. Principal automorphisms can be composed and their set $\text{Aut}(P)$ forms a group. Principal automorphisms above Id_M are usually called **gauge transformations** and form a subgroup $\text{Aut}_M(P)$. All such automorphisms can be considered in a *local* version, defined on the subbundle P_U above open subsets $U \subset M$. The collections $\text{Aut}_{\text{loc}}(P)$ and $\text{Aut}_{M,\text{loc}}(P)$ of local automorphisms form two pseudo-groups with the composition. The elements of $\text{Aut}_{M,\text{loc}}(P)$ are usually called **local gauge transformations**. A principal automorphism $\Phi \in \text{Aut}(P)$ can be identified with the bisection of the gauge groupoid $\mathcal{G}(P)$ given by

$$\sigma_\Phi : M \rightarrow \mathcal{G}(P), \quad x \mapsto \sigma_\Phi(x) = [\Phi(p), p] \quad \text{for any choice of } p \in P_x.$$

This gives a series of group and pseudo-group isomorphisms [Mei17, Examples 3.5]

$$\begin{aligned} \text{Aut}(P) &\cong \mathcal{B}(\mathcal{G}(P)) & \text{Aut}_{\text{loc}}(P) &\cong \mathcal{B}_{\text{loc}}(\mathcal{G}(P)) \\ \text{Aut}_M(P) &\cong \mathcal{B}_M(\mathcal{G}(P)) & \text{Aut}_{M,\text{loc}}(P) &\cong \mathcal{B}_{M,\text{loc}}(\mathcal{G}(P)) \end{aligned}$$

which recover the (local) gauge transformations of P as (local) bisections of $\mathcal{G}(P)$ fixing M . In particular, we have $\mathcal{B}(\text{Pair}(M)) \cong \text{Diff}(M)$ and $\mathcal{B}_{\text{loc}}(\text{Pair}(M)) \cong \text{Diff}_{\text{loc}}(M)$, via the correspondence $\sigma(x) = (\varphi_\sigma(x), x)$. $\square$

A smooth (global) map $\phi : \mathcal{G} \rightarrow \mathcal{G}'$ between two Lie groupoids over M induces a pushforward on their bisections

$$\mathcal{B}(\phi) = \phi_* : \mathcal{B}(\mathcal{G}) \longrightarrow \mathcal{B}(\mathcal{G}'), \quad \sigma \mapsto \phi \circ \sigma, \quad (2)$$

above the same diffeomorphism of M , i.e. $\varphi_{\phi \circ \sigma} = \varphi_\sigma$. Moreover, $\mathcal{B}(\phi)$ is a group morphism if ϕ is a groupoid morphism, in fact

$$\begin{aligned} \mathcal{B}(\phi)(\sigma' \bullet \sigma)(x) &= \phi(\sigma'(\varphi_\sigma(x))\sigma(x)) \\ &= \phi(\sigma'(\varphi_\sigma(x))\phi(\sigma(x))) = (\mathcal{B}(\phi)(\sigma') \bullet \mathcal{B}(\phi)(\sigma))(x), \end{aligned}$$

for any σ, σ' in $\mathcal{B}(\mathcal{G})$ and for any x in M . If $\phi : \mathcal{G} \rightarrow \mathcal{G}'$ is a local map, the composite map $\phi \circ \sigma$ is only defined for a local bisection σ of $\mathcal{G}$ taking values in the domain of ϕ . Therefore, if ϕ is defined on a neighborhood $\mathcal{U}$ of the diagonal $\mathbf{u}(M) \subset \mathcal{G}$, the pushforward

$$\mathcal{B}_{\text{loc}}(\phi) = \phi_* : \mathcal{B}_{\text{loc}}(\mathcal{G}) \longrightarrow \mathcal{B}_{\text{loc}}(\mathcal{G}'), \quad \sigma \mapsto \phi \circ \sigma \quad (3)$$

is a local map defined on the subspace

$$\mathcal{B}_{\text{loc}}(\mathcal{U}) = \{\sigma \in \mathcal{B}_{\text{loc}}(\mathcal{G}) \mid \sigma \text{ is defined on } U \subset M \text{ such that } \sigma(U) \subset \mathcal{U}\}. \quad (4)$$

This is an open neighborhood of the unit bisection $\mathbf{u}$ in $\mathcal{B}_{\text{loc}}(\mathcal{G})$, endowed with the graph topology [Mic80, §4.4] or with the compact open topology [SW15, Definition A.6], because the set of diffeomorphisms φ_σ is open in smooth maps [Mic80, Corollary 5.7] and the map (1) sending bisections to diffeomorphisms is continuous.

2.4 Groupoid actions

Groupoids naturally act on fibre bundles both with fibrewise transformations (as their structure groups) and between fibres on distinct points (as a parallel transport induced by connections).

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $\pi : E \rightarrow M$ a fibre bundle. A left **action** of $\mathcal{G}$ on E is a map [Mac05, Definition 1.6.1], [MM03, §5.3. Semi-direct products]

$$\rho : \mathcal{G} \times_M E \longrightarrow E, (g, e) \longmapsto g \cdot e \quad (5)$$

defined on the set of **composable** pairs in $\mathcal{G} \times E$, namely

$$\mathcal{G} \times_M E := \{(g, e) \in \mathcal{G} \times E, s(g) = \pi(e)\} = \bigcup_{e \in E} \mathcal{G}^{\pi(e)} \times \{e\},$$

such that

- (i) ρ respects the fibres, i.e. $\pi(g \cdot e) = t(g)$ for any $e \in E$ and any $g \in \mathcal{G}$,
- (ii) ρ is unital, i.e. $1_{\pi(e)} \cdot e = e$ for any $e \in E$,
- (iii) ρ is associative, i.e. $g' \cdot (g \cdot e) = (g'g) \cdot e$ for any $e \in E$, any $g \in \mathcal{G}$ composable with e and any $g' \in \mathcal{G}$ composable with g .

In other words, for any $x, y \in M$, the action map $\mathcal{G}_y^x \times E_x \rightarrow E_y$ relates the fibres of E above x to those above y , in pretty the same way as the structure group of the bundle acts on each fibre.

If E is a vector bundle, the action ρ is called **linear** if, for any $g \in \mathcal{G}_y^x$, the induced map $\rho_g : E_x \rightarrow E_y$ is linear. Since g is invertible in $\mathcal{G}$, the map ρ_g is invertible, and therefore it belongs to $\text{GL}(E_x, E_y)$. In other words, linear actions of $\mathcal{G}$ on E are equivalent to groupoid morphisms $\mathcal{G} \rightarrow \text{GL}(E)$ over M [Mac05, Definition 1.7.1] [Mei17, §5.3]. In this case, the bundle E is also called a **linear representation** of the groupoid $\mathcal{G}$. As usual, the representation is called **faithful** if the corresponding groupoid morphism $\mathcal{G} \rightarrow \text{GL}(E)$ is injective.

Definition 2.8. An action $\rho : \mathcal{G} \times_M E \rightarrow E$ is **local** if it is defined on a subspace $\mathcal{U} \times_M E$ of $\mathcal{G} \times_M E$ where $\mathcal{U}$ is an open neighborhood of the units $\mathbf{u}(M) \subset \mathcal{G}$. A local action is denoted by $\rho : \mathcal{G} \times_M E \circrightarrow E$. If E is a vector bundle, a local linear action ρ is equivalent to a local groupoid morphism $\mathcal{G} \circrightarrow \text{GL}(E)$.³

Examples 2.9. Let $E \rightarrow M$ be a vector bundle.

- (a) The frame groupoid $\text{GL}(E) \rightrightarrows M$ has a natural faithful linear representation on E given by the **evaluation**

$$\text{ev} : \text{GL}(E) \times_M E \rightarrow E, (\varphi_y^x, e_x) \mapsto \varphi_y^x(e_x) =: \varphi_y^x \cdot e_x. \quad (6)$$

- (b) If the structure group of E reduces to a Lie group G , let $P \rightarrow M$ be the principal G -bundle obtained by reduction of the frame bundle FE . Then the gauge groupoid $\mathcal{G}(P)$ acts linearly on E , with action $\rho : \mathcal{G}(P) \times_M E \rightarrow E$ given by composing the map $\mathcal{G}(P) \rightarrow \text{GL}(E)$ of Example 2.4 with the evaluation $\text{ev} : \text{GL}(E) \times_M E \rightarrow E$.

³Note that local groupoid maps $\mathcal{G} \rightarrow \text{GL}(E)$ are equivalent to *non-associative* linear actions $\mathcal{G} \times_M E \rightarrow E$, that is, maps of the form (5) satisfying the axioms (i) and (ii) above, but not (iii).

- (c) A linear representation of the pair groupoid $\text{Pair}(M)$ on E is equivalent to a global trivialization of E , because all fibres are identified by the linear isomorphisms ρ_y^x of the associated groupoid morphism $\rho : \text{Pair}(M) \rightarrow \text{GL}(E)$, cf. [Mei17, §5.3]. If M is connected and simply connected, a local linear representation $\rho : \text{Pair}(M) \rightarrow \text{GL}(E)$ is enough to trivialize E , either by extending it to a global one as at the end of Section 2.2, or by identifying far away fibres by composing isomorphisms among nearby fibres covered by the diagonal domain of ρ .

□

2.5 Lie algebroids

Lie algebroids are fibred analogues of Lie algebras over a manifold, and play for Lie groupoids the role that Lie algebras play for Lie groups. They were introduced by J. Pradines in [Pra67] and are nowadays used, for instance, to study foliations [MM03], Poisson geometry [Mar06] and sigma models in string theory [BKS05]. We are mainly concerned with the Lie algebroid defined by a Lie groupoid, but there are many other interesting examples of Lie algebroids which go beyond our scope, cf. [Mac05], [Mei17] and [MM03].

A **Lie algebroid** on a manifold M is a vector bundle $q : A \rightarrow M$ endowed with

- (i) a vector bundle map $a : A \rightarrow TM$ over M called the **anchor**, which can be seen as a $C^\infty(M)$ -map from the sections of A to the vector fields on M ,
- (ii) a Lie bracket $[\cdot, \cdot]_A : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ on its space of sections (that is, an antisymmetric $\mathbb{R}$ -bilinear map satisfying the Jacobi identity), which satisfies the Leibniz rule

$$[\xi, f\eta]_A = f[\xi, \eta]_A + a(\xi)(f)\eta$$

for any $\xi, \eta \in \Gamma(A)$ and $f \in C^\infty(M)$, that is, the bracket $[\cdot, \cdot]_A$ is a derivation.

It then follows that the anchor is a morphism of Lie algebras from $\Gamma(A)$ to the space $\mathfrak{X}(M)$ of vector fields on M , i.e. $a([\xi, \eta]_A) = [a(\xi), a(\eta)]$. The Lie algebroid A is **regular** if a has constant rank, it is **transitive** if a is fibrewise surjective and it is **totally intransitive** if $a = 0$. A Lie algebroid on M is often denoted by its anchor map $A \rightarrow TM$, instead of its bundle map to M .

A **morphism** between two Lie algebroids $A \rightarrow TM$ and $A' \rightarrow TM$ on the same base manifold is a morphism of vector bundles $\phi : A \rightarrow A'$ which commutes with the anchors and preserves the brackets.

Example 2.10 (Atiyah algebroid of a principal bundle). The Atiyah algebroid of a principal G -bundle $\pi : P \rightarrow M$ plays the role of the Lie algebra of the Lie group G : it contains the right invariant vector fields on P , which are then determined by their value on any chosen element p_x of each fibre (which represents the unit in G above the point $x \in M$).

The key fact to implement this analogy is that the group G acts on the tangent bundle $TP \rightarrow P$, with action obtained by differentiating the principal action on P . The space of sections of the quotient bundle

$$A(P) := TP/G \rightarrow M \tag{7}$$

is isomorphic to that of G -invariant vector fields on P [Mac05, §3.5], and therefore it is closed under the Lie bracket of vector fields. Hence $A(P)$ is a Lie algebroid, called the **Atiyah algebroid** of P , with anchor $a : A(P) \rightarrow TM$ induced by $d\pi : TP \rightarrow TM$. The anchor is fibrewise surjective, hence $A(P)$ is transitive. The quotient map $\natural : TP \rightarrow TP/G = A(P)$ is a fibrewise isomorphism

between the two bundles $TP \rightarrow P$ and $A(P) \rightarrow M$ over different base manifolds, and therefore it is fibrewise invertible with multi-valued inverse $\natural^{-1} : A(P) \rightarrow \bigcup_{p \in P} T_p P$. The Atiyah algebroid $A(P)$ is also called the **bundle of connections** of P in [Kob57], because sections of $d\pi : TP \rightarrow TM$ determine horizontal distributions in TP , and the G -invariant ones, which are equivalent to principal connections on P , are exactly the image under $\natural^{-1}$ of the sections of the anchor $a : A(P) \rightarrow TM$.

Being transitive, the Atiyah algebroid fits into a short exact sequence of Lie algebroids over M , called the **Atiyah sequence**,

$$0 \longrightarrow \text{Ker}(a) \cong (P \times \mathfrak{g})/G =: \text{Ad}(P) \longrightarrow A(P) = TP/G \xrightarrow{a} TM \longrightarrow 0, \quad (8)$$

obtained by taking the quotient under the right G -action of the analogue sequence

$$0 \longrightarrow VP = \ker(d\pi) \cong P \times \mathfrak{g} \longrightarrow TP \xrightarrow{d\pi} TM \longrightarrow 0,$$

where the action of G on $P \times \mathfrak{g}$ is obtained by differentiating the action of G on $P \times G$ given by $(p, g) \cdot h = (ph, h^{-1}gh)$ for p in P and g, h in G , cf. [Mac05, §3.2], [MM03, §6.4]. The kernel $\text{Ad}(P)$ of the anchor is the so-called **adjoint bundle** of P and it is a totally intransitive Lie algebroid. It is in fact a **Lie algebra bundle** on M , that is, a bundle with Lie algebras as fibres (isomorphic to $\mathfrak{g}$), which is fibrewise isomorphic to the vertical subbundle $VP \subset TP$. $\square$

Unlike for principal bundles, a Lie groupoid $\mathcal{G} \rightrightarrows M$ does not globally act on itself, nor on its tangent bundle $T\mathcal{G} \rightarrow \mathcal{G}$, because the right multiplication by a given element $g \in \mathcal{G}$ is only defined on the subspace $\mathcal{G}^{\mathbf{t}(g)}$ with source depending on the target of g , and defines a diffeomorphism $\mathcal{G}^{\mathbf{t}(g)} \rightarrow \mathcal{G}^{\mathbf{s}(g)}$ between $\mathbf{s}$ -fibres of $\mathcal{G}$. To obtain a subspace of $\mathcal{G}$ -invariant vector fields, determined by their value at the units $1_x \in \mathcal{G}$ and giving rise to a true bundle over M , the suitable analogue of TP is the **s-tangent** subbundle of $T\mathcal{G}$

$$T^s\mathcal{G} := \ker(ds : T\mathcal{G} \rightarrow TM) = \bigcup_{g \in \mathcal{G}} T_g \mathcal{G}^{\mathbf{s}(g)} \longrightarrow \mathcal{G},$$

made of vectors tangent to the $\mathbf{s}$ -fibres, which inherits the differential of the target $dt|_{T^s\mathcal{G}} : T^s\mathcal{G} \rightarrow TM$ serving as anchor. Indeed, $\mathcal{G}$ acts on $T^s\mathcal{G}$ on the right and the space of sections of the quotient bundle $T^s\mathcal{G}/\mathcal{G}$ on M is isomorphic to that of right-invariant vector fields on $\mathcal{G}$ (with values necessarily in $T^s\mathcal{G}$) and therefore it is closed for the Lie bracket [Mac05, §3.5] [MM03, §4.1]⁴. Finally, the **Lie algebroid** of $\mathcal{G}$ is the quotient bundle $T^s\mathcal{G}/\mathcal{G}$ on M , which can also be described as the pull-back of $T^s\mathcal{G} \rightarrow \mathcal{G}$ along the embedding $\mathbf{u} : M \rightarrow \mathcal{G}$, that is,

$$A(\mathcal{G}) := T^s\mathcal{G}/\mathcal{G} \cong \mathbf{u}^*(T^s\mathcal{G}) = \bigcup_{x \in M} T_{1_x} \mathcal{G}^x.$$

The bundle projection onto M is given by the source, with constant value x on the fibre $T_{1_x} \mathcal{G}^x$. The anchor is the composite map $a = dt|_{T^s\mathcal{G}} \circ i : A(\mathcal{G}) \rightarrow TM$, where $i : \mathbf{u}^*(T^s\mathcal{G}) \rightarrow T^s\mathcal{G} \subset T\mathcal{G}$ is the natural map on the pull-back. The Lie bracket on $\Gamma(A(\mathcal{G}))$ is that of right-invariant vector fields on $\mathcal{G}$.

From the definition it follows that any vector $Y \in A_x(\mathcal{G})$ determines a $\mathbf{t}$ -horizontal lift of $a(Y) = X \in T_x M$ to $T^s\mathcal{G}$ above any $g \in \mathcal{G}^x$, namely the vector

$$\natural_g^{-1}(Y) \in T_g \mathcal{G}^x \subset T^s\mathcal{G},$$

⁴An equivalent definition of the Lie algebroid $A(\mathcal{G})$ is given by [Mei17, §9.2] or again [MM03, §4.1] as the *normal bundle* $T\mathcal{G}|_{\mathbf{u}(M)}/Tu(M)$ of M in $\mathcal{G}$ (cf. [Mei17, §8.2]).

where $\natural^{-1} : A(\mathcal{G}) \rightarrow T^s\mathcal{G}$ is the multi-valued inverse of the quotient map of bundles

$$\begin{array}{ccc} T^s\mathcal{G} & \xrightarrow{\natural} & T^s\mathcal{G}/\mathcal{G} = A(\mathcal{G}) \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{t} & M \end{array},$$

which is a fibrewise isomorphism.

Remark 2.11. As recalled in Example 2.3, transitive Lie groupoids $\mathcal{G}$ on a connected base manifold M are equivalent to gauge groupoids $\mathcal{G}(P)$. The analogue property for Lie algebroids does not hold: there are transitive Lie algebroids A which are *not* Atiyah algebroids $A(P)$ of a principal bundle, cf. [Mei17, Example 6.9]. This is related to the *integrability* of a Lie algebroid A to a Lie groupoid $\mathcal{G}$ such that $A = A(\mathcal{G})$: the cohomological obstruction to integrability is described by C. Mackenzie in [Mac05, Ch. 8] for transitive Lie algebroids and by M. Crainic and R. Fernandes in [CF03] for general ones, see also [Mei17, §10.2]. $\square$

If a Lie algebroid $A(\mathcal{G})$ is *assumed* to come from a Lie groupoid, then the transitivity of $A(\mathcal{G})$ (on a connected base manifold) implies the transitivity of $\mathcal{G}$ [Mac05, Corollary 3.5.18]. Then $\mathcal{G} = \mathcal{G}(P)$ for a principal bundle P and one can compare $A(\mathcal{G}(P))$ to $A(P)$.

Examples 2.12 (Lie algebroids of gauge groupoids). The Lie algebroid of a gauge groupoid $\mathcal{G}(P) \rightrightarrows M$ is isomorphic to the Atiyah algebroid of the principal bundle P [Mei17, Example 9.5 (c)], i.e.

$$A(\mathcal{G}(P)) \cong A(P) \rightarrow TM. \quad (9)$$

Therefore gauge groupoids give rise to various classical examples of Lie algebroids, which all play a role in the theory of connections.

- (a) The Lie algebroid of the pair groupoid $\text{Pair}(M) \rightrightarrows M$ is isomorphic to the tangent bundle $TM \rightarrow M$ equipped with the vector field brackets and the identity $TM \rightarrow TM$ as anchor, i.e. $A(\text{Pair}(M)) \cong TM$ [Mac05, Example 3.5.11].
- (b) If $P = M \times G$ is the trivial principal G -bundle, then $\mathcal{G}(P) = M \times G \times M$ is the trivial groupoid and $A(M \times G \times M) \cong A(M \times G) \cong TM \oplus (M \times \mathfrak{g})$ is called the **trivial Lie algebroid** with structure Lie algebra $\mathfrak{g} = \text{Lie}(G)$. The anchor is given by the projection to TM , and the bracket is

$$[X + v, Y + w] = [X, Y]_{\mathfrak{X}(M)} + (X(w) - Y(v) + [v, w]_{\mathfrak{g}}),$$

where $X, Y \in \mathfrak{X}(M)$, $v, w \in C^\infty(M, \mathfrak{g})$ and $X(w)$ denotes the Lie derivative of w with respect to X [Mac05, Example 3.5.13]. When $G = \{1\}$, this gives back $A(\text{Pair}(M)) \cong TM$.

- (c) Let G and H be two (unrelated) Lie groups, and let Q be a principal H -bundle over M . Then $P = Q \times G$ is a principal $(H \times G)$ -bundle over M , and its gauge groupoid is

$$\mathcal{G}(P) = (Q \times G) \times_{H \times G} (Q \times G) \cong \mathcal{G}(Q) \times G, \quad (10)$$

with structure maps inherited by $\mathcal{G}(Q)$ and multiplication extended to G . Its Lie algebroid is

$$A(P) \cong A(Q) \oplus (M \times G), \quad (11)$$

and *not* $A(Q) \oplus (Q \times G)$, because on the tangent bundle $TP \cong TQ \oplus (Q \times \mathfrak{g})$ the group $H \times G$ acts on both factors of $Q \times \mathfrak{g}$ (and on TQ it acts by H). The anchor is given by that of $A(Q)$ and the bracket is

$$[\xi + v, \eta + w]_{A(P)} = [\xi, \eta]_{A(Q)} + (a(\xi)(w) - a(\eta)(v) + [v, w]_{\mathfrak{g}}),$$

where $\xi, \eta \in \Gamma(A(Q))$, $v, w \in C^\infty(M, \mathfrak{g})$ and $a(\xi) \in \mathfrak{X}(M)$ is the anchor of $A(Q)$. Note that the previous example of the trivial bundle $P = M \times G$ corresponds to $H = \{1\}$, not $G = \{1\}$.

- (d) If $\pi : E \rightarrow M$ is a vector bundle, the Lie algebroid of its frame bundle FE and of the frame groupoid $\text{GL}(E)$ is the **bundle of linear vector fields of E** , $T^{\text{lin}}E \rightarrow M$, with fibre above x given by the linear sections of the canonical projection $TE|_{E_x} \rightarrow E_x$, that is, smooth maps $\xi_x : E_x \rightarrow TE|_{E_x}$ of the form $\xi_x(e) = \frac{d}{dt}(e + t\xi_x)|_{t=0}$ for $e \in E_x$. The projection of ξ_x to M through π is then a well defined tangent vector $X_{\xi_x} = d\pi_e(\xi_x(e)) = \frac{d}{dt}(\pi(e + t\xi_x))|_{t=0}$, independent of the choice of $e \in E_x$. The assignment $a(\xi_x) = X_{\xi_x} \in T_x M$ defines the anchor $a : T^{\text{lin}}E \rightarrow TM$.

In other words, the fibre of $T^{\text{lin}}E$ at $x \in M$ is made of pairs of maps (ξ_x, X_{ξ_x}) which make the following diagram commute

$$\begin{array}{ccc} E_x & \xrightarrow{\xi_x} & TE|_{E_x} \\ \pi \downarrow & & \downarrow d\pi \\ \{x\} & \xrightarrow{X_{\xi_x}} & T_x M \end{array} \quad .$$

The sections of the bundle $T^{\text{lin}}E \rightarrow M$ are then sections of the canonical double bundle

$$\begin{array}{ccc} E & \longleftarrow & TE \\ \pi \downarrow & & \downarrow d\pi \\ M & \longleftarrow & TM \end{array}$$

that is, pairs (ξ, X_ξ) where ξ is a vector field on E and also a bundle map $\xi : E \rightarrow TE$ above the vector field $X_\xi : M \rightarrow TM$ given by $(X_\xi)_x = X_{\xi_x}$ for $\xi_x = \xi|_{E_x}$. These pairs are closed for the Lie bracket of vector fields on E and on M , so $T^{\text{lin}}E$ is a Lie algebroid. Details can be found in [Mac05, §3.4] or in [LGV20, Example 2.3].

Since $d\pi \circ \xi = X_\xi \circ \pi$, the field ξ is a horizontal lift of X_ξ and the anchor $a : T^{\text{lin}}E \rightarrow TM$ is just the projection of a linear vector field ξ on E to its base vector field X_ξ on M . Then, the kernel of the anchor is the bundle $V^{\text{lin}}E \rightarrow M$ of **linear vertical vector fields of E** , that is, the linear vector fields $\xi : E \rightarrow TE$ with values in the vertical subbundle $VE = \ker(d\pi)$ of TE . Since the vertical fibres $V_e E$ are canonically isomorphic to $E_{\pi(e)}$ (by translation of the 0 vector), there is a canonical bundle isomorphisms $V^{\text{lin}}E \cong \text{End}(E)$. This bundle fits in the Atiyah exact sequence of algebroids

$$0 \longrightarrow V^{\text{lin}}E \cong \text{End}(E) \longrightarrow T^{\text{lin}}E \xrightarrow{a} TM \longrightarrow 0. \quad (12)$$

□

Example 2.13 (The algebroid of derivative endomorphisms). The Lie algebroid $A(\text{GL}(E)) = A(FE) = T^{\text{lin}}E \rightarrow TM$ is suitable to describe *horizontal lifts* in TE of vectors tangent to M , i.e. Ehresmann connections on the vector bundle E . Linear connections on E , that is, $\mathbb{R}$ -linear smooth

maps $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$, $(X, \psi) \mapsto \nabla_X \psi$ which are $C^\infty(M)$ -linear in X and $C^\infty(M)$ -derivations in ψ , are *equivalent* but *not equal* to Ehresmann connections, since they describe the *vertical* variation of sections once a horizontal distribution in TE is fixed. Linear connections on E are also related to a Lie algebroid, which is *not* the Lie algebroid of FE or $GL(E)$, although it is isomorphic to $T^{\text{lin}}E$.

A **linear derivation** on E at $x \in M$ is an $\mathbb{R}$ -linear map $D_x : \Gamma(E) \rightarrow E_x$ satisfying the Leibniz rule

$$D_x(f\psi) = X_{D_x}(f)\psi(x) + f(x)D_x\psi \quad \forall f \in C^\infty(M), \forall \psi \in \Gamma(E), \quad (13)$$

for a given tangent vector $X_{D_x} \in T_x M$ associated to D_x , which acts as a derivation of functions on M at x . The **bundle of derivative endomorphisms** of E [Mac05, Example 3.3.4], also called the **bundle of linear derivations** on E [EVT19, §2.1], is the collection $\text{Der}(E)$ of the vector spaces $\text{Der}_x(E)$ of derivations of sections of E at each point of M . Its sections are the first order differential operators $D : \Gamma(E) \rightarrow \Gamma(E)$ satisfying the Leibniz rule (13) for an associated vector field $X_D \in \mathfrak{X}(M)$ defined by $(X_D)_x = X_{D_x}$, with $D_x(\psi) = D(\psi)(x)$. The vector bundle $\text{Der}(E)$ inherits a Lie bracket from the commutator of linear operators, and the anchor $a : \text{Der}(E) \rightarrow TM$ is given by the symbol map $a(D_x) = X_{D_x}$. Therefore $\text{Der}(E) \rightarrow TM$ is a Lie algebroid over M , and it is clearly transitive.

The kernel of the anchor at $x \in M$ is given by the $\mathbb{R}$ -linear maps $D_x : \Gamma(E) \rightarrow E_x$ which are also $C^\infty(M)$ -linear. Hence $\ker(a)$ is isomorphic to the Lie algebra bundle $\text{End}(E)$ and gives rise to a short exact sequence similar to the Atiyah sequence (12),

$$0 \longrightarrow \text{End}(E) \longrightarrow \text{Der}(E) \xrightarrow{a} TM \longrightarrow 0. \quad (14)$$

Note that, for a given vector field $X \in \mathfrak{X}(M)$, the covariant derivative ∇_X of a linear connection on E is precisely a section of $\text{Der}(E)$, with $D = \nabla_X$ and $a(D) = X$. The relationship between the two algebroids $T^{\text{lin}}E$ and $\text{Der}(E)$ mimics the relationship between a horizontal lift to E of $X \in \mathfrak{X}(M)$ and a covariant derivative ∇_X , which is a vertical-valued operator in TE : the isomorphism

$$T^{\text{lin}}E \xrightarrow{\sim} \text{Der}(E), \quad (\xi_x, X_{\xi_x}) \mapsto D_{\xi_x} \quad (15)$$

is given by

$$D_{\xi_x}(\psi) = d\psi|_x(X_{\xi_x}) - \xi_x(\psi(x)) \quad \forall \psi \in \Gamma(E), \quad (16)$$

where $d\psi|_x : T_x M \rightarrow T_{\psi(x)} E$ is the total differential of ψ at x (its tangent map), with symbol $a(D_{\xi_x}) = X_{\xi_x}$. The operator D_{ξ_x} is indeed a linear derivation on E at x , because $d\psi|_x$ is subject to the Leibniz rule and therefore, for any $f \in C^\infty(M)$, we have

$$\begin{aligned} D_{\xi_x}(f\psi) &= d(f\psi)|_x(X_{\xi_x}) - \xi_x(f(x)\psi(x)) \\ &= X_{\xi_x}(f)\psi(x) + f(x)d\psi|_x(X_{\xi_x}) - f(x)\xi_x(\psi(x)) \\ &= X_{\xi_x}(f)\psi(x) + f(x)D_{\xi_x}(\psi). \end{aligned}$$

See [Mac05, Proposition 3.4.4] or [LGV20, Example 2.3] for an alternative definition of this isomorphism using the canonical non-degenerate pairing between E and its dual vector bundle E^* , and [Mac05, Theorem 3.6.6] for an alternative isomorphism based on an exponential map from $A(GL(E))$ to $GL(E)$. $\square$

Remark 2.14. The Lie algebroid $\text{Der}(E) \rightarrow TM$ can be related to the Lie groupoid $\text{GL}(E) \rightrightarrows M$ through an alternative construction of a Lie algebroid $A'(\mathcal{G}) \rightarrow TM$ from any Lie groupoid $\mathcal{G} \rightrightarrows M$, obtained by swapping the source and the target everywhere along the construction of $A(\mathcal{G})$. It is immediate to show that $A(\mathcal{G})$ and $A'(\mathcal{G})$ share the same Lie algebra bundle $L(\mathcal{G})$ as kernel of the respective anchors, i.e. there are two exact sequences

$$\begin{aligned} 0 \longrightarrow L(\mathcal{G}) \cong \text{Ker}(a) \longrightarrow A(\mathcal{G}) \xrightarrow{a} TM \longrightarrow 0, \\ 0 \longrightarrow L(\mathcal{G}) \cong \text{Ker}(a') \longrightarrow A'(\mathcal{G}) \xrightarrow{a'} TM \longrightarrow 0. \end{aligned}$$

When $\mathcal{G} = \text{GL}(E)$, swapping source and target essentially corresponds to swapping *horizontal* and *vertical* vectors tangent to E , at the instar of Eq. (16), which can be written as

$$d\psi|_x(X_x) = \xi_x(\psi) + D_x\psi|_x, \quad a(\xi_x) = a'(D_x) = X_x, \quad (17)$$

and presents the decomposition of the vector $d\psi|_x(X_{\xi_x}) \in T_{\psi(x)}E$ into a horizontal part $\xi_x(\psi) \in T_{\psi(x)}E$ (obtained through $A(\text{GL}(E)) = T^{\text{lin}}E$) and a vertical part $D_x(\psi) = \nabla_{X_x}\psi|_x \in E_x \cong V_{\psi(x)}E$ (obtained through $A'(\text{GL}(E)) = \text{Der}(E)$). $\square$

To conclude this section, we complete the **Lie functor** from groupoids to algebroids by recalling the construction of the algebroid map induced by a groupoid map. Just as the tangent map of a morphism of Lie groups, at the unit, induces a Lie algebra morphism between the corresponding Lie algebras, there exists a tangent map of a local morphism between Lie groupoids, at the units, which gives rise to a Lie algebroid morphism. More generally, one can differentiate at the units any local map between Lie groupoids, and obtain its infinitesimal part along the diagonal.

Let $\phi : \mathcal{G} \multimap \mathcal{G}'$ be a local map between two Lie groupoids on M , defined on an open neighbourhood $\mathcal{U} \subset \mathcal{G}$ of the units. For any $x \in M$, we denote by $d^{\mathbf{t}}\phi|_x$ the **target** or **t-differential** of ϕ at $1_x \in \mathcal{U}$ given by

$$d^{\mathbf{t}}\phi|_x(\dot{\gamma}(0)) = \frac{d}{dt}\phi(\gamma(t))|_{t=0},$$

where $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{U} \cap \mathcal{G}^x$ is any smooth curve living in the source-fibre of x such that $\gamma(0) = 1_x$.

Proposition 2.15. *The t-differential at the units of a local map $\phi : \mathcal{G} \multimap \mathcal{G}'$ between two Lie groupoids induces a vector bundle morphism*

$$d^{\mathbf{t}}\phi|_M : A(\mathcal{G}) \longrightarrow A(\mathcal{G}')$$

between their associated Lie algebroids. If ϕ is a local groupoid morphism, then $d^{\mathbf{t}}\phi|_M$ is a Lie algebroid morphism.

Proof. By assumption, the **t-differential** $d^{\mathbf{t}}\phi$ is defined on vector fields tangent to the source fibres of $\mathcal{G}$ at the units, which span the fibres of the vector bundle $\mathbf{u}^*(T^s\mathcal{G}) = A(\mathcal{G})$. Since ϕ preserves the source and the units, by Definition 2.5, the map $d^{\mathbf{t}}\phi$ maps $\mathbf{u}^*(T^s\mathcal{G})$ to $(\mathbf{u}')^*(T^s\mathcal{G}')$ and gives a vector bundle morphism $d^{\mathbf{t}}\phi|_M : A(\mathcal{G}) \rightarrow A(\mathcal{G}')$. The second assertion is proved in [Mac05, §3.5, (40) and (41)]. $\square$

Remark 2.16. Using the alternative Lie algebroid $A'(\mathcal{G})$ described in Remark 2.14, one can define the **source** or **s-differential** of a local groupoid map $\phi : \mathcal{G} \multimap \mathcal{G}'$ at the units, giving rise to vector bundle morphism

$$d^{\mathbf{s}}\phi|_M : A'(\mathcal{G}) \longrightarrow A'(\mathcal{G}'),$$

by setting

$$d^s\phi|_x(\dot{\gamma}(0)) = \frac{d}{dt}\phi(\gamma(t))|_{t=0},$$

where $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{U} \cap \mathcal{G}_x$ is any smooth curve living in the target-fibre of x such that $\gamma(0) = 1_x$.
 $\square$

If G and G' are two Lie groups, a Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{g}'$ between their Lie algebras always integrates to a Lie group morphism $G \rightarrow G'$ if G is connected and simply connected. The same does *not* hold true for morphisms of Lie algebroids, the obstruction to integrability is studied in [CF03], see also [Mac05, Ch. 8].

3 Direct connections on Lie groupoids

In this section we consider direct connections on Lie groupoids as generalizations of parallel transport along small geodesics, seen as a local map from the pair groupoid of a base manifold to the frame groupoid of a vector bundle, in the spirit of N. Teleman and J. Kubarski [Tel04, Tel07, KT06, Kub08]. We show that a Lie groupoid which admits a direct connection is a gauge groupoid (Proposition 3.4), that direct connections generalise principal connections on principal bundles (Proposition 3.5 and Example 3.8), that they induce infinitesimal connections on the associated Lie algebroid (Proposition 3.13), and consequently that they induce a linear connection on associated vector bundles (Corollary 3.14). Finally, we characterise direct connections that are exactly the parallel transport of their linear connection (Theorem 3.18). Then we introduce the curvature of a direct connection, which measures how much the direct connection deviates from being a groupoid morphism, and show that its antisymmetric component gives rise to the usual curvature tensor of the induced infinitesimal connection (Theorem 3.24), whilst producing finer invariants than just the infinitesimal ones. We conclude with a strong characterization of groupoids admitting a flat direct connection (Proposition 3.26), largely studied by K. Mackenzie as a groupoid morphism [Mac05].

Assumption: We suppose that the base manifold M is equipped with a linear connection ∇^M on TM , so that one can consider geodesics on M . We further assume that ∇^M is torsion-free, and therefore around each point $x \in M$ there exists a uniformly normal neighborhood U_x , where every two points are linked by a unique geodesic [KN63, Theorem 8.7].

3.1 Direct connections and generalized parallel transports

Recall from Example 2.6 that a **diagonal domain** in $\text{Pair}(M)$ is any open neighbourhood $\mathcal{U}$ of the diagonal $\text{Diag}(M) = \{(x, x), x \in M\}$. For further convenience, from now on we assume that $\mathcal{U}$ is **symmetric**, that is, if $(y, x) \in \mathcal{U}$ then also $(x, y) \in \mathcal{U}$. We call **normal diagonal domain** in $\text{Pair}(M)$ a symmetric diagonal domain $\mathcal{U} = \bigcup_{x \in M} U_x \times U_x$ covered by uniformly normal neighborhoods.

Definition 3.1. A **direct connection** on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth local right inverse of the anchor which preserves the units, that is, a smooth local map $\Gamma : \text{Pair}(M) \rightarrow \mathcal{G}$ over M , defined on a symmetric diagonal domain $\mathcal{U} \subset \text{Pair}(M)$, which preserves the source, the target and the units, i.e.

- (i) $\Gamma(y, x) \in \mathcal{G}_y^x$ for all $(y, x) \in \mathcal{U}$,
- (ii) $\Gamma(x, x) = 1_x \in \mathcal{G}_x^x$ for all $x \in M$.

We call the direct connection **antisymmetric** if it also preserves the inversion, i.e.

- (iii) $\Gamma(y, x)^{-1} = \Gamma(x, y)$ for all $(y, x) \in \mathcal{U}$,

transitive if it is a groupoid morphism along geodesics, i.e.

- (iv) $\Gamma(z, y)\Gamma(y, x) = \Gamma(z, x)$ for all $(y, x) \in \mathcal{U}$ and z belonging to the geodesic from x to y , or viceversa: $(z, x) \in \mathcal{U}$ and y belonging to the geodesic from x to z ,

and **flat** if it is a local groupoid morphism, i.e.

- (v) $\Gamma(z, y)\Gamma(y, x) = \Gamma(z, x)$ for all $x, y, z \in M$ such that $(y, x), (z, y), (z, x) \in \mathcal{U}$.

Note that transitivity implies antisymmetry, by taking $z = x$ the initial point of the geodesic from x to y . Therefore we have

$$\text{flatness} \implies \text{transitivity} \implies \text{antisymmetry}.$$

Finally, a direct connection is called **global** if it is defined on the whole pair groupoid $\text{Pair}(M)$.

The only direct connection on the pair groupoid is the identity map $\text{Id} : \text{Pair}(M) \rightarrow \text{Pair}(M)$, and it is clearly flat.

Example 3.2 (Trivial direct connection). Any smooth map $\sigma : M \rightarrow G$ with values in a Lie group G gives rise to a global direct connection Γ_σ on the trivial gauge groupoid $M \times G \times M \rightrightarrows M$, by setting

$$\Gamma_\sigma(y, x) = (y, \sigma(y)\sigma(x)^{-1}, x)$$

for any $(y, x) \in \text{Pair}(M)$. This direct connection is flat, we call it **trivial**. $\square$

Definition 3.3. Let $\mathcal{G}$ be a locally trivial groupoid with section atlas $\sigma_\alpha : U_\alpha \rightarrow \mathcal{G}_{U_\alpha}^{x_0}$ with respect to a fixed point x_0 (cf. Example 2.3). A direct connection Γ on $\mathcal{G}$, defined on a diagonal domain $\mathcal{U}$, is called **locally trivial** if its restrictions to the charts of the section atlas are trivial (cf. Example 3.2), i.e.

$$\Gamma|_{(U_\alpha \times U_\alpha) \cap \mathcal{U}}(y, x) = \sigma_\alpha(y) \sigma_\alpha(x)^{-1}. \quad (18)$$

Such a direct connection is automatically flat.

Before giving more examples of direct connections, let us prove a necessary condition for Lie groupoids to admit direct connections.

Proposition 3.4. *If the base manifold M is connected, a Lie groupoid $\mathcal{G} \rightrightarrows M$ equipped with a direct connection is a gauge groupoid.*

Proof. On a connected base manifold, for $\mathcal{G}$ to be a gauge groupoid it is enough to show that it admits a section atlas, cf. Example 2.3. Let $\Gamma : \text{Pair}(M) \rightarrow \mathcal{G}$ be a direct connection defined on a diagonal domain $\mathcal{U}$, and let $(U_\alpha)_{\alpha \in A}$ be an open cover of M such that $\bigcup_{\alpha \in A} (U_\alpha \times U_\alpha) \subset \mathcal{U}$. Then Γ is well defined on any pair of points laying in the same open set U_α .

Fix an arbitrary point x_0 in M . Since M is connected, for any $x \in M$ one can choose a path $\gamma : [0, 1] \rightarrow M$ connecting x_0 and x , and a finite number k of open sets U_{α_i} 's which cover the path. Order the indices $\alpha_1, \dots, \alpha_k$ in such a way that the consecutive intersections are not empty, with $x_0 \in U_{\alpha_1}$ and $x \in U_{\alpha_k}$, and choose a point $x_i \in U_{\alpha_i} \cap U_{\alpha_{i+1}}$ for any $i = 1, \dots, k$. Then the element

$$\sigma_{\alpha_k}(x) = \Gamma(x, x_k) \Gamma(x_k, x_{k-1}) \dots \Gamma(x_2, x_1) \Gamma(x_1, x_0)$$

belongs to $\mathcal{G}_{x_0}^x$, and gives a section atlas $(\sigma_\alpha : U_\alpha \rightarrow \mathcal{G}_{U_\alpha}^{x_0})_{\alpha \in A}$ based at x_0 . $\square$

If the manifold M is not connected, one can work separately on its connected components. Thus, from now on we only consider connected base manifolds M and gauge groupoids $\mathcal{G}(P) \rightrightarrows M$ associated to a principal G -bundle $P \rightarrow M$.

Proposition 3.5. *Any principal connection on P defines a direct connection on $\mathcal{G}(P)$, through its parallel transport along small geodesics on M . Such a direct connection is transitive, and it is flat if and only if the principal connection on P is.*

Proof. For any two points x and y of M joint by a unique geodesic, let $\mathcal{P}^P(y, x) : P_x \rightarrow P_y$ denote the parallel transport on P along this geodesic. The direct connection Γ^P is defined on the pair (y, x) by

$$\Gamma^P(y, x) = [\mathcal{P}^P(y, x)p, p] \quad \text{for any choice of } p \in P_x. \quad (19)$$

The map Γ^P is well defined on the normal diagonal domain, and has values in $\mathcal{G}(P)$ because for any other element $p' \in P_x$ there exists a unique element g in G such that $p' = pg$, and the equality $[\mathcal{P}^P(y, x)p', p'] = [\mathcal{P}^P(y, x)p, p]$ follows from the G -equivariance of $\mathcal{P}^P(y, x)$.

The direct connection Γ^P is transitive, because parallel transports are. It follows from the Ambrose-Singer theorem [AS53], [KN63, Theorem 8.1], that Γ is flat if and only if the principal connection on P inducing $\mathcal{P}^P$ is flat. $\square$

Principal connections always exist on principal bundles [KN63, Ch.II, Proof of Theorem 2.1], therefore any gauge groupoid on a base manifold M equipped with a linear connection admits a direct connection.

Example 3.6 (Parallel transport on the trivial groupoid induced by a global section). Example 3.2 corresponds to the case of a trivial principal bundle $P = M \times G$ with global section σ . Each fibre P_x is diffeomorphic to G through $\sigma(x) \mapsto 1_G$, so any $p \in P_x$ can be written as $p = \sigma(x)g$ for some $g \in G$. The $\mathfrak{g}$ -valued 1-form on P given by $\omega(\sigma(x)g) := d(\sigma(x)g)(\sigma(x)g)^{-1}$ yields the principal connection on P with parallel transport

$$\mathcal{P}^P(y, x)\sigma(x)g = \sigma(y)g.$$

The induced direct connection Γ_g on $\mathcal{G}(P) = M \times G \times M$ is precisely that of Example 3.2, namely $\Gamma_\sigma(y, x) = (y, \sigma(y)\sigma(x)^{-1}, x)$. $\square$

3.2 Direct connections on frame groupoids

When the principal bundle is the frame bundle FE of a vector bundle $E \rightarrow M$, direct connections on the frame groupoid $\text{GL}(E) = \mathcal{G}(FE) \rightrightarrows M$ are called **linear direct connections** on E by N. Teleman and J. Kubarski [Tel04, Tel07, KT06].

Example 3.7 (Parallel transport on a vector bundle). Let $E \rightarrow M$ be a vector bundle with a linear connection ∇ . Then the parallel transport $\mathcal{P}^E$ of ∇ along small geodesics of M is a linear direct connection on the frame groupoid $\text{GL}(E)$ [Tel04]. The linear connection ∇ is equivalent to a principal connection on the frame bundle FE , and $\mathcal{P}^E$ is precisely the direct connection induced on $\text{GL}(E)$ by the parallel transport $\mathcal{P}^{FE}$ in FE : for any $x \in M$ and any $y \in U_x$, the linear isomorphism

$$\mathcal{P}^E(y, x) : E_x \rightarrow E_y$$

is the composite $\varphi_y \circ \varphi_x^{-1}$ where $\varphi_x : \mathbb{R}^r \rightarrow E_x$ is any frame at x (assuming that E has rank r) and $\varphi_y = \mathcal{P}^{FE}(y, x)(\varphi_x) : \mathbb{R}^r \rightarrow E_y$ is its parallel transported frame at y . $\square$

We borrow from [KT06, Example 5] an easy and illustrative example of a direct connection which is not induced by parallel transport.

Example 3.8 (Direct connections which are not parallel transports). On the base manifold $M = \mathbb{R}$, with global vector field $\partial_x = \frac{d}{dx}$, let us fix the flat linear connection whose geodesics are the segments parametrized by x . Let $E = M \times \mathbb{R}$ be the trivial bundle, with global section $e_1(x) = (x, 1) \in E_x$.

A linear connection $\nabla : \Gamma(TM) \otimes \Gamma(E) \rightarrow \Gamma(E)$ on E (necessarily flat) is given by its Christoffel symbol $\lambda \in C^\infty(M)$ such that $\nabla_{\partial_x} e_1(x) = -\lambda(x) e_1(x)$. The induced parallel transport along geodesics defines a global direct connection $\mathcal{P}^E : \text{Pair}(M) \rightarrow \text{GL}(E)$ by

$$\mathcal{P}^E(y, x) : E_x \rightarrow E_y, \quad e_1(x) \mapsto \mathcal{P}(y, x) e_1(x) = e^{\Lambda(y) - \Lambda(x)} e_1(y), \quad (20)$$

where $\Lambda(x) = \int \lambda(x) dx$ is the primitive function of the Christoffel symbol. Note that $e^{\Lambda(y) - \Lambda(x)} = \sigma(y)\sigma(x)^{-1} = \mathcal{P}^{FE}(y, x)$ is also the parallel transport on the frame bundle of E , which is the trivial principal $GL_1(\mathbb{R})$ -bundle as in Example 3.6, with global section $\sigma(x) = \pm e^{\Lambda(x)}$.

In contrast, the smooth maps of the form

$$\Gamma(y, x) e_1(x) = e^{\Lambda(y) - \Lambda(x) + (y-x)^2 H(y, x)} e_1(y), \quad (21)$$

where $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$ and $\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$ are any smooth functions, and η is not identically zero, define all possible global direct connection on $\text{GL}(E)$ which are not a parallel transport. In fact, if $\Gamma(y, x) = e^{F(y, x)}$ is a global smooth direct connection, the condition $\Gamma(x, x) = 1_x$ and its consequence $(\frac{\partial \Gamma}{\partial y} + \frac{\partial \Gamma}{\partial x})|_{y=x} = 0$ [KT06, Proposition 6] determine F at the 1st order of expansion in $y - x$, in terms of $\Lambda(x) = \int \frac{\partial \lambda}{\partial y}|_{y=x} dx = - \int \frac{\partial \lambda}{\partial x}|_{y=x} dx$. Indeed, if $\tilde{F}$ is of the form $\tilde{F}(y, x) = \Lambda(y) - \Lambda(x) + (y - x)\tilde{H}(y, x)$, with $\tilde{H} : \mathbb{R}^2 \rightarrow \mathbb{R}$ a non-zero smooth function, its expansion at $y = x$ gives

$$\frac{\partial \tilde{h}}{\partial y}|_{y=x} = \Lambda'(x) + \tilde{H}(x, x).$$

Then, if $\tilde{H}(x, x) = 0$ for all $x \in M$, one can write $\tilde{H}(y, x) = (y - x) H(y, x)$ and the direct connection

$$\tilde{\Gamma}(y, x) e_1(x) = e^{\Lambda(y) - \Lambda(x) + (y-x)\tilde{H}(y, x)} e_1(y)$$

can be put in the form (21) of Γ . Instead, if $\tilde{H}(x, x) \neq 0$ for some $x \in M$, then $\tilde{\Gamma}$ is again of the same form (21) for $\tilde{\Lambda}(x) = \Lambda(x) + \int \tilde{H}(x, x) dx$ related to a modified linear connection

$$\tilde{\nabla}_{\partial_x} e_1(x) = -(\lambda(x) + \tilde{H}(x, x)) e_1(x).$$

□

Direct connections on frame groupoids are also related to the notion of parallelism used in [DDD19, Definition 86]. A **parallelism** on E is a smooth local section $U : M \times M \rightarrow \text{Hom}(E)$ of the bundle $\text{Hom}(E) \cong E \boxtimes E^* \rightarrow M \times M$ of linear maps between fibres of E , defined on a diagonal domain and such that $U(x, x) = \text{Id}_{E_x}$. Of course a direct connection on $\text{GL}(E)$ is a parallelism. The converse also holds.

Proposition 3.9. *A parallelism on E yields a direct connection on $\text{GL}(E)$.*

Proof. A parallelism U on E sends a pair $(y, x) \in M \times M$ to a linear map $U(y, x) : E_x \rightarrow E_y$. All we need to prove is that this map is invertible if x and y are sufficiently close.

Suppose that E has rank r , choose a neighbourhood U_x of x trivialising E , so that $E_x \cong \mathbb{R}^r$ and $E_y \cong \mathbb{R}^r$ for any $y \in U_x$. Then compose the map $U(_, x) : U_x \rightarrow \text{End}(\mathbb{R}^r) \cong M_r(\mathbb{R})$ with the determinant map $\det : M_r(\mathbb{R}) \rightarrow \mathbb{R}$. The real valued map $\det U(_, x)$ is continuous and has value 1 on x , because $U(x, x) = \text{Id}_{E_x}$. So there exists an open neighbourhood V_x of x in U_x where $\det U(_, x) > 0$. This means that $U(y, x)$ is invertible for any $y \in V_x$, and therefore U is a direct connection on $\text{GL}(E)$ defined on the diagonal domain $\{(y, x) \mid x \in M, y \in V_x\}$. □

With this result, one can prove in an easy direct way that frame groupoids always have direct connections.

Corollary 3.10. *On a frame groupoid, local trivializations induce a direct connection.*

Proof. Let $\{(U_\alpha, \theta_\alpha)\}_{\alpha \in A}$ be a trivializing atlas of E , with trivializations $\theta_\alpha : E_{U_\alpha} \rightarrow U_\alpha \times \mathbb{R}^r$. For any $x, y \in U_\alpha$, set

$$\Gamma_\alpha(y, x) := (\theta_\alpha|_{E_y})^{-1} \circ \theta_\alpha|_{E_x} \in \text{GL}(E_x, E_y).$$

Then choose a partition of unity $\{\chi_\alpha\}_{\alpha \in A}$ subordinated to the covering $\{U_\alpha\}_{\alpha \in A}$ and glue the maps Γ_α as

$$\Gamma(y, x) := \sum_{\alpha \in A} \chi_\alpha(y) \Gamma_\alpha(y, x) \chi_\alpha(x).$$

This gives a parallelism on E defined on the diagonal domain $\bigcup_{\alpha \in A} U_\alpha \times U_\alpha$, and therefore a direct connection on $\text{GL}(E)$. $\square$

3.3 Infinitesimal connections induced by direct connections

A **connection on a Lie algebroid** $A \rightarrow M$ is a splitting of the anchor $a : A \rightarrow TM$, i.e. a vector bundle map $\delta : TM \rightarrow A$ such that $a \circ \delta = \text{Id}_{TM}$, cf. [Mac05, Definition 5.2.5]. It is called **flat** if it is a morphism of Lie algebroids, i.e. $[\delta(X), \delta(Y)]_A = \delta([X, Y])$ for any $X, Y \in TM$.

The existence of a connection on A forces A to be transitive (the anchor is surjective), however this does not imply that A is the Atiyah algebroid $A(P) = TP/G$ of a principal G -bundle $P \rightarrow M$, cf. Remark 2.11. If we assume that $A = A(\mathcal{G})$ comes from a Lie groupoid, then the existence of a connection on A implies that $\mathcal{G}$ is a gauge groupoid and $A = A(P)$ is an Atiyah algebroid. In this case, a connection on $A(P)$ is also called an **infinitesimal connection on P** and it is equivalent to a principal connection on P [Kob57, Ch. IV]. In fact, δ gives a splitting of the Atiyah sequence given in Eq. (8),

$$0 \begin{array}{c} \xleftarrow{\quad} \end{array} \text{Ad}(P) \begin{array}{c} \xleftarrow{\lambda} \\ \xrightarrow{\iota} \end{array} A(P) \begin{array}{c} \xleftarrow{\delta} \\ \xrightarrow{a} \end{array} TM \begin{array}{c} \xleftarrow{\quad} \end{array} 0$$

where $\lambda : A(P) \rightarrow \text{Ad}(P)$ is a bundle map such that $\iota \circ \lambda + \delta \circ a = \text{Id}_{A(P)}$, that is, $\lambda(\xi) = \xi - \delta(a(\xi))$ for any $\xi \in A(P)$ and in particular λ is the identity on vertical vectors $v \in \text{Ad}(P)$ [Mac05, §5.2, 5.3]. Lifting to TP by $\natural^{-1}$ yields a splitting $TP \cong VP \oplus HP$ into vertical and horizontal subbundles, with

$$VP = \ker(d\pi) = \natural^{-1} \text{Ad}(P) \quad \text{and} \quad HP = \natural^{-1} \delta(TM).$$

In other words, given δ and λ , we recover the *horizontal lift* of any $X \in TM$ to TP as $\natural^{-1} \delta(X)$, and a principal connection $\omega = \natural^{-1} \lambda : TP \rightarrow M \times \mathfrak{g}$. This motivates the alternative name of the Atiyah algebroid $A(P)$ in the literature, as the **bundle of connections** of P , cf. [Kob57, Ch. IV].

Example 3.11 (Connection on the trivial Lie algebroid). If $P = M \times G$ is trivial, then $A(P) = TM \oplus (M \times \mathfrak{g})$ is the trivial Lie algebroid, with $\text{Ad}(P) = M \times \mathfrak{g}$ (cf. Example 2.12 (c)). Then, an infinitesimal connection $\delta : TM \rightarrow A(P)$ splits a vector $X + v \in TM \oplus (M \times \mathfrak{g})$ as $X + v = \delta(X) + \lambda(X + v)$ with

$$\lambda(X + v) = A(X) + v \quad \text{and} \quad \delta(X) = X - A(X),$$

where $-A(X) \in M \times \mathfrak{g}$ represents the *vertical* component of the horizontal lift of X in TP fixed by δ , with respect to the given trivialization of P , and

$$A = \lambda|_{TM} : TM \rightarrow M \times \mathfrak{g}, \quad X \mapsto A(X) = X - \delta(X)$$

gives the connection 1-form on M representing the principal connection ω on P in the given trivialization, that is, a usual *gauge boson*. $\square$

Definition 3.12. Following the classical terminology, we call **infinitesimal connection on a Lie groupoid** $\mathcal{G} \rightrightarrows M$ a connection $\delta : TM \rightarrow A(\mathcal{G})$ on its Lie algebroid. The existence of such a connection implies that $A(\mathcal{G}) = A(P)$ is an Atiyah algebroid, and therefore, again, that $\mathcal{G} = \mathcal{G}(P)$ is a gauge groupoid.

Proposition 3.13. *Let $\mathcal{G}(P) \rightrightarrows M$ be a gauge groupoid endowed with a direct connection $\Gamma : \text{Pair}(M) \ast \rightarrow \mathcal{G}(P)$. Then the $\mathfrak{t}$ -differential of Γ along the diagonal gives an infinitesimal connection $\delta^\Gamma = d^\mathfrak{t}\Gamma|_M : TM \rightarrow A(P)$ defined as*

$$\delta^\Gamma(X) = \frac{d}{dt}\Gamma(c(t), x)|_{t=0}, \quad (22)$$

where $X \in T_x M$ and $c : [0, 1] \rightarrow M$ is the integral curve of X at x . Moreover, Γ is flat (it is a local groupoid morphism) if and only if δ is flat (it is an algebroid morphism).

Proof. Apply Proposition 2.15 to the local map $\Gamma : \text{Pair}(M) \ast \rightarrow \mathcal{G}$. Then, the vector $\delta^\Gamma(X) = d^\mathfrak{t}\Gamma|_x(\dot{c}(0))$ certainly belongs to $A(\mathcal{G}(P)) \cong A(P)$. Moreover, δ^Γ yields a splitting of the anchor $a = d\mathfrak{t}|_{T_{1_x}\mathcal{G}^x}$ because, for $X = \dot{c}(0) \in T_x M$, we have

$$a(\delta^\Gamma(X)) = a\left(\frac{d}{dt}\Gamma(c(t), x)|_{t=0}\right) = \frac{d}{dt}\mathfrak{t}(\Gamma(c(t), x))|_{t=0} = \frac{d}{dt}c(t)|_{t=0} = X.$$

The last statement follows from the fact that the tangent map along the diagonal of a local groupoid morphism is a Lie algebroid morphism [Mac05, §3.5]. $\square$

Corollary 3.14. *With the above notations, if Γ is a direct connection on a frame groupoid $\text{GL}(E) \rightrightarrows M$ or on a reduction $\mathcal{G}(P) \rightarrow \text{GL}(E)$, the local expression*

$$\nabla_X^\Gamma \psi|_x = d^\mathfrak{s}\Gamma(X)\psi|_x = \frac{d}{dt}\Gamma(c(t), x)^{-1} \psi(c(t))|_{t=0}, \quad (23)$$

where ψ is a local smooth section of E around $x \in M$, defines a linear connection on E . Together with the infinitesimal connection of Eq. (22), it yields a decomposition of the tangent map $d\psi|_x : T_x M \rightarrow T_{\psi(x)} E$ into its horizontal and vertical components

$$d\psi|_x(X) = \delta^\Gamma(X)\psi(x) + \nabla_X^\Gamma \psi|_x, \quad (24)$$

where $\delta^\Gamma(X) \in T_x^{\text{lin}} E$ acts linearly on $\psi(x) \in E_x$, and $\nabla_X^\Gamma \in \text{Der}_x(E)$ is a linear derivation of ψ at x .

Proof. Apply Remark 2.16: the expression (23) can be obtained as the $\mathfrak{s}$ -differential of Γ along the diagonal, together with the evaluation of $\text{GL}(E)$ on E of Example 2.9 (a), in the same way as in Proposition 3.13, and finally using the equality

$$\frac{d}{dt}\Gamma(x, c(t)) \psi(c(t))|_{t=0} = \frac{d}{dt}\Gamma(c(t), x)^{-1} \psi(c(t))|_{t=0}$$

of Lemma 3.15. Alternatively, this follows from the general relationship between a horizontal and vertical components of the differential of ψ , as in Remark 2.14. In particular, Eq. (24) is a reformulation of Eq. (17):

$$\begin{aligned}
d\psi|_x(X) &= \frac{d}{dt}\psi(c(t))|_{t=0} \\
&= \frac{d}{dt}\Gamma(c(t), x)\Gamma(c(t), x)^{-1}\psi(c(t))|_{t=0} \\
&= \frac{d}{dt}\Gamma(c(t), x)|_{t=0}\Gamma(x, x)^{-1}\psi(x) + \Gamma(x, x)\frac{d}{dt}\Gamma(c(t), x)^{-1}\psi(c(t))|_{t=0} \\
&= \delta^\Gamma(X)\psi(x) + \nabla_X^\Gamma\psi|_x.
\end{aligned}$$

□

Lemma 3.15. [KT06, Proposition 6] *Any direct connection $\Gamma : \text{Pair}(M) \rightarrow \mathcal{G}$ satisfies the infinitesimal identities*

$$\begin{aligned}
\frac{d}{dt}\Gamma(c(t), x)|_{t=0} &= -\frac{d}{dt}\Gamma(x, c(t))|_{t=0} \\
\frac{d}{dt}\Gamma(c(t), x)^{-1}|_{t=0} &= -\frac{d}{dt}\Gamma(c(t), x)|_{t=0} = \frac{d}{dt}\Gamma(x, c(t))|_{t=0}
\end{aligned}$$

Proof. The first identity is a restatement of the fact that the directional derivative of $\Gamma(y, x)$ along the diagonal vanishes, i.e. $(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i})\Gamma(y, x)|_{y=x} = 0$, which follows from $\Gamma(x, x) = 1_x$. The second identity is the derivative of $\Gamma(c(t), x)^{-1}\Gamma(c(t), x) = \text{Id}_x$. □

A consequence of Proposition 3.13 and Corollary 3.14 is that one can recover the usual Christoffel symbols of a linear connection on E in terms of the matrix elements of a direct connection on $\text{GL}(E)$, cf. also [KT06, Theorem 3]. If $\{e_\alpha(x)\}_{\alpha=1}^r$ is a basis of local sections of E at $x \in M$, and $\psi(x) = \psi^\alpha(x)e_\alpha(x)$ is a local section (with Einstein summation convention), then a direct connection Γ on E is locally given by matrix elements $\Gamma_\alpha^\beta(y, x) \equiv \Gamma(y, x)_\alpha^\beta$ such that

$$\Gamma(y, x)(\psi^\alpha(x)e_\alpha(x)) = \psi^\alpha(x)\Gamma_\alpha^\beta(y, x)e_\beta(y).$$

If $\{\partial_i = \frac{\partial}{\partial x^i}|_x\}_{i=1}^d$ is a basis of local sections of $T_x M$, the expression

$$\nabla_{\partial_i}^\Gamma e_\alpha|_x = \partial_i \Gamma_\alpha^\beta(x, y)|_{y=x} e_\beta(x) := \frac{\partial}{\partial y^i} \Gamma_\alpha^\beta(x, y)|_{y=x} e_\beta(x)$$

defines the usual **Christoffel symbols** of the linear connection ∇^Γ on E as

$$\Gamma_{i\alpha}^\beta(x) := \partial_i \Gamma_\alpha^\beta(x, y)|_{y=x}. \quad (25)$$

Therefore, by (24), for $X = X^i \partial_i \in T_x M$ we have

$$\begin{aligned}
\nabla_X^\Gamma \psi|_x &= X^i \left(\partial_i \psi^\beta(x) + \psi^\alpha(x) \partial_i \Gamma_\alpha^\beta(x, y)|_{y=x} \right) e_\beta(x) \\
\delta(X)\psi(x) &= -X^i \psi^\alpha(x) \partial_i \Gamma_\alpha^\beta(x, y)|_{y=x} e_\beta(x) \\
&= X^i \psi^\alpha(x) \partial_i \Gamma_\alpha^\beta(y, x)|_{y=x} e_\beta(x).
\end{aligned} \quad (26)$$

Proposition 3.16. *Let $\delta : TM \rightarrow A(P)$ be an infinitesimal connection on P and let Γ^δ be the direct connection on $\mathcal{G}(P)$ defined by the parallel transport of δ along small geodesics, as in Proposition 3.5. Then, the infinitesimal connection defined by Γ^δ coincides with δ , that is,*

$$\delta(X) = \frac{d}{dt} \Gamma^\delta(c(t), x)|_{t=0}, \quad (27)$$

where $c : [0, 1] \rightarrow M$ is the integral curve of X at x .

Proof. Let $\mathcal{P}$ be the parallel transport of δ along small geodesics. Then $\Gamma^\delta : \text{Pair}(M) \rightarrow \mathcal{G}(P)$ is given by $\Gamma^\delta(y, x) = [\mathcal{P}(y, x)(p), p]$ for any choice of $p \in P_x$, and its infinitesimal connection is

$$\delta^{\Gamma^\delta}(X) = \frac{d}{dt} \Gamma^\delta(c(t), x)|_{t=0} = \frac{d}{dt} [\mathcal{P}(c(t), x)(p), p]|_{t=0}.$$

By uniqueness of the horizontal lift, the tangent vector at $t = 0$ to the curve $\gamma(t) = \mathcal{P}(c(t), x)(p)$ gives the value of $\delta(\dot{c}(0))$. Taking into account the action of G on TP , which makes the choice of p irrelevant, one can identify $\delta^{\Gamma^\delta}(\dot{c}(0))$ to the class of $\dot{\gamma}(0)$ in TP/G , from which it follows that

$$\delta^{\Gamma^\delta}(\dot{c}(0)) = \mathfrak{h}_p(\dot{\gamma}(0)) = \delta(\dot{c}(0)),$$

for any $\dot{c}(0) \in T_x M$. □

The analogue statement holds for a linear connection ∇ on E and the direct connection Γ^∇ on $\text{GL}(E)$ defined by its parallel transport.

The above result shows that infinitesimal connections can always be integrated to direct connections: it suffices to consider their parallel displacement. However, the correspondence between direct connections and infinitesimal connections is not one-to-one, as shown by next example.

Example 3.17 (Direct connections with equal infinitesimal connections). Consider the trivial vector bundle $E = M \times \mathbb{R}$ on the flat base manifold $M = \mathbb{R}$, as in Example 3.8. The two direct connections on $\text{GL}(E)$ given by Eq. (20) (a parallel transport) and by Eq. (21), namely

$$\begin{aligned} \mathcal{P}(y, x) e_1(x) &= e^{F(y)-F(x)} e_1(y) \\ \Gamma(y, x) e_1(x) &= e^{F(y)-F(x)+(y-x)^2 F'(y,x)} e_1(y), \quad h \neq 0 \end{aligned}$$

have the same infinitesimal connection $\delta(\partial_x) = K'(x) = f(x)$. □

In fact, we can fully characterise the direct connections which are parallel transport: they are exactly the transitive ones.

Theorem 3.18. *Let Γ be a direct connection on a frame groupoid $\text{GL}(E)$ defined on a normal diagonal domain $\mathcal{U}$. If Γ is transitive, then it is the parallel transport of its associated linear connection ∇^Γ on E .*

Proof. For any $(y, x) \in \mathcal{U}$, let $c : [0, 1] \rightarrow M$ be the geodesic from x to y . The direct connection Γ restricted to c gives a linear map $\Gamma(c(t), c(s)) : E_{c(s)} \rightarrow E_{c(t)}$ for any $s, t \in [0, 1]$. Its infinitesimal connection δ^Γ , restricted to c , is given by eq. (22) evaluated at every point $c(t)$, namely

$$\delta^\Gamma(\dot{c}(t)) = \frac{d}{du} \Gamma(c(u), c(t))|_{u=t}.$$

Since Γ is transitive, for any $s, t \in [0, 1]$ we have the composition property

$$\Gamma(c(u), c(t)) \Gamma(c(t), c(s)) = \Gamma(c(u), c(s)).$$

Differentiating it in the variable u at $u = t$, we see that Γ satisfies the linear differential equation

$$\frac{d}{du} \Gamma(c(u), c(s)) \Big|_{u=t} = \frac{d}{du} \Gamma(c(u), c(t)) \Big|_{u=t} \Gamma(c(t), c(s)) = \delta^\Gamma(\dot{c}(t)) \Gamma(c(t), c(s)),$$

with initial condition $\Gamma(c(t), c(t)) = 1_{c(t)}$.

Now, the parallel transport induced by Γ along the geodesic c is also a linear map $\mathcal{P}^\Gamma(c(t), c(s)) : E_{c(s)} \rightarrow E_{c(t)}$ which satisfies the same differential equation

$$\frac{d}{du} \mathcal{P}^\Gamma(c(u), c(s)) \Big|_{u=t} = \delta^\Gamma(\dot{c}(t)) \mathcal{P}^\Gamma(c(t), c(s)),$$

with the same initial condition $\mathcal{P}^\Gamma(c(t), c(t)) = 1_{c(t)}$. In fact, by definition, the parallel transport $\mathcal{P}^\Gamma(c(t), c(s))$ is the linear map such that, for any $e \in E_{c(s)}$, the section $t \mapsto \psi(c(t)) = \mathcal{P}^\Gamma(c(t), c(s))(e)$ along c is parallel, i.e.

$$\nabla_{\dot{c}(t)}^\Gamma \psi|_{c(t)} = d\psi(\dot{c}(t)) - \delta^\Gamma(\dot{c}(t)) \psi(c(t)) = 0.$$

Since $d\psi(\dot{c}(t)) = \frac{d}{du} \mathcal{P}^\Gamma(c(u), c(s)) \Big|_{u=t}(e)$, this gives the above equation.

By uniqueness of the solution of ODEs, then, a transitive direct connection Γ coincides with the parallel transport $\mathcal{P}^\Gamma$ of its infinitesimal connection on every points along the geodesic from x to y , and therefore they coincide on the normal diagonal domain $\mathcal{U}$ in $\text{Pair}(M)$. $\square$

Remark 3.19. The fact that direct connections are *not* in one-to-one correspondence with infinitesimal connections contrasts with *path connections* on groupoids, cf. [Mac05, Definition 6.3.1]. A path connection is a horizontal lift to the groupoid of a curve on the base manifold M . This lift is by definition so constrained that its initial direction (which is an infinitesimal connection on its Lie algebroid) determines it all along the path on M [Mac05, Theorem 6.3.5], and finally identifies it with the exponential map from Lie algebroids to Lie groupoids [Mac05, Corollary 6.3.6].

The situation changes if we consider *flat* connections: there is a one-to-one correspondence between *flat* direct connections, i.e. local groupoid morphisms, and *flat* infinitesimal connections, i.e. algebroid morphisms. This is proved by K. Mackenzie in [Mac05, Corollary 6.2.7] and can be seen as a consequence of [Mac05, Proposition 6.1.8], that we recall in Section 3.5. $\square$

We conclude this section by collecting the characterization of flat connections in classical terms and in terms of groupoids, which follows from Propositions 3.5, 3.13 and Corollary 3.14.

Corollary 3.20. *Let M be a smooth manifold endowed with an affine connection.*

1. *If ω is a principal connection on a principal bundle $P \rightarrow M$ and Γ is the direct connection on $\mathcal{G}(P) \rightrightarrows M$ induced by the parallel transport on P , then ω is flat as a principal connection if and only if Γ is flat as a direct connection.*
2. *If ∇ is a linear connection on a vector bundle $E \rightarrow M$ and Γ is the direct connection on $\text{GL}(E) \rightrightarrows M$ induced by the parallel transport on E , then ∇ is flat as a linear connection if and only if Γ is flat as a direct connection.*

3.4 Curvature of direct connections

The default of a direct connection to be flat is measured, as usual, by a *curvature*. Our definition slightly differs from that given by N. Teleman and J. Kubariski in [Tel04, Tel07, KT06, Kub08] and by A. Kock in [Koc07, Koc17]. The difference measures the default of the direct connection to be antisymmetric and has no infinitesimal counterpart. Therefore our notion of curvature is essentially equivalent to those existing in the literature.

Definition 3.21. Let $\Gamma : \text{Pair}(M) \rightarrow \mathcal{G}$ be a direct connection defined on a symmetric diagonal domain $\mathcal{U} \subset M \times M$. For any $x \in M$, we set

$$\mathcal{U}(x) = \{(z, y) \in \mathcal{U} \mid (y, x), (z, x) \in \mathcal{U}\}$$

and call **curvature of Γ at x** the map $K_x : \mathcal{U}(x) \rightarrow \mathcal{G}_x^x$ given on $(z, y) \in \mathcal{U}(x)$ by

$$K_x(z, y) := \Gamma(z, x)^{-1} \Gamma(z, y) \Gamma(y, x). \quad (28)$$

Then Γ is flat, according to Definition 3.1, if $K_x = 1_x$ for any $x \in M$, and it is transitive if $K_x = 1_x$ along geodesics starting at x . The curvature is **symmetric** if $K_x(z, y) = K_x(y, z)$ for any points, that is, $K_x(y, z)^{-1} K_x(z, y) = 1_x$, and it is **antisymmetric** if $K_x(z, y) = K_x(y, z)^{-1}$, that is, $K_x(y, z) K_x(z, y) = 1_x$. Note that, in general, we have

$$\begin{aligned} K_x(y, z) K_x(z, y) &= \Gamma(y, x)^{-1} \Gamma(y, z) \Gamma(z, y) \Gamma(y, x) \\ &= \text{Ad}_{\Gamma(y, x)^{-1}} (\Gamma(y, z) \Gamma(z, y)), \end{aligned}$$

therefore K is antisymmetric if and only if the direct connection Γ is.

Examples 3.22 (Antisymmetric and flat direct connections). (a) Direct connections on $\mathcal{G}(P)$ of the form $\Gamma(y, x) = g(y)g(x)^{-1}$, induced by global sections g of P as in Example 3.6, are flat.

(b) Direct connections given by a parallel transport $\mathcal{P}^P$ as in Proposition 3.5, or $\mathcal{P}^E$ as in Example 3.8, Eq. (20), are transitive (hence antisymmetric) but not necessarily flat. Moreover, by Theorem 3.18, parallel transports are exactly all transitive direct connections.

(c) The general direct connection Γ on $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ of Example 3.8 Eq. (21) is not flat (and not transitive), and it is antisymmetric only if the function h is antisymmetric. $\square$

The invariants of a direct connection Γ are carried by the curvature operator $K_{x_0} \in \mathcal{G}_{x_0}^{x_0}$ and its derivatives.

Lemma 3.23. *The first derivatives of K_x along a geodesic $y(t)$ starting at x vanish:*

$$\frac{d}{dt} K_x(x, y(t))|_{t=0} = \frac{d}{dt} K_x(y(t), x)|_{t=0} = \frac{d}{dt} K_x(y(t), y(t))|_{t=0} = 0.$$

Proof. The first derivative vanishes because of Lemma 3.15. The second and the third ones because $K_x(y(t), x) = 1_x$ and $K_x(y(t), y(t)) = 1_x$ are constant. $\square$

Note, however, that the first derivatives of K_x computed not along a geodesic do not necessarily vanish, for instance

$$\frac{d}{dt} K_x(z, y(t))|_{t=0}, \quad \frac{d}{dt} K_x(y(t), z)|_{s=0}$$

when $z \neq x$. Let us show that the second derivative of K_x is related to the curvature tensor of the induced infinitesimal connections.

Let P be a principal bundle on M with associated vector bundle E , so that the gauge groupoid $\mathcal{G}(P)$ is a reduction of the frame groupoid $\text{GL}(E)$ (for instance $P \subset FE$). Let $\Gamma : \text{Pair}(M) \multimap \mathcal{G}(P)$ be a direct connection with infinitesimal connection $\delta : TM \rightarrow A(P) \subset T^{\text{lin}}E$ as in Eq. (22) and linear connection $\nabla : TM \rightarrow \text{Der}(E)$ as in Eq. (23). We recall that the **curvature** of δ and of ∇ at a point $x \in M$ is given by

$$\begin{aligned} R_x^\delta(Z, Y)\psi(x) &= [\delta(Z), \delta(Y)]_{A(P)}\psi(x) - \delta([Z, Y])\psi(x) \\ R_x^\nabla(Z, Y)\psi(x) &= [\nabla_Z, \nabla_Y]_{\text{Der}(E)}\psi|_x - \nabla_{[Z, Y]}\psi|_x, \end{aligned}$$

for any $Y, Z \in T_x M$ and for any local section ψ of E around x . In terms of the Lie algebroids, both curvatures are operators with values in the kernel of the anchor, that is,

$$R^\delta, R^\nabla : TM \times TM \rightarrow \text{End}(E) \cong L(T^{\text{lin}}E) \cong L(\text{Der}(E)),$$

and measure how much the algebroid connections deviate from being algebroid morphisms. It was proven in [Mac05, Proposition 5.3.14] that they coincide: $R_x^\delta(Z, Y) = R_x^\nabla(Z, Y)$.

The relationship between the curvature K of the direct connection Γ and the curvatures R^δ and R^∇ of its infinitesimal and linear connections was described by J. Kubarski, N. Teleman in [KT06, Corollary 4] for frame groupoids. We extend their result to general groupoids and give a comprehensive formula in terms of the operator which measures how much K deviates from being symmetric.

Theorem 3.24. *The curvature R^δ of the infinitesimal connection associated to a direct connection Γ is*

$$R_x^\delta(Z, Y) = \frac{\partial^2}{\partial s \partial t} \left(K_x(y(t), z(s))^{-1} K_x(z(s), y(t)) \right) \Big|_{t=s=0},$$

where $Y, Z \in T_x M$ are two vectors tangent to M at x , $y(t) = \exp_x(tY)$ is the integral curve of Y and $z(s) = \exp_x(sZ)$ is the integral curve of Z .

As a consequence, a direct connection which is infinitesimally flat is not necessarily flat: it is only symmetric up to order 2.

Proof. The proof is a simple computation making use of the Leibniz rule, the infinitesimal properties of Γ given in Lemma 3.15 and the dictionary between direct connections and linear connections in terms of Christoffel symbols given in Eqs. (25) and (26). There is however a subtlety in the choice of curves along which we compute the derivatives of Γ , let us give some details. For $\psi = \psi^\alpha e_\alpha$, $Y = Y^j \partial_j$, $Z = Z^k \partial_k$ and $\Gamma_{i\alpha}^\beta(x) = \frac{\partial}{\partial y^i} \Gamma_\alpha^\beta(x, y)|_{y=x}$, we have

$$\begin{aligned} \nabla_Y \psi|_x &= Y^j \partial_j \psi^\beta(x) e_\beta(x) + Y^j \psi^\alpha(x) \frac{\partial}{\partial y^j} \Gamma_\alpha^\beta(x, y)|_{y=x} e_\beta(x) \\ &= \frac{d}{dt} \psi(y(t))|_{t=0} + \frac{d}{dt} \Gamma(x, y(t))|_{t=0} \psi(x). \end{aligned}$$

The derivative of ψ at x disappears in the expression of the curvature $R_x^\delta(Z, Y) = R_x^\nabla(Z, Y)$, which acts linearly on $\psi(x)$, and can be neglected. It remains the linear part in $\psi(x)$, given by the derivative of Γ . Now, as we just showed, the first-order derivative of $\Gamma(x, y)$ in the variable y , at

$y = x$, gives the Christoffel symbol at x . In the curvature $R^\nabla(Z, Y)$, there appears a derivative of the Christoffel symbols in their (unique) variable x : this corresponds to a second-order derivative of $\Gamma(x, y)$ in the variable x . Then, the standard expression of the curvature $R^\nabla(Z, Y)$ in terms of the Christoffel symbols gives

$$\begin{aligned} R_x^\delta(Z, Y) &= Z^k \frac{\partial}{\partial z^k} \Gamma(x, z)|_{z=x} Y^j \frac{\partial}{\partial y^j} \Gamma(x, y)|_{y=x} - Y^j \frac{\partial}{\partial y^j} \Gamma(x, y)|_{y=x} Z^k \frac{\partial}{\partial z^k} \Gamma(x, z)|_{z=x} \\ &\quad + Z^k Y^j \frac{\partial^2}{\partial z^k \partial y^j} \Gamma(z, y)|_{z=y=x} - Y^j Z^k \frac{\partial^2}{\partial y^j \partial z^k} \Gamma(y, z)|_{y=z=x}. \end{aligned}$$

Note that this expression coincides with that given in [KT06, Corollary 4]. To relate this to the formula given above, call $y = \exp_x(Y)$ and $z = \exp_x(Z)$ the end points of the geodesic $y(t)$ and $z(s)$ and consider the geodesic variation of Y along $z(s)$ for small s , denoted Y_s , and the geodesic variation of Z along $y(t)$ for small t , denoted Z_t , as in Figure 1 (cf. [Lee97, Ch. 6] for geodesic variations). The curvature can now be given in terms of the integral curves $y_s(t) = \exp_{z(s)}(tY_s)$ and

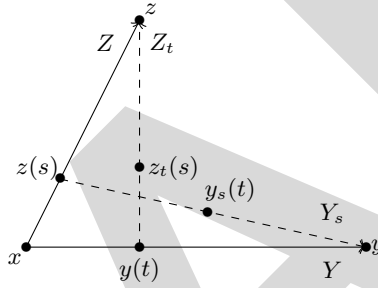


Figure 1: Geodesic variations and related Jacobi fields

$z_t(s) = \exp_{y(t)}(sZ_t)$ for small t, s namely

$$\begin{aligned} R_x^\delta(Z, Y) &= \frac{d}{ds} \Gamma(x, z(s))|_{s=0} \frac{d}{dt} \Gamma(x, y(t))|_{t=0} - \frac{d}{dt} \Gamma(x, y(t))|_{t=0} \frac{d}{ds} \Gamma(x, z(s))|_{s=0} \\ &\quad + \frac{\partial^2}{\partial s \partial t} \Gamma(z(s), y_s(t))|_{s=t=0} - \frac{\partial^2}{\partial t \partial s} \Gamma(y(t), z_t(s))|_{t=s=0} \\ &= \frac{\partial^2}{\partial s \partial t} K_x(z(s), y(t))|_{t=s=0} - \frac{\partial^2}{\partial s \partial t} K_x(y(t), z(s)) \\ &= \frac{\partial^2}{\partial s \partial t} \left(K_x(y(t), z(s))^{-1} K_x(z(s), y(t)) \right)|_{t=s=0}. \end{aligned}$$

The last two equalities are a direct computation from the definition of $K_x(z, y)$ using the Leibniz rule for products and Lemma 3.15. $\square$

3.5 Flat groupoids

A direct connection $\Gamma : \text{Pair}(M) \rightarrow \mathcal{G}$ is flat if and only if it is a local groupoid morphism. Such maps have been extensively studied by K. Mackenzie in [Mac05, Chapter 6], in particular for what concerns the relationship with their infinitesimal Lie algebroid morphisms. We collect here some relevant results, in particular a strong characterization of Lie groupoids which admit a flat direct connection.

Definition 3.25. A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called **flat** if it is locally trivial and it admits an atlas of local decomposition maps $\sigma_\alpha : U_\alpha \rightarrow \mathcal{G}_{U_\alpha}^{x_0}$ (cf. Example 2.3) whose transition functions

$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathcal{G}_{x_0}^{x_0}$ given by $g_{\alpha\beta}(x) = \sigma_\alpha(x)^{-1} \sigma_\beta(x)$ are locally constant, i.e. constant on each connected component. A flat Lie groupoid $\mathcal{G}$ is called **locally exact** if furthermore it admits a locally trivial direct connection $\Gamma : \text{Pair}(M) \rightarrow \mathcal{G}$ (cf. Definition 3.3).

In essence, the subsequent statement is a reformulation of [Mac05, Example 6.1.7] in terms of direct connections.

Proposition 3.26. *A Lie groupoid on a connected base admits a flat direct connection if and only if it is flat. In this case, it is locally exact.*

Proof. ($\Rightarrow$) Assume $\mathcal{G} \rightrightarrows M$ is a Lie groupoid endowed with a local groupoid morphism $\Gamma : \text{Pair}(M) \rightarrow \mathcal{G}$ defined on a diagonal domain $\mathcal{U}$, in which case $\mathcal{G}$ is locally trivial by Proposition 3.4. We choose an open covering $\{U_\alpha\}$ of M such that $\bigcup_\alpha (U_\alpha \times U_\alpha) \subset \mathcal{U}$ and $x_0 \in M$. For every α , fix a point $x_\alpha \in U_\alpha$ together with an element $\xi_\alpha \in \mathcal{G}_{x_\alpha}^{x_0}$ (which exists by the local triviality of $\mathcal{G}$). Then the collection of maps $\sigma_\alpha(x) := \Gamma(x, x_\alpha) \xi_\alpha$ builds a section atlas with constant transition functions $g_{\alpha\beta}(x) = \xi_\alpha^{-1} \Gamma(x_\alpha, x_\beta) \xi_\beta$. Moreover, for fixed α , the map

$$\begin{aligned} \mathcal{G}_{U_\alpha}^{U_\alpha} &\longrightarrow U_\alpha \times \mathcal{G}_{x_0}^{x_0} \times U_\alpha \\ g &\longmapsto (\mathbf{t}(g), \sigma_\alpha(\mathbf{t}(g))^{-1} g \sigma_\alpha(\mathbf{s}(g)), \mathbf{s}(g)) \end{aligned}$$

is an isomorphism under which $\Gamma(y, x)$ is sent to $(y, 1_x, x)$ for all $x, y \in U_\alpha$. Hence, $\mathcal{G}$ is locally exact.

($\Leftarrow$) Conversely, let $\sigma_\alpha : U_\alpha \rightarrow \mathcal{G}_{U_\alpha}^{x_0}$ be a section atlas with constant transition functions $\sigma_\alpha(x)^{-1} \sigma_\beta(x) = g_{\alpha\beta} 1_x$ on the intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and set $\mathcal{U} := \bigcup_\alpha (U_\alpha \times U_\alpha)$. Then, the map $\Gamma : \mathcal{U} \rightarrow \mathcal{G}$ given by Eq. (18), namely $\Gamma(y, x) = \sigma_\alpha(y) \sigma_\alpha(x)^{-1}$ for $x, y \in U_\alpha$, is well defined on $\mathcal{U}$. Indeed, on every intersection $U_{\alpha\beta}$ we have $\sigma_\beta(x) = \sigma_\alpha(x) g_{\alpha\beta}$, therefore

$$\begin{aligned} \Gamma|_{U_\beta \times U_\beta}(y, x) &= \sigma_\beta(y) \sigma_\beta(x)^{-1} \\ &= \sigma_\alpha(y) g_{\alpha\beta} g_{\alpha\beta}^{-1} \sigma_\alpha(x)^{-1} = \Gamma|_{U_\alpha \times U_\alpha}(y, x). \end{aligned}$$

That $\Gamma|_{U_\alpha \times U_\alpha}$ defines a morphism is easily verified. $\square$

Example 3.27. The trivial Lie groupoid $\mathcal{G}(M \times G) = M \times G \times M$ is locally exact, because it is clearly flat and any global section $\sigma : M \rightarrow M \times G$ gives a trivial direct connection $\Gamma_\sigma(x, y) = (y, \sigma(y) \sigma(x)^{-1}, x)$ as in Example 3.2. $\square$

Since we don't really use the fundamental groupoid, we've already cut its example in Section 2.1. Here we propose to cut the Example and the Proposition, or to make them understandable.

Example 3.28. The fundamental groupoid $\Pi(M)$ on a connected manifold M is flat, i.e. it has constant transition maps. Hence, by Proposition 3.26, it admits a flat direct connection $\Gamma_0 : \text{Pair}(M) \rightarrow \Pi(M)$ which can be constructed as follows. Let $\{U_\alpha\}$ be a cover of M by simply connected open subsets, then $\mathcal{U} = \bigcup_\alpha (U_\alpha \times U_\alpha)$ defines a diagonal domain. For any $(y, x) \in \mathcal{U}$, we define $\Gamma_0(y, x)$ as the homotopy class of any path in $\mathcal{U}$ connecting x and y . This is well defined as any two paths in U_α are homotopic, and clearly gives a local groupoid morphism. $\square$

The importance of the direct connection Γ_0 lies in a factorisation result shown in [Mac05], which we reformulate here in the language of direct connections. Let us call two direct connections **germ equivalent** if they are germ equivalent as local maps, i.e. if they agree on some common diagonal domain.

Proposition 3.29. *cut or change the exposition with just a reference to McKenzie [Mac05, Prop. 6.1.8] Let M be connected.*

1. *The flat direct connection $\Gamma_0 : \text{Pair}(M) \circ \longrightarrow \Pi(M)$ is uniquely determined up to germ equivalence.*
2. *Any flat direct connection $\Gamma : \text{Pair}(M) \circ \longrightarrow \mathcal{G}$ on a Lie groupoid $\mathcal{G}$ factorises in a unique way through Γ_0 , i.e. there exists a unique groupoid morphism $H : \Pi(M) \rightarrow \mathcal{G}$, such that $\Gamma = H \circ \Gamma_0$.*

The second result can be used to prove that a flat infinitesimal connection $\delta : TM \rightarrow A(\mathcal{G})$ integrates to a flat direct connection Γ^δ on $\mathcal{G}$, which is unique up to germ equivalence, cf. Remark 3.19.

4 Direct connections on jet groupoids

Jet prolongations of groupoids were first considered by C. Ehresmann [Ehr55] and later revisited by I. Kolář [Kol07]. After reviewing the basic concepts regarding jet prolongation of Lie groupoids and Lie algebroids, in this section we study the jet prolongation of direct connections. We start by introducing the *jet primitive bisection* of any direct connection on a jet groupoid (Definition 4.8). Using this tool, we show that the jet prolongation of a direct connection is determined by those of the trivial direct connection Id on the pair groupoid, and describe their associated infinitesimal connection (Theorem 4.11). We then exhibit three jet prolongations of Id on $\text{Pair}(M)$, based on three Jacobi fields of variation curves through geodesics which differ by the initial value of the field and of its covariant derivative (Theorems 4.15, 4.17 and 4.20). In the three cases we compute the associated infinitesimal connection and show that the first two are, in a sense, orthogonal to each other, while the third one reproduces the exponential jet connection on the Lie algebroid $J^n TM$ given by I. Kolář in [Kol07] and by W. M. Mikulski in [Mik07].

Finally, we provide an example of a direct connection on the frame groupoid of a jet bundle that is not the jet prolongation of a direct connection on the bundle (Theorem 4.34). The idea comes from the theory of regularity structures in stochastic PDEs, introduced by M. Hairer in [Hai14a] for functions on $\mathbb{R}^n$ and further developed for functions on a manifold by A. Dahlqvist, J. Diehl and B. Driver in [DDD19], and for sections of a vector bundle by M. Hairer and H. Singh in [HS23]. To define this higher direct connection, we start with a direct connection Γ on the frame groupoid of a vector bundle E and give a local approximation formula for sections of E with displacement along the fibres ruled by Γ (Theorem 4.29). This requires a generalized notion of Taylor expansion determined by Γ , which contains new integral correction terms (non-local) based on the derivatives of the curvature of Γ along geodesics (Definition 4.24 and Example 4.31). Such derivatives vanish systematically if Γ is the parallel transport of a usual connection, so the non-local correction terms to the Taylor expansion are a new feature of general direct connections. We conclude with the sketch of a *local* model for E -valued jet regularity structures based on the generalized Taylor expansion induced by a general direct connection on E (Remark 4.13).

4.1 Jet prolongation of groupoids

Following [Ehr55], [Kol03, §1], [Mei17, §4.5], we recall the definition and main properties of jet groupoids, using local bisections of Section 2.3. The n -jet **prolongation** of $\mathcal{G}$ is the set of n -jets of local bisections $\sigma : M \rightarrow \mathcal{G}$, that is, the jet space

$$J^n \mathcal{G} := J^n \mathcal{B}_{\text{loc}}(\mathcal{G}) = \{j^n \sigma \in J^n(M, \mathcal{G}) \mid \mathbf{s} \circ \sigma = \text{Id}, \varphi_\sigma = \mathbf{t} \circ \sigma \text{ is a diffeomorphism}\}, \quad (29)$$

together with the structure of a Lie groupoid on M induced by that of $\mathcal{G}$:

- (i) **source** and **target** maps $\mathbf{s}^n, \mathbf{t}^n : J^n \mathcal{G} \rightarrow M$ given respectively by the surjective submersions $\mathbf{s}^n(j_x^n \sigma) = x$ and $\mathbf{t}^n(j_x^n \sigma) = \mathbf{t}(\sigma(x)) = \varphi_\sigma(x)$ for any $x \in M$,
- (ii) the **multiplication** $j_x^n \sigma' j_x^n \sigma = j_x^n(\sigma' \bullet \sigma)$ defined if $x' = \varphi_\sigma(x)$, where $\bullet$ is the product of bisections given in Section 2.3,
- (iii) **unit** $\mathbf{u}^n(x) \equiv 1_x^n = j_x^n \mathbf{u}$, where $\mathbf{u} : M \rightarrow \mathcal{G}$ is the unit bisection of $\mathcal{G}$,
- (iv) **inverse** $(j_x^n \sigma)^{-1} = j_{\varphi_\sigma^{-1}(x)}^n \sigma^{-1}$ where σ^{-1} is the inverse local bisection as given above.

From (i) it follows that

$$(J^n \mathcal{G})_y^x = \{j_x^n \sigma \mid \sigma \text{ is a local bisection on } \mathcal{G} \text{ defined around } x \text{ such that } \varphi_\sigma(x) = y\}. \quad (30)$$

Since $\mathcal{G}$ is fibered over M by the source map, the jet prolongation $J^n \mathcal{G}$ enjoys the same properties as the jet bundle of a fibre bundle described in Section A.4. In particular:

- $J^n \mathcal{G}$ is a closed submanifold of the jet space $J^n(M, \mathcal{G})$ [KMS93, §12.16],
- the jet projections $\pi_{k-1}^k : J^k \mathcal{G} \rightarrow J^{k-1} \mathcal{G}$, $j_x^k \sigma \mapsto j_x^{k-1} \sigma$, for any $1 \leq k \leq n$, give a sequence of Lie groupoid morphisms [Mei17, Example 1.12]

$$J^n \mathcal{G} \rightarrow J^{n-1} \mathcal{G} \rightarrow \dots \rightarrow J^1 \mathcal{G} \rightarrow J^0 \mathcal{G} = \mathcal{G}. \quad (31)$$

In particular, the natural projection $\pi_0^n : J^n \mathcal{G} \rightarrow \mathcal{G}$ is then also a groupoid morphism. In fact, for any σ, σ' in $\mathcal{B}_{\text{loc}}(\mathcal{G})$ and for any x in M such that σ' is defined in $x' = \varphi_\sigma(x)$, we have

$$\begin{aligned} \pi_0^n(j_{x'}^n \sigma' j_x^n \sigma) &= \pi_0^n(j_x^n(\sigma' \bullet \sigma)) = (\sigma' \bullet \sigma)(x) \\ &= \sigma'(x') \sigma(x) = \pi_0^n(j_{x'}^n \sigma') \pi_0^n(j_x^n \sigma). \end{aligned}$$

A smooth map $\phi : \mathcal{G} \rightarrow \mathcal{G}'$ between two Lie groupoids over M can be prolonged to jets using the jet prolongation of the pushforward $\mathcal{B}_{\text{loc}}(\phi)$ in (3) on local bisections, namely

$$J^n \phi : J^n \mathcal{G} \rightarrow J^n \mathcal{G}', \quad j_x^n \sigma \mapsto J^n \phi(j_x^n \sigma) := j_x^n(\phi \circ \sigma). \quad (32)$$

If ϕ is a groupoid morphism, then $J^n \phi$ is also a groupoid morphism, because $\mathcal{B}_{\text{loc}}(\phi)$ is a morphism of pseudo-groups (cf. Section 2.3). Thanks to the locality of the notion of jets, the jet prolongation (32) actually extends to any local map $\phi : \mathcal{G} \rightarrow \mathcal{G}'$. Finally, the n -jet prolongation J^n is a functor on Lie groupoids with smooth local maps, which preserves morphisms.

Proposition 4.1. *Any smooth local map $\phi : \mathcal{G} \rightarrow \mathcal{G}'$ yields a local jet prolongation $J^n \phi : J^n \mathcal{G} \rightarrow J^n \mathcal{G}'$. If ϕ is a groupoid morphism, then $J^n \phi$ is also a groupoid morphism.*

Proof. Suppose that ϕ is defined on an open neighborhood $\mathcal{U} \subset \mathcal{G}$ of the units $\mathbf{u}(M) \subset \mathcal{G}$. We only need to define an open neighborhood $\mathcal{U}^n \subset J^n \mathcal{G}$ of the units $\mathbf{u}^n(M) \subset J^n \mathcal{G}$ on which the jet prolongation (32) is well defined. Note that, being a local map in the sense of Definition 2.5, ϕ preserves the source, the target and the units. Then, since $\mathcal{U}$ is open in $\mathcal{G}$, the subset

$$\mathcal{B}_{\text{loc}}(\mathcal{U}) = \{\sigma \in \mathcal{B}_{\text{loc}}(\mathcal{G}) \mid \sigma \text{ is defined on } U \subset M \text{ such that } \sigma(U) \subset \mathcal{U}\}$$

is open in $\mathcal{B}_{\text{loc}}(\mathcal{G})$, by the topological properties in [Mic80, §4.4]. Therefore, the set

$$\mathcal{U}^n := J^n \mathcal{B}_{\text{loc}}(\mathcal{U}) = \{j_x^n \sigma \in J^n \mathcal{G} \mid \sigma \text{ is defined on } U \subset M \text{ such that } \sigma(U) \subset \mathcal{U}\}$$

satisfies the required properties, and moreover the jet projection π_0^n restricted to $\mathcal{U}^n$ has image in $\mathcal{U}$. Indeed, $\mathcal{U}^n$ contains the units $\mathbf{u}^n(M) = \{j_x^n \mathbf{u} \mid x \in M\}$, because $\mathbf{u}(M) \subset \mathcal{U}$, and it is open in $J^n \mathcal{G}$ in the Fréchet topology induced by the supremum norm in all derivatives of order no larger than n on compact subsets, which corresponds to compact open topology in [Mic80, §4.3]. Finally, the fact that J^n preserves groupoid morphisms can be easily verified with a direct computation. $\square$

Example 4.2 (Jet prolongation of gauge groupoids). It was proved in [Kol07, §4] that the n -jet prolongation of a gauge groupoid $\mathcal{G}(P) \rightrightarrows M$ is isomorphic to the gauge groupoid of the n -jet principal bundle $W^n P$, namely

$$J^n \mathcal{G}(P) \cong \mathcal{G}(W^n P). \quad (33)$$

Furthermore, the jet prolongation of a morphism $\phi : \mathcal{G}(P) \rightarrow \mathcal{G}(P')$ yields a morphism of gauge groupoids $J^n \phi : \mathcal{G}(W^n P) \rightarrow \mathcal{G}(W^n P')$. This gives rise to several examples:

- (a) For $P = M \times G$ we have $W^n(M \times G) \cong F^n M \times T_d^n G \cong F^n M \times_{GL_d^n} W^n G$, hence

$$J^n(M \times G \times M) \cong F^n M \times T_d^n G \times F^n M \cong F^n M \times_{GL_d^n} W^n G \times_{GL_d^n} F^n M. \quad (34)$$

- (b) In particular, for $G = \{1\}$ we have $T_d^n(\{1\}) = \{1\}$ and $W^n(M \times \{1\}) = F^n M$, therefore

$$J^n \text{Pair}(M) = \text{inv} J^n(M, M) \cong \mathcal{G}(F^n M) = F^n M \times_{GL_d^n} F^n M. \quad (35)$$

For $n = 1$, since $GL_d^1 = GL_d(\mathbb{R})$ and $F^1 M = F(TM)$ is a frame bundle, the 1-jet groupoid is a frame groupoid [Mei17, Example 1.12]

$$J^1 \text{Pair}(M) \cong \mathcal{G}(FM) \cong GL(TM). \quad (36)$$

- (c) The frame groupoid $GL(E)$ of a vector bundle E with rank r is the gauge groupoid of the frame bundle FE with fibre $GL_r := GL_r(\mathbb{R})$. It follows that

$$J^n GL(E) \cong \mathcal{G}(W^n FE) = W^n FE \times_{W_d^n GL_r} W^n FE. \quad (37)$$

- (d) Suppose that a frame bundle FE admits a reduction first to a principal G -bundle P , and further to the trivial principal bundle $M \times \{1\}$ (which forces E to be trivial, i.e. $E \cong M \times \mathbb{R}^r$). Then we have a sequence of gauge subgroupoids

$$\mathcal{G}(M \times \{1\}) = \text{Pair}(M) \subset \mathcal{G}(P) \subset \mathcal{G}(FE) = GL(E).$$

The jet bundle $J^n E$ has structure group $GL(P_{d,r}^n)$, frame bundle $F(J^n E)$ and frame groupoid $GL(J^n E)$, which all admit reductions to the jet prolongation of the starting objects: we have a sequence of subgroups

$$\begin{aligned} W_d^n \{1\} &= GL_d^n \subset W_d^n G = GL_d^n \ltimes T_d^n G \\ &\subset W_d^n GL_r = GL_d^n \ltimes T_d^n GL_r \subset GL(P_{d,r}^n), \end{aligned} \quad (38)$$

a corresponding sequence of reduced principal bundles

$$\begin{aligned} W^n(M \times \{1\}) &= F^n M \subset W^n P = F^n M \times_M J^n P \\ &\subset W^n FE = F^n M \times_M J^n FE \subset F(J^n E), \end{aligned} \quad (39)$$

and an induced sequence of subgroupoids

$$\begin{aligned} \mathcal{G}(F^n M) &= J^n \text{Pair}(M) \subset \mathcal{G}(W^n P) = J^n \mathcal{G}(P) \\ &\subset \mathcal{G}(W^n FE) = J^n GL(E) \subset \mathcal{G}(F(J^n E)) = GL(J^n E). \end{aligned} \quad (40)$$

□

Example 4.3 (Frame groupoid of a jet bundle). Let $E \rightarrow M$ be a vector bundle. The frame groupoid $\text{GL}(J^n E)$ of the jet bundle of E is the gauge groupoid of the frame bundle $F(J^n E)$ and it is *not* a jet prolongation, since it contains $J^n \text{GL}(E) = \mathcal{G}(W^n FE)$ as a proper subgroupoid. Indeed, by Lemma A.1, $GL(P_{d,r}^n)$ contains $W_d^n GL_r$ as a proper subgroup, hence the frame bundle $F(J^n E)$ contains the jet prolongation $W^n FE$ as a proper principal subbundle. Then, by (124) and (33) we have

$$\begin{aligned} \text{GL}(J^n E) &\cong \mathcal{G}(F(J^n E)) = F(J^n E) \times_{GL(P_{d,r}^n)} F(J^n E) \\ &\cong W^n FE \times_{W_d^n GL_r} F(J^n E) \\ &\cong W^n FE \times_{W_d^n GL_r} GL(P_{d,r}^n) \times_{W_d^n GL_r} W^n FE, \end{aligned} \quad (41)$$

while by (37) we have $J^n \text{GL}(E) \cong W^n FE \times_{W_d^n GL_r} W^n FE$. The inclusion $J^n \text{GL}(E) \hookrightarrow \text{GL}(J^n E)$ is given by $[A, B]_\sim \mapsto [A, 1^n, B]_\sim$ for any $A, B \in W^n FE$, where 1^n is the unit in the group $GL(P_{d,r}^n)$, with equivalence relations $(AH, B) \sim (A, B H^{-1})$ and $(AH, G, BH') \sim (A, HG(H')^{-1}, B)$ for any $G \in GL(P_{d,r}^n)$ and $H, H' \in W_d^n GL_r$. □

Example 4.4 (Action of a jet frame groupoid on the jet bundle). The inclusion $J^n \text{GL}(E) \hookrightarrow \text{GL}(J^n E)$ of Example 4.3 can also be understood as the canonical action of a jet $j_{x_0}^n \sigma \in J_{x_0}^n \text{GL}(E)$ on the bundle $J^n E$, where $\sigma : U \rightarrow \text{GL}(E)$ is a local bisection defined on an open neighborhood of $x_0 \in M$, that is, as a linear map $j_{x_0}^n \sigma : J_{x_0}^n E \rightarrow J_{y_0}^n E$, where $y_0 = \varphi_\sigma(x_0)$. For any $j_{x_0}^n f \in J_{x_0}^n E$, where $f : U \rightarrow E_U$ is a local section around x_0 , this action is given by

$$\begin{aligned} j_{x_0}^n \sigma \cdot j_{x_0}^n f &:= j_{y_0}^n \left(y \mapsto \sigma(\varphi_\sigma^{-1}(y)) \cdot f(\varphi_\sigma^{-1}(y)) \right) \\ &= j_{x_0}^n \left(x \mapsto \sigma(x) \cdot f(x) \right) \circ j_{y_0}^n \varphi_\sigma^{-1}, \end{aligned} \quad (42)$$

where φ_σ^{-1} is the inverse diffeomorphism of φ_σ and where we denote by $\cdot$ the linear action of $\sigma(x) \in \text{GL}(E)_{\varphi_\sigma(x)}^x$ on $f(x) \in E_x$, as in eq. (6), and by $\circ$ the jet composition as in eq. (100). Indeed, the expression $\tilde{f}(y) = \sigma(\varphi_\sigma^{-1}(y)) \cdot f(\varphi_\sigma^{-1}(y))$ of eq. (42) gives the section of E on $\varphi_\sigma(U)$ making the following diagram commutative:

$$\begin{array}{ccc} E_U & \xrightarrow{\sigma} & E_{\varphi_\sigma(U)} \\ f \uparrow & & \uparrow \tilde{f} \\ U & \xrightarrow{\varphi_\sigma} & \varphi_\sigma(U) \end{array}$$

□

4.2 Jet prolongation of Lie algebroids

The n -jet bundle of a Lie algebroid $A \rightarrow TM$ is the vector bundle $J^n A \rightarrow M$ of its underlying vector bundle $A \rightarrow M$. It turns out [Kum15, Proposition 1] [Mei17, Example 6.11] that this jet bundle is again a Lie algebroid $J^n A \rightarrow TM$, with anchor and bracket given by

$$a^n(j_x^n X) = a(X_x) \quad \text{and} \quad [j_x^n X, j_x^n Y]_{J^n A} = j_x^n([X, Y]_A) \quad (43)$$

for any smooth local sections X, Y of A defined around x in M . If $\phi : A \rightarrow A'$ is a smooth map between Lie algebroids which commutes with the anchors, then there exists a jet prolongation

$$J^n \phi : J^n A \rightarrow J^n A', \quad j_x^n X \mapsto J^n \phi(j_x^n X) := j_x^n(\phi \circ X) \quad (44)$$

which commutes with the jet prolongation of the anchors. If ϕ also preserves the bracket, hence it is a morphism of Lie algebroids, then by (43) the map $J^n \phi$ is also a morphism of Lie algebroids.

Example 4.5 (Jet prolongation of Atiyah algebroids). For a principal G -bundle P , the n -jet prolongation of its Atiyah algebroid $A(P)$ is isomorphic to the Atiyah algebroid of its n -jet principal bundle $W^n P$ [Kol08, §2 Eq. (8)]

$$J^n A(P) \cong A(W^n P). \quad (45)$$

A morphism $\phi : P \rightarrow P'$ of principal bundles over M induces a morphism $A(\phi) : A(P) \rightarrow A(P')$ between their algebroids [Mac05, Definition 3.2.8] and a principal prolongation (121) $W^n \phi : W^n P \rightarrow W^n P'$. One can check that the jet prolongation (44) $J^n A(\phi) : J^n A(P) \rightarrow A(P')$ coincides with the algebroid map $A(W^n \phi) : A(W^n P) \rightarrow A(W^n P')$. $\square$

Example 4.6 (Lie algebroid of jet groupoids). If $\mathcal{G}$ is a Lie groupoid with Lie algebroid $A(\mathcal{G})$, there is an isomorphism of Lie algebroids [Kum15, Proposition 1 and Theorem 1], [Mei17, Example 9.5 (e)]

$$A(J^n \mathcal{G}) \cong J^n A(\mathcal{G}). \quad (46)$$

A smooth local map $\phi : \mathcal{G} \rightarrow \mathcal{G}'$ of Lie groupoids over M induces a vector bundle map $d^t \phi|_M : A(\mathcal{G}) \rightarrow A(\mathcal{G}')$ between their Lie algebroids by Proposition 2.15, and a local jet prolongation $J^n \phi : J^n \mathcal{G} \rightarrow J^n \mathcal{G}'$ by Proposition 4.1. One can check that the jet prolongation $J^n(d^t \phi|_M) : J^n A(\mathcal{G}) \rightarrow J^n A(\mathcal{G}')$ coincides with the $\mathbf{t}$ -differential $d^t(J^n \phi)|_M : A(J^n \mathcal{G}) \rightarrow A(J^n \mathcal{G}')$. $\square$

Examples 4.7 (Summary on jet algebroids and jet groupoids). The previous results relate in many ways Atiyah algebroids, gauge groupoids and their jet prolongations:

$$J^n A(\mathcal{G}(P)) \stackrel{(46)}{\cong} A(J^n \mathcal{G}(P)) \stackrel{(33)}{\cong} A(\mathcal{G}(W^n P)) \stackrel{(9)}{\cong} A(W^n P) \stackrel{(45)}{\cong} J^n A(P).$$

These isomorphisms give some key examples:

- (a) [Kol09, Proposition 1], [Kol08, §2 Eq. (6)] For $P = M \times \{1\}$, we have $A(M \times \{1\}) = TM$, $W^n(M \times \{1\}) = F^n M$ and $\mathcal{G}(M \times \{1\}) = \text{Pair}(M)$, therefore

$$A(J^n \text{Pair}(M)) \cong A(F^n M) \cong J^n TM. \quad (47)$$

The case $n = 1$ is particularly important for connections, and gives a sequence of key isomorphisms from $J^1 TM$ to the bundle $T^{\text{lin}} TM$ of Example 2.12 (e):

$$J^1 TM \stackrel{(47)}{\cong} A(FM) \stackrel{(9)}{\cong} A(J^1 \text{Pair}(M)) \stackrel{(36)}{\cong} A(\text{GL}(TM)) = T^{\text{lin}} TM. \quad (48)$$

Using the isomorphism of Example 2.13, we also have $J^1 TM \cong \text{Der}(TM)$.

- (b) For $P = M \times G$, we have $A(M \times G) = TM \oplus (M \times \mathfrak{g})$, $W^n(M \times G) = F^n M \times T_d^n G$ and $\mathcal{G}(M \times G) = M \times G \times M$, therefore

$$J^n(TM \oplus (M \times \mathfrak{g})) \cong A(J^n(M \times G \times M)) \cong A(F^n M \times T_d^n G).$$

Since $F^n M$ is a principal GL_d^n -bundle, this algebroid fits in Example 2.12 (d), and we have

$$J^n(TM \oplus (M \times \mathfrak{g})) \cong A(F^n M) \oplus (M \times \text{Lie}(T_d^n G)) \cong J^n TM \oplus (M \times T_d^n \mathfrak{g}). \quad (49)$$

- (c) If $P = FE$, then $A(FE) = T^{\text{lin}} E$ is the Lie algebroid of linear vector fields of E , which is isomorphic to that of derivative endomorphisms $\text{Der}(E)$, $W^n(FE) \cong F^n M \times^{\pi} J^n FE$ and $\mathcal{G}(FE) = \text{GL}(E)$. Therefore

$$J^n T^{\text{lin}} E \cong A(J^n \text{GL}(E)) \cong A(F^n M \times_M J^n FE) \cong J^n \text{Der}(E).$$

□

4.3 Jet prolongation of direct connections

Let $\Sigma : \text{Pair}(M) \rightarrow J^n \mathcal{G}$ be a direct connection on the jet prolongation of a Lie groupoid, for $n \geq 1$, defined on a diagonal domain $\mathcal{U} \subset \text{Pair}(M)$. From the description of $(J^n \mathcal{G})_y^x$ given in (30), it follows that for any $(y, x) \in \mathcal{U}$ the groupoid element $\Sigma(y, x) \in (J^n \mathcal{G})_y^x$ is necessarily of the form $\Sigma(y, x) = j_x^n \sigma_y^x$, where σ_y^x is a local bisection of $\mathcal{G}$ defined around x whose associated local diffeomorphism $\varphi_y^x = \mathbf{t} \circ \sigma_y^x$ satisfies $\varphi_y^x(x) = y$ (which follows from (30)) and $\varphi_x^x(z) = z$ (which follows from $\Sigma(x, x) = 1_x^n$). Direct connections on $J^n \mathcal{G}$ can be studied via the collection of their associated local bisections.

Definition 4.8. Consider on $\mathcal{B}_{\text{loc}}(\mathcal{G})$ the graph topology [Mic80, §4.4] or the compact open topology [SW15, Definition A.6], as in Section 2.3. A **jet primitive bisection** of Σ is a smooth local map

$$\sigma_{\Sigma} : \text{Pair}(M) \rightarrow \mathcal{B}_{\text{loc}}(\mathcal{G}), (y, x) \mapsto \sigma_y^x$$

defined on the diagonal domain $\mathcal{U} \subset \text{Pair}(M)$ of Σ and such that, for every $(y, x) \in \mathcal{U}$,

- (i) $\Sigma(y, x) = j_x^n \sigma_y^x$
- (ii) the local diffeomorphisms $\varphi_y^x = \mathbf{t} \circ \sigma_y^x$ satisfy $\varphi_y^x(x) = y$ and $\varphi_x^x = \text{id}$.

A direct connection Σ on $J^n \mathcal{G}$ determines:

- a direct connection on $\mathcal{G}$ given by the composite map

$$\Sigma_0 = \pi_0^n \circ \Sigma : \text{Pair}(M) \rightarrow \mathcal{G}, (y, x) \mapsto \sigma_y^x(x) \in \mathcal{G}_y^x, \quad (50)$$

where $\pi_0^n : J^n \mathcal{G} \rightarrow J^0 \mathcal{G} = \mathcal{G}$ is the jet projection described in §4.1,

- an n -th order direct connection on $\text{Pair}(M)$ given by the composite map

$$\Delta_{\Sigma} = J^n \mathbf{a} \circ \Sigma : \text{Pair}(M) \rightarrow J^n \text{Pair}(M), (y, x) \mapsto j_x^n \tilde{\sigma}_y^x, \quad (51)$$

where $\mathbf{a} : \mathcal{G} \rightarrow \text{Pair}(M)$ is the anchor of $\mathcal{G}$, $J^n \mathbf{a} : J^n \mathcal{G} \rightarrow J^n \text{Pair}(M)$ is its transform under the jet functor J^n defined in (32), and $\tilde{\sigma}_y^x(z) = (\varphi_y^x(z), z)$ is the jet primitive bisection on $J^n \text{Pair}(M)$ induced by σ_y^x on $J^n \mathcal{G}$.

These two direct connections are related by the equality

$$\mathbf{a} \circ \Sigma_0 = \text{Id}_{\text{Pair}(M)} = \mathbf{a}_M^n \circ \Delta_\Sigma, \quad (52)$$

where $\mathbf{a}_M^n : J^n \text{Pair}(M) \rightarrow \text{Pair}(M)$ is the anchor of $J^n \text{Pair}(M)$, as defined in section 4.1. In other words, we have a commutative diagram

Definition 4.9. Given a direct connection $\Gamma : \text{Pair}(M) \ast \rightarrow \mathcal{G}$ on the groupoid $\mathcal{G}$ and a direct connection $\Delta^n : \text{Pair}(M) \ast \rightarrow J^n \text{Pair}(M)$ on the jet pair groupoid, we call **n -jet prolongation of Γ along Δ^n** the direct connection on $J^n \mathcal{G}$ given by composite local map

$$\Gamma^n = J^n \Gamma \circ \Delta^n, \quad (53)$$

where $J^n \Gamma : J^n \text{Pair}(M) \rightarrow J^n \mathcal{G}$ is the transformed of Γ under the jet functor J^n given in (32). We then have $(\Gamma^n)_0 = \Gamma$ and

$$\Delta_{\Gamma^n} = J^n \mathbf{a} \circ \Gamma^n = J^n \mathbf{a} \circ J^n \Gamma \circ \Delta^n = J^n (\mathbf{a} \circ \Gamma) \circ \Delta^n = \Delta^n.$$

Moreover, the jet prolongation Γ^n is compatible with the filtration on $J^n \mathcal{G}$, that is,

$$\pi_{n-1}^n \Gamma^n = \Gamma^{(n-1)}, \quad (54)$$

where $\pi_{n-1}^n : J^n \mathcal{G} \rightarrow J^{n-1} \mathcal{G}$ are the canonical projections.

Hence a direct connection on a groupoid $\mathcal{G}$ admits several jet prolongations to $J^n \mathcal{G}$, determined by direct connections on $J^n \text{Pair}(M)$.

Example 4.10 (Direct connections on the jet pair groupoid). Since the anchor on $\text{Pair}(M)$ is the identity map, any direct connection $\Delta^n : \text{Pair}(M) \ast \rightarrow J^n \text{Pair}(M)$ is the jet prolongation of $\Delta^0 = \text{Id} : \text{Pair}(M) \rightarrow \text{Pair}(M)$ (along itself), and it is uniquely determined by its jet primitive bisection σ_y^x , that is, by a smooth collection of local diffeomorphisms φ_y^x from a neighborhood of $x \in M$ to a neighborhood of a nearby point y such that $\sigma_y^x(z) = (\varphi_y^x(z), z)$. In the next section we give three examples of direct connections Δ^n on $J^n \text{Pair}(M)$, associated to three different jet primitive diffeomorphisms. $\square$

The jet primitive bisections of direct connections on $J^n \text{Pair}(M)$ allow us to give explicit formulas for the jet prolongations of direct connections on any groupoid $\mathcal{G}$, and for their infinitesimal connections on $A(J^n \mathcal{G}) \cong J^n A(\mathcal{G})$, cf. eq. (46).

Theorem 4.11. A direct connection Σ on $J^n \mathcal{G}$ is the jet prolongation of a direct connection Γ on $\mathcal{G}$ if and only if its jet primitive bisections σ_y^x are (up to contact order n) of the form

$$\sigma_y^x(z) = \Gamma(\varphi_y^x(z), z),$$

where φ_y^x are the local diffeomorphisms of some jet primitive bisections of Δ_Σ on $J^n\text{Pair}(M)$, that is,

$$\Delta_\Sigma(y, x) = j_x^n(z \mapsto (\varphi_y^x(z), z)).$$

In this case, its infinitesimal connection $\delta^\Sigma : TM \rightarrow A(J^n\mathcal{G}) \cong J^n A(\mathcal{G})$ is given on any $X \in T_x M$ by

$$\delta^\Sigma(X) = j_x^n \left(z \mapsto \delta^\Gamma \left(\frac{d}{dt} \varphi_{y(t)}^x(z) \Big|_{t=0} \right) \right) = J^n \delta^\Gamma \circ \delta^{\Delta_\Sigma}(X),$$

where δ^Γ is the infinitesimal connection of Γ , $y(t)$ is a small geodesic with $y(0) = x$ and $\dot{y}(0) = X$, and

$$\delta^{\Delta_\Sigma}(X) = j_x^n \left(z \mapsto \frac{d}{dt} \varphi_{y(t)}^x(z) \Big|_{t=0} \right) \quad (55)$$

is the infinitesimal connection $\delta^{\Delta_\Sigma} : TM \rightarrow A(J^n\text{Pair}(M)) \cong J^n TM$ of the direct connection Δ_Σ on $J^n\text{Pair}(M)$.

Proof. The first assertion is an easy direct computation: on one side we have $\Sigma(y, x) = j_x^n \sigma_y^x$, and on the other side we have

$$J^n \Gamma(\Delta_\Sigma^n(y, x)) = J^n \Gamma(j_x^n \tilde{\sigma}_y^x) = J^n \Gamma(j_x^n(z \mapsto (\varphi_y^x(z), z))) = j_x^n(z \mapsto \Gamma(\varphi_y^x(z), z)).$$

For the second assertion, let $y(t)$ be a small geodesic starting at x in the direction of $X \in T_x M$. Then, according to Proposition 3.13, we have

$$\begin{aligned} \delta^\Sigma(X) &= \frac{d}{dt} \Sigma(y(t), x) \Big|_{t=0} = \frac{d}{dt} j_x^n \left(z \mapsto \Gamma(\varphi_{y(t)}^x(z), z) \right) \Big|_{t=0} \\ &= j_x^n \left(z \mapsto \frac{d}{dt} \Gamma(\varphi_{y(t)}^x(z), z) \Big|_{t=0} \right) = j_x^n \left(z \mapsto \delta^\Gamma \left(\frac{d}{dt} \varphi_{y(t)}^x(z) \Big|_{t=0} \right) \right), \end{aligned}$$

where the equality between the first and the second line is due to the fact that the derivatives in t and in the coordinates of z commute since all the maps are smooth and the variables t and z are mutually independent. $\square$

Example 4.12 (Jet prolongation of direct connections on frame groupoids). Let $\Gamma : \text{Pair}(M) \multimap \text{GL}(E)$ be a direct connection on a frame groupoid defined on a diagonal domain $\mathcal{U} \subset \text{Pair}(M)$. Then, for any (y_0, x_0) in $\mathcal{U}$, $\Gamma(y_0, x_0) : E_{x_0} \rightarrow E_{y_0}$, $e_{x_0} \mapsto \Gamma(y_0, x_0) \cdot e_{x_0}$ is a linear isomorphism.

Let $\Delta^n : \text{Pair}(M) \multimap J^n\text{Pair}(M)$ be a direct connection on the jet pair groupoid defined on the same diagonal domain $\mathcal{U}$, with jet primitive diffeomorphisms $\varphi_{y_0}^{x_0}$.

- (a) The jet prolongation $\Gamma^n : \text{Pair}(M) \multimap J^n\text{GL}(E)$ along Δ^n is best described by linear operators acting on the jet bundle $J^n E$, as in Example 4.4. For $(y_0, x_0) \in \mathcal{U}$, the linear isomorphism $\Gamma^n(y_0, x_0) : J_{x_0}^n E \rightarrow J_{y_0}^n E$ is given by

$$\begin{aligned} \Gamma^n(y_0, x_0) \cdot j_{x_0}^n f &= j_{y_0}^n \left(y \mapsto \Gamma(y, \psi_{x_0}^{y_0}(y)) \cdot f(\psi_{x_0}^{y_0}(y)) \right) \\ &= j_{x_0}^n \left(x \mapsto \Gamma(\varphi_{y_0}^{x_0}(x), x) \cdot f(x) \right) \circ j_{y_0}^n \psi_{x_0}^{y_0}, \end{aligned}$$

where f is a local section of E around x_0 and $\psi_{x_0}^{y_0}$ is the inverse diffeomorphism of $\varphi_{y_0}^{x_0}$.

- (b) If $E = M \times \mathbb{R}^r$ is the trivial bundle on $M = \mathbb{R}^d$, with constant global section $\mathbf{1} : x \mapsto \mathbf{1}_x = (x, (1, \dots, 1))$, consider the trivial direct connection on $\text{GL}(E)$ given by $\Gamma(y, x) \cdot \mathbf{1}_x = \mathbf{1}_y$. Then, for a section $f : M \rightarrow E$, $x \mapsto f(x) \mathbf{1}_x$, we have $\Gamma(y, x) \cdot f(x) \mathbf{1}_x = f(x) \mathbf{1}_y$. It follows that

$$\begin{aligned} \Gamma^n(y_0, x_0) \cdot j_{x_0}^n f \mathbf{1}_{x_0} &= j_{y_0}^n (y \mapsto \Gamma(y, \psi_{x_0}^{y_0}(y)) \cdot f(\psi_{x_0}^{y_0}(y)) \mathbf{1}_y) \\ &= j_{y_0}^n (y \mapsto f(\psi_{x_0}^{y_0}(y)) \mathbf{1}_{\psi_{x_0}^{y_0}(y)}) \\ &= (j_{x_0}^n f \circ j_{y_0}^n \psi_{x_0}^{y_0}) \mathbf{1}_{y_0} \in J_{y_0}^n E. \end{aligned}$$

- (c) If moreover $M = \mathbb{R}^d$ is equipped with the trivial connection ∇ , then $\psi_{x_0}^{y_0}(y) = y + x_0 - y_0$ and the above formula boils down to:

$$\Gamma^n(y_0, x_0) \cdot j_{x_0}^n f \mathbf{1}_{x_0} = j_{x_0}^n f \mathbf{1}_{y_0}. \quad (56)$$

□

We conclude this section by showing that the existence of a direct connection on a jet groupoid forces it to be a gauge groupoid, but it does not force the direct connection to be a jet prolongation.

Remark 4.13. If a jet groupoid $J^n \mathcal{G}$ admits a direct connection Σ , then by Proposition 3.4 it is the gauge groupoid $J^n \mathcal{G} \cong \mathcal{G}(Q)$ of some principal bundle $Q \rightarrow M$. On the other hand, since Σ induces a direct connection Σ_0 on $\mathcal{G}$, the latter is also the gauge groupoid $\mathcal{G} \cong \mathcal{G}(P)$ of some principal bundle $P \rightarrow M$. The isomorphism (33) then implies that $Q \cong W^n P$, and therefore $J^n \mathcal{G}$ is the gauge groupoid of a jet prolonged principal bundle.

However, this result does not ensure that the direct connection Σ on $J^n \mathcal{G}$ is necessarily the n -jet prolongation of its 0-th order projection Σ_0 on $\mathcal{G}$. In fact, the identity (53) is non-trivial: using the fact that $\Sigma_0 = \Gamma$ and the definition of $\Delta_\Sigma = \Delta^n$, the equality

$$\Sigma = J^n \Gamma \circ \Delta^n = J^n (\pi_0^n \circ \Sigma \circ a) \circ \Sigma$$

implies that $J^n (\pi_0^n \circ \Sigma \circ a) = \text{Id}_{J^n \mathcal{G}}$ and therefore that $\pi_0^n \circ \Sigma \circ a = \text{Id}_{\mathcal{G}}$ locally on $\mathcal{G}$ and up to order n , while the definition of direct connections only ensures that $a \circ \Gamma = \text{Id}_{\text{Pair}(M)}$ and $a^n \circ \Sigma = \text{Id}_{\text{Pair}(M)}$. A counterexample, inspired by the theory of regularity structures, is given in section 4.7. □

Before that, we show that there are canonical direct connections on $J^n \text{Pair}(M)$ constructed from torsion-free linear connections ∇ on TM . The key idea is to exploit the isomorphism $\text{GL}(TM) \cong J^1 \text{Pair}(M)$ given in (36): we find a jet primitive bisection of ∇ seen as a direct connection on $J^1 \text{Pair}(M)$, and use its jets to define the direct connections on $J^n \text{Pair}(M)$. Then, by the defining equation (53), n -jet prolongations of a direct connection Γ on $\mathcal{G}$ are also determined canonically from torsion-free linear connections ∇ on TM .

4.4 Jet prolongation of direct connections on the pair groupoid

In this section we construct three canonical direct connections on $J^n \text{Pair}(M)$ starting from a given torsion-free linear connection ∇^M on TM , and their associated infinitesimal connections.

To define them, we adopt the general strategy outlined in Theorem 4.11, which amounts to find jet primitive bisections $\sigma_y^x(z) = (\varphi_y^x(z), z)$ of Δ^n on $J^n \text{Pair}(M)$, given by local diffeomorphisms φ_y^x of M such that $\varphi_y^x(x) = y$ and $\varphi_x^x(z) = z$. In particular, Δ^0 is necessarily the identity map

on $\text{Pair}(M)$, and $\Delta^1 : \text{Pair}(M) \rightarrow J^1\text{Pair}(M) \cong \text{GL}(TM)$ is given by the differential of φ_y^x at x , namely

$$\Delta^1(y, x) = (d\varphi_y^x)_x : T_x M \xrightarrow{\cong} T_y M. \quad (57)$$

Let us fix some notations. For any point $x \in M$, denote by $\exp_x : V_{0,x} \subset T_x M \rightarrow M$ the exponential map along geodesics. As in Section 3.1, let U_x be a uniformly normal neighborhood of x , so that any $y \in U_x$ is linked to x by a unique geodesic: denote by $\mathcal{P}(y, x) : T_x M \rightarrow T_y M$ the parallel transport of ∇ along that geodesic, and by $\mathcal{U} = \bigcup_{x \in M} U_x \times U_x$ the normal diagonal domain in $\text{Pair}(M)$. In the sequel, we will use *Jacobi vector fields* and related *variational curves through geodesics*, see [Lee97, Ch.6 and Ch.10] or [dC92, Ch.5]⁵. For this, let us denote by R^M the curvature tensor of ∇^M , and by $D_t V(t) = \nabla_{\dot{y}(t)} V(t)$ the covariant derivative along a curve $y(t)$ of a vector field $V(t) = V(y(t))$ defined along the curve. We recall that a **Jacobi field** is any vector field $J(t)$ along a geodesic $y(t) = \exp_x(tX)$ such that

$$D_t^2 J(t) + R_{y(t)}^M(J(t), \dot{y}(t)) \dot{y}(t) = 0 \quad \text{for all } t \in [0, 1], \quad (58)$$

which is uniquely determined by its initial conditions $J(0)$ and $D_t J(0)$ [dC92, Definition 2.1]. In the sequel, we denote $J_{X,Z}(t)$ the Jacobi field (along a given geodesic) with initial conditions $J(0) = X$ and $D_t J(t)|_{t=0} = Z$.

m

4.4.1 Parallel jet direct connection

Definition 4.14. We call **parallel direct connection** on $J^n\text{Pair}(M)$ the local map $\mathcal{P}^n : \text{Pair}(M) \rightarrow J^n\text{Pair}(M)$ defined on any $(y, x) \in \mathcal{U}$ by

$$\mathcal{P}^n(y, x) = j_x^n(z \mapsto \alpha_y^x(z)), \quad (59)$$

with jet primitive bisections $\alpha_y^x : U_x \rightarrow \text{Pair}(M)$, $z \mapsto (\alpha_y^x(z), z)$ given by the local diffeomorphisms

$$\alpha_y^x : U_x \rightarrow M, \quad z \mapsto \alpha_y^x(z) = \exp_y(\mathcal{P}(y, x)Z), \quad (60)$$

where $Z \in T_x M$ is defined by $z = \exp_x(Z)$.

Theorem 4.15. *The parallel direct connection $\mathcal{P}^n$ on $J^n\text{Pair}(M)$ is the jet prolongation of the parallel transport along geodesics, that is, $\mathcal{P}^1(y, x) = \mathcal{P}(y, x) : T_x M \rightarrow T_y M$.*

Its infinitesimal connection $\delta^{\mathcal{P}^n} : TM \rightarrow J^n TM$ is the jet prolongation of the Jacobi transport map given for any $X \in T_x M$ by

$$\delta^{\mathcal{P}^n}(X) = j_x^n(z \mapsto J_{X,0}(1)), \quad (61)$$

where $J_{X,0}(s)$ is the Jacobi field along the geodesic $z(s) = \exp_x(sZ)$ from x to $z \in U_x$ with initial conditions $J(0) = X$ and $D_s J(s)|_{s=0} = 0$.

Proof. For the first assertion we use eq. (57). For $z \in U_x$, let $z(s) = \exp_x(sZ)$ be the geodesic from x to z with $Z = \dot{z}(0) \in T_x M$. Then we have

$$\mathcal{P}^1(y, x)Z = (da_y^x)_x(\dot{z}(0)) = \frac{d}{ds} \left(\alpha_y^x(z(s)) \right) \Big|_{s=0} = \frac{d}{ds} \left(\exp_y(\mathcal{P}(y, x) sZ) \right) \Big|_{s=0} = \mathcal{P}(y, x) Z.$$

⁵The references [Lee97, Ch.6 and Ch.10] or [dC92, Ch.5] treat Jacobi fields for Riemannian manifolds and the related Levi-Civita connection ∇ , but the results hold for any torsion-free linear connection on a manifold.

Note also that, by the isomorphisms of groupoids $\mathcal{G}(FM) \cong \text{GL}(TM)$, this can equivalently be seen as the jet composition with $j_x^1 a_y^x$ and gives the parallel transport $\mathcal{P}^{FM}(y, x) : F_x M \rightarrow F_y M$ on the frame bundle of M , that is,

$$\mathcal{P}^{FM}(y, x)(\psi) = j_x^1 a_y^x \circ j_0^1 \tilde{\psi} = j_0^1 (a_y^x \circ \tilde{\psi})$$

where a frame $\psi : \mathbb{R}^d \xrightarrow{\cong} T_x M$ is seen as the differential at 0 of a local diffeomorphism $\tilde{\psi} : \mathbb{R}^d \xrightarrow{\sim} M$ such that $\tilde{\psi}(0) = x$, and therefore it is seen as a jet $j_0^1 \tilde{\psi}$.

For the second assertion we use eq. (55), and we need to compute the derivative $\frac{d}{dt} a_{y(t)}^x(z) \big|_{t=0}$. To do this, we consider the variational curve of the geodesic $z(s)$ along the geodesic $y(t) = \exp_x(tX)$ from x to y (cf. Fig. 2) given by

$$\alpha(t, s) = a_{y(t)}^x(z(s)) = \exp_{y(t)}(sZ(t)), \quad (t, s) \in [0, 1] \times [0, 1] \quad (62)$$

where we choose $Z(t) = \mathcal{P}(y(t), x)Z$. This variational curve has

- initial data $\alpha(0, s) = z(s)$, $\alpha(t, 0) = y(t)$ and end-point $\alpha(1, 1) = \exp_y(\mathcal{P}(y, x)Z) = a_y^x(z)$,
- geodesic main curves $s \mapsto \alpha(t, s)$ in the direction of $\partial_s \alpha(t, s) \big|_{s=0} = Z(t)$,
- geodesic transversal curves $t \mapsto \alpha(t, s)$ in the direction of the Jacobi field $J_\alpha(s) = \partial_t \alpha(t, s) \big|_{t=0}$ along $z(s)$, so that $\alpha(t, s) = \exp_{z(s)}(tJ_\alpha(s))$.

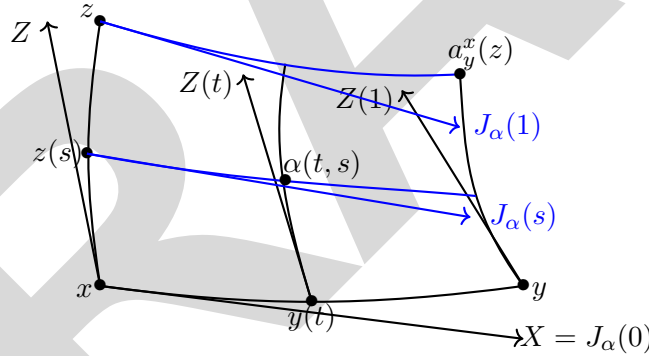


Figure 2: Variational curve $\alpha(t, s)$ of the geodesic $z(s)$, with vertical main curves in the direction of $Z(t) = \exp_{y(t)}(\mathcal{P}(y(t), x)Z)$ and horizontal transversal curves in the direction of the Jacobi field $J_\alpha(s)$.

Using eq. (55), we then have

$$\delta^{\mathcal{P}^n}(X) = j_x^n \left(z \mapsto \frac{d}{dt} a_{y(t)}^x(z) \big|_{t=0} \right) = j_x^n (z \mapsto \partial_t \alpha(t, 1) \big|_{t=0}) = j_x^n (z \mapsto J_\alpha(1)).$$

The Jacobi field $J_\alpha(s)$ satisfies the initial conditions $J_\alpha(0) = \partial_t \alpha(t, 0) \big|_{t=0} = X$ and

$$D_s J_\alpha(s) \big|_{s=0} = D_s (\partial_t \alpha(t, s) \big|_{t=0}) \big|_{s=0} = D_t (\partial_s \alpha(t, s) \big|_{s=0}) \big|_{t=0} = D_t Z(t) = 0,$$

where we used [Lee97, Lemma 6.3] and the fact that $Z(t)$ is parallel along $y(t)$. Therefore J_α coincides with $J_{X,0}$. $\square$

4.4.2 Jacobi jet direct connection

Definition 4.16. Similarly, we call **Jacobi direct connection** on $J^n\text{Pair}(M)$ the local map $\Delta^n : \text{Pair}(M) \rightarrow J^n\text{Pair}(M)$ defined on any $(y, x) \in \mathcal{U}$ by

$$\Delta^n(y, x) = j_x^n(z \mapsto \beta_y^x(z)), \quad (63)$$

with jet primitive bisections $\beta_y^x : U_x \rightarrow \text{Pair}(M)$, $z \mapsto \beta_y^x(z) = (b_y^x(z), z)$ given by the local diffeomorphisms

$$b_y^x : U_x \rightarrow M, \quad z \mapsto b_y^x(z) = \exp_z(\mathcal{P}(z, x)X), \quad (64)$$

where $X \in T_x M$ is defined by $y = \exp_x(X)$.

Theorem 4.17. *The Jacobi direct connection Δ^n on $J^n\text{Pair}(M)$ is the jet prolongation of the Jacobi transport map $\Delta^1(y, x) : T_x M \rightarrow T_y M$, $Z \mapsto J_{Z,0}(1)$, where $J_{Z,0}(t)$ is the Jacobi field along the geodesic $y(t) = \exp_x(tX)$ from x to y with initial conditions $J(0) = Z$ and $D_t J(t)|_{t=0} = 0$.*

Its infinitesimal connection $\delta^n : TM \rightarrow J^n TM$ is the jet prolongation of the parallel transport along geodesics, that is,

$$\delta^n(X) = j_x^n(z \mapsto \mathcal{P}(z, x)X).$$

Proof. For the first assertion, we use again eq. (57) and get $\Delta^1(y, x) = (db_y^x)_x : T_x M \rightarrow T_y M$. For any $Z \in T_x M$ small enough to have $z = \exp_x(Z) \in U_x$ and a geodesic $z(s) = \exp_x(sZ)$ from x to z (if Z is not small enough we rescale it and consider a geodesic on a smaller interval of s), we consider the variational curve of $y(t)$ along $z(s)$ (cf. Fig. 3) given by

$$\beta(s, t) = b_{y(t)}^x(z(s)) = \exp_{z(s)}(tX(s)), \quad (s, t) \in [0, 1] \times [0, 1] \quad (65)$$

where we choose $X(s) = \mathcal{P}(z(s), x)X$. This variational curve has

- initial data $\beta(0, t) = y(t)$, $\beta(s, 0) = z(s)$ and end-point $\beta(1, 1) = \exp_z(\mathcal{P}(z, x)X) = b_y^x(z)$,
- geodesic main curves $t \mapsto \beta(s, t)$ in the direction of $\partial_t \beta(s, t)|_{t=0} = X(s)$,
- geodesic transversal curves $s \mapsto \beta(s, t)$ in the direction of the Jacobi field $J_\beta(t) = \partial_s \beta(s, t)|_{s=0}$ along the geodesic $y(t)$, so that $\beta(s, t) = \exp_{y(t)}(sJ_\beta(t))$.

Then we have

$$\Delta^1(y, x)(Z) = (db_y^x)_x(\dot{z}(0)) = \frac{d}{ds} b_y^x(z(s))|_{s=0} = \partial_s \beta(s, 1)|_{s=0} = J_\beta(1).$$

Since $J_\beta(t)$ has initial values $J_\beta(0) = Z$ and $D_t J_\beta(t)|_{t=0} = D_s X(s)|_{s=0} = D_s \mathcal{P}(z(s), x)X|_{s=0} = 0$, it coincides with $J_{Z,0}$.

Finally, let us show that the map $\Delta^1(y, x) : T_x M \rightarrow T_y M$, $Z \mapsto J_{Z,0}(1)$ defines a direct connection on $\text{GL}(TM) \cong J^1\text{Pair}(M)$. In fact, it is well defined for $(y, x) \in \mathcal{U}$, it is linear in Z because $J_{Z,0}(t) = J_\beta(t)$ is the solution of an ODE whose initial conditions are linear in Z , and finally it is invertible: its inverse is the linear map $\Phi(x, y) : T_y M \rightarrow T_x M$, $U \mapsto \tilde{J}(1)$ where $\tilde{J}(t) := \partial_s \tilde{\beta}(s, t)$ is the Jacobi field determined by the inverse variational curve $\tilde{\beta}(s, t) = \beta(s, 1-t)|_{s=0}$ of the inverse geodesic $x(t) = y(1-t)$ from y to x , along the geodesic $u(s) = \exp_y(sU)$, hence $\tilde{J}(0) = U$, and with second initial condition

$$D_t \tilde{J}(t)|_{t=0} = D_s [-\mathcal{P}(u(s), z(s)) \mathcal{P}(z(s), x) X]|_{s=0}$$

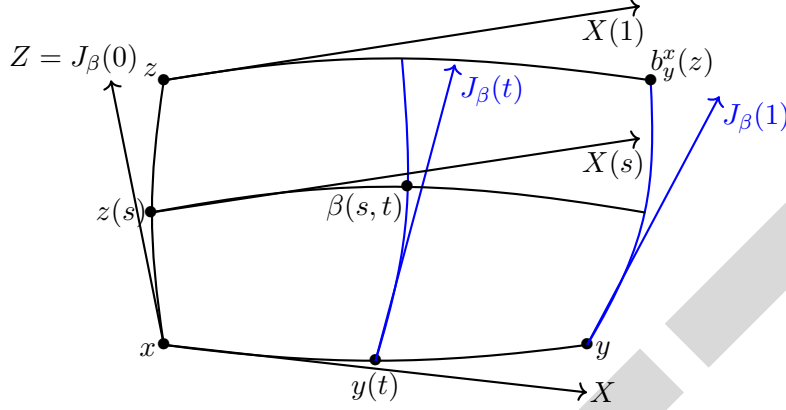


Figure 3: Variational curve $\beta(s, t)$ of the geodesic $y(t)$, with horizontal main curves in the direction of $X(s) = \exp_{z(s)}(\mathcal{P}(z(s), x)X)$ and vertical transversal curves in the direction of the Jacobi field $J_\beta(t)$.

instead of $D_s[-\mathcal{P}(u(s), y) \mathcal{P}(y, x) X]|_{s=0}$. The map $\Phi(x, y)$ is indeed the inverse of $\Delta^1(y, x)$, because $\tilde{J}(t) = J_\beta(1 - t)$: if $U = J_\beta(1) = \tilde{J}(0)$, then $\tilde{J}(1) = J_\beta(0) = Z$.

For the second assertion, we use again eq. (55) and get

$$\delta^n(X) = j_x^n \left(z \mapsto \frac{d}{dt} b_{y(t)}^x(z) \Big|_{t=0} \right) = j_x^n \left(z \mapsto \partial_t \beta(1, t) \Big|_{t=0} \right) = j_x^n (z \mapsto X(1)) = j_x^n (z \mapsto \mathcal{P}(z, x) X).$$

□

Proposition 4.18. *The local maps $\mathcal{P}^n$ and Δ^n define two generally distinct direct connections on $J^n \text{Pair}(M)$. The two direct connections coincide if and only if ∇ is flat.*

Proof. As shown in the previous proofs, the primitive bisections of the two direct connections are associated to two variational curves $\alpha(s, t)$ and $\beta(t, s)$ with the same initial data but distinct end-points, velocity vectors and Jacobi fields. The two variational curves are necessarily distinct, that is, $\alpha(t, s) \neq \beta(s, t)$ except for $s = 0$ or $t = 0$. In fact, the equality $\exp_{y(t)}(sZ(t)) = \exp_{z(s)}(tX(s))$ means that $X(s) = J_\alpha(s)$ is the variation field of $\alpha(s, t)$ and $Z(t) = J_\beta(t)$ is the variation field of β . These equalities in general do not hold, because for generic points $y, z \in U_x$ (hence generic "small" tangent vectors $X, Z \in T_x M$) the fields $X(s) = \mathcal{P}(z(s), x)X$ and $Z(t) = \mathcal{P}(y(t), x)Z$ are not Jacobi fields:

$$\begin{aligned} D_s^2 X(s) + R_{z(s)}^M(X(s), \dot{z}(s)) \dot{z}(s) &= \mathcal{P}(z(s), x) R_x^M(X, Z) Z \neq 0, \\ D_t^2 Z(t) + R_{y(t)}^M(Z(t), \dot{y}(t)) \dot{y}(t) &= \mathcal{P}(y(t), x) R_x^M(Z, X) X \neq 0. \end{aligned}$$

In particular, the two end-points $a_y^x(z) = \exp_y(\mathcal{P}(y, x)Z)$ and $b_y^x(z) = \exp_z(\mathcal{P}(z, x)X)$ are distinct.

Finally, the two direct connections $\mathcal{P}^n$ and Δ^n coincide if and only if we have $\mathcal{P}^n(y, x) = \Delta^n(y, x)$ for any points $x, y \in M$ sufficiently close to each other. This means that $a_y^x(z) = b_y^x(z)$ for any z in a neighborhood $U \subset U_x$ of x , or equivalently that the two variational curves $\alpha(t, s)$ and $\beta(s, t)$ coincide. Applying the argument above, this holds if and only if the curvature tensor $R_x^M(X, Z)$ vanishes for the fixed tangent vector $X = \exp_x(y) \in T_x M$ and for any other (small) tangent vector $Z = \exp_x(z) \in T_x M$ corresponding to any $z \in U$. If this holds for any $x \in M$ and $y \in U_x$, it means that ∇ is flat. □

4.4.3 Exponential jet direct connection

Definition 4.19. We call **exponential direct connection** on $J^n\text{Pair}(M)$ the local map $\text{Exp}^n : \text{Pair}(M) \rightarrow J^n\text{Pair}(M)$ defined on any $(y, x) \in \mathcal{U}$ by

$$\text{Exp}^n(y, x) = j_x^n(z \mapsto \gamma_y^x(z)), \quad (66)$$

with jet primitive bisections $\gamma_y^x : U_x \rightarrow \text{Pair}(M)$, $z \mapsto (c_y^x(z), z)$ given by the local diffeomorphisms

$$c_y^x : U_x \rightarrow M, \quad z \mapsto c_y^x(z) = \exp_x(X + Z), \quad (67)$$

where $X, Z \in T_x M$ are defined by $y = \exp_x(X)$ and $z = \exp_x(Z)$.

Theorem 4.20. The exponential direct connection Exp^n on $J^n\text{Pair}(M)$ is the jet prolongation of the Jacobi transport map $\text{Exp}^1(y, x) = T_x M \rightarrow T_y M$, $Z \mapsto J_{0,Z}(1)$, where $J_{0,Z}(t)$ is the Jacobi field along the geodesic $y(t) = \exp_x(tX)$ from x to y with initial conditions $J_{0,Z}(0) = 0$ and $D_t J_{0,Z} = Z$.

Its infinitesimal connection $\delta^{\text{Exp}^n} : TM \rightarrow J^n TM$ is the Lie algebroid exponential jet prolongation $\varepsilon^n : TM \rightarrow J^n TM$ given in [Mik07, §3] and [Kol09, §5].

Proof. For the first assertion, by eq. (57) we have $\text{Exp}^1(y, x) = (dc_y^x)_x : T_x M \rightarrow T_y M$. Consider the variational curve of $y(t)$ by a spray of geodesics starting at x in directions $X(s) = X + sZ$ (cf. Fig. 4), given by

$$\gamma(s, t) = \exp_x(tX(s)) = \exp_x(tX + stZ), \quad (s, t) \in [0, 1] \times [0, 1]. \quad (68)$$

This variational curve has

- initial data $\gamma(0, t) = y(t)$, $\gamma(s, 0) = x$ and end-point $\gamma(1, 1) = \exp_x(X + Z) = c_y^x(z)$,
- geodesic main curves $t \mapsto \gamma(s, t)$ in the direction of $\partial_t \gamma(s, t)|_{t=0} = X(s)$,
- geodesic transversal curves $s \mapsto \gamma(s, t)$ in the direction of the Jacobi field $J_\gamma(t) = \partial_s \gamma(s, t)|_{s=0}$ along $y(t)$, so that $\gamma(s, t) = \exp_{y(t)}(sJ_\gamma(t))$.

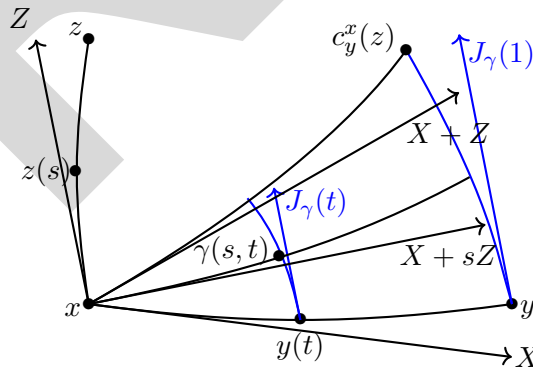


Figure 4: Variational curve $\gamma(s, t)$ of the geodesic $y(t)$, with radial main curves in the direction of $X(s) = \exp_x(X + sZ)$ and vertical transversal curves in the direction of the Jacobi field $J_\gamma(t)$.

Since $c_y^x(z(s)) = \exp_x(X + sZ) = \gamma(s, 1)$, we have

$$\text{Exp}^1(y, x)(Z) = (dc_y^x)_x(\dot{z}(0)) = \frac{d}{ds} c_y^x(z(s))|_{s=0} = \partial_s \gamma(s, 1)|_{s=0} = J_\gamma(1).$$

Since $J_\gamma(t)$ has initial values $J_\gamma(0) = 0$ and $D_t J_\gamma(t)|_{t=0} = D_s(X + sZ)|_{s=0} = Z$, it coincides with $J_{0,Z}$.

Now let us look at its infinitesimal connection δ^{Exp^n} . Let us recall that W.M. Mikulski [Mik07, §3] and I. I. Kolář [Kol09, §5] introduced a canonical **exponential jet prolongation** of the Lie algebroid connection $\varepsilon : TM \rightarrow J^1 TM \cong A(\text{GL}(TM))$ associated to a torsion-free linear connection ∇ on TM : it is the Lie algebroid connection $\varepsilon^n : TM \rightarrow J^n TM$ defined on a tangent vector $X \in T_x M$ as the n -jet

$$\varepsilon^n(X) := j_{0_x}^n \left(d \exp_x(\tilde{X}) \right), \quad (69)$$

where $d \exp_x : T(T_x M) \rightarrow TM$ is the differential of $\exp_x$ in a neighborhood of the null vector 0_x in $T_x M$ and $\tilde{X}$ is the vector field on $T_x M$ obtained by translation of X , that is, $\tilde{X}(Z) = X + Z \in T_X(T_x M)$ for any $Z \in T_x M \cong T_{0_x} T_x M$. This infinitesimal connection extends $\varepsilon^0 = \text{Id} =: TM \rightarrow TM$ because $\varepsilon^0(X) = (d \exp_x(\tilde{X}))(0_x) = (d \exp_x)_X(0_x) = X$. At higher orders, since $Z = D_t J_\gamma(t)|_{t=0}$, by [dC92, Corollary 2.5] we have

$$\varepsilon^n(X) = j_{0_x}^n(Z \mapsto (d \exp_x)_X(Z)) = j_{0_x}^n(Z \mapsto J_\gamma(1)),$$

where the jet is computed using the derivatives of the curve $s \mapsto sZ$ with values in $T_x M$. On the other side, using eq. (55), we have

$$\delta^{\text{Exp}^n}(X) = j_x^n \left(z \mapsto \frac{d}{dt} c_{y(t)}^x(z)|_{t=0} \right) = j_x^n(z \mapsto \partial_t \gamma(1, t)|_{t=0}) = j_x^n(z \mapsto J_\gamma(1)),$$

where the jet is computed using the derivatives of the curve $s \mapsto z(s)$ on M . Since $z(s) = \exp_x(sZ)$, the two infinitesimal connections ε^n and δ^{Exp^n} coincide. $\square$

Remark 4.21. For a general manifold M and a generic linear connection ∇ , the exponential jet direct connection Exp^n is clearly different from the parallel one $\mathcal{P}^n$ and from the Jacobi one J^n . As in Proposition 4.18, they coincide when ∇ is flat. $\square$

4.5 Flat jet groupoids

According to Proposition 3.26, a jet groupoid $J^n \mathcal{G}$ on a connected manifold M is flat if and only if it admits a flat direct connection. Suppose that $\mathcal{G}$ is flat. Then, is $J^n \mathcal{G}$ necessarily flat? In this section we show that in general this is not the case, and give a sufficient condition (on M) to have flat jet groupoids.

Example 4.22 (Non flat jet groupoid). **Example of jet prolongation of a flat direct connection which is not flat?** $\square$

Theorem 4.23. *Assume that the manifold M is affine, i.e. TM is equipped with a flat torsion-free linear connection. Then:*

1. *the groupoid $J^n \text{Pair}(M)$ is flat,*
2. *the jet prolongation $J^n \mathcal{G}$ of any flat groupoid $\mathcal{G}$ on M is flat.*

Proof. 1. If TM is endowed with a flat and torsion-free linear connection ∇ , then the parallel direct connection $\mathcal{P}^n$ on $J^n\text{Pair}(M)$ given in Definition 4.14 is flat, that is, it is a groupoid morphism. In fact, consider three nearby points $x, y, z \in M$ and

$$\begin{aligned} \mathcal{P}^n(y, x) &= j_x^n(u \mapsto (a_y^x(u), u)) & \text{with} & & a_y^x(u) &= \exp_y(\mathcal{P}(y, x)U), & u &= \exp_x(U) \\ \mathcal{P}^n(z, y) &= j_y^n(v \mapsto (a_z^y(v), v)) & \text{with} & & a_z^y(v) &= \exp_z(\mathcal{P}(z, y)V), & v &= \exp_y(V). \end{aligned}$$

Since ∇ is flat, we have $\mathcal{P}(z, y)\mathcal{P}(y, x) = \mathcal{P}(z, x)$ and therefore

$$a_z^y(a_y^x(u)) = \exp_z(\mathcal{P}(z, y)\mathcal{P}(y, x)U) = \exp_z(\mathcal{P}(z, x)U) = a_z^x(u).$$

Then, the multiplication in $J^n\text{Pair}(M)$ gives

$$\mathcal{P}^n(z, y)\mathcal{P}^n(y, x) = j_x^n(u \mapsto (a_z^y(a_y^x(u)), u)) = j_x^n(u \mapsto (a_z^x(u), u)) = \mathcal{P}^n(z, x).$$

This proves that the groupoid $J^n\text{Pair}(M)$ is flat.

2. Now let $\mathcal{G}$ be a flat groupoid, equipped with a flat direct connection Γ . Then its jet prolongation along $\mathcal{P}^n$ is the composition $\Gamma^n = J^n\Gamma \circ \mathcal{P}^n$. Since $J^n\Gamma$ is also a groupoid morphism, by Proposition 4.1, the direct connection Γ^n is flat. Therefore $J^n\mathcal{G}$ is a flat groupoid. $\square$

4.6 Taylor expansion with respect to direct connections

Inspired by regularity structures on a manifold [DDD19], in this section we introduce a covariant notion of Taylor expansion of local sections of a vector bundle E induced by a generic direct connection Γ on E , which turns out to present the usual *local* terms given by higher covariant derivatives and new *non-local* integral corrections, cf. Theorem 4.29. These extra integral terms vanish if Γ is transitive, that is, by Theorem 3.18, if Γ is the parallel transport of its associated linear connection, as in [DDD19].

As before, consider a base manifold M equipped with a torsion-free linear connection ∇^M on TM , and a vector bundle $E \rightarrow M$ of finite rank equipped with a direct connection $\Gamma : \text{Pair}(M) \rightarrow \text{GL}(E)$ on its frame groupoid. Suppose that Γ is defined on a normal diagonal domain $\mathcal{U} = \bigcup_{x \in M} U_x \times U_x$ covered by uniformly normal neighborhoods. Denote by $\nabla := \nabla^\Gamma$ the linear connection on E induced by Γ , see eq. (23), by $\nabla^n : \Gamma(E) \rightarrow T^*M^{\otimes n} \otimes E$ the higher covariant derivatives induced by ∇^M and ∇ , recalled in (127), and by $D_\nabla^n : \Gamma(E) \rightarrow S^n(T^*M) \otimes E$ its associated symmetrized covariant derivatives, as in (129).

Definition 4.24. Let $x_0 \in M$ be a given point, U_{x_0} a uniformly normal neighborhood of x_0 and $\psi \in \Gamma_{U_{x_0}}(M, E)$ a smooth local section defined on U_{x_0} . We call **generalized Taylor section** of order n of ψ at x_0 induced by Γ the local section $\text{GTay}_{x_0}^n \psi$ of E defined on any $x \in U_{x_0}$ as

$$\text{GTay}_{x_0}^n \psi(x) = \text{Tay}_{x_0}^n \psi(x) + \text{Int}_{x_0}^n \psi(x), \quad (70)$$

where we set

- **n -th order Taylor polynomial of ψ at x_0**

$$\text{Tay}_{x_0}^n \psi(x) = \Gamma(x, x_0) \sum_{k=0}^n \frac{1}{k!} D_\nabla^k \psi|_{x_0}(X^{\otimes k}), \quad (71)$$

- n -th order integral correction of ψ at x_0

$$\text{Int}_{x_0}^n \psi(x) = \Gamma(x, x_0) \sum_{k=0}^n \frac{1}{n!} \binom{n+1}{k} \int_0^1 (1-t)^n Q_{n+1-k}(t) \Gamma(x(t), x_0)^{-1} D_{\nabla}^k \psi|_{x(t)} (\dot{x}(t)^{\otimes k}) dt, \quad (72)$$

- n -th order remainder of ψ at x_0

$$\text{Rem}_{x_0}^n \psi(x) = \Gamma(x, x_0) \frac{1}{n!} \int_0^1 (1-t)^n \Gamma(x(t), x_0)^{-1} D_{\nabla}^{n+1} \psi|_{x(t)} (\dot{x}(t)^{\otimes n+1}) dt, \quad (73)$$

where $X \in T_{x_0}M$ is the initial velocity of the geodesic $x(t) = \exp_{x_0}(tX)$ from x_0 to x and the operator $Q_m(t) \in \text{End}(E_{x_0})$ is a polynomial in the correction terms

$$\begin{aligned} C_1(t) &:= \frac{d}{ds} K_{x_0}(x(s), x(t))|_{s=t} \\ C_2(t) &:= \frac{d}{du} C_1(u)|_{u=t} = \frac{\partial^2}{\partial u \partial s} K_{x_0}(x(s), x(u))|_{s=u=t} \\ C_n(t) &:= \frac{d}{dv} C_{n-1}(v)|_{v=t} \quad n \geq 3 \end{aligned} \quad (74)$$

given by the recurrence relation

$$Q_{m+1}(t) = \frac{d}{dt} Q_m(t) + Q_m(t) C_1(t) \quad (75)$$

with initial condition $Q_0(t) = 1_{x_0}$. Note that the argument of the integrals in (72) and (73) are usual vector-valued functions in E_{x_0} , so the integrals are considered component-wise. Finally note also that the symmetrized covariant derivatives $D_{\nabla}^k \psi$ are evaluated on the same tangent vector k times, therefore they coincide with the higher covariant derivatives $\nabla^k \psi$.

The expression of the polynomials $Q_m(t)$ in terms of the corrections $C_n(t)$ can be given in an explicit closed form, but this requires heavy combinatorial notations and we omit it because we don't need it. Instead, we give some explicit examples.

Example 4.25 (Integral corrections). The first terms of the polynomial operators $Q_m(t)$ solving the recurrence (75) are:

$$\begin{aligned} Q_0(t) &= 1_{x_0} & Q_1(t) &= C_1 & Q_2(t) &= C_2 + C_1^2 \\ Q_3(t) &= C_3 + (2 C_2 C_1 + C_1 C_2) + C_1^3 \\ Q_4(t) &= C_4 + (3 C_3 C_1 + 3 C_2^2 + C_1 C_3) + (3 C_2 C_1^2 + 2 C_1 C_2 C_1 + C_1^2 C_2) + C_1^4 \\ Q_5(t) &= C_5 + (4 C_4 C_1 + 6 C_3 C_2 + 4 C_2 C_3 + C_1 C_4) \\ &\quad + (6 C_3 C_1^2 + 3 C_1 C_3 C_1 + C_1^2 C_3) + (8 C_2^2 C_1 + 4 C_2 C_1 C_2 + 3 C_1 C_2^2) \\ &\quad + (4 C_2 C_1^3 + 3 C_1 C_2 C_1^2 + 2 C_1^2 C_2 C_1 + C_1^3 C_2) + C_1^5 \end{aligned}$$

where all the terms C_n depend on t and do not necessarily vanish for a generic direct connection Γ .

The integral corrections for the expansions up to order 4 are:

$$\begin{aligned}
\text{Int}_{x_0}^0 \psi(x) &= \Gamma(x, x_0) \int_0^1 Q_1(t) \Gamma(x(t), x_0)^{-1} \psi(x(t)) dt \\
\text{Int}_{x_0}^1 \psi(x) &= \Gamma(x, x_0) \int_0^1 (1-t) \left[Q_2(t) \Gamma(x(t), x_0)^{-1} \psi(x(t)) + 2 Q_1(t) \Gamma(x(t), x_0)^{-1} \nabla \psi|_{x(t)}(\dot{x}(t)) \right] dt \\
\text{Int}_{x_0}^2 \psi(x) &= \Gamma(x, x_0) \int_0^1 (1-t)^2 \left[\frac{1}{2} Q_3(t) \Gamma(x(t), x_0)^{-1} \psi(x(t)) + \frac{3}{2} Q_2(t) \Gamma(x(t), x_0)^{-1} \nabla \psi|_{x(t)}(\dot{x}(t)) \right. \\
&\quad \left. + \frac{3}{2} Q_1(t) \Gamma(x(t), x_0)^{-1} \nabla^2 \psi|_{x(t)}(\dot{x}(t)^{\otimes 2}) \right] dt \\
\text{Int}_{x_0}^3 \psi(x) &= \Gamma(x, x_0) \int_0^1 (1-t)^3 \left[\frac{1}{6} Q_4(t) \Gamma(x(t), x_0)^{-1} \psi(x(t)) + \frac{2}{3} Q_3(t) \Gamma(x(t), x_0)^{-1} \nabla \psi|_{x(t)}(\dot{x}(t)) \right. \\
&\quad \left. + Q_2(t) \Gamma(x(t), x_0)^{-1} \nabla^2 \psi|_{x(t)}(\dot{x}(t)^{\otimes 2}) + \frac{2}{3} Q_1(t) \Gamma(x(t), x_0)^{-1} \nabla^3 \psi|_{x(t)}(\dot{x}(t)^{\otimes 3}) \right] dt \\
\text{Int}_{x_0}^4 \psi(x) &= \Gamma(x, x_0) \int_0^1 (1-t)^4 \left[\frac{1}{24} Q_5(t) \Gamma(x(t), x_0)^{-1} \psi(x(t)) + \frac{5}{24} Q_4(t) \Gamma(x(t), x_0)^{-1} \nabla \psi|_{x(t)}(\dot{x}(t)) \right. \\
&\quad \left. + \frac{5}{12} Q_3(t) \Gamma(x(t), x_0)^{-1} \nabla^2 \psi|_{x(t)}(\dot{x}(t)^{\otimes 2}) + \frac{5}{12} Q_2(t) \Gamma(x(t), x_0)^{-1} \nabla^3 \psi|_{x(t)}(\dot{x}(t)^{\otimes 3}) \right. \\
&\quad \left. + \frac{5}{24} Q_1(t) \Gamma(x(t), x_0)^{-1} \nabla^4 \psi|_{x(t)}(\dot{x}(t)^{\otimes 4}) \right] dt
\end{aligned}$$

It should be noted that, unlike the Taylor polynomials $\text{Tay}_{x_0}^n \psi$, the integral correction $\text{Int}_{x_0}^{n+1} \psi$ of order $n+1$ is not simply that of order n plus an additional term. $\square$

Lemma 4.26. *With the above notation, we have*

$$\text{Rem}_{x_0}^n \psi(x) = \mathcal{O}(\|X\|^{\otimes n+1})$$

and therefore $\lim_{x \rightarrow x_0} \text{Rem}_{x_0}^n \psi(x) = 0$ at order $n+1$, as usual.

Proof. Assume that x_0 is a fixed point in M , $x \in U_{x_0}$ goes to x_0 and let $x(t) = \exp_{x_0}(tX)$ be the (unique) geodesic from x_0 to x , which has compact support in M . Let us give an upper bound for every factor appearing in the definition (73) of the remainder:

- The operator $\Gamma(x, x_0)$ is an isomorphism of vector spaces from E_{x_0} to E_x . Since x is the end-point of the geodesic $x(t)$, the operator norm of $\Gamma(x, x_0)$ is bounded by a finite constant $\Lambda(\Gamma, x_0, x) := \max_{t \in [0,1]} \|\Gamma(x(t), x_0)\|$.
- Similarly, the operator $\Gamma(x(t), x_0)^{-1}$ is an isomorphism of vector spaces from $E_{x(t)}$ to E_{x_0} , therefore its operator norm is bounded by a finite constant $\Lambda(\Gamma^{-1}, x_0, x) := \max_{t \in [0,1]} \|\Gamma(x(t), x_0)^{-1}\|$.
- The higher covariant derivatives $\nabla^k \psi|_{x(t)}(\dot{x}(t)^{\otimes k})$ are vectors in $E_{x(t)}$ with bounded norm

$$\|\nabla^k \psi|_{x(t)}(\dot{x}(t)^{\otimes k})\| = \|\nabla^k \psi|_{x(t)}\| \|\dot{x}(t)\|^k \leq \max_{t \in [0,1]} \|\nabla^k \psi|_{x(t)}\| \max_{t \in [0,1]} \|\dot{x}(t)\|^k.$$

On one side, set $\Lambda(\nabla^k, x_0, x) := \max_{t \in [0,1]} \|\nabla^k \psi|_{x(t)}\| < \infty$ the upper bound of the operator norm of the covariant derivatives of ψ on the geodesic. On the other side, the velocity of the geodesic at the point $x(t)$ is $\dot{x}(t) = \mathcal{P}(x(t), x_0)(X)$, and since the parallel transport is an isometry (here we consider on the vector bundle E the canonical Euclidean metric), we have $\|\dot{x}(t)\| = \|X\|$.

Putting all these bounds together, and knowing that $\int_0^1 (1-t)^n dt = \frac{1}{n+1}$ we obtain an overall estimate of the remainder:

$$\|\text{Rem}_{x_0}^n \psi(x)\| \leq \frac{1}{(n+1)!} \Lambda(\Gamma, x_0, x) \Lambda(\Gamma^{-1}, x_0, x) \Lambda(\nabla^{n+1}, x_0, x) \|X\|^{n+1}.$$

Now, in the limit $x \rightarrow x_0$, the velocity X of the geodesic from x_0 to x goes to 0. Since Γ is smooth, we also have $\|\Gamma(x(t), x_0)\| \rightarrow \|1_{x_0}\| = 1$, hence $\Lambda(\Gamma, x_0, x) \rightarrow 1$, and similarly $\Lambda(\Gamma^{-1}, x_0, x) \rightarrow 1$. We have no explicit estimate for the limit of the upper bound $\Lambda(\nabla^{n+1}, x_0, x)$, but we can bound it uniformly in x considering its sup for x in a compact set $K \subset U_{x_0}$ containing all small geodesics starting at x_0 . Finally, the above estimate shows that on every such compact set K there exist constants Λ independent of x such that

$$\|\text{Rem}_{x_0}^n \psi(x)\| \leq \frac{1}{(n+1)!} \Lambda(\Gamma, x_0, K) \Lambda(\Gamma^{-1}, x_0, K) \Lambda(\nabla^k, x_0, K) \|X\|^{n+1},$$

and therefore $\text{Rem}_{x_0}^n \psi(x) = \mathcal{O}(\|X\|^{n+1})$ for $x \rightarrow x_0$. $\square$

Remark 4.27. Suppose Γ is a transitive direct connection, which is equivalent to the parallel transport of a linear connection ∇ on E by Theorem 3.18. Then all the correction terms $C_n(t)$ vanish, and with them all the polynomials $Q_m(t)$ for $m \geq 1$ and therefore the integral corrections $\text{Int}_{x_0}^n \psi$. Only the term for $Q_0(t) = 1_{x_0}$ survives in the remainder, which then coincides with that given in [DDD19, Definition 65] for the trivial bundle $E = M \times \mathbb{R}$ (where the displacement from E_{x_0} to E_x can be omitted, since it is induced by the global trivialization of the bundle).

As in [DDD19, Theorem 62], for transitive direct connections one can then prove an approximation theorem for a section ψ in terms of the Taylor polynomial, which only contain local data given by the jet of ψ at x_0 :

$$\psi(x) = \text{Tay}_{x_0}^n \psi(x) + \text{Rem}_{x_0}^n \psi(x)$$

where $x(t) = \exp_{x_0}(tX)$ and $\text{Rem}_{x_0}^n \psi(x) = \mathcal{O}(\|X\|^{n+1})$. $\square$

We now prove an approximation theorem for a section ψ in the case of a general direct connection. The proof is presented in two steps: first we prove an approximation formula with a modified Taylor expansion, which contains all the coefficients coming from successive derivatives, then we prove the final approximation formula, by showing that some of the previous coefficients vanish.

Lemma 4.28. *Let Γ be a direct connection on $\text{GL}(E)$ defined on a diagonal domain covered by uniformly normal neighborhood, and let ∇ be the linear connection on E induced by Γ . Set*

$$\widetilde{\text{Tay}}_{x_0}^n \psi(x) = \Gamma(x, x_0) \sum_{k=0}^n \frac{1}{k!} a_k^{(n)}(x_0) \nabla^k \psi|_{x_0} (X^{\otimes k}), \quad (76)$$

where the coefficient $a_k^{(n)}(x_0)$ is an operator in $\text{End}(E_{x_0})$ defined by

$$a_k^{(n)}(x_0) := 1_{x_0} + \sum_{m=1}^{n-k} \frac{1}{m!} Q_m(0), \quad (77)$$

where $Q_m(0)$ is the evaluation of the polynomial $Q_m(t)$ given in eq. (75) at $t = 0$. Then, for any $x \in U_{x_0}$ and for any $n \geq 0$ we have

$$\psi(x) = \widetilde{\text{Tay}}_{x_0}^n \psi(x) + \text{Int}_{x_0}^n \psi(x) + \text{Rem}_{x_0}^n \psi(x).$$

Proof. For any $x \in U_{x_0}$, let $x(t) = \exp_{x_0}(tX)$ be the geodesic from x_0 to x , with initial velocity $X \in T_{x_0}M$. For any $t \in [0, 1]$, set

$$F(t) = \Gamma(x(t), x_0)^{-1} \psi(x(t)) \in E_{x_0},$$

so that $\psi(x) = \Gamma(x, x_0) F(1)$, $\psi(x_0) = F(0)$ and $\nabla_X \psi|_{x_0} = \frac{d}{dt} \Gamma(x(t), x_0)^{-1} \psi(x(t))|_{t=0} = F'(0)$. The usual Taylor's theorem with integral remainder applies to the vector-valued function F and gives

$$F(1) = \sum_{m=0}^n \frac{1}{m!} F^{(m)}(0) + \frac{1}{n!} \int_0^1 (1-t)^n F^{(n+1)}(t) dt. \quad (78)$$

1) Determination of $\widetilde{\text{Tay}}_{x_0}^n \psi(x)$. The first task is to evaluate the expansion of $F^{(m)}(0)$ in terms of the homogeneous polynomials $\nabla^k \psi|_{x_0}(X^{\otimes k})$. For this, we expand the successive derivatives at generic $t \in [0, 1]$

$$F^{(m)}(t) = \frac{d^m}{dt^m} (\Gamma(x(t), x_0)^{-1} \psi(x(t))) = \frac{d}{dt} F^{(m-1)}(s)|_{s=t} \quad (79)$$

in terms of the homogeneous polynomials of eq. (128)

$$\nabla^k \psi|_{x(t)}(\dot{x}(t)^{\otimes k}) = \nabla_{\dot{x}(t)} (\nabla_{\dot{x}(t)} \cdots (\nabla_{\dot{x}(t)} \psi) \cdots)|_{x(t)}$$

and of some corrections coming from the curvature of Γ along the geodesic $x(t)$, which can only appear as coefficients of the homogeneous terms in degrees smaller than m . The computation of the expansion of $F^{(m)}(t)$ is a non-trivial combinatorial issue, let us proceed by steps. Set

$$A_k(t) := \Gamma(x(t), x_0)^{-1} \nabla^k \psi|_{x(t)}(\dot{x}(t)^{\otimes k}) \in E_{x_0}, \quad k \geq 0 \quad (80)$$

$$F^{(m)}(t) =: \sum_{k=0}^m P_k^{(m)}(t) A_k(t), \quad m \geq 0 \quad (81)$$

where $P_k^{(m)}(t)$ are operators acting on E_{x_0} that depend on the correction terms. With these notations, we have $A_0(t) = F^{(0)}(t) = F(t)$ and $A_k(0) = \nabla^k \psi|_{x_0}(X^{\otimes k})$, so the principal part of the expansion (78) becomes

$$\sum_{m=0}^n \frac{1}{m!} F^{(m)}(0) = \sum_{m=0}^n \frac{1}{m!} \sum_{k=0}^m P_k^{(m)}(0) A_k(0) = \sum_{k=0}^n \frac{1}{k!} \left(\sum_{m=k}^n \frac{k!}{m!} P_k^{(m)}(0) \right) \nabla^k \psi|_{x_0}(X^{\otimes k}).$$

and the finite terms

$$\tilde{a}_k^{(n)}(x_0) := \sum_{m=k}^n \frac{k!}{m!} P_k^{(m)}(0)$$

give the coefficients of the Taylor expansion of $\psi(x)$ near $\psi(x_0)$ with the corrections produced by the direct connection Γ , if it is not transitive. Let us prove that $\tilde{a}_k^{(n)}(x_0)$ agrees with the coefficient $a_k^{(n)}(x_0)$ given in (77). For this, we use equations (79), (80) and (81) to determine the generating corrections and the recurrence relations for $P_k^{(m)}(t)$.

Order $m = 0$. We have $F^{(0)}(t) = A_0(t) = F(t) \in E_{x_0}$, so $P_0^{(0)}(t) = 1_{x_0}$. It follows that $\tilde{a}_0^{(0)}(x_0) = P_0^{(0)}(0) = 1_{x_0}$.

Order $m = 1$. The derivative $F'(t) = \frac{d}{ds}F(s)|_{s=t} = \frac{d}{ds}(\Gamma(x(s), x_0)^{-1}\psi(x(s)))|_{s=t}$ cannot be computed directly in terms of $\nabla\psi$, because x_0 is not the value of $x(s)$ at $s = t$. We introduce a trick that we will apply at any order of derivation.

Trick to derive sections along a geodesic. Since $x(s)$ and $x(t)$ are points on the same geodesic, using the transitive property of $\exp$ in time we can consider $x(t)$ as the starting point of the geodesic leading to $x(s) = \exp_{x(t)}((s-t)\dot{x}(t))$, and therefore we know the derivative

$$\frac{d}{ds}(\Gamma(x(s), x(t))^{-1}\psi(x(s)))|_{s=t} = \nabla_{\dot{x}(t)}\psi|_{x(t)}.$$

For any section ψ of E , this gives the value of a new section of E at the point $x(t)$, and $\Gamma(x(t), x_0)^{-1}\nabla_{\dot{x}(t)}\psi|_{x(t)}$ brings its value back to x_0 . To apply this derivative, we transform $\Gamma(x(s), x_0)^{-1}$ into $\Gamma(x(s), x(t))^{-1}$ using the curvature $K_{x_0}(x(s), x(t))$, cf. eq. (28), and get

$$\Gamma(x(s), x_0)^{-1} = K_{x_0}(x(s), x(t)) \Gamma(x(t), x_0)^{-1} \Gamma(x(s), x(t))^{-1}.$$

Finally, we can write

$$\Gamma(x(s), x_0)^{-1}\psi(x(s)) = K_{x_0}(x(s), x(t)) \Gamma(x(t), x_0)^{-1} \Gamma(x(s), x(t))^{-1} \psi(x(s)) \quad (82)$$

and apply the Leibniz rule to compute its derivative at $s = t$.

Since $\Gamma(x(t), x_0)^{-1}$ does not depend on s , the derivative of $F(s)$ expressed using (82) has two terms:

$$\begin{aligned} F'(t) &= \frac{d}{ds}K_{x_0}(x(s), x(t))|_{s=t} \Gamma(x(t), x_0)^{-1} \Gamma(x(t), x(t))^{-1} \psi(x(t)) \\ &\quad + K_{x_0}(x(t), x(t)) \Gamma(x(t), x_0)^{-1} \frac{d}{ds}(\Gamma(x(s), x(t))^{-1} \psi(x(s)))|_{s=t} \\ &= \frac{d}{ds}K_{x_0}(x(s), x(t))|_{s=t} \Gamma(x(t), x_0)^{-1} \psi(x(t)) + \Gamma(x(t), x_0)^{-1} \nabla_{\dot{x}(t)}\psi|_{x(t)} \\ &= \frac{d}{ds}K_{x_0}(x(s), x(t))|_{s=t} F(t) + \Gamma(x(t), x_0)^{-1} \nabla_{\dot{x}(t)}\psi|_{x(t)} \\ &= C_1(t) F^{(0)}(t) + A_1(t), \end{aligned} \quad (83)$$

where $C_1(t)$ is the first correction term defined in (74), which does not necessarily vanish for $t \neq 0$ (it does for instance if Γ is transitive, cf. Definition 3.1 (iv)). Eq. (83) then gives the first term of the recurrence:

$$F^{(1)}(t) = C_1(t) A_0(t) + A_1(t). \quad (84)$$

In other words, we have

$$P_0^{(1)}(t) = C_1(t) \quad \text{and} \quad P_1^{(1)}(t) = 1_{x_0}. \quad (85)$$

It follows that $\tilde{a}_0^{(1)}(x_0) = P_0^{(0)}(0) + P_0^{(1)}(0) = 1_{x_0} + C_1(0)$ and $\tilde{a}_1^{(1)}(x_0) = P_1^{(1)}(0) = 1_{x_0}$.

Order $m = 2$. Now we compute $F''(t) = \frac{d}{du}F'(u)|_{u=t}$. To make the recurrence explicit, start from $F'(u)$ given by eq. (83) and apply the Leibniz rule. On one side we need the derivative of $C_1(t)$,

which is $C_2(t)$ by definition (74), and the derivative of $F^{(0)}(t)$, which is $F^{(1)}(t)$. On the other side, we need the derivative of $A_1(t) = \Gamma(x(t), x_0)^{-1} \nabla \psi|_{x(t)}(\dot{x}(t))$, that we compute using again the trick (82), thus obtaining, as in eq. (83)

$$\begin{aligned} \frac{d}{dt} A_1(t) &= \frac{d}{ds} \left(\Gamma(x(s), x_0)^{-1} \nabla \psi|_{x(s)}(\dot{x}(s)) \right) \Big|_{s=t} \\ &= \frac{d}{ds} K_{x_0}(x(s), x(t)) \Big|_{s=t} \Gamma(x(t), x_0)^{-1} \nabla_{\dot{x}(t)} \psi|_{x(t)} + \Gamma(x(t), x_0)^{-1} \nabla_{\dot{x}(t)} \nabla_{\dot{x}(t)} \psi|_{x(t)} \\ &= C_1(t) A_1(t) + A_2(t). \end{aligned} \quad (86)$$

Putting all the terms together, we get

$$\begin{aligned} F^{(2)}(t) &= C_2(t) F^{(0)}(t) + C_1(t) F^{(1)} + C_1(t) A_1(t) + A_2(t) \\ &= (C_2(t) + C_1(t)^2) A_0(t) + 2 C_1(t) A_1(t) + A_2(t), \end{aligned} \quad (87)$$

that is,

$$P_0^{(2)}(t) = C_2(t) + C_1(t)^2, \quad P_1^{(2)}(t) = 2 C_1(t) \quad \text{and} \quad P_2^{(2)}(t) = 1_{x_0}. \quad (88)$$

It follows that

$$\begin{aligned} \tilde{a}_0^{(2)}(x_0) &= P_0^{(0)}(0) + P_0^{(1)}(0) + P_0^{(2)}(0) = 1_{x_0} + C_1(0) + (C_2(0) + C_1(0)^2) \\ \tilde{a}_1^{(2)}(x_0) &= P_0^{(1)}(0) + \frac{1}{2} P_0^{(2)}(0) = 1_{x_0} + C_1(0) \\ \tilde{a}_2^{(2)}(x_0) &= P_2^{(2)}(0) = 1_{x_0}. \end{aligned}$$

Order m . It is now clear that the operators $P_k^{(m)}(t)$ defined by eq. (81) are polynomials of the correction terms $C_1(t)$, $C_2(t)$ and their derivatives $C_n(t)$ of eq. (74). They satisfy a recurrence relation computed by applying the derivation rules of polynomials to obtain $F^{(m+1)}(t)$ from $F^{(m)}(t) = \sum_{k=0}^m P_k^{(m)}(t) A_k(t)$. For this, we need the derivative of $A_k(t) = \Gamma(x(t), x_0)^{-1} \nabla^k \psi|_{x(t)}(\dot{x}(t)^{\otimes k})$, that we compute again using the trick (82) as for (86), and gives

$$\frac{d}{dt} A_k(t) = C_1(t) A_k(t) + A_{k+1}(t). \quad (89)$$

We then have

$$\begin{aligned} \frac{d}{dt} F^{(m)}(t) &= \sum_{k=0}^m \left(\frac{d}{dt} P_k^{(m)}(t) A_k(t) + P_k^{(m)}(t) (C_1(t) A_k(t) + A_{k+1}(t)) \right) \\ &= \sum_{k=0}^m \left(\frac{d}{dt} P_k^{(m)}(t) A_k(t) + P_k^{(m)}(t) C_1(t) A_k(t) \right) + \sum_{k=1}^{m+1} P_{k-1}^{(m)}(t) A_k(t). \end{aligned}$$

We set $P_{-1}^{(m)}(t) = 0$, which is meaningful because there are no possible terms $A_k(t)$ for negative k , and assume as inductive hypothesis that $P_m^{(m)}(t) = 1_{x_0}$ for any $m \geq 0$ (hence $\frac{d}{dt} P_m^{(m)}(t) = 0$), which is already confirmed for $m = 0, 1, 2$. Then we can write

$$\frac{d}{dt} F^{(m)}(t) = \sum_{k=0}^{m+1} \left(\frac{d}{dt} P_k^{(m)}(t) + P_k^{(m)}(t) C_1(t) + P_{k-1}^{(m)}(t) \right) A_k(t).$$

Comparing this with the expansion $F^{(m+1)}(t) = \sum_{k=0}^{m+1} P_k^{(m+1)}(t) A_k(t)$, and knowing that $C_1 = P_0^{(1)}$, gives the recurrence relation

$$P_k^{(m+1)}(t) = P_{k-1}^{(m)}(t) + \frac{d}{dt} P_k^{(m)}(t) + P_k^{(m)}(t) P_0^{(1)}(t). \quad (90)$$

Now let us prove that this identity implies

$$P_k^{(m)}(t) = \binom{m}{k} P_0^{(m-k)}(t). \quad (91)$$

Since $P_{-1}^{(m)}(t) = 0$ and $P_{m+1}^{(m)}(t) = 0$, because there are no possible covariant derivatives of ψ of order $m+1$ if we derivative F at order m , from (90) we deduce that $P_k^{(k)}(t) = 1_{x_0} = \binom{k}{k} P_0^{(0)}(t)$. This, in turn, implies that

$$\begin{aligned} P_k^{(k+1)}(t) &= P_{k-1}^{(k)}(t) + \frac{d}{dt} P_k^{(k)}(t) + P_k^{(k)}(t) P_0^{(1)}(t) = P_{k-1}^{(k)}(t) + P_0^{(1)}(t) \\ &= P_{k-2}^{(k-1)}(t) + 2 P_0^{(1)}(t) = \dots = (k+1) P_0^{(1)}(t) = \binom{k+1}{1} P_0^{(1)}(t). \end{aligned}$$

By induction, suppose that (91) holds for m and k . First observe that, by (90), we have

$$P_0^{(m-k+1)}(t) = \frac{d}{dt} P_0^{(m-k)}(t) + P_0^{(m-k)}(t) P_0^{(1)}(t).$$

Then, replacing the terms $P_k^{(m)}(t)$ in the expression of $P_k^{(m+1)}(t)$ we get

$$\begin{aligned} P_k^{(m+1)}(t) &= P_{k-1}^{(m)}(t) + \binom{m}{k} \left[\frac{d}{dt} P_0^{(m-k)}(t) + P_0^{(m-k)}(t) P_0^{(1)}(t) \right] = P_{k-1}^{(m)}(t) + \binom{m}{k} P_0^{(m-k+1)}(t) \\ &= P_{k-2}^{(m-1)}(t) + \binom{m-1}{k-1} P_0^{(m-k)}(t) + \binom{m}{k} P_0^{(m-k+1)}(t) = \dots \\ &= \left[\binom{m-k}{0} + \dots + \binom{m-1}{k-1} + \binom{m}{k} \right] P_0^{(m-k+1)}(t) = \binom{m-k+1}{k} P_0^{(m-k+1)}(t) \end{aligned}$$

by the Chu-Vandermonde identity of binomial coefficients

$$\binom{k+p+1}{k} = \sum_{q=0}^k \binom{q+p}{q}.$$

In conclusion, we have

$$F^{(m)}(t) = \sum_{k=0}^m P_k^{(m)}(t) A_k(t) = \sum_{k=0}^m \binom{m}{k} Q_{m-k}(t) A_k(t), \quad (92)$$

where the polynomials $Q_m(t) := P_0^{(m)}(t)$ satisfy the simplified recurrence given by eq. (75), obtained by inserting (91) in (90).

Conclusion. Finally we come to the equality $\tilde{a}_k^{(n)}(x_0) = a_k^{(n)}(x_0)$. From the previous results, it follows that

$$\begin{aligned} \tilde{a}_k^{(n)}(x_0) &= \sum_{m=k}^n \frac{k!}{m!} P_k^{(m)}(0) = \sum_{m=k}^n \frac{k!}{m!} \frac{m!}{k!(m-k)!} P_0^{(m-k)}(0) \\ &= \sum_{j=0}^{n-k} \frac{1}{j!} Q_j(0), \end{aligned}$$

where we set $j = m - k$. The equality follows from $Q_0(0) = P_0^{(0)}(t) = 1_{x_0}$.

2) Determination of $\text{Int}_{x_0}^n \psi(x)$ and $\text{Rem}_{x_0}^n \psi(x)$. To conclude the proof of the lemma, the second task is to compute the integral remainder in eq. (78). We expand the derivative $F^{(n+1)}(t)$ inside the integral using eq. (92) and separate the last term, which depends on the derivative $\nabla^{n+1} \psi|_{x(t)}(\dot{x}^{\otimes n+1})$:

$$F^{(n+1)}(t) = \sum_{k=0}^n \binom{n+1}{k} Q_{n+1-k}(t) A_k(t) + \binom{n+1}{n+1} Q_0(t) A_{n+1}(t).$$

Substituting the terms $A_k(t)$ given in eq. (80) and $Q_0(t) = 1_{x_0}$, we get

$$\begin{aligned} \frac{1}{n!} \int_0^1 (1-t)^n F^{(n+1)}(t) dt &= \sum_{k=0}^n \frac{1}{n!} \binom{n+1}{k} \int_0^1 (1-t)^n Q_{n+1-k}(t) \Gamma(x(t), x_0)^{-1} \nabla^k \psi|_{x(t)}(\dot{x}(t)^{\otimes k}) dt \\ &\quad + \frac{1}{n!} \binom{n+1}{n+1} \int_0^1 (1-t)^n Q_0(t) \Gamma(x(t), x_0)^{-1} \nabla^{n+1} \psi|_{x(t)}(\dot{x}(t)^{\otimes n+1}) dt \\ &= \Gamma(x, x_0)^{-1} \text{Int}_{x_0}^n \psi(x) + \Gamma(x, x_0)^{-1} \text{Rem}_{x_0}^n \psi(x), \end{aligned}$$

which concludes the proof of the lemma. $\square$

Theorem 4.29. *Let Γ be a direct connection on $\text{GL}(E)$ defined on a diagonal domain covered by uniformly normal neighborhood. With the notations of Definition 4.24, for any $x \in U_{x_0}$ and for any $n \geq 0$ we have*

$$\psi(x) = \text{Tay}_{x_0}^n \psi(x) + \text{Int}_{x_0}^n \psi(x) + \text{Rem}_{x_0}^n \psi(x).$$

In other words, the generalized Taylor section $\text{GTay}_{x_0}^n \psi(x) = \text{Tay}_{x_0}^n \psi(x) + \text{Int}_{x_0}^n \psi(x)$ is an approximation of $\psi(x)$ of order n for $x \rightarrow x_0$.

Since for a general direct connection Γ on $\text{GL}(E)$ the term $\text{Int}_{x_0}^n \psi(x)$ does not necessarily vanish, this result says that the approximation of a local section ψ at a point x_0 using a general direct connection is not given by its *local* expansion $\text{Tay}_{x_0}^n \psi(x)$, but requires the *non-local* correction term $\text{Int}_{x_0}^n \psi(x)$.

Proof. After Lemma 4.28, the only thing which remains to prove is that $\widetilde{\text{Tay}}_{x_0}^n \psi = \text{Tay}_{x_0}^n \psi$, i.e. that all the coefficients $a_k^{(n)}(x_0)$ are equal to 1_{x_0} . This means that the polynomials $Q_m(0)$ vanish at $t = 0$ for any $m \geq 1$. To see this, let us expand $\psi(x)$ at x_0 using Lemma 4.28 and evaluate the expansion in $x = x_0$. Since, by Lemma 4.26, we have $\text{Rem}_{x_0}^n \psi(x_0) = 0$, for any $n \geq 0$ we have

$$\widetilde{\text{Tay}}_{x_0}^n \psi(x_0) + \text{Int}_{x_0}^n \psi(x_0) = \psi(x_0). \quad (93)$$

For $x = x_0$, both the initial velocity X of the geodesic $x(t)$ from x_0 to x and the parallel transported velocity $\dot{x}(t)$ become the null vector. Therefore the terms on the LHS containing derivatives $\nabla^k \psi|_{x_0}(X^{\otimes k})$ and $\nabla^k \psi|_{x(t)}(\dot{x}(t)^{\otimes k})$ vanish for $k > 0$.

The integrals appearing in $\text{Int}_{x_0}^n \psi(x_0)$ must be understood as the limits for $s \rightarrow 0$ of the integrals along the new geodesic (in time t) from x_0 to the point $x(s) = \exp_{x_0}(sX)$ of the original geodesic

from x_0 to x . The new geodesic can be parametrized as $x(st) = \exp_{x_0}(sX)$, for $t \in [0, 1]$. Then, since $\Gamma(x_0, x_0) = 1_{x_0}$ and $a_0^{(n)}(x_0) = 1_{x_0} + \sum_{m=1}^n \frac{1}{m!} Q_m(0)$, the identity (93) becomes

$$\begin{aligned} \psi(x_0) &= \psi(x_0) + \sum_{m=1}^n \frac{1}{m!} Q_m(0) \psi(x_0) + \lim_{s \rightarrow 0} \frac{n+1}{n!} \int_0^1 (1-t)^n Q_{n+1}(st) \psi(x(st)) dt \\ &= \psi(x_0) + \sum_{m=1}^n \frac{1}{m!} Q_m(0) \psi(x_0) + \frac{1}{n!} Q_{n+1}(0) \psi(x_0). \end{aligned}$$

Now we observe that $Q_1(0) = C_1(0) = 0$, by Lemma 3.15. Therefore the sum over m can start at $m = 2$, and we have

$$\sum_{m=2}^n \frac{1}{m!} Q_m(0) + \frac{(n+1)^2}{n!} Q_{n+1}(0) = 0.$$

By induction on $n \geq 1$, it follows that $Q_m(0) = 0$ for any $m \geq 2$, and therefore $\widetilde{\text{Tay}}_{x_0}^n \psi = \text{Tay}_{x_0}^n \psi$. $\square$

Corollary 4.30. *The correction terms $C_n(0)$ due to the curvature of Γ along geodesics, evaluated at $t = 0$, all vanish.*

Proof. Lemma 3.15 directly implies that $C_1(0) = 0$. For $n \geq 2$, this follows by induction on n from the expression (75) of the vanishing polynomials $Q_m(0)$, which are of the form $C_m(0)$ plus terms in lower orders. For instance, for $m = 2$ and $m = 3$ we have

$$\begin{aligned} Q_2(0) &= C_2(0) + C_1(0)^2 = 0 \\ Q_3(0) &= C_3(0) + 2C_2(0)C_1(0) + C_1(0)C_2(0) + C_1^3(0) = 0 \end{aligned}$$

giving $C_2(0) = 0$, then $C_3(0) = 0$, and so on. $\square$

Example 4.31 (Approximation of sections). Let $\psi : M \rightarrow E$ be a smooth section, $x_0 \in M$ a given point and U_{x_0} a uniformly normal neighborhood. The expansion of order 0 of ψ at x_0 is given, at $x \in U_{x_0}$, by

$$\begin{aligned} \psi(x) &= \underbrace{\overbrace{\Gamma(x, x_0) \psi(x_0)}^{\text{local but not approximation}} + \Gamma(x, x_0) \int_0^1 C_1(t) \Gamma(x(t), x_0)^{-1} \psi(x(t)) dt}_{\text{good approximation but non local}} \\ &\quad + \underbrace{\Gamma(x, x_0) \int_0^1 \Gamma(x(t), x_0)^{-1} \nabla \psi|_{x(t)}(\dot{x}(t)) dt}_{\mathcal{O}(\|X\|)}. \end{aligned}$$

The expansion of order 1 of ψ at x_0 is given, at $x \in U_{x_0}$, by

$$\begin{aligned}
\psi(x) &= \underbrace{\Gamma(x, x_0) \psi(x_0) + \Gamma(x, x_0) \nabla \psi|_{x_0}(X)}_{\text{local terms but not a good approximation}} \\
&+ \underbrace{\Gamma(x, x_0) \int_0^1 (1-t) \left[(C_2(t) + C_1(t)^2) \Gamma(x(t), x_0)^{-1} \psi(x(t)) + \overbrace{2 C_1(t) \Gamma(x(t), x_0)^{-1} \nabla \psi|_{x(t)}(\dot{x}(t))}^{\mathcal{O}(\|X\|)} \right] dt}_{\text{extra non-local terms for good approximation}} \\
&+ \underbrace{\Gamma(x, x_0) \int_0^1 (1-t) \Gamma(x(t), x_0)^{-1} \nabla^2 \psi|_{x(t)}(\dot{x}(t)^{\otimes 2}) dt}_{\mathcal{O}(\|X\|^2)}.
\end{aligned}$$

Note that the term $\mathcal{O}(\|X\|)$ in the second line could be affected to the reminder. We keep it in the principal part of the expansion so that the reminder cancels out at the usual order $n+1$. Other choices of an integral correction and of a remainder could be done directly in Definition 4.24. $\square$

4.7 Higher-order direct connections and jet regularity structures

The jet bundle $J^n E$ contains the higher covariant derivatives of the sections of E and therefore it is of special interest in the context of regularity structures [Hai14a, HS23]. Inspired by [DDD19, Definition 80], in this section we give an example of a higher-order direct connection on $J^n E$ which is not a jet prolongation of a direct connection on E . We then show that it can be used to define a covariant *local* model for an E -valued jet regularity structure.

Definition 4.32. Let Γ be a direct connection on $\text{GL}(E)$ defined on the diagonal domain $\mathcal{U}$. For any $n \geq 0$, we call **Taylor direct connection induced by Γ** the collection of linear map $\Sigma^n(y_0, x_0) : J_{x_0}^n E \rightarrow J_{y_0}^n E$ defined for any $x_0 \in M$ and any $y_0 \in U_{x_0}$ such that $(y_0, x_0) \in \mathcal{U} \subset \text{Pair}(M)$, by

$$\Sigma^n(y_0, x_0) j_{x_0}^n \psi := j_{y_0}^n (\text{GTay}_{x_0}^n \psi),$$

where we recall from Definition 4.24 that $\text{GTay}_{x_0}^n \psi(x) = \text{Tay}_{x_0}^n \psi(x) + \text{Int}_{x_0}^n \psi(x)$ is a local section of E defined for any $x \in U_{x_0}$.

Example 4.33. At order $n=0$, we recover the value of ψ transported by Γ plus a non-local correction due to the curvature of Γ along the geodesic $x(t) = \exp_{x_0}(tX)$ from x_0 to y_0 :

$$\Sigma^0(y_0, x_0) \psi(x_0) = \Gamma(y_0, x_0) \psi(x_0) + \Gamma(y_0, x_0) \int_0^1 C_1(t) \Gamma(x(t), x_0)^{-1} \psi(x(t)) dt.$$

To describe Σ^n at order $n=1$, we use the representation of jets by covariant derivatives recalled in eqs. (131) and (132). We have

$$\Sigma^1(y_0, x_0) j_{x_0}^1 \psi = j_{y_0}^1 (\text{GTay}_{x_0}^1 \psi) = \left[y_0, \text{GTay}_{x_0}^1 \psi(y_0), \nabla(\text{GTay}_{x_0}^1 \psi)|_{y_0} \right],$$

where

$$\begin{aligned}
\text{GTay}_{x_0}^1 \psi(y_0) &= \Gamma(y_0, x_0) \psi(x_0) + \Gamma(y_0, x_0) \nabla \psi|_{x_0}(X) \\
&+ \Gamma(y_0, x_0) \int_0^1 (1-t) (C_2(t) + C_1(t)^2) \Gamma(x(t), x_0)^{-1} \psi(x(t)) dt \\
&+ \Gamma(y_0, x_0) \int_0^1 (1-t) 2 C_1(t) \Gamma(x(t), x_0)^{-1} \nabla \psi|_{x(t)}(\dot{x}(t)) dt.
\end{aligned}$$

FINISH: compute $\nabla(\text{GTay}_{x_0}^1 \psi)|_{y_0}(Y)$ for any (small) $Y \in T_{y_0}M$ with Jacobi fields for sprays of geodesics centered at x_0 in starting direction of $y_0 = \exp_{x_0}(X)$ with slope in direction of $z_0 = \exp_{y_0}(Y)$.

I tried to compute it, but it is more difficult then the previous cases and I don't know how to do it...

We fix three points $x_0, y_0 \in U_{x_0}$ and $z_0 \in U_{y_0}$, the starting geodesic $x(t) = \exp_{x_0}(tX)$ from x_0 to y_0 and the geodesic $y(s) = \exp_{y_0}(sY)$ from y_0 to z_0 along which we want to differentiate (with respect to s) the function of s with values in E_{y_0} given by

$$\begin{aligned} \Gamma(y(s), y_0)^{-1} \text{GTay}_{x_0}^1 \psi(y(s)) &= \psi(x_0) + \nabla \psi|_{x_0}(sY) \\ &+ \int_0^1 (1-t) (C_2(\gamma(s, t)) + C_1(\gamma(s, t))^2) \Gamma(\gamma(s, t), x_0)^{-1} \psi(\gamma(s, t)) dt \\ &+ \int_0^1 (1-t) 2 C_1(\gamma(s, t)) \Gamma(\gamma(s, t), x_0)^{-1} \nabla \psi|_{\gamma(s, t)}(\partial_t \gamma(s, t)) dt. \end{aligned}$$

Here the integrals are computed along the geodesics (in t) from x_0 to $y(s)$, given by the variational curve of $x(t)$

$$\gamma(s, t) = \exp_{x_0}(tX(s))$$

such that $\gamma(s, 1) = \exp_{x_0}(X(s)) = y(s)$. Applying the theory of Jacobi fields, we have

- main curves $t \mapsto \gamma(t, s)$ with unknown initial velocity $\partial_t \gamma(s, t)|_{t=0} = X(s)$,
- transversal curves $s \mapsto \gamma(t, s)$ with initial velocity $\partial_s \gamma(s, t)|_{s=0} = J(t)$ giving the Jacobi field along the initial geodesic $x(t)$, satisfying

$$D_t^2 J(t) + R_{x(t)}(J(t), \dot{x}(t)) \dot{x}(t) = 0 \quad \text{for any } t \in [0, 1].$$

Note that we don't know the initial conditions of this Jacobi field, namely $J(0)$ and $D_t J(t)|_{t=0} = D_s X(s)|_{s=0}$, because we don't know $X(s)$. Instead, we know the final value $J(1) = Y$, but I don't know how to use it to perform the derivative of the integrands with respect to s . $\square$

Theorem 4.34. *The collection of maps $\Sigma^n(y_0, x_0)$ defines a direct connection on $\text{GL}(J^n E)$ which is not a jet prolongation of a direct connection on E .*

Proof. For the first assertion, by Proposition 3.9 it is enough to show that Σ is a parallelism. The map $\Sigma^n(y_0, x_0)$ is clearly a linear operator from $J_{x_0}^n E$ to $J_{y_0}^n E$, because the two operators $\text{Tay}_{x_0}^n$ and $\text{Int}_{x_0}^n$ are linear on the derivatives of ψ . We only have to show that $\Sigma(x_0, x_0) = \text{Id}_{E_{x_0}}$, that is,

$$\Sigma^n(x_0, x_0) j_{x_0}^n \psi = j_{x_0}^n \psi.$$

Since, by eqs. (131) and (132), we can represent jets by higher covariant derivatives, this means

$$\nabla^k(\text{GTay}_{x_0}^n \psi)|_{x_0} = \nabla^k \psi|_{x_0} \quad \text{for any } k \leq n. \quad (94)$$

Theorem 4.29 and Lemma 4.26 say that this equality holds for $k = 0$, and therefore we have

$$\text{GTay}_{x_0}^n \psi(x_0) = \psi(x_0) \quad \text{for any } n \geq 0. \quad (95)$$

Let us prove eq. (94) by induction. For $n = 0$, it means that $\text{GTay}_{x_0}^0 \psi(x_0) = \psi(x_0)$, which holds by eq. (95). Now suppose that eq. (94) holds up to order n and consider it at order $n + 1$. We prove separately, in the Lemma 4.35, that

$$\nabla^k(\text{GTay}_{x_0}^{n+1} \psi)|_{x_0} = \nabla^k(\text{GTay}_{x_0}^n \psi)|_{x_0} \quad (96)$$

for any $k \leq n$, therefore it only remains to compute the derivative at order $n + 1$. Using the recursive definition (128) of higher covariant derivative, the equality (96) for $k = n$ and the inductive hypothesis, we have

$$\nabla^{n+1}(\text{GTay}_{x_0}^{n+1} \psi)|_{x_0} = \nabla(\nabla^n(\text{GTay}_{x_0}^n \psi))|_{x_0} = \nabla(\nabla^n \psi)|_{x_0} = \nabla^{n+1} \psi|_{x_0},$$

which concludes the proof.

To show that Σ^n is not the jet prolongation of a direct connection on E , it suffices to show that it does not take value in the subgroupoid $J^n \text{GL}(E)$. This is due to two reasons: the first is the presence in $\Sigma^n(y_0, x_0) j_{x_0}^n \psi$ of the integral correction $\text{Int}_{x_0}^n \psi(x)$, which is a non-local term, i.e. it cannot be described by derivatives of a direct connection on E . This is clear already for the 0th order direct connection Σ^0 given above.

The second reason is that $\Sigma^n j_{x_0}^n \psi$ is not filtered in the order of jets of ψ , that is, the invertible matrix associated to Σ^n in the linear basis of $J_x^n E$ spanning jets $j_x^n \psi$ by the covariant derivatives $\nabla^k \psi|_{x_0}$ is not a triangular matrix. This can be seen in $\Sigma^1(y_0, x_0)$, since the 0-th order component of the jet $j_{y_0}^1(\text{GTay}_{x_0}^1 \psi)$ is

$$\begin{aligned} \text{GTay}_{x_0}^1 \psi(y_0) &= \Gamma(y_0, x_0) \psi(x_0) + \Gamma(y_0, x_0) \nabla \psi|_{x_0}(X) \\ &\quad + \Gamma(y_0, x_0) \int_0^1 (1-t) (C_2(t) + C_1(t)^2) \Gamma(x(t), x_0)^{-1} \psi(x(t)) dt \end{aligned}$$

which already contains $\nabla \psi|_{x_0}(X)$, that is, the full data of $j_{x_0}^1 \psi$ (beside the extra non-local term already mentioned). $\square$

Lemma 4.35. *With the above notations, we have*

$$\nabla^k(\text{GTay}_{x_0}^{n+1} \psi)|_{x_0} = \nabla^k(\text{GTay}_{x_0}^n \psi)|_{x_0}$$

for any $n \geq 0$ and any $0 \leq k \leq n$.

Proof. For any $x \in U_{x_0}$, related to x_0 by the geodesic $x(t) = \exp_{x_0}(tX)$, and for any $n \geq 0$, set

$$H_n(x) = \text{GTay}_{x_0}^{n+1} \psi(x) - \text{GTay}_{x_0}^n \psi(x).$$

We prove that $\nabla^k H_n|_{x_0}(X^{\otimes k}) = 0$ for any $k \geq 0$. For $k = 0$, we have $H_n(x_0) = \psi(x_0) - \psi(x_0) = 0$ by Theorem 4.29. For $k = 1$, we have $\nabla H_n|_{x_0}(X) = \frac{d}{ds} \Gamma(x(s), x_0)^{-1} H_n(x(s))|_{s=0}$, where

$$\begin{aligned} \Gamma(x(s), x_0)^{-1} H_n(x(s)) &= \frac{1}{(n+1)!} \nabla^{n+1} \psi|_{x_0}((sX)^{\otimes n+1}) \\ &\quad + \sum_{k=0}^{n+1} \frac{1}{n+1} \binom{n+2}{k} \int_0^1 (1-t)^{n+1} Q_{n+2-k}(ts) A_k(ts) dt \\ &\quad - \sum_{k=0}^n \frac{1}{n} \binom{n+1}{k} \int_0^1 (1-t)^n Q_{n+1-k}(ts) A_k(ts) dt. \end{aligned}$$

Here, as in the proof of Theorem 4.29, we integrate along the geodesic from x_0 to $x(s)$, parametrized by $x(ts) = \exp_{x_0}(tsX)$ for $t \in [0, 1]$. The polynomials $Q_m(t)$ are defined by eq. (75) and the $A_k(t)$ by eq. (80). Let us compute separately the derivative of the three lines:

- The derivative in s of the first line makes appearing powers of s and keeps the presence of higher covariant derivatives evaluated in sX , therefore the whole term vanishes for $s = 0$.
- The derivative in s of the second line affects the integrand $Q_{n+2-k}(ts) A_k(ts)$: using eq. (75) for the derivative of $Q_{n+2-k}(ts)$, and eq. (89) for the derivative of $A_k(ts)$, for every $k \leq n+1$ we obtain

$$\begin{aligned} \frac{d}{ds} Q_{n+2-k}(ts) A_k(ts) \Big|_{s=0} &= \frac{d}{ds} Q_{n+2-k}(ts) \Big|_{s=0} A_k(0) + Q_{n+2-k}(0) \frac{d}{ds} A_k(ts) \Big|_{s=0} \\ &= t (Q_{n+3-k}(0) - Q_{n+2-k}(0)) A_k(0) + Q_{n+2-k}(0) t (C_1(0) A_k(0) + A_{k+1}(0)) \end{aligned}$$

which vanishes because $Q_m(0) = 0$ for any $m > 0$, by Theorem 4.29.

- The third line is treated as the second.

For $k > 1$, this mechanism repeats and produces either higher covariant derivatives of ψ at x_0 evaluated in the null tangent vector, or vanishing polynomials $Q_m(0) = 0$ for increasing indices m . $\square$

I CHECKED on [HS23] and compared with [DDD19]:

- Personal comment: Hairer is very aggressive and does not recognize the value of others' work. This is all he says about DDD's paper: "At this point we mention the work [DDD19] which builds a minimal framework to solve Equation (g-PAM) in the special case $f = 0$ and $g = \text{id}$ within regularity structures.", in [HS23, page 3]. Of course, he never quotes our unpublished notes.
- DDD do not use jets, but properly consider only *local* functions defined on open neighborhoods U_p of $p \in M$, and define models Π as producing *local* solutions of the PDE living in $\mathcal{D}'(U_p)$, cf. [DDD19, Definition 14], using parallelisms and higher covariant derivatives on the trivial bundle, cf. [DDD19, Definitions 76 and 80] and all intermediate Lemmas.

As a result, their polynomial model on a manifold is local, but completely covariant.

- Instead, Hairer-Singh use jets, but a general model for a regularity structure is a map Π producing *global* E -valued distributions $\mathcal{D}'(E)$ (cf. [HS23, Definition 4.13]).

They define a model for a jet regularity structure (in [HS23, Definition 4.20]) using an *admissible realization* of $J^k E$ [HS23, Definition 3.21] which produces *global* sections of E by reading in local coordinates the usual functions defined by the Taylor expansion extended globally with a bump-like transition function.

- Hairer-Singh assume M to be a Riemannian manifold endowed with the Levi-Civita connection (as DDD), and E to be a metric vector bundle endowed with a compatible linear connection. The connection on E is used to define the topology on E -valued distributions [HS23, Par. 2.2, p.9], but it is not used to construct admissible realizations of $J^k E$. Therefore their model for jet regularity structures is definitely not covariant.
- If the key point is how to produce approximated *global* solutions, the good covariant approach is to ask how to glue the local covariant solutions produced by DDD. Instead, Hairer-Singh produce directly global solutions (with compact support) by importing those on $\mathbb{R}^n$ via local charts.

Remark 4.36. Given a direct connection Γ on the frame groupoid $\mathrm{GL}(E)$, the generalized Taylor expansion GTay^n of sections of E and the higher direct connection Σ^n induced by Γ give a *local* model for E -valued jet regularity structures, in the sense of [HS23, Definitions 4.13 and 4.20], but with maps $\Pi_{x_0}^\Gamma : J_{x_0}^n E \rightarrow \Gamma(U_{x_0}, E|_{U_{x_0}})$, defined for any $x_0 \in M$ by

$$\Pi_{x_0}^\Gamma(j_{x_0}^n \psi)(x) := \mathrm{GTay}_{x_0}^n \psi(x), \quad x \in U_{x_0},$$

which create *local* sections of E around x_0 instead of global sections, and with *local* transport maps given, for any $x_0 \in M$ and any $y_0 \in U_{x_0}$, by the direct connection $\Sigma^n(y_0, x_0) : J_{x_0}^n E \rightarrow J_{y_0}^n E$, that we recall to be

$$\Sigma^n(y_0, x_0) j_{x_0}^n \psi := j_{y_0}^n (\mathrm{GTay}_{x_0}^n \psi).$$

In fact, the generalized Taylor expansion GTay^n is a (local) admissible realisation, in the sense of [HS23, Definition 3.21], where the first condition $j_{x_0}^n (\mathrm{GTay}_{x_0}^n \psi) = j_{x_0}^n \psi$ holds by Theorem 4.34. $\square$

A Appendix: Jet bundles and jet prolongations

In this appendix we review the basic definitions and results on jet bundles and jet prolongation of principal bundles. Our main references are the standard books by P.W. Michor [Mic80], by D.J. Saunders [Sau89], and by I. Kolář, P. Michor, and J. Slovák [KMS93].

A.1 Jets and Taylor polynomials of smooth local functions

Let M and M' be two manifolds of dimension respectively d and d' . Given a local map $f : M \rightarrow M'$ defined around a point $x \in M$, the n -**jet** of f in x , denoted by $j_x^n f$, is the equivalence class of local maps from M to M' having the same **contact of order** n of f in x , that is, the same value and the same derivatives in x up to order n . Jets of functions can be defined for any integer order $n \geq 0$, and $J^0(M, M') = M \times M'$. Denote by $J_x^n(M, M')$ the space of n -jets of local maps $f : M \rightarrow M'$ defined around x and such that $y = f(x)$, and set $J_x^n(M, M') := \bigcup_{y \in M'} J_x^n(M, M')_y$. The **jet space**

$$J^n(M, M') := \bigcup_{(x,y) \in M \times M'} J_x^n(M, M')_y \quad (97)$$

has two natural projections to M (the **source** π^n) and to M' (the **target** π_0^n).

As equivalence classes, jets are by definition independent of local coordinates in M and in M' , but a representative of a jet involves derivatives which do depend on the choice of local coordinates. Given a choice of coordinates on M and on M' , the identification of a local map $f : M \rightarrow M'$ fixing $f(x) = y$ to its local coordinates expression $\tilde{f} : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$, fixing $\tilde{f}(0) = 0$, yields an isomorphism [KMS93, §12.6] between $J_x^n(M, M')_y$ and the real vector space

$$L_{d,d'}^n := J_0^n(\mathbb{R}^d, \mathbb{R}^{d'})_0 \cong \bigoplus_{k=1}^n S^k((\mathbb{R}^d)^*) \otimes \mathbb{R}^{d'} \cong \overline{\mathbb{R}_n[X_1, \dots, X_d]} \otimes \mathbb{R}^{d'} \quad (98)$$

of dimension $\left[\binom{d+n}{d} - 1 \right] d'$ which prolongs the matrix space $L_{d,d'}^1 \cong M_{d,d'}(\mathbb{R})$ for $n = 1$ and contains the matrix coefficients $a_\alpha^j = \frac{1}{|\alpha|!} \partial^\alpha \tilde{f}^j(0) \in \mathbb{R}$. The space $S^k((\mathbb{R}^d)^*)$ is that of symmetric k -covectors on $\mathbb{R}^d$, cf. [KMS93, §12.10] and [Sau89, Theorems 5.1.7 and 6.2.9]. The collection $(a_\alpha^j)_{j,\alpha}$, with

$j = 1, \dots, d'$ and $|\alpha| \geq 1$, together with $f(x) = (a_0^j)_j$, gives the Taylor polynomial of f at x in the chosen coordinates. Therefore $J^n(\mathbb{R}^d, \mathbb{R}^{d'}) \cong \mathbb{R}^d \times L_{d,d'}^n \times \mathbb{R}^{d'}$. It follows that the jet space $J^n(M, M')$ is a smooth manifold [Mic80, Theorem 1.10]: if⁶ M has an atlas of local charts $\{(U, u : U \rightarrow \mathbb{R}^d)\}$ and M' has an atlas of local charts $\{(U', u' : M' \rightarrow \mathbb{R}^{d'})\}$, then $J^n(M, M')$ has atlas $\{(J^n(U, U'), J^n(u^{-1}, u))\}$ with local charts

$$J^n(u^{-1}, u) : J^n(U, U') \rightarrow J^n(u(U), u'(U')) \cong u(U) \times L_{d,d'}^n \times u'(U') \quad (99)$$

given by $J^n(u^{-1}, u)(j_x^n f) = j_x^n(u' \circ f \circ u^{-1})$.

A.2 Higher jet groups and higher frame bundles

Jet composition is the key operation to express the invariance of jets under change of local coordinates. For two local maps $f : M \rightarrowtail M'$ around x and $f' : M' \rightarrowtail M''$ around $f(x) \in M'$, the **jet composition** of $j_x^n f$ and $j_{f(x)}^n f'$ is the n -jet [KMS93, §12.3]

$$j_{f(x)}^n f' \circ j_x^n f = j_x^n (f' \circ f). \quad (100)$$

The jet composition is associative, it preserves the inversion, i.e. if $h : M \rightarrowtail M'$ is invertible then $(j_x^n h)^{-1} = j_{h(x)}^n h^{-1}$, and for $M' = M$ it has a two-sided unit $1_x := j_x^n \text{Id}_M$.

Consequently, we can read the jets of local diffeomorphisms $h : M \rightarrowtail M'$ as elements in the **space of invertible jets from M to M'**

$$\text{inv}J^n(M, M') := \text{set of invertible } n\text{-jets } j_x^n h \in J^n(M, M') \text{ with respect to jet composition.} \quad (101)$$

In particular, this gives the **n -jet group in dimension d** [KMS93, §12.6]

$$\begin{aligned} GL_d^n &:= \text{inv}L_{d,d}^n = \text{inv}J_0^n(\mathbb{R}^d, \mathbb{R}^d)_0 \\ &\cong \text{group of } n\text{-jets at } 0 \text{ of local diffeomorphisms } h : \mathbb{R}^d \rightarrowtail \mathbb{R}^d \text{ preserving } 0, \text{ with} \\ &\quad \text{jet composition.} \end{aligned} \quad (102)$$

This group extends the trivial group $GL_d^0 \cong \{1_{GL_d(\mathbb{R})}\}$ and the general linear group $GL_d^1 \cong GL_d(\mathbb{R})$, where a jet $j_0^1 h \in GL_d^1$ is identified with the differential $dh_0 \in GL_d(\mathbb{R})$.

The jet of (the inverse of) a local chart $\varphi : \mathbb{R}^d \rightarrowtail M$, which represents a choice of local coordinates of M at $x = \varphi(0)$, belongs to the **n -frame bundle of M** [KMS93, §12.12]

$$\begin{aligned} F^n M &:= \text{inv}J_0^n(\mathbb{R}^d, M) \xrightarrow{\pi_0^n} M \\ &\cong \text{set of } n\text{-jets at } 0 \text{ of local diffeomorphisms } \varphi : \mathbb{R}^d \rightarrowtail M. \end{aligned} \quad (103)$$

This is a principal GL_d^n -bundle with right action given by the jet composition $j_0^n \varphi \circ j_0^n h = j_0^n (\varphi \circ h)$, where h represents a change of local coordinates on M at x and $\varphi \circ h$ the new local chart. The n -frame bundle is trivial for a flat manifold $M = \mathbb{R}^d$, on which we can decompose $\varphi = \tau_x \circ h$ into a diffeomorphism h preserving 0 and the translation $\tau_x(y) = x + y$ bringing 0 to x with jet $j_0^n \tau_x = (x, \text{Id})$: this gives the isomorphism $F^n \mathbb{R}^d \cong \mathbb{R}^d \times GL_d^n$ which identifies $j_0^n \varphi \equiv (x, j_0^n h)$. The n -frame bundle is also trivial for $n = 0$, since $F^0 M \cong M \times \{1\}$, but in general it is not. For instance, $F^1 M \cong FM := F(TM)$ is the usual frame bundle, since a jet $j_0^1 \varphi \in F_x^1 M$ is equivalent to the pair

⁶[we could decide to cancel this explanation]

$(\varphi(0), d\varphi_0) \in F_x M$ with $\varphi(0) = x$, where the differential $d\varphi_0 : \mathbb{R}^d \xrightarrow{\cong} T_x M$ gives a linear frame of TM at x .

The simultaneous choice of a local chart φ on M and φ' on M' is then represented on jets by the two n -frame bundles $F^n M$ and $F^n M'$, and the change of local coordinates h and h' is represented by the two n -jet groups GL_d^n and $GL_{d'}^n$. With the notations of the diagram

$$\begin{array}{ccc}
 M & \xrightarrow{*} & M' \\
 \uparrow \varphi & \nearrow \bar{f} & \uparrow \varphi' \\
 \mathbb{R}^d & \xrightarrow{*} & \mathbb{R}^{d'} \\
 h \circlearrowleft & & \circlearrowright h'
 \end{array} \quad (104)$$

the group $GL_d^n \times GL_{d'}^n$ acts on $L_{d,d'}^n$ from the left by $(j_0^n h, j_0^n h') \cdot j_0^n \bar{f} = j_0^n (h' \circ \bar{f} \circ h^{-1})$, and the jet space $J^n(M, M')$ can be seen as a fibre bundle over $M \times M'$ associated to $F^n M \times F^n M'$ with fibre $L_{d,d'}^n$ [KMS93, §12.12], that is

$$J^n(M, M') \cong (F^n M \times F^n M') \times_{GL_d^n \times GL_{d'}^n} L_{d,d'}^n \quad j_x^n f \mapsto [j_0^n \varphi, j_0^n \varphi', j_0^n \bar{f}]_{\varphi(0)=x}. \quad (105)$$

The isomorphism holds because the local charts φ and φ' can be chosen arbitrarily (with $\varphi(0) = x$) to define $\bar{f} = (\varphi')^{-1} \circ f \circ \varphi$ from f and viceversa.

A.3 Higher tangent bundle

The local map $\bar{f} : \mathbb{R}^d \rightarrow M'$ of diagram (104) reads $f : M \rightarrow M'$ in local coordinates on M . Its jet belongs to a very useful jet space, the **n -tangent bundle of M' in dimension d** , also called the bundle of **n -velocities on M' in dimension d** [KMS93, §12.8],

$$\begin{aligned}
 T_d^n M' &:= J_0^n(\mathbb{R}^d, M') \xrightarrow{\pi_0^n} M' \\
 &\cong \text{set of } n\text{-jets at } 0 \text{ of local maps } \bar{f} : \mathbb{R}^d \rightarrow M'.
 \end{aligned} \quad (106)$$

This is the fibre bundle⁷ with fibre $L_{d,d'}^n$ associated to the principal bundle $F^n M'$ [KMS93, §12.12], that is

$$T_d^n M' \cong F^n M' \times_{GL_{d'}^n} L_{d,d'}^n \quad j_0^n \bar{f} \mapsto [j_0^n \varphi', j_0^n \bar{f}]. \quad (107)$$

Since $T_d^n M'$ admits a left GL_d^n -action given by $j_0^n h \cdot j_0^n \bar{f} = j_0^n (\bar{f} \circ h^{-1})$, it gives rise to a second presentation of the jet bundle $J^n(M, M')$ as the fibre bundle associated to $F^n M$ over M with fibre $T_d^n M'$ (from which it inherits the bundle projection to M'):

$$J^n(M, M') \cong F^n M \times_{GL_d^n} T_d^n M' \quad j_x^n f \mapsto [j_0^n \varphi, j_0^n \bar{f}]_{\varphi(0)=x}. \quad (108)$$

The isomorphisms (107) and (108) together imply (105).

⁷The higher tangent bundle $T_d^n M'$ is not a vector bundle for $n > 1$, unless M' is a vector space [KMS93, §12.8]. To have a vector bundle, one can consider the dual of the higher cotangent bundle $T_d^{n*} M' = J^n(M', \mathbb{R}^d)_0$, which is a vector bundle over M' [KMS93, §12.14].

A.4 Jet prolongation of fibre bundles and vector bundles

For a fibre bundle $\pi : E \rightarrow M$, denote by $\Gamma_{\text{loc}}^\infty(M, E)$ the set of its smooth local sections $\psi : M \multimap E$, with domain denoted by $U_\psi \subset M$. The **n -jet bundle** of E is the set

$$J^n E := J^n \Gamma_{\text{loc}}^\infty(M, E) = \{j_x^n \psi \in J^n(M, E) \mid \pi \circ \psi = \text{Id}_{U_\psi}\}. \quad (109)$$

The space $J^n E$ is a closed submanifold of the jet space $J^n(M, E)$ [KMS93, §12.16] and it is a bundle over M [Sau89, §6.2]. Moreover, the target jet projection $\pi_0^n : J^n E \rightarrow E$ yields a fibre bundle also on E . Forgetting the highest derivatives, one gets a filtered tower of affine bundles above E

$$J^n E \xrightarrow{\pi_{n-1}^n} J^{n-1} E \xrightarrow{\pi_{n-2}^{n-1}} \dots \xrightarrow{\pi_1^2} J^1 E \xrightarrow{\pi_0^1} J^0 E = E. \quad (110)$$

A bundle map $\phi : E \rightarrow E'$ between fibre bundles over M induces a bundle map between their n -jet prolongations

$$J^n \phi : J^n E \longrightarrow J^n E', \quad j_x^n \psi \mapsto J^n \phi(j_x^n \psi) := j_x^n (\phi \circ \psi). \quad (111)$$

If $\pi : E \rightarrow M$ is a vector bundle of rank r , then the source jet projection $\pi^n : J^n E \rightarrow M$ is also a vector bundle [KMS93, §12.17], while the target jet projection $\pi_0^n : J^n E \rightarrow E$ is an affine bundle. The fibre of $J^n E$ over M is modelled on the vector space

$$\begin{aligned} P_{d,r}^n &:= T_d^n \mathbb{R}^r = J_0^n(\mathbb{R}^d, \mathbb{R}^r) \cong \mathbb{R}^r \oplus L_{d,r}^n \\ &\cong \bigoplus_{k=0}^n S^k((\mathbb{R}^d)^*) \otimes \mathbb{R}^r \cong \mathbb{R}_n[X_1, \dots, X_d] \otimes \mathbb{R}^r \end{aligned} \quad (112)$$

of dimension $\binom{d+n}{d}r$, that is, the space of $\mathbb{R}^r$ -valued polynomials in d variables, which represent the Taylor polynomials of sections of E in given local coordinates. With the notations of fig. (104), an element of $P_{d,r}^n$ can be written as the jet $j_0^n \tilde{f} = \tilde{f}(0) + j_0^n \tilde{f}$ where $j_0^n \tilde{f} \in L_{d,r}^n$. In summary, $J^n E$ is the vector bundle associated to its frame bundle $F(J^n E)$ on M with fibre $P_{d,r}^n$, that is, $J^n E \cong F(J^n E) \times_{GL(P_{d,r}^n)} P_{d,r}^n$.

A.5 Jet prolongation of structure groups

The structure group of the jet bundle $J^n E$ is the subgroup of $GL(P_{d,r}^n)$ which hosts its transition functions, in particular the jets of transitions functions of E . If E has trivializing atlas $\{U_\alpha\}$ with associated local sections $\{f_\alpha\}$ and structure group $G \subset GL_r(\mathbb{R})$, the transition functions are the G -equivariant maps $\hat{g}_{\alpha\beta} : U_{\beta|U_{\alpha\beta}} \times G \rightarrow U_{\alpha|U_{\alpha\beta}} \times G$, defined on the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ of overlapping charts, which relate the local sections by $f_\alpha|_{U_{\alpha\beta}} = \hat{g}_{\alpha\beta} f_\beta|_{U_{\alpha\beta}}$. Reading the points $x \in U_\alpha$ in local coordinates $\vec{x}$ via the chart $\varphi = \varphi_\alpha^{-1} : \mathbb{R}^d \multimap U_\alpha$, and those of U_β in local coordinates $\vec{x}' = h(\vec{x})$ via the chart $\varphi_\beta^{-1} = \varphi \circ h : \mathbb{R}^d \multimap U_\beta$, where $h = \varphi_\alpha \circ \varphi_\beta^{-1} : \mathbb{R}^d \multimap \mathbb{R}^d$ is the change of coordinates, leads to define the **n -jet prolongation of G in dimension d** as the jet space [KMS93, §15.2]

$$W_d^n G \cong \text{set of } n\text{-jets at } (0, 1) \in \mathbb{R}^d \times G \text{ of } G\text{-equivariant local diffeomorphisms} \quad (113)$$

$$\hat{h} : \mathbb{R}^d \times G \xrightarrow{\sim} \mathbb{R}^d \times G \text{ defined around } (0, 1) \text{ and preserving } 0 \in \mathbb{R}^d.$$

Such maps $\hat{h}$ are necessarily of the form $\hat{h}(\vec{x}, a) = (h(\vec{x}), g(\vec{x})a) = (h, \text{Id}) \circ \hat{g}(\vec{x}, a)$ for a local map $g : \mathbb{R}^d \multimap G$ defined around 0 and $\hat{g} : \mathbb{R}^d \times G \xrightarrow{\sim} \mathbb{R}^d \times G$ given by $\hat{g}(\vec{x}, a) = (\vec{x}, g(\vec{x})a)$. Then the

jet $j_{(0,1)}^n \hat{h} = (j_0^n h, 1) \circ j_{(0,1)}^n \hat{g}$ is equivalent to the pair of jets $(j_0^n h, j_0^n g) \in GL_d^n \times T_d^n G$ (cf. (102) and (106)). The n -tangent space $T_d^n G$ is a group with the pointwise product

$$j_0^n g \cdot j_0^n g' = j_0^n (gg'), \quad (114)$$

and GL_d^n acts on $T_d^n G$ on the right by composition. The jet composition of two jets $j_{(0,1)}^n \hat{h}$ and $j_{(0,1)}^n \hat{h}'$ becomes the semidirect group law on their representative pairs

$$(j_0^n h, j_0^n g) \cdot (j_0^n h', j_0^n g') = (j_0^n (h \circ h'), j_0^n ((g \circ h') g')). \quad (115)$$

In other words, there is a group isomorphism [KMS93, §15.2]

$$W_d^n G \cong GL_d^n \ltimes T_d^n G, \quad j_{(0,1)}^n \hat{h} = (j_0^n h, 1) \circ j_{(0,1)}^n \hat{g} \equiv (j_0^n h, j_0^n g). \quad (116)$$

The semidirect product law ensures also that $W_d^n G$ acts on $P_{d,r}^n$, with left action given for $j_{(0,1)}^n \hat{h} = (j_0^n h, j_0^n g)$ and $j_0^n \bar{f} = \bar{f}(0) + j_0^n \tilde{f}$ by

$$j_{(0,1)}^n \hat{h} \cdot j_0^n \bar{f} := g(0) \bar{f}(0) + j_0^n (g \circ h^{-1}) \cdot j_0^n (\tilde{f} \circ h^{-1}). \quad (117)$$

Lemma A.1. *For any subgroup $G \subseteq GL_r(\mathbb{R})$, there is a proper inclusion of groups*

$$W_d^n G = GL_d^n \ltimes T_d^n G \hookrightarrow GL(P_{d,r}^n). \quad (118)$$

Proof. Expressing jets in the standard basis of polynomials representing $P_{d,r}^n$, as after eq. (98), shows that the action (117) is a matrix product and the action is linear. This defines the inclusion (118). If we order this basis by increasing order of derivatives, we mimic the affine filtration (110) of jets: the matrices associated to the jets $j_0^n (g \circ h^{-1})$ and $j_0^n (\tilde{f} \circ h^{-1})$ are then upper triangular. This corresponds to the results in [KMS93, §13, §21] on the filtration of jet subgroups and of their induced actions. In other words, this means that the image of $W_d^n G$ by the inclusion (118) is a proper subgroup of $GL(P_{d,r}^n)$. $\square$

A.6 Jet prolongation of principal bundles

The jet bundle $J^n E$ on M gives rise to the frame bundle $F(J^n E)$, which is a principal $GL(P_{d,r}^n)$ -bundle on M . On $J^n E$, we can consider to reduce the structure group to the subgroup $W^n GL_r(\mathbb{R})$, and more generally, if $E = P \times_G \mathbb{R}^k$ is the vector bundle associated to a principal G -bundle P , we can consider a reduction of the structure group of $J^n E$ to $W_d^n G$. The corresponding reduction of the principal bundle $F(J^n E)$ is obtained by considering the jet prolongation of (the inverse of) the local trivializations $\theta_\alpha : U_\alpha \times G \rightarrow P$, which are local G -equivariant bundle automorphisms giving rise, on overlapping charts, to the transition functions $\hat{g}_{\alpha\beta}$. Reading these maps in local coordinates on M leads to define the **principal n -jet prolongation** of P as the bundle [KMS93, §15.3]

$$W^n P = \text{set of } n\text{-jets } j_{(0,1)}^n \theta \text{ of local bundle automorphisms } \theta : \mathbb{R}^d \times G \rightarrow P \quad (119)$$

above base maps $\varphi : \mathbb{R}^d \rightarrow M$,

which is naturally a bundle over P with the target projection π_0^n , and becomes a bundle over M with projection $\pi \circ \pi_0^n$. The jet group $W_d^n G$ acts on $W^n P$ on the right by jet composition

$$j_{(0,1)}^n \theta \cdot j_{(0,1)}^n \hat{h} = j_{(0,1)}^n (\theta \circ \hat{h}) = j_{(0,1)}^n (\theta \circ (h, \text{Id}) \circ \hat{g}), \quad (120)$$

and $W^n P$ is a principal $W_d^n G$ -bundle over M [KMS93, §15.3]. A bundle map $\phi : P \rightarrow P'$ between two principal bundles over M has a jet prolongation

$$W^n \phi : W^n P \rightarrow W^n P', \quad j_{(0,1)}^n \theta \mapsto W^n \phi(j_{(0,1)}^n \theta) = j_{(0,1)}^n (\phi \circ \theta). \quad (121)$$

Local trivializations of P are in bijection with local sections of P . This bijection survives on jet prolongations up to a choice of local coordinates: since the choice of the local chart φ is totally arbitrary (because of the quotient by the action of GL_d^n), the principal prolongation $W^n P$ can be expressed as the pullback bundle of the two bundles $F^n M$ and $J^n P$ over M , that is [KMS93, §15.4] [Kol09, Eq. (33)]

$$W^n P \cong F^n M \times_M J^n P, \quad j_{(0,1)}^n \theta \mapsto (j_0^n \varphi, j_x^n p) \quad (122)$$

where θ is the local trivialization determined by the local section p , modulo a local chart φ on M such that $\varphi(0) = x$, i.e. $\theta(\vec{x}, a) = p(\varphi(\vec{x}))a$ for any $\vec{x} \in \mathbb{R}^d$ close to $\vec{0}$ and for any $a \in G$.

Examples A.2 (Jet prolongation of principal bundles). Some easy examples:

- (a) When $M = \mathbb{R}^d$, every local chart at a point x is of the form $\varphi = \tau_x \circ h$, where $\tau_x(y) = x+y$ is the translation sending 0 to x and h is a change of coordinates respecting 0 [KMS93, §15.3]. This trivializes the frame bundle $F\mathbb{R}^d$. Since $j_0^n \tau_x = x$, the expression $\theta(\vec{x}, a) = (\varphi(\vec{x}), g(\vec{x})a) = (x + h(\vec{x}), g(\vec{x})a)$ gives

$$\begin{aligned} W^n(\mathbb{R}^d \times \{1\}) &\cong F^n \mathbb{R}^d \cong \mathbb{R}^d \times GL_d^n, & j_{(0,1)}^n \theta &\mapsto j_0^n \varphi \mapsto (x, j_0^n h) \\ W^n(\mathbb{R}^d \times G) &\cong F^n \mathbb{R}^d \times_{\mathbb{R}^d} J^n(\mathbb{R}^d \times G) \\ &\cong \mathbb{R}^d \times W_d^n G & j_{(0,1)}^n \theta &\mapsto (x, j_0^n h, j_0^n g) \\ W^n(F\mathbb{R}^d) &= W^n(\mathbb{R}^d \times GL_d(\mathbb{R})) \cong \mathbb{R}^d \times W_d^n GL_d(\mathbb{R}). \end{aligned}$$

- (b) For general M , the n -frame bundle $F^n M$ is not necessarily the trivial GL_d^n -bundle, the expression $\theta(\vec{x}, a) = (\varphi(\vec{x}), g(\vec{x})a)$ gives

$$\begin{aligned} W^n(M \times \{1\}) &\cong F^n M \times_M J^n(M \times \{1\}) \cong F^n M, & j_{(0,1)}^n \theta &\mapsto j_0^n \varphi, \\ W^n(M \times G) &\cong F^n M \times_{GL_d^n} W_d^n G, & j_{(0,1)}^n \theta &\mapsto [j_0^n \varphi, (j_0^n h, j_0^n g)]. \end{aligned}$$

In fact, we have

$$\begin{aligned} W^n(M \times G) &\cong F^n M \times_M J^n(M \times G) \\ &\cong F^n M \times_M J^n(M, G) \\ &\cong F^n M \times_M (F^n M \times_{GL_d^n} T_d^n G) && \text{follows from (108)} \\ &\cong F^n M \times_{GL_d^n} W_d^n G \end{aligned}$$

The last line follows from the explicit isomorphism $R : (\varphi, [\varphi h, w]) \mapsto [\varphi, (h^{-1}, w)]$ from $F^n M \times_M (F^n M \times_{GL_d^n} T_d^n G)$ to $F^n M \times_{GL_d^n} W_d^n G$.

Moreover, since $J^n(FM)$ is a principal $T_d^n GL_d(\mathbb{R})$ -bundle on M , cf. [BDK14, Theorem 1], it follows that

$$W^n(FM) \cong F^n M \times_M J^n(FM) \cong F^n M \times T_d^n GL_d(\mathbb{R}).$$

□

A.7 Associated jet bundles and reductions

Let $E = P \times_G V$ be a fibre bundle with fibre V associated to a principal G -bundle $P \rightarrow M$. Then $J^n E$ is the fibre bundle with fibre $T_d^n V$ associated to the principal $W_d^n G$ -bundle $W^n P$ [KMS93, §15.5], that is,

$$J^n E \cong W^n P \times_{W_d^n G} T_d^n V. \quad (123)$$

Example A.3. Let E be a vector bundle with typical fibre $\mathbb{R}^r$ and structure group $GL_r(\mathbb{R})$. We can consider three interesting jet bundles by reducing the structure group of $J^n E$.

- (a) Since the jet bundle $J^n E$ is a vector bundle with typical fibre $P_{d,r}^n = T_d^n(\mathbb{R}^r)$, its frame bundle $F(J^n E)$ is a principal $GL(P_{d,r}^n)$ -bundle and we have

$$J^n E \cong F(J^n E) \times_{GL(P_{d,r}^n)} P_{d,r}^n.$$

- (b) The frame bundle FE is a principal $GL_n(\mathbb{R})$ -bundle such that $E \cong FE \times_{GL_n(\mathbb{R})} \mathbb{R}^r$. Then, by (123), we have $J^n E \cong W^n FE \times_{W_d^n GL_r(\mathbb{R})} P_{d,r}^n$. This means that the structure group $GL(P_{d,r}^n)$ of $J^n E$ can be reduced to its subgroup $W_d^n GL_r(\mathbb{R})$, therefore we have

$$F(J^n E) \cong W^n FE \times_{W_d^n GL_r(\mathbb{R})} GL(P_{d,r}^n). \quad (124)$$

- (c) If the structure group of E can be reduced to $G \subset GL_r(\mathbb{R})$, so that there is a principal G -bundle P such that $E \cong P \times_G \mathbb{R}^r$, then (123) says that $J^n E \cong W^n P \times_{W_d^n G} P_{d,r}^n$. In other words we can further reduce the structure group of $J^n E$ to $W_d^n G \subset W_d^n GL_r(\mathbb{R})$, and therefore we have

$$W^n FE \cong W^n P \times_{W_d^n G} W_d^n GL_r(\mathbb{R}), \quad (125)$$

$$F(J^n E) \cong W^n P \times_{W_d^n G} GL(P_{d,r}^n). \quad (126)$$

Note that by Lemma A.1 there is a proper inclusion of groups $W_d^n G \subsetneq W_d^n GL_r(\mathbb{R}) \subsetneq GL(P_{d,r}^n)$, and therefore a proper inclusion of principal bundles $W^n P \subsetneq W^n FE \subsetneq F(J^n E)$. $\square$

A.8 Jets and covariant derivatives

Let M be equipped with a torsion-free linear connection ∇^M on TM , and let $E \rightarrow M$ be a vector bundle of finite rank equipped with a linear connection ∇ . The **higher covariant derivatives**

$$\nabla^n : \Gamma(E) \rightarrow \Gamma(T^* M^{\otimes n} \otimes E) \quad (127)$$

are defined as follows, cf. [KN63, Ch.III, Section 2] or [JL14, eq. (21)]. Let ψ be a section of E . For $n=0$ set $\nabla^0 \psi = \psi$. For $n=1$ let $\nabla^1 \psi = \nabla \psi$ be the given covariant derivative. For $n > 1$, first recall that the linear connection ∇^M on TM induces a linear connection ∇^* on $T^* M$ such that the Lie derivative $\mathcal{L}_Y$ satisfies the Leibniz rule

$$\mathcal{L}_Y \langle \alpha, X \rangle = \langle \nabla_Y^* \alpha, X \rangle + \langle \alpha, \nabla_Y X \rangle,$$

where $\alpha \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$. Then extend the tensor product $\nabla^* \otimes \nabla$ to a linear connection $\nabla^{(n+1)}$ on $T^* M^{\otimes n} \otimes E$ satisfying another Leibniz rule

$$\nabla^{(n+1)}(\alpha_1 \otimes \cdots \otimes \alpha_n \otimes \psi) = \sum_{j=1}^n \alpha_1 \otimes \cdots \otimes \nabla^* \alpha_j \otimes \cdots \otimes \alpha_n \otimes \psi + \alpha_1 \otimes \cdots \otimes \alpha_n \otimes \nabla \psi,$$

for $\alpha_1, \dots, \alpha_n \in \Omega^1(M)$. Finally, compose these connections into

$$\nabla^n \psi := \nabla^{(n)} \nabla^{(n-1)} \dots \nabla^{(2)} \nabla \psi.$$

This section can be contracted with n vector fields and evaluated at a point x , or evaluated at x and contracted with n tangent vectors at x , since we have

$$\nabla^n \psi|_x (X_x, Y_x, \dots, Z_x) = \nabla^n \psi(X, Y, \dots, Z)|_x.$$

Note that, when $\nabla^n \psi$ is evaluated along a geodesic $x(t) = \exp_{x_0}(tX)$ and contracted n times with the velocity vector $\dot{x}(t)$, applying successively [KN63, Ch.III, Proposition 2.12] or [Lee97, Exercice 4.5, p.54] (or directly [DDD19, Definition 56]), we have

$$\begin{aligned} \nabla^2 \psi|_{x(t)}(\dot{x}(t), \dot{x}(t)) &= \nabla_{\dot{x}(t)}(\nabla_{\dot{x}(t)} \psi)|_{x(t)} - \nabla \psi|_{x(t)}(\nabla_{\dot{x}(t)}^M \dot{x}(t)) = \nabla_{\dot{x}(t)}(\nabla_{\dot{x}(t)} \psi)|_{x(t)} \\ \nabla^n \psi|_{x(t)}(\dot{x}(t)^{\otimes n}) &= \nabla_{\dot{x}(t)}(\nabla_{\dot{x}(t)} \dots (\nabla_{\dot{x}(t)} \psi) \dots)|_{x(t)} \quad n \geq 2. \end{aligned} \quad (128)$$

The **symmetrized covariant derivatives**

$$D_{\nabla}^n : \Gamma(E) \rightarrow \Gamma(S^n(T^*M) \otimes E) \quad (129)$$

are then defined as

$$D_{\nabla}^n \psi := [\text{Sym}^n \otimes \text{Id}_E] (\nabla^n \psi),$$

where $\text{Sym}^n : T^*M^{\otimes n} \rightarrow S^n(T^*M) = (T^*M^{\otimes n})^{S_n}$ is the usual symmetrizing map given by

$$\text{Sym}^n(\alpha_1 \otimes \dots \otimes \alpha_n) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \alpha_{\sigma^{-1}(1)} \otimes \dots \otimes \alpha_{\sigma^{-1}(n)}.$$

The operators D_{∇}^n provide a splitting of the jet bundle exact sequence [Pal65, Ch. IV, Theorem 7]

$$0 \rightarrow S^n(T^*M) \otimes E \longrightarrow J^n E \xrightarrow{\pi_{n-1}^n} J^{n-1} E \rightarrow 0.$$

As a consequence, see [Pal65, Ch. IV, Corollary, p.90], the given torsion-free linear connection ∇^M on TM , and the linear connection ∇ on E , provide a canonical isomorphism of vector bundles

$$J^n E \cong \bigoplus_{k=0}^n S^k(T^*M) \otimes E \quad (130)$$

under which jets are represented by covariant derivatives as

$$j_{x_0} \psi \mapsto [x_0, \psi(x_0), D_{\nabla} \psi|_{x_0}, \dots, D_{\nabla}^n \psi|_{x_0}]. \quad (131)$$

Note that $D_{\nabla} \psi|_{x_0} = \nabla \psi|_{x_0}$, and when $D_{\nabla}^n \psi|_{x_0}$ is contracted with n equal vector fields X , it coincides with $\nabla^n \psi|_{x_0}$, that is,

$$D_{\nabla}^n \psi|_{x_0}(X, \dots, X) = \nabla^n \psi|_{x_0}(X, \dots, X). \quad (132)$$

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