Self-dual gravity revisited

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Abstract. Reconsidering the harmonic space description of the self-dual Einstein equations, we streamline the proof that all self-dual pure gravitational fields allow a local description in terms of an unconstrained analytic prepotential in harmonic space. Our formulation yields a simple recipe for constructing self-dual metrics starting from any explicit choice of such prepotential; and we illustrate the procedure by producing a metric related to the Taub-NUT solution from the simplest monomial choice of prepotential.

1. Introduction

Many years ago Penrose [1] pointed out that the twistor transform in flat space [2] remarkably yielded itself to a deformation to curved space, providing a construction (in principle) of the general self-dual solution of the Einstein equations. Several classes of solutions have been explicitly constructed using the twistor technique (see, e.g. [3]). Further classes of explicit self-dual metrics have been found by finding classes of particular solutions to the second-order partial differential equation to which Plebanski [4] reduced the self-dual Einstein equations (see, e.g. the review [5] for details of this approach).

   The aim of this paper is to describe an alternative version of the 'curved twistor construction' of local solutions using the harmonic space description. Harmonic spaces were originally devised [6] as tools for the construction of unconstrained off-shell $N = 2$ and $N = 3$ supersymmetric theories. This involved the 'harmonization' of the internal unitary automorphism groups G of the Poincaré supersymmetry algebra, i.e. the inclusion of harmonics on some coset of G as auxiliary variables; quantities in conventional superspace being recoverable as coefficients in a harmonic decomposition. Subsequent applications have involved harmonic spaces in which the rotation group (rather than some internal symmetry group) is 'harmonized', i.e. these harmonic spaces are cosets of the Poincaré group by a subgroup H of the rotation group. Ordinary four-dimensional space, recall, is the coset of the Poincaré group by the *entire* rotation group, so these harmonic spaces are basically an enlargement of four-dimensional space by the coset of the rotation group by H. Conventional four-dimensional fields are recoverable from fields in such harmonic spaces by performing an expansion in the harmonics on the coset space. Such harmonic spaces are basically manifestly covariant versions of twistor spaces [7]; and they can be used to construct explicit local solutions by reformulating the Penrose–Ward twistor

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† **V I Ogievetsky (1928–96).** Victor Isaakovich Ogievetsky sadly died on 23rd March, 1996, while this paper was being refereed.

transform in harmonic space language. Moreover, just like twistor spaces, they are amenable to supersymmetrization. Many four-dimensional integrable systems have hitherto yielded themselves to this harmonic-twistor method of describing the general solution: the Yang–Mills self-duality equations [2, 8–10], all their supersymmetric extensions [11], and the full $N = 3$ super-Yang–Mills equations [12]. Twistor theory, moreover, affords adaptation to curved spaces (reviewed in e.g. [13]). Specifically, it yields a method of describing self-dual solutions of Einstein equations (with or without a cosmological constant [14]). The harmonic space variant similarly allows itself to be applied to the field equations describing hyper-Kähler [15, 16] and quaternionic spaces [17].

The harmonic-twistor method for self-dual theories uses a presentation of the equations in a harmonic space with S^2 harmonics as auxiliary variables; and the essence of this version of the twistor transform is a transformation to a system of coordinates, an 'analytic' frame, in which an invariant 'analytic' subspace exists and in which the equations take the form of Cauchy–Riemann-like (CR) equations. The method, reviewed in, e.g. [18], therefore encodes the solution of nonlinear equations in certain 'analytic' functions (by which we mean functions depending only on coordinates of the invariant 'analytic' subspace). For instance, the Yang–Mills self-duality equations $F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ take the form (see, e.g. [9, 10]) of the following system in harmonic space with coordinates $\{x^{\pm\alpha} \equiv x^{\alpha a}u_a^{\pm}, u_a^{\pm} ; u^{+a}u_a^- = 1, u_a^{\pm} \sim e^{\pm i\gamma}u_a^{\pm}\}$:

$$[\mathcal{D}_\alpha^+, \mathcal{D}_\beta^+] = 0 = [\mathcal{D}^{++}, \mathcal{D}_\alpha^+] \,,$$

where $\mathcal{D}_\alpha^+ \equiv \partial/\partial x^{-\alpha} + A_\alpha^+ \equiv u^{+a}(\partial/\partial x^{\alpha a} + A_{\alpha a})$, $\mathcal{D}^{++} \equiv u_a^+\partial/\partial u_a^-$. Here α, a are 2-spinor indices and the signs denote the conserved U(1) charge. The harmonics $u_a^{\pm}$ are fundamental SU(2) spinors; any representation of SU(2) allowing presentation as a symmetrized product of them. Being defined up to a U(1) transformation, these harmonics parametrize $S^2 \simeq$ SU(2)/U(1). Moreover, since they do so globally, these $u_a^{\pm}$'s are much more convenient objects than their Euler angle or stereographic parametrizations, allowing the avoidance of the Riemann–Hilbert problem. Having written the self-duality equations in the above harmonic space language, the Frobenius argument allows the crucial transformation to an 'analytic' frame, in which the covariant derivative $\mathcal{D}_\alpha^+$ takes the form of the flat derivative $\partial/\partial x^{-\alpha}$, completely trivializing the first commutation relation above; and in the process shifting all the unsolved (dynamical) data to the harmonic derivative, which loses its flatness by acquiring a connection: $\mathcal{D}^{++} \rightarrow u_a^+\partial/\partial u^{-a} + V^{++}$. This connection V^{++} then carries all the dynamical information and the equation of motion for the connection A_α^+ is replaced by the Cauchy–Riemann-like analyticity condition: $[\mathcal{D}_\alpha^+, \mathcal{D}^{++}] = (\partial/\partial x^{-\alpha})V^{++} = 0$. So the general local solution is encoded in an arbitrary analytic $V^{++} = V^{++}(x^{+\alpha}, u_a^{\pm})$. 'Integrability' therefore becomes manifest, though the problem of constructing specific explicit self-dual vector potentials reduces to that of inverting the above transformation for any specified analytic V^{++}.

This method has already been considered as a means of solving self-dual gravity [15], where the presence of vielbeins as well as connections requires a suitable adaptation of the above flat-space Yang–Mills strategy. In this paper we reconsider this problem and show that even in the curved case the essential features of the above strategy can be maintained. In particular, we show that a special 'half-flat' analytic coordinate frame exists in which two of the four-dimensional covariant derivatives become completely flat; and in which all the dynamics become concentrated in the covariant harmonic derivative $\mathcal{D}^{++}$, just as in the Yang–Mills case, but now in both the connection *and* vielbein parts of the latter. All these parts of $\mathcal{D}^{++}$, moreover, can be solved for in terms of a *single* arbitrary analytic prepotential, which therefore encodes the general solution of the gravitational self-duality conditions. In

the next section we discuss the harmonization: the formulation of the self-duality conditions for the Riemann tensor in harmonic space. Following [15] we then introduce (in section 3) the class of analytic coordinate frames which are distinguished by the property of having an invariant analytic subspace and we discuss the corresponding transformation rules in section 4. We then list (in section 5) the self-duality equations, 'the CR system', for the vielbein and connection fields in these analytic frames. This system has sufficient gauge freedom, affording the choice (in section 6) of a particular analytic frame, the 'half-flat' gauge, in which a great deal of the simplicity of the flat-space (Yang–Mills) construction outlined above is recovered; and as a consequence (section 7) the general local solution of the CR system (which in this gauge takes a manifestly Cauchy–Riemann form) follows in terms of a single arbitrary (i.e. unconstrained) analytic prepotential which encapsulates the dynamics. Our refinement of the construction of [15] considerably streamlines the procedure for the explicit construction of self-dual metrics and corresponding spin connections. We describe this procedure in section 8; and in section 9 we demonstrate it for the particular case of the simplest non-trivial monomial choice of analytic prepotential which we explicitly decode to reveal a metric related to the self-dual Taub-NUT solution. In fact, the form of the metric we obtain is precisely that obtained by the alternative construction of hyper-Kähler metrics using the harmonic superspace construction [16] of $N = 2$ supersymmetric sigma models, which have hyper-Kähler manifolds as target spaces. In section 10 we discuss the relation to the alternative Plebanski approach [4], requiring solution of a second-order differential equation. As a byproduct, our method yields a prescription for the production of solutions to Plebanski's second 'heavenly' equation, though our construction is independent of this equation and does not require its solution for the construction of self-dual metrics. By virtue of its generality, our method is a promising one for the explicit construction of new local solutions to self-dual gravity, for reviews on self-dual gravity see, e.g. [19]. Further, it paves the way towards the solution of self-dual supergravity theories and the explicit construction of supersymmetric hyper-Kähler manifolds.

Our considerations are good for complexified space or for real spaces of signature $(4, 0)$ or $(2, 2)$ (with appropriate handling of the latter as a restriction of complexified space). For concreteness however, we shall deal with the real Euclidean version, with tangent space (structure) group being the direct product $SU(2) \times SU(2)$; the first $SU(2)$ having Greek spinor indices $\alpha, \beta, \ldots$, whereas we denote the second $SU(2)$ by Latin spinor indices $a, b, \ldots$, $(\alpha, a = 1, 2)$. The covariant derivative $\mathcal{D}_{\alpha a}$ takes values in the tangent space algebra and defines the components of the Riemann curvature tensor by virtue of the commutation relations [20]

$$[\mathcal{D}_{\beta b}, \mathcal{D}_{\alpha a}] = \epsilon_{ab} R_{\alpha \beta} + \epsilon_{\alpha \beta} R_{ab},$$

with

$$R_{\alpha \beta} \equiv C_{(\alpha \beta \gamma \delta)} \Gamma^{\gamma \delta} + R_{(\alpha \beta)(cd)} \Gamma^{cd} + \tfrac{1}{6} R \Gamma_{\alpha \beta},$$

$$R_{ab} \equiv C_{(abcd)} \Gamma^{cd} + R_{(\gamma \delta)(ab)} \Gamma^{\gamma \delta} + \tfrac{1}{6} R \Gamma_{ab},$$

where round brackets denote symmetrization and, in this spinor notation, $C_{(abcd)} (C_{(\alpha \beta \gamma \delta)})$ are the (anti-) self-dual components of the Weyl tensor, $R_{(\alpha \beta)(cd)}$ are the components of the trace-free Ricci tensor, R is the scalar curvature, $(\Gamma^{\gamma \delta}, \Gamma^{cd})$ are generators of the tangent space gauge algebra. We raise and lower all tangent space indices using the antisymmetric invariant tensors $\epsilon^{\alpha \beta}$, ϵ^{ab} and $\epsilon_{\alpha \beta}, \epsilon_{ab}$, respectively, e.g. $\epsilon^{ab} \psi_b = \psi^a$, $\epsilon_{ab} \psi^b = \psi_a$; $\epsilon_{12} = 1$. We take 'self-duality' to mean that the Riemann tensor is self-dual, which in spinor notation means that

$$R_{ab} = 0.$$

By virtue of the above (irreducible) decomposition this is tantamount to the self-duality of the Weyl tensor ($C_{(abcd)} = 0$) and the vanishing of both the tracefree Ricci tensor and the scalar curvature ($R_{(\gamma\delta)(ab)} = R = 0$). These conditions clearly imply the source-free Einstein field equations. In the present work we shall restrict ourselves to this case of zero cosmological constant. Evidently, we may write these self-duality conditions in the form of the constraints

$$[\mathcal{D}_{\beta b}, \mathcal{D}_{\alpha a}] = \epsilon_{ab} R_{\alpha\beta}. \tag{1}$$

Now, since $R_{(\gamma\delta)(ab)} = R = 0$, we have that $R_{\alpha\beta} = C_{(\alpha\beta\gamma\delta)}\Gamma^{\gamma\delta}$. The self-dual part of the curvature, $R_{\alpha\beta}$, therefore takes values only in the SU(2) algebra with generators $\Gamma^{\gamma\delta}$. This means that we may work in a 'self-dual gauge' (see, e.g. [21]) in which only this half of the tangent space group is localized: just the SU(2) labelled by indices $\alpha, \beta, \ldots$, while the second SU(2) (indices $a, b, \ldots$) remains rigid. Correspondingly, the space coordinate $x^{\mu a}$ only has one 'world' spinor index μ, the second one being identified with the tangent space spinor index a. Covariant derivatives therefore take the form

$$\mathcal{D}_{\alpha a} = E^{\mu b}_{\alpha a}\partial_{\mu b} + \omega_{a\alpha} , \tag{2}$$

where in this gauge the spin connections take values only in the SU(2) algebra (generators Γ^{δ}_{β}) of the local structure group, namely

$$(\omega_{\alpha a}) = \omega^{\delta}_{\alpha a\sigma}(\Gamma^{\sigma}_{\delta}), \tag{3}$$

a restriction which clearly implies that the curvatures also take values only in this restriction of the tangent space algebra, i.e. are then automatically self-dual. So with the connection in the form (3), equation (1) is no longer a constraint on the curvature; and the problem reduces to that of finding vierbeins $E^{\mu b}_{\alpha a}$ satisfying the conditions of zero torsion implicit in (1). This is the principal difference from the Yang–Mills case, where there are neither vierbeins nor torsions and the entire problem is that of solving the analogue of (1) as curvature constraints on the connection (as opposed to torsion constraints on the vierbein).

In the above 'self-dual gauge', not only is the curvature automatically self-dual, but since both connection and curvature are restricted to take values in $SU(2) \simeq Sp(1)$, the metric is manifestly hyper-Kähler. This way of considering hyper-Kähler spaces, as solutions of the self-duality conditions (1), may immediately be generalized to higher dimensions. Equations describing higher ($4n$)-dimensional hyper-Kähler spaces may be obtained by simply replacing the $SU(2) \simeq Sp(1)$ index α in (1) by an $Sp(n)$ one [15]. We shall for now restrict ourselves, however, to pure self-dual gravity in four dimensions, the treatment of $4n$-dimensional hyper-Kähler manifolds for $n > 1$ is evident (see section 10).

2. The self-duality equations in harmonic space

Having an 'ungauged' SU(2) part of the tangent space algebra at our disposal, we 'harmonize' it by introducing S^2 harmonics $\{u^{\pm a}; u^{+a}u^-_a = 1, u^{\pm}_a \sim e^{\pm i\gamma}u^{\pm}_a\}$, where a is an SU(2) spinor index and $\pm$ denote U(1) charges [6]. We begin by defining the covariant derivatives in the *central* coordinate basis of *harmonic space* thus:

$$\mathcal{D}^{\pm}_{\alpha} \equiv D^{\pm}_{\alpha} + \omega^{\pm}_{\alpha} = u^{\pm a}\mathcal{D}_{\alpha a}, \qquad \mathcal{D}^{++} = \partial^{++} = u^+_a\frac{\partial}{\partial u^-_a} \tag{4}$$

where the harmonic derivative $\mathcal{D}^{++}$ is a partial derivative acting only on harmonics and is connectionless (this being the characteristic feature of this basis).

The following system in harmonic space is equivalent to the self-duality conditions (1):

$$[\mathcal{D}_\alpha^+, \mathcal{D}_\beta^+] = 0 \tag{5a}$$

$$[\mathcal{D}^{++}, \mathcal{D}_\alpha^+] = 0 \tag{5b}$$

$$[\mathcal{D}_\alpha^+, \mathcal{D}_\beta^-] = 0 \qquad (\text{modulo } R_{\alpha\beta}) \tag{5c}$$

$$[\mathcal{D}^{++}, \mathcal{D}_\alpha^-] = \mathcal{D}_\alpha^+. \tag{5d}$$

We obtain (5a) on multiplying (1) by $u^{+a}u^{+b}$. Conversely, (5b) ensures linearity of $\mathcal{D}_\alpha^+$ in the harmonics, so the harmonic (i.e. u^+-) expansion of (5a) yields (1) up to possible torsion terms containing ϵ_{ab}, namely $\epsilon_{ab}T_{\alpha\beta}^{\gamma c}\mathcal{D}_{\gamma c}$. (In the flat-space Yang–Mills case, recall, the relations (5a, b) *are* equivalent to the self-duality conditions.) To exclude the existence of such torsion terms we need to include the commutation relations (5c); and (5d) is then required in order to ensure that in the present central basis the $\mathcal{D}_\alpha^-$ contain harmonics only linearly (as in (4)). The system of commutators (5a–d) is then equivalent to the original self-duality relations (1)†. The system (5) is in fact a Cauchy–Riemann-like (CR) system [1, 8]. Only the coordinate frame needs to be changed in order to make its CR nature manifest.

3. The analytic frames

The choice (4) of covariant derivatives corresponds to what we have called the *central frame* with coordinates $\{x^{\mu\pm} \equiv x^{\mu a}u_a^\pm, u_a^\pm\}$. The system of commutators (5), however, describes self-duality covariantly, without reference to the particular form (4) of $\mathcal{D}^{++}$ and $\mathcal{D}_\alpha^\pm$, these covariant derivatives in general contain connections *and* vielbeins, providing covariance under both SU(2) local frame transformations and general coordinate transformations $\delta x^{\mu a} = \tau^{\mu a}(x)$. In the above central coordinates the equivalence (5) $\Leftrightarrow$ (1) is manifest, however this covariance may be exploited in order to choose an alternative special coordinate system, the 'half-flat' gauge mentioned in the introduction, in which the CR nature of (5) becomes manifest instead. The latter coordinate system belongs to a select (gauge equivalent) class of *analytic frames*, which have the distinguishing feature of an invariant 'analytic' subspace under general coordinate transformations. We call an object 'analytic' if it is independent of a subset (namely $\{x_h^{\mu-}\}$) of some new set of coordinates $\{x_h^{\mu\pm}, u_a^\pm\}$, with the invariant 'analytic' subspace having 'holomorphic'‡ coordinates $x_h^{\mu+}$. Any such new coordinates $x_h^{\mu\pm}$ are of course some nonlinear functions of the central frame (or customary $x^{\mu a}$) coordinates and of the harmonics $u_a^\pm$:

$$x^{\mu a} \rightarrow x_h^{\mu\pm} = x_h^{\mu\pm}(x^{\mu a}u_a^\pm, u_a^\pm). \tag{6}$$

The necessity of determining these h-coordinates as (invertible) functions of the central frame ones is the main novel feature of the curved-space construction and this is the crucial difference from the flat self-dual Yang–Mills case, where we have simply the linear relationship of the central coordinates $x^{\mu\pm} = x^{\mu a}u_a^\pm$.

In order to have an invariant 'analytic' subspace, the functions $x_h^{\mu\pm}(x^{\mu a}u_a^\pm, u_a^\pm)$ in (6) are clearly required to have the crucial property that under the mapping (6) the covariant

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† In this paper we shall deal only with those covariant derivatives ($\mathcal{D}_\alpha^\pm, \mathcal{D}^{++}$) which enter the system (5); we shall not use the additional harmonic covariant derivative $\mathcal{D}^{--}$ on which the discussion of [15] was based.

‡ This terminology, borrowed from complex analysis, is to be understood only in this sense. We take our $x^{\mu a}, x_h^{\mu\pm}$ to be *real* coordinates. Appropriate Hermiticity conditions for the harmonics are discussed in the harmonic space literature [6, 9, 10]. Of course, all coordinates can also be complexified (see [10]).

derivatives D_α^+ in (4) contain derivations with respect to $x_h^{\mu-}$ only. In general, (6) induces the mapping:

$$D_\alpha^+ \to (D_\alpha^+ x_h^{\mu-})\partial_{h\mu}^+ + (D_\alpha^+ x_h^{\mu+})\partial_{h\mu}^-,$$

where $\partial_{h\mu}^+ = \partial/\partial x_h^{\mu-}$, $\partial_{h\mu}^- = \partial/\partial x_h^{\mu+}$, so the requirement that only $\partial_{h\mu}^+$-terms appear on the right is tantamount to the condition that the holomorphic coordinates preserve the flat space relation

$$D_\alpha^+ x_h^{\mu+} = 0. \tag{7}$$

We take this to be the *defining condition* for analytic frame coordinates. This yields

$$D_\alpha^+ = (D_\alpha^+ x_h^{\mu-})\partial_{h\mu}^+ \equiv f_\alpha^\mu \partial_{h\mu}^+, \tag{8}$$

where we have defined an analytic frame zweibein f_α^μ. Having only derivatives with respect to $x^{\mu-}$ on the right, the conditions $D_\alpha^+ \Psi = 0$ then imply (for invertible zweibein f_α^μ) the required 'analyticity' of Ψ, i.e. the independence of $x_h^{\mu-}$, $\partial_{h\mu}^+ \Psi = 0$.

On the other hand, the negatively charged derivatives D_α^- in (4) contain derivations with respect to all the new coordinates:

$$D_\alpha^- = -e_\alpha^\mu \partial_{h\mu}^- + e_\alpha^{--\mu} \partial_{h\mu}^+, \tag{9}$$

where we have defined a further neutral zweibein,

$$e_\alpha^\mu = -D_\alpha^- x_h^{\mu+}, \tag{10}$$

(the minus sign is chosen so as to have $e_\alpha^\mu = \delta_\alpha^\mu$ in the flat-space limit) and a doubly-negatively charged one

$$e_\alpha^{--\mu} = D_\alpha^- x_h^{\mu-}. \tag{11}$$

We now come to the harmonic derivative. Being flat in the central coordinates, $\mathcal{D}^{++} = \partial^{++}$, it acquires vielbeins in the holomorphic ones:

$$\partial^{++} \to \Delta^{++} = \partial^{++} + H^{++\mu+}\partial_{h\mu}^- + (x_h^{\mu+} + H^{++\mu-})\partial_{h\mu}^+, \tag{12}$$

where the vielbeins are defined in terms of the holomorphic coordinates thus:

$$H^{++\mu+} = \partial^{++} x_h^{\mu+}, \tag{13}$$

$$H^{++\mu-} = \partial^{++} x_h^{\mu-} - x_h^{\mu+}. \tag{14}$$

Note that these vielbeins are defined so as to have $H^{++\mu+} = H^{++\mu-} = 0$ in the flat-space limit.

Now, the curvature in (5a) being zero, we may perform an SU(2) structure group transformation making the covariant derivatives $\mathcal{D}_\alpha^+$ connectionless, as in the Yang–Mills case discussed in the introduction. Namely, (5a) implies the existence of an invertible matrix $\varphi_\beta^{\check\beta}$ (having inverse $(\varphi^{-1})_{\check\beta}^\beta$; $(\varphi^{-1})_{\check\beta}^\beta \varphi_{\check\alpha}^\beta = \delta_{\check\alpha}^{\check\beta}$, $\varphi_\beta^{\check\beta}(\varphi^{-1})_{\check\beta}^\alpha = \delta_\beta^\alpha$) satisfying the equations

$$\mathcal{D}_\alpha^+ \varphi \equiv (D_\alpha^+ + \omega_\alpha^+)\varphi = 0 \tag{15}$$

which imply the pure-gauge form of the connection in terms of φ:

$$(\mathcal{D}_\alpha^+)_\beta^\gamma = D_\alpha^+ \delta_\beta^\gamma - D_\alpha^+ \varphi_\beta^{\check\beta} \varphi_{\check\beta}^{-1\gamma}. \tag{16}$$

We may therefore use a solution of (15) to perform an SU(2) transformation in order to 'gauge away' the connection in $\mathcal{D}_\alpha^+$:

$$\mathcal{D}_\alpha^+ \to \varphi^{-1} \mathcal{D}_\alpha^+ \varphi = D_\alpha^+$$

with D_a^+ as in (8) above. Now, under this transformation, just as in the Yang–Mills case, the harmonic derivative $\mathcal{D}^{++}$ acquires a connection:

$$D^{++} \to \mathcal{D}^{++} = \varphi^{-1}[D^{++}]\varphi \equiv \Delta^{++} + \omega^{++}, \tag{17}$$

where

$$\omega^{++} = \varphi^{-1}\Delta^{++}\varphi. \tag{18}$$

We shall show that in a particular analytic frame the equations implied by the CR system (5) for the analytic frame vielbeins and connections defined above may be solved (treating the h-coordinates $\{x_h^{\mu\pm}, u_a^{\pm}\}$ as independent variables) in terms of an arbitrary analytic prepotential. Our strategy will then be to solve (13), (14), (18) for $x_h^{\mu+}$, $x_h^{\mu-}$, and φ, respectively, for some specific choice of analytic prepotential (treating, in turn, the central coordinates $\{x^{\mu a}, u_a^{\pm}\}$ as independent variables). Having determined the latter data, we shall invert the mapping (6) and obtain the vierbein in (2) explicitly. The self-dual metric will then afford immediate construction. The problem of the explicit construction will therefore reduce to that of solving (13), (14), (18) for fields $H^{++\mu\pm}$ and ω^{++} determined by some specified choice of the analytic prepotential.

4. Transformation rules

There exist two kinds of the tangent-space transformations; central frame ones with local parameters $\tau_\alpha^\beta(x^{\mu a})$ and analytic frame ones with local parameters $\lambda_{\breve{\beta}}^{\breve{\alpha}}(x_h^{\mu+}, u)$ (we distinguish λ-transforming indices by a 'breve' accent). The matrix φ is defined up to the local gauge equivalence

$$\varphi_\alpha^{\breve{\alpha}} \sim \tau_\alpha^\beta(x^{\mu a})\varphi_\beta^{\breve{\beta}}\lambda_{\breve{\beta}}^{\breve{\alpha}}(x_h^{\mu+}, u). \tag{19}$$

The connection in (15) does not transform under the analytic transformations parametrized by λ, these being 'pregauge' in the central frame; whereas for the analytic frame connection ω^{++} in (18) the tangential transformations parametrized by τ are 'pregauge' and therefore leave this connection *invariant*. Consider some spinor F_α, which has tangent transformation

$$\delta F_\alpha = \tau_\alpha^\beta(x^{\mu a})F_\beta.$$

The parameters $\tau_\alpha^\beta(x)$ being non-analytic, this transformation would not be consistent if F_α were analytic, i.e. the τ-transformation does not preserve the analytic subspace. However using φ we can pass to a spinor having breved indices, $F_{\breve{\alpha}} = (\varphi^{-1})_{\breve{\alpha}}^\beta F_\beta$, $F_\alpha = \varphi_\alpha^{\breve{\beta}} F_{\breve{\beta}}$, which has tangent transformations preserving the analytic subspace:

$$\delta F_{\breve{\alpha}} = \lambda_{\breve{\alpha}}^{\breve{\beta}}(x_h^+, u)F_{\breve{\beta}}.$$

So φ is clearly a bridge taking us from central frame tangent-space indices (α) to analytic frame tangent-space ones ($\breve{\alpha}$). In the analytic frame we shall use only quantities with breved tangent-space spinor indices, using as many φ's as are necessary in order to obtain the suitably transforming quantity. In the CR system (5) therefore, we pass to appropriately transforming covariant derivatives:

$$\mathcal{D}_{\breve{\alpha}}^+ = (\varphi^{-1})_{\breve{\alpha}}^\beta \mathcal{D}_\beta^+ \qquad \mathcal{D}_{\breve{\alpha}}^- = (\varphi^{-1})_{\breve{\alpha}}^\beta \mathcal{D}_\beta^-.$$

In gravitation theory the most important transformations are the world ones. For the central frame coordinates these form the diffeomorphism group with local parameters $\tau^{\mu a}(x)$, i.e. $\delta x^{\mu a} = \tau^{\mu a}(x)$. In an analytic frame, by definition (see section 3), harmonic

space diffeomorphisms preserve analyticity, i.e. leave the analytic subspace (the analytic planes with coordinates $x_h^{\mu+}$) of harmonic space invariant. Namely,

$$\delta x_h^{\mu+} = \lambda^{\mu+}(x_h^+, u), \tag{20}$$

whereas

$$\delta x_h^{\mu-} = \lambda^{\mu-}(x_h^+, x_h^-, u). \tag{21}$$

The important property being of course that *the positively charged parameters are analytic, whereas the negatively charged ones are not*, implying that the most general diffeomorphism preserves the analytic subspace. The harmonics also allow transformation, requiring consideration of the complexified picture [10]. However we do not need to consider these transformations since they do not affect the CR system (5). Such transformations, however, are necessary in the case of non-zero cosmological constant (see [17]).

From the covariance of the covariant derivatives (8), (9), (12) under the transformations (20), (21), we obtain the following transformation rules for the vielbeins introduced in section 3:

$$\delta f_{\check{\alpha}}^{\mu} = f_{\check{\alpha}}^{\nu} \partial_{h\nu}^+ \lambda^{\mu-} + \lambda_{\check{\alpha}}^{\check{\beta}} f_{\check{\beta}}^{\mu}, \tag{22}$$

$$\delta e_{\check{\alpha}}^{\mu} = e_{\check{\alpha}}^{\nu} \partial_{h\nu}^- \lambda^{\mu+} + \lambda_{\check{\alpha}}^{\check{\beta}} e_{\check{\beta}}^{\mu}, \tag{23}$$

$$\delta e_{\check{\alpha}}^{--\mu} = -e_{\check{\alpha}}^{\nu} \partial_{h\nu}^- \lambda^{\mu-} + e_{\check{\alpha}}^{--\nu} \partial_{h\nu}^+ \lambda^{\mu-} + \lambda_{\check{\alpha}}^{\check{\beta}} e_{\check{\beta}}^{--\mu}, \tag{24}$$

$$\delta H^{++\mu+} = \Delta^{++} \lambda^{\mu+}, \tag{25}$$

$$\delta H^{++\mu-} = \Delta^{++} \lambda^{\mu-} - \lambda^{\mu+}. \tag{26}$$

5. The Cauchy–Riemann system of equations

We now examine the content of the CR system (5) in an analytic frame, i.e. with the covariant derivatives taking the explicit form

$$\mathcal{D}_{\check{\alpha}}^+ = f_{\check{\alpha}}^{\mu} \partial_{h\mu}^+$$

$$\mathcal{D}_{\check{\alpha}}^- = -e_{\check{\alpha}}^{\mu} \partial_{h\mu}^- + e_{\check{\alpha}}^{--\mu} \partial_{h\mu}^+ + \omega_{\check{\alpha}}^-$$

$$\mathcal{D}^{++} = \partial^{++} + H^{++\mu+} \partial_{h\mu}^- + (x_h^{\mu+} + H^{++\mu-}) \partial_{h\mu}^+ + \omega^{++}.$$

Not all the equations implied by (5) for the vielbein and connection fields in these covariant derivatives are 'dynamical' in character, in the sense of requiring solution for the determination of the metric. We shall first extract the set of such 'dynamical' equations, the remaining equations basically determining redundant fields.

For the zweibein $f_{\check{\beta}}^{\mu}$ we have from (5a) the equations

$$f_{[\check{\alpha}}^{\nu} \partial_{h\nu}^+ f_{\check{\beta}]}^{\mu} = 0. \tag{27}$$

The vanishing of the torsion coefficients of $\partial_{h\mu}^-$ in (5c) and (5b) requires the vielbeins $e_{\check{\alpha}}^{\mu}$ and $H^{++\mu+}$, respectively, to be *analytic*:

$$D_{\check{\alpha}}^+ e_{\check{\beta}}^{\mu} = 0, \tag{28}$$

$$D_{\check{\alpha}}^+ H^{++\mu+} = 0. \tag{29}$$

The vanishing of the curvature in (5*b*) yields a further analyticity condition; for the connection ω^{++},

$$D_{\breve{\alpha}}^{+}\omega^{++} = 0 \, . \tag{30}$$

The solution of this equation is however not independent of the solution of the previous two analyticity conditions; the equation

$$-\mathcal{D}^{++}e_{\breve{\alpha}}^{\mu} - \mathcal{D}_{\breve{\alpha}}^{-}H^{++\mu+} = 0, \tag{31}$$

which is a consequence of the vanishing of the torsion coefficients of $\partial_{h\mu}^{-}$ in (5*d*), provides an important constraint among the three analytic fields $e_{\breve{\beta}}^{\mu}$, $H^{++\mu+}$ and ω^{++}. Furthermore, these fields determine $H^{++\mu-}$ by virtue of the equation

$$D_{\breve{\alpha}}^{+}H^{++\mu-} = \mathcal{D}^{++}f_{\breve{\alpha}}^{\mu} \, , \tag{32}$$

which arises from the requirement of the vanishing of the torsion coefficients of $\partial_{h\mu}^{+}$ in constraint (5*b*).

In order to solve self-dual gravity (1), it suffices to solve the above set of equations (27)–(32). The remaining equations from (5) are basically conditions determining consistent expressions for the fields $e_{\breve{\beta}}^{--\mu}$ and $\omega_{\breve{\alpha}}^{-}$, whose determination is actually not necessary in order to find self-dual metrics. These fields represent the same degrees of freedom as the fields $\{f_{\breve{\alpha}}^{\mu}, e_{\breve{\alpha}}^{\mu}, H^{++\mu\pm}, \omega^{++}\}$ and therefore represent redundant degrees of freedom. The field $e_{\breve{\alpha}}^{--\mu}$ is determined by the equation following from the equality of the coefficients of $\partial_{h\mu}^{+}$ in (5*d*), namely,

$$\mathcal{D}^{++}e_{\breve{\alpha}}^{--\mu} = f_{\breve{\alpha}}^{\mu} + \mathcal{D}_{\breve{\alpha}}^{-}(H^{++\mu-} + x_{h}^{\mu+}). \tag{33}$$

The vanishing of torsion coefficients of $\partial_{h\mu}^{+}$ in (5*c*) yields

$$D_{\breve{\alpha}}^{+}e_{\breve{\beta}}^{--\mu} = \mathcal{D}_{\breve{\beta}}^{-}f_{\breve{\alpha}}^{\mu} \, , \tag{34}$$

which together with the condition obtained from the requirement that the antisymmetric part of the curvature in (5*c*) vanishes, i.e.

$$D_{[\breve{\alpha}}^{+}\omega_{\breve{\beta}]}^{-} = 0, \tag{35}$$

determines $\omega_{\breve{\beta}}^{-}$, which satisfies the final equation contained in (5), namely the vanishing of the curvature in (5*d*)

$$\mathcal{D}^{++}\omega_{\breve{\alpha}}^{-} - \mathcal{D}_{\breve{\alpha}}^{-}\omega^{++} = 0, \tag{36}$$

automatically, by virtue of (33).

6. The 'half-flat' gauge

The set of fields satisfying the system of equations listed in the previous section is actually highly redundant, possessing the gauge invariances (22)–(26). The system may therefore be reduced by fixing the parameters $\lambda_{\breve{\alpha}}^{\breve{\beta}}(x_{h}^{+}, u)$, $\lambda^{\mu+}(x_{h}^{+}, u)$, and $\lambda^{\mu-}(x_{h}^{\pm}, u)$, and thereby specifying local coordinates. Remarkably, in a suitable coordinate gauge, this system of equations becomes manifestly soluble. Firstly, since $e_{\breve{\alpha}}^{\mu}$ is analytic (28), the gauge invariance (22) with analytic parameters $\lambda_{\breve{\alpha}}^{\breve{\beta}}$, allows us to choose local coordinates $x_{h}^{\mu+}$ such that $e_{\breve{\alpha}}^{\mu}$ is a unit matrix. Furthermore, the torsion constraint (5*a*) essentially says, by Frobenius' integrability theorem, that $\mathcal{D}_{\breve{\alpha}}^{+}$ is gauge-equivalent to the partial derivative $\partial_{\breve{\alpha}}^{+}$. There therefore exists a coordinate gauge in which (27) is solved by virtue of the zweibein $f_{\breve{\alpha}}^{\mu}$ also

taking the form of a unit matrix. Choosing such a gauge in which $\mathcal{D}_{\breve{\alpha}}^+$ is completely flat is therefore tantamount to changing coordinates (using gauge degrees of freedom parameterized by $\lambda^{\mu-}$ (23)) thus: $x_h^{\mu-} \rightarrow y^{\mu-} = y^{\mu-}(x_h^{\nu+}, x_h^{\nu-})$, $x_h^{\mu+} \rightarrow y^{\mu+} = y^{\mu+}(x_h^{\nu+})$, such that in the old coordinates

$$f_{\breve{\alpha}}^\mu = \frac{\partial x_h^{\mu-}}{\partial y^{\breve{\alpha}-}} , \tag{37}$$

which manifestly solves (27) since

$$\frac{\partial x_h^{\nu-}}{\partial y^{[\breve{\alpha}-}} \frac{\partial}{\partial x_h^{\nu-}} \frac{\partial}{\partial y^{\breve{\beta}]-}} x_h^{\mu-} = \frac{\partial}{\partial y^{[\breve{\alpha}-}} \frac{\partial}{\partial y^{\breve{\beta}]-}} x_h^{\mu-} \equiv 0.$$

With this zweibein $\mathcal{D}_{\breve{\alpha}}^+$ is clearly flat in the new coordinates:

$$\mathcal{D}_{\breve{\alpha}}^+ = \frac{\partial}{\partial y^{\breve{\alpha}-}},$$

the constraint (5a) becoming an identity and Frobenius' theorem becoming manifest. We shall henceforth use this gauge,

$$f_{\breve{\alpha}}^\mu = \delta_{\breve{\alpha}}^\mu, \qquad e_{\breve{\alpha}}^\mu = \delta_{\breve{\alpha}}^\mu , \tag{38}$$

although we shall continue to call the coordinates in this particular gauge $x_h^{\mu\pm}$ rather than $y^{\breve{\alpha}\pm}$. In this special 'half-flat' gauge the distinction between world and tangent indices has evidently been eliminated and only the set of vielbein and connection fields $\{H^{++\mu\pm}, e_{\breve{\alpha}}^{--\mu}, \omega^{++}, \omega_{\breve{\alpha}}^-\}$ remain, of which, as we shall see, those contained in $\mathcal{D}^{++}$, namely $\{H^{++\mu\pm}, \omega^{++}\}$ contain all the dynamical information. This Yang–Mills-like feature is the distinguishing one of this particular analytic frame.

In this gauge residual gauge transformations have parameters constrained by relations from (22), (23), namely

$$\partial_{h\breve{\alpha}}^- \lambda^{\mu+} + \lambda_{\breve{\alpha}}^\mu = 0, \qquad \partial_{h\breve{\alpha}}^+ \lambda^{\mu-} + \lambda_{\breve{\alpha}}^\mu = 0.$$

So the residual diffeomorphism parameters $\lambda^{\mu\pm}(x_h^+, u)$ are no longer arbitrary but are constrained by the relations

$$\partial_{h\mu}^- \lambda^{\mu+} = 0, \qquad \partial_{h\mu}^+ \lambda^{\mu-} = 0 ,$$

since the tangent parameters $\lambda_{\breve{\alpha}}^\mu$ are traceless. It follows that the thus constrained $\lambda^{\mu+}$ can be expressed in terms of an unconstrained doubly charged *analytic* parameter λ^{++}:

$$\lambda_{\text{res}}^{\mu+}(x_h^+, u) = \partial_h^{\mu-} \lambda^{++}(x_h^+, u) . \tag{39a}$$

These diffeomorphism parameters in turn determine the Lorentz ones, the residual tangent transformations actually being induced by the world ones:

$$(\lambda_{\breve{\alpha}}^\mu)_{\text{res}} = -\partial_{h\breve{\alpha}}^- \lambda_{\text{res}}^{\mu+}(x_h^+, u) . \tag{39b}$$

As for the remaining $\lambda^{\mu-}$ transformations, these have parameters:

$$(\lambda^{\mu-})_{\text{res}} = \partial_{h\nu}^- \lambda^{\mu+}(x_h^+, u) x_h^{\nu-} + \tilde{\lambda}^{\mu-}(x_h^+, u) , \tag{39c}$$

where $\tilde{\lambda}^{\mu-}(x_h^+, u)$ is an arbitrary *analytic* parameter†. The remaining vielbeins $H^{++\mu+}$, $H^{++\mu-}$, and $e_{\breve{\alpha}}^{--\mu}$ still transform according to (25), (26), (24), respectively, with the parameters being the residual ones (39).

† When considering self-dual gravity in [15] the alternative gauge condition

$$\Delta^{++} \lambda^{\mu-} = \lambda^{\mu+}$$

corresponding to the preservation of the flat space relation $\partial^{++} x^{\mu-} = x^{\mu-}$ was adopted. This condition completely fixes the gauge parameters $\lambda^{\mu-}$. We choose to avoid this lack of freedom, preferring the more convenient gauge (38).

7. The analytic frame solution

We shall now show that in the 'half-flat' gauge (38) with covariant derivatives taking the form

$$\mathcal{D}_{\breve{\alpha}}^+ = \partial_{h\breve{\alpha}}^+$$

$$\mathcal{D}_{\breve{\alpha}}^- = -\partial_{h\breve{\alpha}}^- + e_{\breve{\alpha}}^{--\mu}\partial_{h\mu}^+ + \omega_{\breve{\alpha}}^- \tag{40}$$

$$\mathcal{D}^{++} = \partial^{++} + H^{++\mu+}\partial_{h\mu}^- + (x_h^{\mu+} + H^{++\mu-})\partial_{h\mu}^+ + \omega^{++},$$

the system of equations (27), (28), or equivalently the self-duality system (5) becomes manifestly soluble. Moreover, our main claim is that:

An unconstrained analytic prepotential, $\mathcal{L}^{+4}$, encodes all local information on self-dual gravity.

To prove this claim we begin by recalling that in the 'half-flat' gauge the difference between world and tangent indices has become rather conventional, all essential information about the manifold having moved to the vielbeins and connections in $\mathcal{D}^{++}$. Equations (27), (28) clearly drop out in this gauge, leaving, from the dynamical set (27)–(32), only the analyticity conditions for $H^{++\mu+}$ and ω^{++} (29), (30) and the relationships (31) and (32). These four equations, and therefore the analytic frame self-duality conditions, can be consistently solved in terms of a single arbitrary analytic prepotential of charge $+4$. To prove that such a prepotential exists (at least locally) we begin with an arbitrary *analytic* $H^{++\breve{\beta}+}$ (satisfying (29)). The relation (31) then yields an expression for the harmonic connection which is manifestly analytic, automatically satisfying its equation of motion (30),

$$\omega_{\breve{\alpha}}^{++\breve{\beta}} = \partial_{h\breve{\alpha}}^- H^{++\breve{\beta}+}. \tag{41}$$

Now the requirement of tracelessness of this connection yields a constraint on $H^{++\breve{\beta}+}$ which has a local solution in terms of the sought unconstrained analytic prepotential $\mathcal{L}^{+4}$, i.e.

$$H^{++\mu+} = \partial_h^{-\mu}\mathcal{L}^{+4} = \epsilon^{\mu\nu}\frac{\partial\mathcal{L}^{+4}}{\partial x_h^{+\nu}}. \tag{42}$$

The transformation rule (25) for $H^{++\mu+}$ induces the gauge transformation

$$\delta\mathcal{L}^{+4} = \Delta^{++}\lambda^{++} + \partial_h^{\mu-}\lambda^{++}\partial_{h\mu}^-\mathcal{L}^{+4} = \partial^{++}\lambda^{++}, \tag{43}$$

where λ^{++} is the unconstrained analytic gauge parameter in (39a). So prepotentials differing by the harmonic partial derivative of an analytic function correspond to gauge equivalent solutions of the CR system. Equation (32) remains and yields a relationship, using (41), between the two vielbeins in $\mathcal{D}^{++}$:

$$\partial_{h\breve{\alpha}}^+ H^{++\breve{\beta}-} = \omega_{\breve{\alpha}}^{++\breve{\beta}} = \partial_{h\breve{\alpha}}^- H^{++\breve{\beta}+}. \tag{44}$$

Integrating this equation, we obtain

$$H^{++\breve{\beta}-} = x_h^{\breve{\alpha}-}\partial_{h\breve{\alpha}}^- H^{++\breve{\beta}+} = x_h^{\breve{\alpha}-}\partial_{h\breve{\alpha}}^-\partial_h^{-\breve{\beta}}\mathcal{L}^{+4}, \tag{45}$$

up to an arbitrary analytic function absorbed by the gauge freedom (26).

We can therefore determine all the required fields ($H^{++\mu\pm}$ and ω^{++}) consistently, i.e. solve the dynamical content of (5) in terms of the unconstrained (i.e. arbitrary) analytic prepotential $\mathcal{L}^{+4}$. For the sake of completeness we show in appendix A that all the other

equations from (5) are also solved in terms of $\mathcal{L}^{+4}$ and determine the remaining analytic frame fields $(e_{\breve{\alpha}}^{--\mu}, \omega_{\breve{\alpha}}^{-})$ as functionals of $\mathcal{L}^{+4}$.

The conventional vierbeins and connections in the central basis may now be constructed according to the procedure we give in the next section. The correspondence between self-dual metrics and prepotentials $\mathcal{L}^{+4}$ is, however, not unique, since prepotentials related by the gauge transformation (43) correspond to equivalent metrics. We may fix this freedom by using the *normal gauge* [15] in which $\mathcal{L}^{+4}$ has no explicit dependence on u_a^+, i.e. $\mathcal{L}^{+4} = \mathcal{L}^{+4}(x_h^{\mu+}, u_a^-)$. In other words any explicit u^+-dependence may always be gauged away using the freedom (43). Consider the harmonic expansion of an arbitrary prepotential having explicit u^+-dependence,

$$\mathcal{L}^{+4}(x_h^+, u^+, u^-) = \mathcal{L}^{+4}_{\text{normal}}(x_h^+, u^-) + \sum_{n,m} u^{+a_1} \ldots u^{+a_n} u^{-b_1} \ldots u^{-b_m} f^{4-n+m}_{(a_1 \ldots a_n b_1 \ldots b_m)}(x_h^+) \, ,$$

(46)

where the coefficients $f^{4-n+m}_{(a_1 \ldots a_n b_1 \ldots b_m)}(x_h^+)$ are monomials of degree $(4 - n + m)$ in x_h^{1+}, x_h^{2+}. Now every term in the sum on the right may easily be shown to be a harmonic partial derivative of an analytic function, so the entire sum has the form $\partial^{++}\lambda^{++}(x_h^+, u^+, u^-)$, which may be absorbed by the gauge freedom (43), yielding the normal gauge form of $\mathcal{L}^{+4}$.

8. The reconstruction of vierbeins and connections in the central frame

As we have seen, the analytic prepotential $\mathcal{L}^{+4}$ encodes all the analytic basis dynamical information. But how does one extract the self-dual metric in the original central basis from it? We now outline the procedure for doing this starting from some specified analytic prepotential $\mathcal{L}^{+4}$.

Step A.   From (42) and (45) obtain the vielbeins of $\mathcal{D}^{++}$:

$$H^{++\mu+} = \partial_h^{-\mu}\mathcal{L}^{+4}$$
$$H^{++\mu-} = x_h^{\breve{\alpha}-}\partial_{h\breve{\alpha}}^{-}\partial_h^{-\mu}\mathcal{L}^{+4}.$$

Step B.   Consider (13) as equations for the holomorphic coordinates $x_h^{\mu+}$:

$$\partial^{++}x_h^{\mu+} = \partial_h^{-\mu}\mathcal{L}^{+4}. \tag{47}$$

Integrating these first-order equations, find $x_h^{\mu+}$ as functions of the *central* frame coordinates $x^{\mu\pm}(\equiv x^{\mu a}u_a^{\pm})$ and the harmonics.

Step C.   Having obtained $x_h^{\mu+}$, similarly solve (14), i.e.

$$\partial^{++}x_h^{\mu-} = x_h^{\mu+} + x_h^{\breve{\alpha}-}\partial_{h\breve{\alpha}}^{-}\partial_h^{-\mu}\mathcal{L}^{+4} \tag{48}$$

in order to determine $x_h^{\mu-}$ as a function of the central frame coordinates.

Step D.   From (41) obtain the connection of $\mathcal{D}^{++}$:

$$\omega_{\breve{\alpha}}^{++\breve{\beta}} = \partial_{h\breve{\alpha}}^{-}\partial_h^{-\breve{\beta}}\mathcal{L}^{+4}, \tag{49}$$

and using the results of steps B and C, express it explicitly in terms of central coordinates.

Step E. With the ω^{++} obtained in step D, solve equation (18) rewritten in the central frame $\partial^{++}\varphi = \varphi\omega^{++}$, i.e.

$$\partial^{++}\varphi_\alpha^{\check\beta} = \varphi_\alpha^{\check\alpha}\partial_{h\check\alpha}^-\partial_h^{-\check\beta}\mathcal{L}^{+4} \tag{50}$$

in order to obtain the bridge φ in central coordinates.

Step F. Using results of steps B and C evaluate the partial derivatives $\partial x^{\nu\pm}/\partial x_h^{\mu-}$.

The above data affords the immediate construction of explicit self-dual vierbeins, metrics, and connections as follows:

Step G. Multiply the bridge φ obtained in step E with one of the matrices of coordinate differentials from step F, and extract the self-dual vierbein from the relation

$$\varphi_\alpha^\nu \frac{\partial x^{\mu-}}{\partial x_h^{\nu-}} = u^{+a} E_{\alpha a}^{\mu b} u_b^- \,, \tag{51}$$

using the completeness relation $u^{+a}u_b^- - u^{-a}u_b^+ = \delta_b^a$. Invert this vierbein and obtain the self-dual metric

$$ds^2 = \epsilon_{ac}\epsilon_{\alpha\beta} E_{\mu b}^{\alpha a} E_{\nu d}^{\beta c}\, dx^{\mu b}\, dx^{\nu d} \,. \tag{52}$$

The proof of the relation (51) is as follows. The central frame vierbeins are given by the equation

$$(\varphi^{-1})_{\check\alpha}^\alpha u^{+a} E_{\alpha a}^{\mu b} \frac{\partial x_h^{\nu-}}{\partial x^{\mu b}} = f_{\check\alpha}^\nu$$

which follows from equations (2), (4), (8) and (17). Fixing the gauge (38) and introducing quantities

$$Z_\alpha^\mu = u^{+a} E_{\alpha a}^{\mu b} u_b^-, \qquad Z_\alpha^{\mu++} = u^{+a} E_{\alpha a}^{\mu b} u_b^+ \tag{53}$$

we obtain the system of equations

$$Z_\alpha^\mu \frac{\partial x_h^{\nu-}}{\partial x^{\mu-}} + Z_\alpha^{\mu++} \frac{\partial x_h^{\nu-}}{\partial x^{\mu+}} = \varphi_\alpha^\nu,$$

$$Z_\alpha^\mu \frac{\partial x_h^{\nu+}}{\partial x^{\mu-}} + Z_\alpha^{\mu++} \frac{\partial x_h^{\nu+}}{\partial x^{\mu+}} = 0$$

which have solution

$$Z_\alpha^\mu = \varphi_\alpha^\nu \frac{\partial x^{\mu-}}{\partial x_h^{\nu-}} \,, \qquad Z_\alpha^{\mu++} = \varphi_\alpha^\nu \frac{\partial x^{\mu+}}{\partial x_h^{\nu-}} \,. \tag{54}$$

By the construction (53), as functions of the central frame coordinates $\{x^{\mu a} = x^{\mu+}u^{-a} - x^{\mu-}u^{+a}, u_a^\pm\}$, these are bilinear in the harmonics, affording immediate extraction of the vierbeins $E_{\alpha a}^{\mu b} = E_{\alpha a}^{\mu b}(x^{\nu c})$ in the customary space.

Step H. The connection ω_α^+ is given in terms of the bridge by the formula (16), which therefore yields $\omega_{\alpha a} = \omega_{\alpha a}(x^{\mu b})$, since ω_α^+, as a function of central frame coordinates $\{x^{\mu a}, u_a^\pm\}$, is by construction (see (4)) linear in the harmonics u^{+a}.

Therefore, extract the self-dual spin connection from the central frame formula

$$(\omega_\alpha^+)_\beta^\gamma = (\omega_{\alpha a})_\beta^\gamma u^{+a}$$

$$= -D_\alpha^+ \varphi_\beta^{\check\beta}(\varphi^{-1})_{\check\beta}^\gamma$$

$$= -\varphi_\alpha^{\check\gamma} D_{\check\gamma}^+ \varphi_\beta^{\check\beta}(\varphi^{-1})_{\check\beta}^\gamma$$

$$= -\varphi_\alpha^{\check\gamma} \frac{\partial \varphi_\beta^{\check\beta}}{\partial x_h^{\check\gamma -}}(\varphi^{-1})_{\check\beta}^\gamma$$

$$= -\left(Z_\alpha^{++\mu} \frac{\partial \varphi_\beta^{\check\beta}}{\partial x^{\mu +}} + Z_\alpha^\mu \frac{\partial \varphi_\beta^{\check\beta}}{\partial x^{\mu -}} \right)(\varphi^{-1})_{\check\beta}^\gamma \; . \tag{55}$$

The thus constructed connection $(\omega_{\alpha a})_\beta^\gamma$ is also a solution of an SU(2) self-dual Yang–Mills theory in a curved space with metric (52). This is obvious in our formulation since this ω_α^+, by construction, satisfies (5a, b), which for this connection are precisely the *Yang–Mills* CR conditions in a self-dual background. This formulation therefore makes manifest the observation of [22] that the spin connection $(\omega_{\alpha a})_\beta^\gamma$ corresponding to a self-dual gravitational solution is such a self-dual Yang–Mills vector potential.

9. An example

We now explicitly illustrate the procedure outlined above for the simplest example of $\mathcal{L}^{+4}$: a monomial of fourth degree in the holomorphic coordinates:

$$\mathcal{L}^{+4} = g x_h^{1+} x_h^{1+} x_h^{2+} x_h^{2+}, \tag{56}$$

where g is a dimensionful parameter. This is invariant under the symmetry transformation

$$x_h^{1+'} = e^\gamma x_h^{1+}, \qquad x_h^{2+'} = e^{-\gamma} x_h^{2+}, \tag{57}$$

which plays an important role in the explicit solubility of this example. The simplest quantity invariant under this symmetry is $\rho^{++} \equiv x_h^{1+} x_h^{2+}$, in terms of which this choice of $\mathcal{L}^{+4}$ can be expressed.

Step A. From this prepotential, we get the harmonic vielbeins $H^{++\mu+}$ using (42)

$$H^{++\mu+} = \epsilon^{\mu\nu} \frac{\partial \mathcal{L}^{+4}}{\partial x_h^{+\nu}} = 2g\rho^{++}(\sigma_3)_\nu^\mu x_h^{\nu+}, \tag{58}$$

where σ_3 is the Pauli matrix. Using these expressions, we find from (45) that

$$H^{++\mu-} = x_h^{-\check\alpha} \partial_{h\check\alpha}^- H^{++\check\beta+} = 2g(\sigma_3)_\nu^\mu (x_h^{\nu-}\rho^{++} + x_h^{\nu+}(x_h^{1-}x_h^{2+} + x_h^{1+}x_h^{2-})). \tag{59}$$

It is worth emphasizing once again that $H^{++\mu-}$ are non-analytic, they contain the antiholomorphic coordinates $x_h^{\mu-}$ explicitly.

Step B. From the definition (13) and having the explicit form for $H^{++\mu+}$ (58) we can write down the equations for the holomorphic coordinates $x_h^{\mu+}$:

$$\partial^{++} x_h^{\mu+} = 2g\rho^{++}(\sigma_3)_\nu^\mu x_h^{\nu+} \; . \tag{60}$$

It follows from these equations that the invariant ρ^{++} is actually the conserved current corresponding to the symmetry (57), in the sense that

$$\partial^{++}\rho^{++} = 0. \tag{61}$$

Due to this conservation law, equations (60) are effectively *linear* equations having solutions

$$x_h^{1+} = e^{2g\rho}x^{1+}, \qquad x_h^{2+} = e^{-2g\rho}x^{2+}, \tag{62}$$

where ρ and $x^{\mu+}$ are solutions of the equations

$$\partial^{++}\rho = \rho^{++} , \tag{63}$$

$$\partial^{++}x^{\mu+} = 0 . \tag{64}$$

The latter equation allows the natural identification of the central frame coordinates from the harmonic expansion $x^{\mu+} = x^{\mu a}u_a^+$; and using (62) ρ^{++} may be seen to have the same form in these central coordinates as in holomorphic ones, i.e. $\rho^{++} = x_h^{1+}x_h^{2+} = x^{1+}x^{2+}$. Now (63) may be seen to have the following solution in terms of central coordinates:

$$\rho = \tfrac{1}{2}(x^{1+}x^{2-} + x^{1-}x^{2+}),$$

up to the addition of a solution of the homogeneous part of (63), which can clearly be absorbed by redefinition of $x^{+\mu}$, which also satisfies a homogeneous equation, (64). The expressions (62) are therefore indeed the required ones for the holomorphic coordinates $x_h^{\mu+}$ in terms of the central ones.

Step C. The equations for the negatively charged coordinates are given by (14), which we solve using the positively charged coordinates already found above. Inserting the explicit expressions (59) we have

$$\partial^{++}x_h^{1-} = (1 + 2gx_h^{1+}x_h^{2-} + 4gx_h^{1-}x_h^{2+})x_h^{1+},$$
$$\partial^{++}x_h^{2-} = (1 - 2gx_h^{1-}x_h^{2+} - 4gx^{1+}x_h^{2-})x_h^{2+} , \tag{65}$$

equations linear in $x_h^{\mu-}$ as they stand. Moreover, together with (62), they imply that

$$\partial^{++}(x_h^{1-}x_h^{2+} + x_h^{1+}x_h^{2-}) = 2x_h^{1+}x_h^{2+},$$

so comparing with (63) and using the linearity of $x^{\mu\pm}$ in the harmonics, we may make the identification

$$\rho = \tfrac{1}{2}(x^{1+}x^{2-} + x^{1-}x^{2+}) = \tfrac{1}{2}(x_h^{1+}x_h^{2-} + x_h^{1-}x_h^{2+}). \tag{66}$$

So ρ, like ρ^{++}, has the same form in both coordinate systems. We may now present the equations in the form of a system linear in the holomorphic coordinates,

$$\partial^{++}x_h^{1-} = x_h^{1+} + 4g\rho x_h^{1+} + 2g\rho^{++}x_h^{1-} ,$$

$$\partial^{++}x_h^{2-} = x_h^{2+} - 4g\rho x_h^{2+} - 2g\rho^{++}x_h^{2-}$$

having solution:

$$x_h^{1-} = x^{1-}(1 + 2gx^{1+}x^{2-})e^{2g\rho}, \qquad x_h^{2-} = x^{2-}(1 - 2gx^{1-}x^{2+})e^{-2g\rho}. \tag{67}$$

Now since ρ is the same in both central and holomorphic coordinates, the relationships (62) and (67) are readily invertible, yielding the following expressions for the central coordinates as functions of the holomorphic ones:

$$x^{1+} = e^{-2g\rho}x_h^{1+}, \qquad\qquad x^{2+} = e^{2g\rho}x_h^{2+}$$

$$x^{1-} = e^{-2g\rho}(x_h^{1-} - 2g\rho^{--}x_h^{1+}), \qquad x^{2-} = e^{2g\rho}(x_h^{2-} + 2g\rho^{--}x_h^{2+}),$$

where

$$\rho^{--} \equiv x^{1-}x^{2-} = \frac{2x_h^{1-}x_h^{2-}}{(1 - gr_h^2 + \sqrt{1 - 2gr_h^2 + 16g^2\rho^2})} ;$$

and $r_h^2 \equiv 2(x_h^{1-}x_h^{2+} - x_h^{1+}x_h^{2-})$.

Step D. Using (41) we find the connection ω^{++} to have the form

$$\omega^{++} = \partial_{h\check{\alpha}}^- H^{++\check{\beta}+} = e^{-2g\sigma_3\rho}\hat{\omega}^{++}e^{2g\sigma_3\rho}$$

where

$$\hat{\omega}^{++} = 2g \begin{pmatrix} 2x^{1+}x^{2+} & -x^{2+}x^{2+} \\ x^{1+}x^{1+} & -2x^{1+}x^{2+} \end{pmatrix} \equiv 2g(\sigma_3\rho^{++} + X^{++}),$$

where X^{++} is the matrix defined in appendix B.

Step E. To find the bridge it is useful to present it in the form,

$$\varphi = \hat{\varphi}e^{2g\sigma_3\rho},$$

where $\hat{\varphi}$ satisfies, by virtue of (50), the equation

$$\partial^{++}\hat{\varphi} = 2g\hat{\varphi}X^{++}.$$

Now inserting the ansatz $\hat{\varphi} = e^{f(r^2)X^{+-}}$, where $r^2 \equiv 2(x^{1-}x^{2+} - x^{1+}x^{2-})$ and X^{+-} satisfies $\partial^{++}X^{+-} = X^{++}$ and is given explicitly in appendix B, into the relation $X^{++} = (1/2g)\hat{\varphi}^{-1}\partial^{++}\hat{\varphi}$, the function $f(r^2)$ is determined to be $f(r^2) = -\ln(1 - gr^2)/gr^2$, yielding

$$\varphi = (1 - gr^2)^{-\frac{1}{2}}(1 + 2gX^{+-})e^{2g\sigma_3\rho}, \tag{68}$$

a result which may also be obtained (and easily checked) simply by linear algebra from a careful consideration of the algebra (B1) in appendix B. The bridge φ is defined by (50) up to multiplication by a factor whose ∂^{++} derivative vanishes and the unimodularity requirement, $\det \varphi = 1$, yields the normalization in (68).

By virtue of the algebra of the matrices $X^{\pm\mp}$, X^{++} described in appendix B, the inverse bridge may immediately be written down:

$$\varphi^{-1} = (1 - gr^2)^{-\frac{1}{2}}e^{-2g\sigma_3\rho}(1 - 2gX^{-+}) .$$

Step F. It follows from formulae (65) and (67) that

$$\frac{\partial x^{\nu+}}{\partial x_h^{\mu-}} = -ge^{-2g\rho\sigma_3}X^{++}$$

and that

$$\frac{\partial x^{\nu-}}{\partial x_h^{\mu-}} = \frac{e^{-2g\rho\sigma_3}}{(1 - gr^2)}\left(1 - g(3 - gr^2)X^{-+}\right) .$$

Step G. Substituting the latter expression and (68) in (51) we obtain:

$$Z_\alpha^\mu = \varphi_\alpha^\nu \frac{\partial x^{\mu-}}{\partial x_h^{\nu-}} = (1 - gr^2)^{-\frac{1}{2}}(1 - gX^{-+}) .$$

As expected, the harmonic expansion of Z^μ_α contains only bilinear pieces and the self-dual vierbein consequently has the form

$$E^{\mu b}_{\alpha a} = (1 - gr^2)^{-\frac{1}{2}} \begin{pmatrix} \delta^b_a - gx^2_a x^{1b} & gx^2_a x^{2b} \\ -gx^1_a x^{1b} & \delta^b_a + gx^1_a x^{2b} \end{pmatrix}$$

having inverse

$$E^{\alpha b}_{\mu a} = (1 - gr^2)^{-\frac{1}{2}} \begin{pmatrix} \delta^b_a(1 - gr^2) + gx^2_a x^{1b} & -gx^2_a x^{2b} \\ gx^1_a x^{1b} & \delta^b_a(1 - gr^2) - gx^1_a x^{2b} \end{pmatrix}.$$

Inserting this in (52), using the parametrization $x^{\mu a} = \begin{pmatrix} y & -\bar{z} \\ z & \bar{y} \end{pmatrix}$ and denoting $r^2 = 2(y\bar{y} + z\bar{z})$, we find

$$ds^2 = 2(1 - gr^2)(dy\, d\bar{y} + dz\, d\bar{z}) + gr^4 \frac{(2 - gr^2)}{(1 - gr^2)} \sigma^2_z, \tag{69}$$

where σ_z is one of the three differential 1-forms [21] related to the Maurer–Cartan forms on SU(2),

$$\sigma_x = \frac{1}{r^2}(\bar{z}\, dy - y\, d\bar{z} + z\, d\bar{y} - \bar{y}\, dz),$$

$$\sigma_y = \frac{1}{r^2}(z\, dy - y\, dz + \bar{y}\, d\bar{z} - \bar{z}\, d\bar{y}),$$

$$\sigma_z = \frac{1}{r^2}(\bar{y}\, dy - y\, d\bar{y} + \bar{z}\, dz - z\, d\bar{z}).$$

Setting $g = 0$ in (69) (corresponding to $\mathcal{L}^{+4} = 0$) we clearly obtain the flat metric

$$ds^2_{\text{flat}} = 2(dy\, d\bar{y} + dz\, d\bar{z}) = dr^2 + r^2(\sigma^2_x + \sigma^2_y + \sigma^2_z). \tag{70}$$

Using the second equality in (70), we obtain precisely the form of the metric obtained using the harmonic space formulation of $N = 2$ sigma models [16], namely

$$ds^2 = (1 - gr^2)\, dr^2 + r^2(1 - gr^2)(\sigma^2_x + \sigma^2_y) + \frac{r^2}{(1 - gr^2)} \sigma^2_z. \tag{71}$$

This is a form of the self-dual Euclidean Taub-NUT metric. Denoting the parameter $g = 1/4m^2$, the variable change $r^2 \to 2m(\rho - m)$ yields the form of the Taub-NUT metric in, e.g. [21]:

$$ds^2 = \frac{\rho + m}{4(\rho - m)}\, d\rho^2 + (\rho^2 - m^2)(\sigma^2_x + \sigma^2_y) + 4m^2 \frac{\rho - m}{\rho + m} \sigma^2_z.$$

Unlike (71), the latter form of the metric is only defined in the domain $\rho > m$; and therefore does not have a well defined flat limit ($m \to \infty$).

Step H. Substituting derivatives of the central frame coordinates and bridge φ from steps E and F into (55), we obtain

$$(\omega^+_1)^\gamma_\beta = -2g(1 - gr^2)^{-\frac{3}{2}} \begin{pmatrix} x^{2+}(1 - \frac{1}{2}gr^2) & 0 \\ x^{1+} & -x^{2+}(1 - \frac{1}{2}gr^2) \end{pmatrix}$$

$$(\omega^+_2)^\gamma_\beta = -2g(1 - gr^2)^{-\frac{3}{2}} \begin{pmatrix} x^{1+}(1 - \frac{1}{2}gr^2) & -x^{2+} \\ 0 & -x^{1+}(1 - \frac{1}{2}gr^2) \end{pmatrix},$$

expressions manifestly linear in the harmonics, allowing us to extract the connection components (3) immediately:

$$(\omega_{\alpha a})^\gamma_\beta = 2g(1 - gr^2)^{-\frac{3}{2}} \begin{pmatrix} (1 - \frac{1}{2}gr^2)(x^2_a, x^1_a) & (0, -x^2_a) \\ (x^1_a, 0) & -(1 - \frac{1}{2}gr^2)(x^2_a, x^1_a) \end{pmatrix}.$$

10. Conclusions and further remarks

We have shown that *all* self-dual gravitational fields allow local description in terms of an unconstrained analytic prepotential $\mathcal{L}^{+4}$ in harmonic space. The explicit performance of our construction relies only on the solution of first-order differential equations on S^2. Our method therefore promises to be a fruitful one for the explicit construction of self-dual metrics.

Whether global characteristics (e.g. topological invariants) and singularity properties of the manifold can be *determined* by a prescient choice of $\mathcal{L}^{+4}$ remains an open question. The curvature tensor, however, may indeed be evaluated from the analytic frame connection $\omega_{\breve{\beta}}^-$ by virtue of

$$R_{\breve{\alpha}\breve{\beta}} = \partial^+_{h(\breve{\alpha}}\omega^-_{\breve{\beta})} \ , \tag{72}$$

or equivalently

$$C_{\breve{\alpha}\breve{\beta}\breve{\gamma}}{}^{\breve{\delta}} = \partial^+_{h(\breve{\alpha}}\omega^-_{\breve{\beta})\breve{\gamma}}{}^{\breve{\delta}} = \partial^+_{h\breve{\alpha}}\partial^+_{h\breve{\beta}}e^{--\breve{\delta}}_{\breve{\gamma}} \ ,$$

where the connection $\omega_{\breve{\alpha}}^-$ is determined in terms of the vielbein $e^{--\mu}_{\breve{\beta}}$, which in turn is determined in terms of $\mathcal{L}^{+4}$ (see appendix A). The expression (72), which is by construction u-independent, immediately yields the manifestly total-derivative form of the Pontryagin density:

$$R^{\breve{\alpha}\breve{\beta}} R_{\breve{\alpha}\breve{\beta}} = \partial^{+\breve{\alpha}}_h \partial^{+\breve{\beta}}_h e^{--\breve{\delta}}_{\breve{\gamma}} \partial^+_{h\breve{\alpha}}\partial^+_{h\breve{\beta}} e^{--\breve{\gamma}}_{\breve{\delta}} = \partial^{+\breve{\alpha}}_h(\partial^{+\breve{\beta}}_h e^{--\breve{\delta}}_{\breve{\gamma}} \partial^+_{h\breve{\alpha}}\partial^+_{h\breve{\beta}} e^{--\breve{\gamma}}_{\breve{\delta}}).$$

In our framework the fields $\omega_{\breve{\alpha}}^-$ and $e^{--\mu}_{\breve{\beta}}$ are redundant, carrying no additional dynamical information; all their equations of motion being identically satisfied by virtue of what we have called the 'dynamical' subset of the CR equations. Alternatively, one could choose to ignore the relations in (5) involving $\mathcal{D}^{++}$ and attempt instead to solve the equations for $e^{--\mu}_{\breve{\beta}}$, which describes the same degrees of freedom. These equations imply the form (see (A5) in appendix A) $e^{--\breve{\gamma}}_{\breve{\beta}} = \partial^+_{h\breve{\beta}}\partial^{+\breve{\gamma}}_h e^{-4}$, where e^{-4} is a non-analytic prepotential, which, in this alternative framework, carries all the dynamical information. Indeed the equation

$$[\mathcal{D}_\alpha^-, \mathcal{D}_\beta^-] = 0, \tag{73}$$

which completes the algebra of covariant differential operators in (5), and which needs to be included if one wishes to exclude (5*b, d*) (this equation is an identity in our framework), may easily be seen to contain the dynamical equation for e^{-4},

$$\partial^{\breve{\alpha}+}\partial^-_{\breve{\alpha}} e^{-4} + \tfrac{1}{2}\partial^{\breve{\alpha}+}\partial^{\breve{\beta}+} e^{-4}\partial^+_{\breve{\alpha}}\partial^+_{\breve{\beta}} e^{-4} = 0 \ , \tag{74}$$

as a consequence of the vanishing of the torsion coefficient of $\partial^+_{h\breve{\alpha}}$. (These torsion constraints actually imply that the left-hand side of (74) is equal to some arbitrary analytic function, which however may be set to zero using a pregauge symmetry of e^{-4}, namely $\delta e^{-4} = x^{\mu-} G_\mu^{-3}(x^+)$, for arbitrary analytic G_μ^{-3}.) In the alternative approaches to the self-dual Einstein equations, which do not introduce the auxiliary $\mathcal{D}^{++}$ (e.g. [4, 23]), the dynamics is indeed described by (74) or transformations thereof. The field e^{-4} is precisely

Plebanski's second 'heavenly function' [4], for which (73) is the 'zero-curvature' or Lax representation†. Indeed the contravariant form of the basis (40) with Cartan 1-forms

$$\Omega^{\mu+} = \mathrm{d}x_h^{\mu+}$$
$$\Omega^{\mu-} = \mathrm{d}x_h^{\mu-} + e_\nu^{--\mu}\,\mathrm{d}x_h^{\nu+}$$

yields the (u-independent) invariant $\mathrm{d}x_{h\nu}^+(\mathrm{d}x_h^{\nu-} + e_\mu^{--\nu}\,\mathrm{d}x_h^{\mu+})$, which is precisely the form of the metric given in [5]. We emphasize, however, that the advantage of introducing $\mathcal{D}^{++}$ and working with harmonics as independent variables, is that (74) is then automatically satisfied. If explicit solutions of (74) are for some reason required, they may also be constructed in our framework from a first-order equation ((A2) in appendix A).

The alternative Monge–Ampère form of Plebanski's equation [4] corresponds to a different gauge to (38), i.e. a different choice of analytic frame coordinates; our method may also be used to construct solutions to that form of Plebanski's equation. We hope to return to a discussion of both of Plebanski's equations in the harmonic space setting in a future publication.

The self-duality conditions are well known to be differential equations whose solutions are automatically hyper-Kähler metrics. Our construction of solutions generalizes to $4m$-dimensions, where hyper-Kähler metrics [15] may similarly be thought of as solutions to the generalized self-duality conditions [24]

$$[\mathcal{D}_{Bb}, \mathcal{D}_{Aa}] = \epsilon_{ab} R_{AB}, \tag{75}$$

where A is an Sp(m) index and a, as above, is an Sp(1) index. These equations manifestly break the $4m$-dimensional rotation group to Sp(m) $\times$ Sp(1). Delocalizing the Sp(1) yields connections and curvatures manifestly taking values in the Sp(m) subalgebra. In other words, the holonomy group is Sp(m), so (75) indeed describe higher-dimensional hyper-Kähler spaces. Our entire construction generalizes to these higher-dimensional cases on replacing the Sp(1) indices α, β by Sp(m) indices A, B. The further generalization with indices A, B in (75) considered as 'superindices' of the superalgebra OSp($N|m$), yields constraints in chiral superspace which may be thought of as equations for N-extended supersymmetric $(4m|2N)$-dimensional hyper-Kähler spaces. In this supersymmetric case, our construction requires suitable modifications in order to accommodate the intricacies of superalgebras. The case $m = 1$ corresponds to N-extended supersymmetric self-dual supergravity. In fact, the present work was motivated by our initial attempts to supersymmetrize the harmonic-twistor construction for self-dual gravity and the construction in this paper is indeed amenable to supersymmetrization, yielding a general solution of all extended self-dual Poincaré supergravity theories. This is under preparation for publication.

Acknowledgments

† The conjugation of harmonics which yields real $x^\pm$ of course does not hold if the harmonics are given fixed numerical values and (74) is then an equation in $\mathbb{C}^4$.

Appendix A. The redundancy of the degrees of freedom in $\mathcal{D}_\alpha^-$

In this appendix we prove the claim in section 7 that the fields $e_{\breve{\beta}}^{--\mu}$ and $\omega_{\breve{\alpha}}^-$ are redundant degrees of freedom and are indeed entirely determined in terms of the 'dynamical' fields $H^{++\mu+}$ and ω^{++}, which in turn are determined by the analytic prepotential $\mathcal{L}^{+4}$.

Equation (33) reduces to

$$\mathcal{D}^{++}e_{\breve{\alpha}}^{--\mu} = -\partial_{h\breve{\alpha}}^- H^{++\mu-} + e_{\breve{\alpha}}^{--\nu}\partial_{h\nu}^+ H^{++\mu-}. \tag{A1}$$

Now recalling the first expression for ω^{++} in (44), we note that this equation is actually just the equation

$$\mathcal{D}^{++}e_{\breve{\alpha}}^{--\breve{\beta}} = -\partial_{h\breve{\alpha}}^- H^{++\breve{\beta}-}$$

with the connection acting in the gauge (38) on *both* spinor indices of $e_{\breve{\alpha}}^{--\breve{\beta}}$. Now in the central basis this equation reads simply

$$\partial^{++}e_\alpha^{--\beta} = -\varphi_\alpha^{\breve{\alpha}}\partial_{h\breve{\alpha}}^- H^{++\breve{\beta}-}(\varphi^{-1})_{\breve{\beta}}^\beta, \tag{A2}$$

a first-order linear equation which uniquely determines $e_\alpha^{--\beta}$ in terms of φ and $H^{++\beta-}$, which in turn are determined in terms of the arbitrary analytic prepotential $\mathcal{L}^{+4}$ by virtue of (41), (45).

For the Taub-NUT example of section 9, explicit integration of (A2) yields

$$e_\alpha^{--\beta} = \frac{4g}{(1-gr^2)}\big(X^{--}\{(1-\tfrac{1}{2}gr^2)^2 - \tfrac{1}{2} - 4g^2\rho^2\}$$
$$+ \sigma_3(\tfrac{1}{2}\rho^{--}I - 2g\rho X^{--}) + 2g\rho\rho^{--}I\big)_\alpha^\beta, \tag{A3}$$

where X^{--} is the matrix in (B2).

From (34) we obtain an expression for the connection in $\mathcal{D}_{\breve{\beta}}^-$:

$$\omega_{\breve{\beta}\breve{\alpha}}^{-\breve{\gamma}} = \partial_{h\breve{\alpha}}^+ e_{\breve{\beta}}^{--\breve{\gamma}}, \tag{A4}$$

for which (35) implies the constraint $\omega_{[\breve{\beta}\breve{\alpha}]}^{-\breve{\gamma}} = 0$. The latter together with the constraint of tracelessness of this SU(2) connection have a local solution in terms of an unconstrained non-analytic prepotential $e^{-4} = e^{-4}(x_h^\pm, u^\pm)$, in terms of which

$$e_{\breve{\beta}}^{--\breve{\gamma}} = \partial_{h\breve{\beta}}^+ \partial_h^{+\breve{\gamma}} e^{-4},$$
$$\omega_{\breve{\beta}\breve{\alpha}}^{-\breve{\gamma}} = \partial_{h\breve{\alpha}}^+ \partial_{h\breve{\beta}}^+ \partial_h^{+\breve{\gamma}} e^{-4}. \tag{A5}$$

The only remaining equation for the connection fields is the equation (36), which is an identity by virtue of the other equations. Namely, acting on (A1) by $\partial_{\breve{\beta}}^+$ we obtain precisely (36) with the connection components written in the forms (44) and (A4).

All equations in the CR system therefore allow solution in terms of the unconstrained analytic prepotential $\mathcal{L}^{+4}$, proving our claim.

Appendix B. The quadratic matrices $X^{\pm\pm}$, $X^{\pm\mp}$

The explicit solubility of the example of section 9 relies on remarkable properties of matrices quadratic in the central coordinates $x^{\mu\pm} \equiv x^{\mu a}u_a^\pm$. Consider

$$X^{++} = \begin{pmatrix} x^{1+}x^{2+} & -x^{2+}x^{2+} \\ x^{1+}x^{1+} & -x^{2+}x^{1+} \end{pmatrix}.$$

This is actually the harmonic derivative of two possible matrices, X^{+-}, X^{-+} ,

$$X^{++} = \partial^{++}X^{+-} = \partial^{++}X^{-+} \; ;$$

$$X^{+-} = \begin{pmatrix} x^{1+}x^{2-} & -x^{2+}x^{2-} \\ x^{1+}x^{1-} & -x^{2+}x^{1-} \end{pmatrix}, \qquad X^{-+} = \begin{pmatrix} x^{1-}x^{2+} & -x^{2-}x^{2+} \\ x^{1-}x^{1+} & -x^{2-}x^{1+} \end{pmatrix},$$

together with which it obeys the algebra

$$X^{++}X^{++} = X^{+-}X^{++} = X^{++}X^{-+} = 0$$

$$X^{+-}X^{-+} = X^{-+}X^{+-} = 0$$

$$X^{++}X^{+-} = -\tfrac{1}{2}r^2 X^{++} = -X^{-+}X^{++} \tag{B1}$$

$$X^{+-}X^{+-} = -\tfrac{1}{2}r^2 X^{+-} \; , \qquad X^{-+}X^{-+} = \tfrac{1}{2}r^2 X^{-+}$$

$$X^{+-} - X^{-+} = -\tfrac{1}{2}r^2 I.$$

where $r^2 \equiv 2(x^{1-}x^{2+} - x^{1+}x^{2-})$. Furthermore, the matrix

$$X^{--} = \begin{pmatrix} x^{1-}x^{2-} & -x^{2-}x^{2-} \\ x^{1-}x^{1-} & -x^{2-}x^{1-} \end{pmatrix} \tag{B2}$$

satisfies the equation $\partial^{++}X^{--} = X^{+-} + X^{-+}$.

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