The Hirota conditions

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The condition on the polynomial P for a Hirota equation $P\tau \cdot \tau = 0$ to have an N-soliton solution for arbitrary N is examined and simplified.

I. INTRODUCTION

While the role of affine Lie algebras in explaining many of the miracles of soliton mathematics is understood,[1–3] the Hirota conditions have so far eluded interpretation. These relations express the condition under which a given partial differential equation, when expressed in Hirota or quadratic (homogeneous) form, has an N-soliton or N-phase rational solution. It is generally agreed, although not rigorously proved, that, if these conditions hold for arbitrary N, the evolution equation is completely integrable and belongs to a commuting family, each of whose members is also a completely integrable soliton equation. The goal of this paper is to simplify the Hirota conditions and to express them in a way that may lead to an algebraic interpretation. In particular, we build on the idea, first expressed by one of the authors in Ref. 4, that the *phase shift function* plays a central role in identifying the members of a particular family of soliton equations. This function is common to each of the equations in the commuting family and measures the phase shift experienced by two colliding solitons. The fact that the same phase shift, which is a function of the two-soliton amplitudes, is shared by each of the members of the family is a simple consequence of the commutability of the flows.

The Hirota formalism homogenizes the partial differential equation by converting it into a bilinear, and in some cases a quadratic, equation. For example, the transformation

$$q(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\tau \tag{1.1}$$

converts the Korteweg–de Vries equation

$$q_t + 6qq_x + q_{xxx} = 0 \tag{1.2}$$

into the form

$$\tau\tau_{xt} - \tau_x\tau_t + \tau\tau_{xxxx} - 4\tau_x\tau_{xxx} + 3\tau_{xx}^2 = 0. \tag{1.3}$$

Hirota developed a very neat way of writing this equation by introducing a derivative operator D_{x_j}, which acts on ordered pairs of functions as follows:

$$D_x\sigma(x)\cdot\tau(x) = \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\sigma(x+\epsilon)\tau(x-\epsilon) \tag{1.4}$$

and in general

$$D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}\sigma(x_r)\cdot\tau(x_r) = \prod_{r=1}^{n}\lim_{\epsilon_r\to 0}\frac{\partial^{a_r}}{\partial\epsilon_r^{\alpha_r}}\sigma(x_r+\epsilon_r)\tau(x_r-\epsilon_r). \tag{1.5}$$

The right-hand sides of (1.4) and (1.5) are exactly the same as the Leibnitz formula for derivatives of products except for certain sign changes. Using this notation, Eq. (1.3) may be written (call $t = t_3$)

$$(D_xD_{t_3} + D_x^4)\tau\cdot\tau = 0. \tag{1.6}$$

Associated with the Korteweg–de Vries equation is the polynomial $x_1x_3 + x_1^4$ ($x_1 = x$, $x_3 = t_3$). Each member of the Korteweg–de Vries (KdV) family of equations may be written in quadratic form. The next member in the family, designated KdV 5, is

$$q_{t_5} - q_{xxxxx} - 20q_xq_{xx} - 10qq_{xxx} - 30q^2q_x = 0, \tag{1.7}$$

and, using (1.5), this may be written

$$(D_xD_{t_5} - D_x^6 + \tfrac{3}{5}(D_xD_{t_3} + D_{t_3}^2))\tau\cdot\tau = 0. \tag{1.8}$$

Notice that in order to write KdV 5 in quadratic form, one needs to include the KdV 3 time variable t_3 in addition to the time t_5 that appears in (1.7). The Hirota equations for KdV 5 are the pair of equations (1.6) and (1.8). We also observe that these two examples of Hirota equations are even and homogeneous under the weight assignment $W(D_{t_{2k+1}}) = 2k+1$.

Other well-known equations that have Hirota form are the Sawada–Kotera equation, $t_1 = x$,

$$(D_{t_1}^6 + 9D_{t_1}D_{t_5})\tau\cdot\tau = 0; \tag{1.9}$$

the Ramani equation

$$(D_{t_1}^6 - 5D_{t_1}^3D_{t_3} - 5D_{t_3}^2)\tau\cdot\tau = 0; \tag{1.10}$$

the Ito equation

$$(D_{t_3}^2 + 2D_{t_3}D_{t_1}^3)\tau\cdot\tau = 0; \tag{1.11}$$

and the Kadomtsev–Petviashvili equation

$$(u_t + uu_x + u_{xxx})_x + u_{yy} = 0, \tag{1.12}$$

which is transformed by (1.1) into

$$(\tfrac{3}{4}D_y^2 - D_xD_{t_3} + \tfrac{1}{4}D_x^4)\tau\cdot\tau = 0. \tag{1.13}$$

One of the advantages of the Hirota formalism is that it is relatively easy to find expressions for the multisoliton solution. The reason for this is that the N-phase multisoliton solution, which for the KdV family is given by

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$$\tau(x=t_1,t_3,t_5,...) = \sum_{\mu_j,\mu_l=0,1} \exp\left(\sum_{j=1}^{N}\mu_j\theta_j + \sum_{1<j<l<N} A_{jl}\mu_j\mu_l\right), \tag{1.14}$$

consists of sums of exponentials and the Hirota operator D acts in a simple way on ordered pairs of exponentials, e.g.,

$$D_x^m e^{k_1x}\cdot e^{k_2x} = (k_1-k_2)^m e^{k_1x+k_2x}. \tag{1.15}$$

In (1.14),

$$\theta_j = \sum_0^\infty (-1)^n k_j^{2n+1} t_{2n+1}, \quad t_1 = x, \tag{1.16}$$

The phase shift A_{jl} is given by

$$e^{A_{jl}} = ((k_j-k_l)/(k_j+k_l))^2, \tag{1.17}$$

and the first sum is taken over all configurations of the μ_j, $j=1,...,N$, each choice being either a zero or a one.

We emphasize that (1.14) provides the common N-soliton solution for all members of the KdV family. As we have mentioned, they all share the same phase shift, a property that can be deduced readily from the fact that the flows $q_{t_{2r+1}}$, $r=0,1,...$, commute. The general formula analogous to (1.15) is

$$P(D_{t_1},D_{t_3},...)e^{\theta_j}\cdot e^{\theta_l} = P(\mathbf{k}_j-\mathbf{k}_l)e^{\theta_j+\theta_l}, \tag{1.18}$$

where

$$P(\mathbf{k}_j-\mathbf{k}_l) = P(k_j-k_l,...,(-1)^r(k_j^{2r+1}-k_l^{2r+1}),...). \tag{1.19}$$

We now ask a natural question. Given an even homogeneous polynomial $P_{2L}(D_{t_1},D_{t_3},...,D_{t_{2k+1}})$ of weight $2L$, under what conditions does the corresponding Hirota equation

$$P_{2L}(D_{t_1},D_{t_3},...,D_{t_{2k+1}})\tau\cdot\tau = 0 \tag{1.20}$$

having an N-soliton solution for arbitrary N? The one-soliton form

$$\tau = 1+e^\theta, \tag{1.21}$$

with

$$\theta = \sum_0^\infty k^{(2r+1)} t_{2r+1}, \quad t_1 = x, \tag{1.22}$$

is a solution provided that the vector $\{k^{(2r+1)}\}_0^\infty$ lies on the manifold (which we call the dispersion relation)

$$P_{2L}(k^{(1)},k^{(3)},...,k^{(2r+1)},...) = 0. \tag{1.23}$$

We are going to confine ourselves in this paper to the class of Hirota equation for which (1.23) is satisfied by $k^{(2r+1)}$ being a power of a single parameter k:

$$k^{(1)} = k, \quad k^{(2r+1)} = (-1)^r k^{2r+1}. \tag{1.24}$$

This corresponds to evolution equations like the Korteweg–de Vries equation, which describe how a function of $x=t_1$ evolves with respect to a sequence of times t_{2r+1}. The Kadomtsev–Petviashvili (KP) equation, on the other hand, is part of a family for which the equations describe the evolution in times $t_3,t_4,...$ of a function $q(x=t_1, y=t_2,t_3,t_4,...)$ of $x=t_1$ and $y=t_2$. The dispersion relative for (1.13) is satisfied by expressing each $k^{(r)}$ (here $\theta=\Sigma k^{(r)}t_r$) as a function of two parameters

$$k^{(1)} = u-v, \quad k^{(2)} = u^2-v^2, \quad k^{(3)} = u^3-v^3,... .$$

These equations are associated with the Lie algebra gl(∞) corresponding to the infinite-dimensional linear group. On the other hand, the KdV hierarchy, which is recovered from the KP hierarchy by setting $v=-u=k/2$ and writing t_{2r+1} as $(-1)^r 2^{r+1} t_{2r+1}$, is associated with a subalgebra of gl(∞), namely the Kac–Moody algebra $A_1^{(1)}$ associated with sl(2).

The two-soliton solution

$$\tau = 1+e^{\theta_1}+e^{\theta_2}+e^{A_{12}+\theta_1+\theta_2} \tag{1.25}$$

is a solution of (1.20) with

$$\theta_j = \sum_0^\infty (-1)^r k_j^{2r+1} t_{2r+1}$$

provided the phase shift is chosen as

$$e^{A_{12}} = -P_{2L}(\mathbf{k}_1-\mathbf{k}_2)/P_{2L}(\mathbf{k}_1+\mathbf{k}_2), \tag{1.26}$$

where $P(\mathbf{k}_1\pm\mathbf{k}_2)$ is defined by (1.19). The coefficients of $e^{2\theta_1}$ and $e^{2\theta_2}$ are zero because of (1.18) and the fact that $P_{2L}(\mathbf{0})$ is zero. The coefficient of $e^{2\theta_1+\theta_2}$ is zero because $P_{2L}(\mathbf{k}_1) = P_{2L}(k_1,-k_1^3,k_1^5,...)$ is zero. Thus, Hirota equations in quadratic form always have a two-soliton solution. For a three-soliton solution, there is an additional constraint, obtained by demanding that the coefficient of $e^{\theta_1+\theta_2+\theta_3}$ in the expression

$$P(D_{t_1},D_{t_3},...)\tau\cdot\tau$$

be zero. This condition can be written

$$\begin{aligned} p_{123}\{&P_{2L}(\mathbf{k}_2-\mathbf{k}_3)P_{2L}(\mathbf{k}_3+\mathbf{k}_1) \\ &\times P_{2L}(\mathbf{k}_2+\mathbf{k}_1)P_{2L}(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\} \\ &+P_{2L}(\mathbf{k}_2-\mathbf{k}_3)P_{2L}(\mathbf{k}_3-\mathbf{k}_1) \\ &\times P_{2L}(\mathbf{k}_2-\mathbf{k}_1)P(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) = 0, \end{aligned} \tag{1.27}$$

where

$$\begin{aligned} P(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) &= P(k_1+k_2+k_3,-k_1^3-k_2^3-k_3^3,..., \\ &\times(-1)^r(k_1^r+k_2^r+k_3^r),...), \end{aligned}$$

and p_{123} is the permutation over 1,2,3. For an N-soliton solution, the condition, originally derived by Hirota,[5] is

$$\sum_{\mu_j} P\left(\sum_1^N \mu_j\mathbf{k}_j\right)\prod_{m>l} P(\mu_m\mathbf{k}_m-\mu_l\mathbf{k}_l)\mu_l\mu_m = 0. \tag{1.28}$$

The summation in (1.28) is over all sequences $(\mu_1,\mu_2,...,\mu_N)$, where $\mu_j=\pm 1$, $j=1,...,N$. In each term of the summation,

$$P\left(\sum_1^N \mu_j\mathbf{k}_j\right)\prod_{m>l} P(\mu_m\mathbf{k}_m-\mu_l\mathbf{k}_l)\mu_l\mu_m,$$

all the μ's are determined once a particular choice of the sequence $(\mu_1,\mu_2,...,\mu_N)$ of plus and minus ones is made. Equation (1.28) is known as the Hirota condition and we call a homogeneous polynomial P of even degree that satisfies this condition for all *N a Hirota polynomial.* It is the expression (1.28) that we aim to simplify. In particular, we would like to find an algorithm to determine all polynomials of weight $2L$ that satisfy it.

II. DISCUSSION OF RESULTS

The first curious fact about (1.28) is that it is not, on the surface, linear. And it should be, because integrable evolution equations come in families and therefore linear combinations of these flows should also be integrable and satisfy (1.28) for every N. However, recall that all the members of a commuting family share the same phase shift

$$e^{A_{ml}} = -\frac{P(\mathbf{k}_m - \mathbf{k}_l)}{P(\mathbf{k}_m + \mathbf{k}_l)} = -\frac{P_{2M}(\mathbf{k}_m - \mathbf{k}_l)}{P_{2M}(\mathbf{k}_m + \mathbf{k}_l)}, \tag{2.1}$$

where $2M$ is the lowest weight of any number of the integrable family. For example, the lowest weight of the KdV family (1.9) is that of the KdV equation itself, namely 4. The Sawada–Kotera family of integrable equations begins at level 6. Therefore in (1.28), we can replace the second P, which contains differences on two k's only, with P_{2M}, because dividing (1.28) across by $P_{2L}(\mathbf{k}_l + \mathbf{k}_m)$ gives an equation linear in $P_{2L}(\Sigma_1^N \mu_l \mathbf{k}_l)$ with coefficients of functions of the phase shifts, which are the same for every L in the commuting family of Hirota equations. With this observation, the Hirota condition for a given P_{2L} can now be written

$$Q(k_1,...,k_N) = \sum P_{2L}\left(\sum_1^N \mu_j \mathbf{k}_j\right) \times \prod_{m>l} P_{2M}(\mu_m \mathbf{k}_m - \mu_l \mathbf{k}_l)\mu_l\mu_m = 0. \tag{2.2}$$

What we will show is that if $Q(k_1,...,k_s) = 0$ for $s \leqslant N-1$, then (2.2) has a factor

$$k_1^{N+1} \cdots k_N^{N+1} \prod_{m>l}^N (k_m^2 - k_l^2)^2$$

of degree $3N^2 - N$. But, from (2.2), a straightforward count shows that $Q(k_1,...,k_N)$ has degree $2L + MN(N-1)$. Thus if

$$3N^2 - N > 2L + MN(N-1), \tag{2.3}$$

$Q(k_1,...,k_N)$ must be identically zero. For cases in which $M = 2$ or 3, that is, in those cases for which the lowest weight member of the integrable sequence is 4 or 6, this condition is nontrivial and tells us that after one establishes that P_{2L} has an r-soliton solution, $3 \leqslant r \leqslant N_0$, where N_0 is the maximum integer for which (2.3) holds, then it has an N-soliton solution for arbitrary N. For $M = 2$, $N_0 = [(-1+\sqrt{1+8L})/2]$ and for $M = 3$, $N_0 = L$. In actual fact one simply has to establish that P_{2L} has an N_0-soliton solution because, by simply allowing several soliton amplitudes to decay or their locations to move to infinity, the fact that P_{2L} has an N_0-soliton solution implies that it has an r-soliton solution $3 \leqslant r < N_0$.

Let us look at several consequences of this result. Denote by $P_{2L}^{(2M)}$ the polynomial weight $2L$, which has the phase shift function given by (2.1). Then we have the following.

(i) If $M = L = 2$, (2.3) holds for every $N \geqslant 2$. It follows that

$$P_4(D_{t_1}, D_{t_3})\tau \cdot \tau = 0$$

has an N-soliton solution for all N. This is the well-known result that the KdV 3 equation has an N-soliton solution.

(ii) If $M = L = 3$, then (2.3) is satisfied for any $N > 3$. This implies that if

$$P_6(D_{t_1}, D_{t_3}, D_{t_5})\tau \cdot \tau = 0$$

has a three-soliton solution, then it has an N-soliton solution for arbitrary N.

(iii) In the case $M = 3$, when $N > L$, (2.3) holds. Therefore, if

$$P_{2L}^{(6)}(D_{t_1}, D_{t_3}, ...)\tau \cdot \tau = 0$$

has an L-soliton solution, then it has an N-soliton solution for arbitrary N.

(iv) In the case $M = 2$, (2.3) is satisfied provided

$$N > [(-1+\sqrt{1+8L})/2] = N_0.$$

Therefore, if

$$P_{2L}^{(4)}(D_{t_1}, D_{t_3}, ...)\tau \cdot \tau = 0 \tag{2.4}$$

has a N_0-soliton solution, then it has an N-soliton solution for arbitrary N. For instance, when $L = 3,4,5$, (2.4) has an N-soliton solution for arbitrary N; when $L = 6,7,8,9$, if (2.4) has a three-soliton solution, then it has an N-soliton solution for arbitrary N.

Because we have assumed the dispersion relation (1.23) is satisfied by (1.24), this last result refers to members of the KdV family. It shows that, contrary to the conjecture stated by the first author in Ref. 4, the Hirota polynomials (that is, the polynomials that have N-soliton solutions for arbitrary N) are not completely determined by the phase shift function. Namely, just because P_{2L} satisfies (2.1) with $M = 2$ is not sufficient to guarantee it is a Hirota polynomial. As we have just mentioned, it is sufficient for $L = 3,4,5$ that is, for polynomials of weights 6, 8, and 10. For polynomials of weights 12–18, P_{2L} needs also to satisfy the three-soliton condition. For a polynomial of general weight $2L$, P_{2L} must satisfy (2.2) for all N up to $[(-1+\sqrt{1+8L})/2]$.

Since the general form of the polynomial at any weight level is a linear combination of all products of odd weights that add to $2L$, these constraints leads to a set of homogeneous linear algebraic equations on the W_L coefficients, where W_L is the number of ways an even number $2L$ can be decomposed into a sum of odd numbers less than $2L$. It is reasonable to conjecture that these equations will contain information about the underlying algebraic structure of the equation family whose phase shift function is given by (2.1).

III. PROOF OF MAIN RESULT

Consider the equation in Hirota form

$$P_{2L}(D_{t_1}, D_{t_3}, ...)\tau \cdot \tau = 0, \tag{3.1}$$

with the phase shift function given by

$$e^{A_{12}} = -\frac{P_{2L}(\mathbf{k}_1 - \mathbf{k}_2)}{P_{2L}(\mathbf{k}_1 + \mathbf{k}_2)} = -\frac{P_{2M}(\mathbf{k}_1 - \mathbf{k}_2)}{P_{2M}(\mathbf{k}_1 + \mathbf{k}_2)}, \quad M \leqslant L, \tag{3.2}$$

where P_{2L} and P_{2M} satisfy the conditions

$$P_{2i}(-D_{t_1}, -D_{t_3},...) = P_{2i}(D_{t_1}, D_{t_3},...), \quad (3.3)$$

$$P_{2i}(0,0,...) = 0, \quad (3.4)$$

$$P_{2i}(\mathbf{k}) = P_{2i}(k, -k^3, k^5,...) = 0, \quad (3.5)$$

and we define

$$P_{2i}(\mathbf{k}_1 \pm \mathbf{k}_2) = P_{2i}(k_1 \pm k_2, -(k_1^3 \pm k_2^3),..., \times(-1)^r(k_1^{2r+1} \pm k_2^{2r+1}),...). \quad (3.6)$$

The condition that (3.1) has an N-soliton solution is

$$\sum P_{2L}\left(\sum_1^N \mu_i \mathbf{k}_i\right) \prod_{j>i} P_{2L}(\mu_j \mathbf{k}_j - \mu_i \mathbf{k}_i)\mu_i\mu_j = 0,$$

which is equivalent to [using (3.2)]

$$\sum P_{2L}\left(\sum_1^N \mu_i \mathbf{k}_i\right) \prod_{j>i} P_{2M}(\mu_j \mathbf{k}_j - \mu_i \mathbf{k}_i)\mu_i\mu_j = 0, \quad (3.7)$$

where the sum is taken over all sequences $(\mu_i)_{i=1}^N$ of plus and minus ones. It is easy to see that (3.7) can be rewritten as

$$Q_N = Q(k_1,...,k_N) = \sum_{\mu_j = -1,1} P_{2L}\left(\sum_1^N \mu_i \mathbf{k}_i\right) \times \prod_{j>i} P_{2M}(\mu_j \mathbf{k}_j - \mu_i \mathbf{k}_i) \prod_i \mu_i^{N-1} = 0. \quad (3.8)$$

In the following proof, we follow closely the ideas used by Hirota[6] for proving that the KdV equation has an N-soliton solution. The Q_N has the following properties.

(i) When N is odd, Q_N is even in the k_i; when N is even, Q_N is odd in each of the k_i, i.e.,

$$Q(k_1,..., -k_i,...,k_N) = (-1)^{N-1}Q(k_1,...,k_i,...,k_N). \quad (3.9)$$

(ii) Q_N is a homogeneous symmetric polynomial in the k_i's, i.e.,

$$Q(k_1,...,k_i,...,k_j,...,k_N) = Q(k_1,...,k_j,...,k_i,...,k_N). \quad (3.10)$$

The result (3.9) is easily seen by replacing k_i by $-k_i$ and μ_i by $-\mu_i$ (dummy index) in (3.8). Also, (3.10) can be verified by interchanging k_i, k_j, and μ_i, μ_j.

It is clear from (3.5) and (3.8) that

$$Q(k_1) = P_{2L}(\mathbf{k}_1) = 0, \quad (3.11)$$

$$Q(k_1,k_2) = P_{2L}(\mathbf{k}_1 + \mathbf{k}_2)P_{2M}(\mathbf{k}_1 - \mathbf{k}_2) - P_{2L}(\mathbf{k}_1 - \mathbf{k}_2)P_{2M}(\mathbf{k}_1 + \mathbf{k}_2) = 0. \quad (3.12)$$

Theorem: Provided

$$Q(k_1,...,k_i) = 0, \quad i < N-1, \quad (3.13)$$

then

$$Q(k_1,...,k_N) = k_1^{N+1} \cdots k_N^{N+1} \prod_{j>i}^N (k_j^2 - k_i^2)^2 \tilde{Q}(k_1,...,k_N), \quad (3.14)$$

and if N,L,M satisfy

$$3N^2 - N > 2L + MN(N-1), \quad (3.15)$$

then

$$Q(k_1,...,k_N) = 0. \quad (3.16)$$

Proof: (3.5) implies

$$P_{2M}(\mu_j \mathbf{k}_j) = 0, \quad (3.17)$$

which yields

$$P_{2M}(\mu_j \mathbf{k}_j - \mu_i \mathbf{k}_i) = k_j k_i \bar{P}_{2M}(\mu_j, \mu_i, k_j, k_i).$$

Hence we obtain from (3.8) that

$$Q(k_1,...,k_N) = k_1^{N-1} \cdots k_N^{N-1} \bar{Q}(k_1,...,k_N). \quad (3.18)$$

By using (3.13) and (3.17) and noting that (3.3) implies

$$P(\mu_j \mathbf{k}_j - \mu_i \mathbf{k}_i) = P(\mathbf{k}_j - \mu_i\mu_j \mathbf{k}_i), \quad (3.19)$$

and

$$\left.\frac{dP_{2M}(\mu_j \mathbf{k}_j - \mu_i \mathbf{k}_i)}{dk_i}\right|_{k_i=0} = \left.\frac{dP_{2M}(\mathbf{k}_j - \mu_i\mu_j \mathbf{k}_i)}{dk_i}\right|_{k_i=0} = \mu_i\mu_j \left.\frac{dP_{2M}(\mathbf{k}_j - \mathbf{k}_i)}{dk_i}\right|_{k_i=0}, \quad (3.20)$$

we find

$$\left.\frac{d^{N-1}Q(k_1,...,k_N)}{dk_1^{N-1}}\right|_{k_1=0} = \sum_{\mu_j=-1,1} P_{2L}\left(\sum_{i=2}^N \mu_i \mathbf{k}_i\right) \prod_{j>i\geq 2} P_{2M}(\mu_j \mathbf{k}_j - \mu_i \mathbf{k}_i) \prod_{i=1}^N \mu_i^{N-1} \prod_{j=2}^N \left.\frac{dP_{2M}(\mu_j \mathbf{k}_j - \mu_1 \mathbf{k}_1)}{dk_1}\right|_{k_1=0}$$

$$= 2\prod_{j=2}^N \left.\frac{dP_{2M}(\mathbf{k}_j - \mathbf{k}_1)}{dk_1}\right|_{k_1=0} \sum_{\substack{\mu_j=-1,1 \\ j>1}} P_{2L}\left(\sum_{i=2}^N \mu_i \mathbf{k}_i\right) \prod_{j>i\geq 2} P_{2M}(\mu_j \mathbf{k}_j - \mu_i \mathbf{k}_i) \prod_{i=2}^N \mu_i^{N-2}$$

$$= 2\prod_{j=2}^N \left.\frac{dP_{2M}(\mathbf{k}_j - \mathbf{k}_1)}{dk_1}\right|_{k_1=0} Q(k_2,...,k_N) = 0. \quad (3.21)$$

According to (3.9) and (3.18), Q_N can be written as

$$Q(k_1,...,k_N) = k_1^{N-1}R_1(k_2,...,k_N) + k_1^{N+1}R_2(k_2,...,k_N) + k_1^{N+3}R_3(k_2,...,k_N) + \cdots.$$

Using (3.21), we obtain

$$R_1(k_2,...,k_N) = 0$$

and

$$Q(k_1,...,k_N) = k_1^{N+1}R_2(k_2,...,k_N) + k_1^{N+3}R_3(k_2,...,k_N) + \cdots = k_1^{N+1}R(k_1,k_2,...,k_N).$$

Therefore, from the properties (3.9) and (3.10), it follows that

$$Q(k_1,...,k_N) = k_1^{N+1} \cdots k_N^{N+1}\hat{Q}(k_1,...,k_N), \tag{3.22}$$

where the polynomial $\hat{Q}(k_1,...,k_N)$ is even and symmetric in the k_i's.

Next, evaluate Q_N when $k_1 = k_2$:

$$Q(k_1,k_1,k_3,...,k_N) = \sum_{\substack{\mu_2=\mu_1\\ \mu_j=-1,1}} P_{2L}\left(\sum_1^N \mu_i \mathbf{k}_i\right) \prod_{i=3}^N \mu_i^{N-1} \prod_{\substack{j>i\\ j>2}} P_{2M}(\mu_j\mathbf{k}_j - \mu_i\mathbf{k}_i)P_{2M}(\mu_1\mathbf{k}_2 - \mu_1\mathbf{k}_1)\Big|_{k_1=k_2}$$

$$+ \sum_{\substack{\mu_2=-\mu_1\\ \mu_j=-1,1}} P_{2L}\left(\sum_1^N \mu_i \mathbf{k}_i\right) \prod_{i=3}^N \mu_i^{N-1}(-1)^{N-1} \prod_{j>i\geqslant 3} P_{2M}(\mu_j\mathbf{k}_j - \mu_i\mathbf{k}_i)$$

$$\times \prod_{j\geqslant 3} [P_{2M}(\mu_j\mathbf{k}_j + \mu_1\mathbf{k}_2)P_{2M}(\mu_j\mathbf{k}_j - \mu_1\mathbf{k}_1)]P_{2M}(\mu_1\mathbf{k}_2 + \mu_1\mathbf{k}_1)\Big|_{k_1=k_2}$$

$$= \sum_{\substack{\mu_2=-\mu_1\\ \mu_j=-1,1}} P_{2L}\left(\sum_{i=3}^N \mu_i \mathbf{k}_i\right) \prod_{i=3}^N \mu_i^{N-1} \prod_{j>i\geqslant 3} P_{2M}(\mu_j\mathbf{k}_j - \mu_i\mathbf{k}_i)$$

$$\times(-1)^{N-1}\prod_{j\geqslant 3} [P_{2M}(\mathbf{k}_j + \mu_1\mu_j\mathbf{k}_1)P_{2M}(\mathbf{k}_j - \mu_1\mu_j\mathbf{k}_1)]P_{2M}(\mu_1\mathbf{k}_1 + \mu_1\mathbf{k}_1)$$

$$= (-1)^{N-1}2\prod_{j\geqslant 3} [P_{2M}(\mathbf{k}_j + \mathbf{k}_1)P_{2M}(\mathbf{k}_j - \mathbf{k}_1)]P_{2M}(\mathbf{k}_1 + \mathbf{k}_1)Q(k_3,...,k_N) = 0. \tag{3.23}$$

Since Q_N is a symmetric polynomial in the k_i's, (3.23) implies that for any i,j,

$$Q(k_1,...,k_N)|_{k_i=k_j} = 0,$$

and from (3.22) this yields

$$\hat{Q}(k_1,...,k_N)|_{k_i=k_j} = 0.$$

Hence $\hat{Q}_N$ is certainly factorized by $(k_i - k_j)$ and therefore $\hat{Q}_N$, as a symmetric polynomial in the k_i's, must be factorized by

$$\prod_{\substack{i,j=1\\ i\neq j}} (k_i - k_j) \quad \text{or} \quad \prod_{j>l} (k_j - k_i)^2.$$

But since $\hat{Q}_N$ is even in the k_i, $\hat{Q}_N$ must be factorized by

$$\prod_{j>l} (k_i^2 - k_j^2)^2.$$

This implies that (3.14) holds. So the order of Q_N must be at least $3N^2 - N$. However the order of P_{2L} is $2L$ and the order of the polynomial product

$$\prod_{j>i}^N P_{2M}(\mu_j\mathbf{k}_j - \mu_i\mathbf{k}_i)$$

is $MN(N-1)$; hence according to (3.8) the order of Q_N must be at most $2L + MN(N-1)$. Clearly, if $Q_N \neq 0$, it must satisfy that

$$3N^2 - N \leqslant \text{order}(Q_N) \leqslant 2L + MN(N-1).$$

Therefore, if $3N^2 - N > 2L + MN(N-1)$, there is a contradiction and we must conclude that $Q(k_1,...,k_N) = 0$.

IV. FURTHER SIMPLIFICATION AND EXAMPLES

The previous section has pointed out that in order to see whether the equations

$$P_{2L}^{(4)}(D_{t_1},D_{t_3},...)\tau\cdot\tau = 0 \tag{4.1}$$

or

$$P_{2L}^{(6)}(D_{t_1},D_{t_3},...)\tau\cdot\tau = 0 \tag{4.2}$$

has an N-soliton solution for arbitrary N, we only need to check whether it has an r-soliton solution $r \leqslant N_0$. However, it is not trivial to check the condition (3.8) for some r and therefore it is useful to simplify it further.

If $H(k_1,...,k_N)$ is a polynomial of $k_1,...,k_N$,

$$H(k_1,...,k_N) = \sum a(l_1,...,l_N)k_1^{2l_1}\cdots k_N^{2l_N}$$

$$+ \sum b(j_1,...,j_N)k_1^{2j_1+1}\cdots k_N^{2j_N+1}$$

$$+ \sum c(i_1,...,i_N)k_1^{i_1}\cdots k_N^{i_N},$$

where some of the $i_1,...,i_N$ in the last sum are even and some of them are odd.

Define the operators L_e and L_o as follows:

$$L_eH(k_1,...,k_N) = \sum a(l_1,...,l_N)k_1^{2l_1}\cdots k_N^{2l_N},$$

$$L_oH(k_1,...,k_N) = \sum b(j_1,...,j_N)k_1^{2j_1+1}\cdots k_N^{2j_N+1}.$$

Proposition: The condition (3.8) that (3.1) has an N-soliton solution is equivalent to

$$L_e\left[P_{2L}\left(\sum_1^N \mathbf{k}_i\right)\prod_{j>i}P_{2M}(\mathbf{k}_j-\mathbf{k}_i)\right]=0,\quad \text{when } N \text{ is odd,} \tag{4.3}$$

$$L_o\left[P_{2L}\left(\sum_1^N \mathbf{k}_i\right)\prod_{j>i}P_{2M}(\mathbf{k}_j-\mathbf{k}_i)\right]=0,\quad \text{when } N \text{ is even.} \tag{4.4}$$

Proof: It is easy to see from the definition for the operators L_e and L_o that

$$L_e\left[P_{2L}\left(\sum_1^N \mu_i\mathbf{k}_i\right)\prod_{j>i}P_{2M}(\mu_j\mathbf{k}_j-\mu_i\mathbf{k}_i)\right]$$
$$=L_e\left[P_{2L}\left(\sum_1^N \mathbf{k}_i\right)\prod_{j>i}P_{2M}(\mathbf{k}_j-\mathbf{k}_i)\right]$$

and

$$L_o\left[P_{2L}\left(\sum_1^N \mu_i\mathbf{k}_i\right)\prod_{j>1}P_{2M}(\mu_j\mathbf{k}_j-\mu_i\mathbf{k}_i)\right]$$
$$=\prod_{i=1}^N \mu_i L_o\left[P_{2L}\left(\sum_1^N \mathbf{k}_i\right)\prod_{j>i}P_{2M}(\mathbf{k}_j-\mathbf{k}_i)\right].$$

Since Q_N is even in k_i when N is odd we have, for N odd,

$$Q(k_1,...,k_N)$$
$$=L_eQ(k_1,...,k_N)$$
$$=\sum_{\mu=-1,1}L_e\left[P_{2L}\left(\sum_1^N \mu_i\mathbf{k}_i\right)\prod_{j>i}P_{2M}(\mu_j\mathbf{k}_j-\mu_i\mathbf{k}_i)\right]$$
$$=2^NL_e\left[P_{2L}\left(\sum_1^N \mathbf{k}_i\right)\prod_{j>i}P(\mathbf{k}_j-\mathbf{k}_i)\right].$$

When N is even, Q_N is odd in k_i, hence we get

$$Q(k_1,...,k_N)$$
$$=L_oQ(k_1,...,k_N)$$
$$=\sum_{\mu_j=-1,1}L_o\left[P_{2L}\left(\sum_1^N \mu_i\mathbf{k}_i\right)\right.$$
$$\left.\times\prod_{j>i}P_{2M}(\mu_j\mathbf{k}_j-\mu_i\mathbf{k}_i)\prod_{i=1}^N\mu_i\right]$$
$$=\sum_{\mu_j=-1,1}L_o\left[P_{2L}\left(\sum_1^N \mathbf{k}_i\right)\prod_{j>i}P_{2M}(\mathbf{k}_j-\mathbf{k}_i)\right]\prod_{i=1}^N\mu_i^2$$
$$=2^NL_o\left[P_{2L}\left(\sum_1^N \mathbf{k}_i\right)\prod_{j>i}P_{2M}(\mathbf{k}_j-\mathbf{k}_i)\right].$$

As an example, we will use these simplifications to identify all equations with weight level 6 with the Hirota property. The most general form of P_6 is

$$D_{t_1}D_{t_5}+aD_{t_3}^2+bD_{t_1}^3D_{t_3}+cD_{t_1}^6. \tag{4.5}$$

From (3.5), a,b,c, must satisfy

$$1+a-b+c=0. \tag{4.6}$$

The theorem given in the previous section told us that (4.5) has an N-soliton solution for arbitrary N if it has a three-soliton solution. Therefore the condition that (4.5) has an N-soliton solution reads

$$L_e[P_6(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)P_6(\mathbf{k}_3-\mathbf{k}_2)$$
$$\times P_6(\mathbf{k}_3-\mathbf{k}_1)P_6(\mathbf{k}_2-\mathbf{k}_1)]=0. \tag{4.7}$$

Notice that this expression is considerably simpler than (1.7). Using (4.6) and (4.7), a little calculation shows

$$(3a+6c+1)(9c-1)$$
$$\times[a^2+(7c+2)a+c^2+2c+1)]=0. \tag{4.8}$$

This implies that all the Hirota equations at weight level 6 are the following equations: (i) KdV equation,

$$(D_{t_1}D_{t_5}+(-\tfrac{1}{3}-2c)D_{t_3}^2$$
$$+(\tfrac{2}{3}-c)D_{t_1}^3D_{t_3}+cD_{t_1}^6)\tau\cdot\tau=0; \tag{4.9}$$

and

$$\text{(ii) } (D_{t_1}D_{t_5}+aD_{t_3}^2+(\tfrac{10}{9}+a)$$
$$\times D_{t_1}^3D_{t_3}+\tfrac{1}{9}D_{t_1}^6)\tau\cdot\tau=0. \tag{4.10}$$

Taking $a\to\infty$, we get Ito's equation from (4.10),

$$(D_{t_3}^2+2D_{t_1}^3D_{t_3})\tau\cdot\tau=0.$$

$$\text{(iii) } (D_{t_1}D_{t_5}+(-\tfrac{7}{2}c-1\pm\tfrac{1}{2}\sqrt{45c^2+20c})D_{t_3}^2$$
$$+(-\tfrac{5}{2}c\pm\tfrac{1}{2}\sqrt{45c^2+20c})D_{t_1}^3D_{t_3}$$
$$+cD_{t_1}^6)\tau\cdot\tau=0. \tag{4.11}$$

Taking $c=-1$, (4.11) yields the Sawada–Kotera equation after rescaling the variables

$$(D_{t_1}^6+9D_{t_1}D_{t_5})\tau\cdot\tau=0.$$

We obtain the Ramani equation by taking $c\to\infty$ in (4.11),

$$(D_{t_1}^6-5D_{t_1}^3D_{t_3}-5D_{t_3}^2)\tau\cdot\tau=0. \tag{4.12}$$

We emphasize that this is a complete list of all Hirota polynomials of weight 6 that satisfy the conditions (3.3)–(3.5).

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