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Democratic supersymmetry

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We present generalizations of N-extended supersymmetry algebras in four dimensions, using Lorentz covariance and invariance under permutation of the N supercharges as selection criteria.
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I. INTRODUCTION

Recent developments in string theory have revealed the need to study generalizations of supersymmetry which lie beyond the realm of existing classifications of space–time supersymmetry algebras. Space–time supersymmetry algebras are $\mathbb{Z}_2$-graded super Lie algebras $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ having even $\mathfrak{g}_0$ and odd $\mathfrak{g}_1$ subspaces, where the even part $\mathfrak{g}_0=\langle M\rangle\oplus\langle P\rangle\oplus\mathfrak{h}$ includes the generators of spacetime Lorentz transformations M, translations P, and a subspace of additional "internal symmetries" $\mathfrak{h}$. The usual relation between spin and statistics implies that generators of $\mathfrak{g}_1$ transform as half-integer spin representations under the Lorentz transformations. Traditional classifications of spacetime supersymmetries were based on assumptions arising from the additional requirement that the supersymmetries act on either S-matrix elements[1] or on some physical Hilbert space of particle states.[2] In particular these restrict the maximum spin of the generators to be one and require the internal symmetries to be "central" in the sense that they commute with all other generators. Moreover, in four dimensional space–time, the realization of these algebras on physical states restricts finite dimensional representations to contain fields of spin less than or equal to two and the maximal number N of independent supercharges in $\mathfrak{g}_1$ to eight.

There are several instances in which spacetime supersymmetries and representations more general than those allowed in traditional settings occur. In M-theory, for instance, the internal symmetries $\mathfrak{h}$ do not commute with the Lorentz generators (see e.g., Ref. 3). In N-extended super self-dual theories in four dimensional Euclidean space, finite dimensional representations containing fields of spin higher than two do occur and there are consistent theories for any choice of N.[4] In $N=2$ string theory,[5] the absence of the usual relation between spin and statistics gives rise to a realization of a purely even variant of supersymmetry[6] on an infinite dimensional space of string states. There are indications that this statistics-twisted version of supersymmetry is related to an $N\to\infty$ extension of the super Poincaré algebra, which has a realization on an $N=\infty$ self-dual Yang–Mills supermultiplet.[4] These examples show that there seems to be room for the study of more general superalgebras containing the (N-extended) super Poincaré algebra or the super de Sitter algebra as a subalgebra or as a contraction. The work of Fradkin and Vasiliev (e.g., Ref. 7) on higher spin superalgebras on anti de Sitter space is also noteworthy in this respect. The present paper is a further contribution in this direction.

In a series of recent papers[8] we recently developed an approach to the study of generalized super-Poincaré algebras containing generators having spins higher than one. We showed that, contrary to common belief, such superalgebras indeed exist and are realizable in terms of vector

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fields on generalized superspaces having coordinates of higher spin which commute or anticommute in accordance with their statistics. We constructed numerous examples of generalized superalgebras with generators having spins up to two.

In the present paper, we address ourselves to another type of generalization, concerning the question of higher multiplicities of certain representations in the superalgebra. Theories with N supercharges are of special interest. In these, there does not seem to be any principle which distinguishes some of the supercharges from the others and field theories containing such supercharges are usually taken to be invariant under permutation of the supercharges. In this paper, we impose this permutation invariance at the level of the superalgebra, introducing what we will call *democratic superalgebras*. Our purpose here is not a complete classification of possibilities; rather, we aim to show that under the imposition of *democracy*, even in the widely familiar four-dimensional case, an investigation of super Jacobi identities yields some potentially interesting democratic spacetime superalgebras which lie beyond known classifications. The main novel feature which arises in our approach is that the algebra of Lorentz scalars $\mathfrak{h}$ generated by the superderivations is no longer either Abelian or in the center of $\mathfrak{g}$. Although democracy implies the Coleman–Mandula requirement[9] that the scalars commute with (even) translations, they possibly rotate spinor derivations among themselves.

II. DEMOCRATIC SUPERALGEBRAS

A. Four-dimensional space–time supersymmetry

Since our aim is to generalize traditional discussions and since our considerations are purely algebraic, we restrict ourselves to the general complex setting. The question of the appropriate real form depends in any case on the signature of the space–time on which the superalgebra is to be realized; and this depends on the specific context of the application. We consider the Lorentz group to be SO$(4,\mathbb{C})$, with complex generators $M_{\alpha\beta}, M_{\dot\alpha\dot\beta}$, where there is of course no conjugation between dotted and undotted spinor indices.

We shall consider $\mathbb{Z}_2$-graded N-extended complex supersymmetry algebras of the form $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$, with even part

$$\mathfrak{g}_0=\langle M_{\alpha\beta}, M_{\dot\alpha\dot\beta}, \nabla_{\alpha\dot\alpha}\rangle\oplus\mathfrak{h}, \tag{1}$$

where $\nabla_{\alpha\dot\alpha}$ denotes the derivative vector fields generating translations, and $\mathfrak{h}$ is the subspace of internal symmetries,

$$\mathfrak{h}=\left\langle Y^i, Z^{ij}=-Z^{ji};\ \sum_i Y^i=0,\ \sum_i Z^{ij}=0,\ i,j=1,\ldots,N\right\rangle, \tag{2}$$

spanned by a set of Lorentz scalar generators, $(N-1)$ Y's and $(N-1)(N-2)/2$ Z's.

The odd subspace $\mathfrak{g}_1$ is spanned by N copies of the two types of spinor representations of so$(4,\mathbb{C})$, namely, the $2N$ fermionic operators $\nabla^i_\alpha, \nabla^i_{\dot\alpha}$ $(i=1,\ldots,N)$, which together with the bosonic vectorial operator $\nabla_{\alpha\dot\beta}$, form the set of superderivations acting on an N-extended superspace. We denote the vector space of superderivations,

$$\mathcal{D}=\langle \nabla_{\alpha\dot\beta}, \nabla^i_\alpha, \nabla^i_{\dot\alpha}\rangle=\mathcal{D}_0\oplus\mathcal{D}_1,$$

where the even and odd parts are spanned by the vector and spinor derivations, respectively. The vector space $\mathcal{D}$ may be extended to include vector fields having higher spins on the lines of the consideration in Ref. 8. For simplicity, however, we restrict ourselves here, to the consideration of operators having spin less than or equal to one.

We shall assume that all the elements in $\mathfrak{g}$ have commutation or anticommutation relations in agreement with their statistics and with covariance under the Lorentz transformations with generators $M_{\alpha\beta}$, $M_{\dot\alpha\dot\beta}$, $\alpha,\beta,\dot\alpha,\dot\beta=1,2$ satisfying

$$[M_{\alpha\beta},M_{\gamma\delta}]=\epsilon_{\beta\gamma}M_{\alpha\delta}+\epsilon_{\alpha\gamma}M_{\beta\delta}+\epsilon_{\beta\delta}M_{\alpha\gamma}+\epsilon_{\alpha\delta}M_{\beta\gamma},$$
$$[M_{\dot\alpha\dot\beta},M_{\dot\gamma\dot\delta}]=\epsilon_{\dot\beta\dot\gamma}M_{\dot\alpha\dot\delta}+\epsilon_{\dot\alpha\dot\gamma}M_{\dot\beta\dot\delta}+\epsilon_{\dot\beta\dot\delta}M_{\dot\alpha\dot\gamma}+\epsilon_{\dot\alpha\dot\delta}M_{\dot\beta\dot\gamma}, \tag{3}$$
$$[M_{\alpha\beta},M_{\dot\gamma\dot\delta}]=0.$$

Lorentz covariance, in particular, determines all commutators of the basic operators with the M, namely,

$$[M_{\alpha\beta},\nabla^i_\gamma]=\epsilon_{\alpha\gamma}\nabla^i_\beta+\epsilon_{\beta\gamma}\nabla^i_\alpha,\ \ [M_{\alpha\beta},\nabla^i_{\dot\gamma}]=0,\ \ [M_{\alpha\beta},Y_i]=0,\ \ [M_{\alpha\beta},Z^{ij}]=0,$$
$$[M_{\dot\alpha\dot\beta},\nabla^i_{\dot\gamma}]=\epsilon_{\dot\alpha\dot\gamma}\nabla^i_{\dot\beta}+\epsilon_{\dot\beta\dot\gamma}\nabla^i_{\dot\alpha},\ \ [M_{\dot\alpha\dot\beta},\nabla^i_\gamma]=0,\ \ [M_{\dot\alpha\dot\beta},Y_i]=0,\ \ [M_{\dot\alpha\dot\beta},Z^{ij}]=0. \tag{4}$$

Given these commutation rules, all Jacobi identities involving at least two M's are automatically satisfied. Lorentz covariance also yields restrictions on the (anti)commutators of any two elements of $\mathfrak{g}$. These guarantee that the Jacobi identities involving at least one M are also automatically satisfied.

The spinor derivations $\nabla^i_\alpha,\nabla^i_{\dot\alpha}$ are taken to transform under some group of automorphisms T of the superalgebra $\mathfrak{g}$,

$$T\nabla^i_\alpha T^{-1}=U^i_j\nabla^j_\alpha,\ \ T\nabla^i_{\dot\alpha}T^{-1}=V^i_j\nabla^j_{\dot\alpha}, \tag{5}$$

where the matrices U,V are representations of the group element T. In this paper, we make particular use of discrete transformations, taking U and V to be permutation matrices on the index i. When the automorphism group is continuous, the action of the group can be expressed in the form of commutation relations with the generators of the group: for instance, the scalar generators Y or Z which appear in (32)–(37).

We shall also allow the possibility of generating scalars by anticommuting spinor derivations, e.g., $\{\nabla^i_\alpha,\nabla^j_\beta\}\sim\epsilon_{\alpha\beta}Z^{ij}$. Traditionally,[1] such Lorentz scalars are always taken to be central with respect to $\mathcal{D}$. In our approach we do not *a priori* restrict the Lorentz scalars to be central. In fact they rotate the spinor derivations $\nabla^i_\alpha,\nabla^i_{\dot\alpha}$ just as the automorphisms (5). This is the main source of our novel examples of spacetime supersymmetries.

B. Democracy

1. Permutation invariants

We shall impose what we call democracy: we require the supercommutation relations to be invariant under the combined permutations of the i-indices of ∇^i_α and of $\nabla^i_{\dot\alpha}$. The group generating democracy S_N is the diagonal group of two groups of permutations acting independently on the two sets of spinors, with permutation matrices $U=V$ in (5).

The Clebsch–Gordon coefficients of the democratic group may be described as follows. The permutation invariant coupling among p $(p>1)$ i type indices can be associated to Young-type diagrams. Given a Young diagram with p $(p>0)$ boxes denoted $[m]=[m_1,m_2,\ldots,m_p]$, with m_j boxes in the jth row $(\Sigma_j m_j=p, m_{i+1}\leqslant m_i)$, we associate with it a p-index tensor $\theta^{i_1 i_2\cdots i_p}_{[m_1 m_2\cdots m_p]}$ defined by

$$\theta^{\overbrace{i_1\ldots i_{m_1}}^{m_1}\overbrace{j_1\ldots j_{m_2}}^{m_2}\overbrace{k_1\ldots k_{m_3}}^{m_3}\cdots}_{[m_1 m_2\ldots m_p]}=1\quad \text{if}\ \ i_1=\ldots=i_{m_1},\ \ j_1=\ldots=j_{m_2},\ \ k_1=\ldots=k_{m_3},\ldots=0\quad \text{otherwise}. \tag{6}$$

Note that these tensors clearly do not have the standard Young diagram symmetries. From these θ tensors, by permuting indices, all the invariant tensors of the permutation group can be constructed. For a Young-type diagram with p boxes, if n_l is the number of rows having length $m_k = l$, the number of independent invariant tensors is given by $p!/(\Pi_k m_k! \Pi_l n_l!)$.

Some of these tensors have a simple interpretation in terms of the familiar Kronecker tensor δ^{ij}. In particular,

$$\theta^{ij}_{[2]} = \delta^{ij}, \quad \theta^{ijkl}_{[22]} = \delta^{ij}\delta^{kl}, \quad \theta^{ijklmn}_{[222]} = \delta^{ij}\delta^{kl}\delta^{mn}, \tag{7}$$

correspond to the invariant tensors of so(N). These are special cases of the identities,

$$\theta^{i_1 \cdots i_{m_1} i_{m_1+1} \cdots i_{m_1+m_2} \cdots}_{[m_1 m_2 m_3 \cdots]} = \theta^{i_1 \cdots i_{m_1}}_{[m_1]} \theta^{i_{m_1+1} \cdots i_{m_1+m_2}}_{[m_2]} \theta^{\cdots}_{[m3]} \cdots. \tag{8}$$

One further useful identity (with summation over repeated indices assumed) is

$$\theta^{i}_{[1]} \theta^{ijk\cdots}_{[m]} = \theta^{jk\cdots}_{[m-1]}, \tag{9}$$

which is valid for all $m > 0$ if we define

$$\theta_{[0]} := N. \tag{10}$$

2. Trace conditions

We note that the tensor $\theta^{i}_{[1]}$ can be used to decompose tensors into their permutation irreducible parts. In particular, a vector V^i has two irreducible components given by the scalar projection S,

$$S = \theta^{\,j}_{[1]} V^j \tag{11}$$

and its complementary piece, of dimension $N-1$,

$$Y^j = V^i - \frac{1}{N} \theta^{i}_{[1]} S. \tag{12}$$

Similarly, a general antisymmetric tensor T^{ij} can be decomposed under the permutation group into two irreducible pieces. A piece of the form Y^j is obtained by the projection

$$Y^j = \theta^{i}_{[1]} T^{ij}, \tag{13}$$

which satisfies

$$\theta^{j}_{[1]} Y^j = 0. \tag{14}$$

The other irreducible piece Z^{ij}, of dimension $(N-1)(N-2)/2$, can be defined by

$$Z^{ij} = T^{ij} - \frac{1}{N} (\theta^{i}_{[1]} Y^j - \theta^{j}_{[1]} Y^i) \tag{15}$$

and satisfies

$$\theta^{i}_{[1]} Z^{ij} = 0. \tag{16}$$

We will generically call conditions imposed on the structure constants which guarantee the irreducibility of the relevant tensors *trace conditions*. The tensors Y in (12) and Z in (15) will be called *trace-free*.

Since the Y^k and the Z^{kl} need to satisfy the trace-free conditions (16) and (14), it is convenient to use some partially trace-free combinations of the invariant θ-tensors with certain symmetries,

$$
\begin{aligned}
t^{ij}_{[2]} &:= \theta^{ij}_{[2]} - \frac{1}{N}\theta^{ij}_{[11]},\\
t^{ijk}_{[21]} &:= \theta^{ijk}_{[21]} - \frac{1}{N}\theta^{ijk}_{[111]},\\
t^{ijk}_{[3]} &:= \theta^{ijk}_{[3]} - \frac{1}{N}(\theta^{kji}_{[21]} + \theta^{ijk}_{[21]} + \theta^{ikj}_{[21]}),\\
t^{ijkl}_{[22]} &:= \theta^{ijkl}_{[22]} - \theta^{kjil}_{[22]} - \frac{1}{N}(\theta^{ijkl}_{[211]} - \theta^{kjil}_{[211]} - \theta^{ilkj}_{[211]} + \theta^{klij}_{[211]}),\\
t^{ijklm}_{[32]} &:= \theta^{jklim}_{[32]} - \theta^{ikljm}_{[32]} - \theta^{jkmil}_{[32]} + \theta^{ikmjl}_{[32]} - \frac{1}{N}(\theta^{jklim}_{[311]} - \theta^{ikljm}_{[311]} - \theta^{jkmil}_{[311]} + \theta^{ikmjl}_{[311]})\\
&\quad + \frac{1}{N}(\theta^{jmilk}_{[221]} - \theta^{imjlk}_{[221]} + \theta^{jkilm}_{[221]} - \theta^{ikjlm}_{[221]} - \theta^{jkiml}_{[221]} + \theta^{ikjml}_{[221]} - \theta^{lkimj}_{[221]} + \theta^{lkjmi}_{[221]} + \theta^{mkilj}_{[221]}\\
&\quad - \theta^{mkjli}_{[221]}) - \frac{3}{N^2}(\theta^{jmkil}_{[2111]} - \theta^{imkjl}_{[2111]} - \theta^{jlkim}_{[2111]} + \theta^{ilkjm}_{[2111]}),\\
t^{jkmiln}_{[222]} &:= \theta^{jkmiln}_{[222]} - \theta^{ikmjln}_{[222]} - \theta^{jlmikn}_{[222]} + \theta^{ilmjkn}_{[222]} - \theta^{jknilm}_{[222]} + \theta^{iknjlm}_{[222]} + \theta^{jlnikm}_{[222]} - \theta^{ilnjkm}_{[222]} - \frac{1}{N}(\theta^{jkmiln}_{[2211]}\\
&\quad - \theta^{ikmjln}_{[2211]} - \theta^{jlmikn}_{[2211]} + \theta^{ilmjkn}_{[2211]} - \theta^{jknilm}_{[2211]} + \theta^{iknjlm}_{[2211]} + \theta^{jlnikm}_{[2211]} - \theta^{ilnjkm}_{[2211]} - \theta^{limkjn}_{[2211]} + \theta^{kimljn}_{[2211]}\\
&\quad + \theta^{ljmkin}_{[2211]} - \theta^{kjmlin}_{[2211]} + \theta^{linkjm}_{[2211]} - \theta^{kinljm}_{[2211]} - \theta^{ljnkim}_{[2211]} + \theta^{kjnlim}_{[2211]} + \theta^{jmknil}_{[2211]} - \theta^{imknjl}_{[2211]} - \theta^{jmlnik}_{[2211]}\\
&\quad + \theta^{imlnjk}_{[2211]} - \theta^{jnkmil}_{[2211]} + \theta^{inkmjl}_{[2211]} + \theta^{jnlmik}_{[2211]} - \theta^{inlmjk}_{[2211]}).
\end{aligned}
\tag{17}
$$

These satisfy the useful identities,

$$t^{jnim}_{[22]} + t^{jmin}_{[22]} \equiv 0, \tag{18}$$

$$t^{jkmiln}_{[222]} + t^{jmnilm}_{[222]} \equiv 0, \tag{19}$$

$$t^{ij}_{[2]}t^{km}_{[2]} - t^{kj}_{[2]}t^{im}_{[2]} - \tfrac{1}{2}t^{knip}_{[22]}t^{pjnm}_{[22]} \equiv 0, \tag{20}$$

$$t^{minl}_{[22]}t^{lqjp}_{[22]} - t^{mjnl}_{[22]}t^{lqip}_{[22]} + \tfrac{1}{2}t^{ikjl}_{[22]}t^{mkpnlq}_{[222]} \equiv 0, \tag{21}$$

$$t^{ijk}_{[3]}t^{knm}_{[3]} - t^{njk}_{[3]}t^{kim}_{[3]} + \frac{1}{2N}t^{npiq}_{[22]}t^{qjpm}_{[22]} \equiv 0, \tag{22}$$

$$t^{kmin}_{[22]}t^{njmp}_{[22]} + t^{kmjn}_{[22]}t^{minp}_{[22]} + t^{imjn}_{[22]}t^{nkmp}_{[22]} \equiv 0, \tag{23}$$

$$t^{qimpjn}_{[222]}t^{nkrmls}_{[222]} + t^{qkmpln}_{[222]}t^{jmrins}_{[222]} + t^{jkmiln}_{[222]}t^{qmspnr}_{[222]} \equiv 0. \tag{24}$$

C. The supercommutators of the superderivations

Using the invariant θ and t tensors, the most general permutation invariant and Lorentz covariant supercommutation relations of the superderivations may be expressed,

$$\{\nabla^i_\alpha,\nabla^j_{\dot\alpha}\}=(a_2 t^{ij}_{[2]}+a_{11}\theta^{ij}_{[11]})\nabla_{\alpha\dot\alpha}, \tag{25}$$

$$\{\nabla^i_\alpha,\nabla^j_\beta\}=(b_2 t^{ij}_{[2]}+b_{11}\theta^{ij}_{[11]})M_{\alpha\beta}+\epsilon_{\alpha\beta}b_{21}(t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+\epsilon_{\alpha\beta}b_{22}t^{ikjl}_{[22]}Z^{kl}, \tag{26}$$

$$\{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\}=(\overline{b_2} t^{ij}_{[2]}+\overline{b_{11}}\theta^{ij}_{[11]})M_{\dot\alpha\dot\beta}+\epsilon_{\dot\alpha\dot\beta}\overline{b_{21}}(t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+\epsilon_{\dot\alpha\dot\beta}\overline{b_{22}}t^{ikjl}_{[22]}Z^{kl}, \tag{27}$$

$$[\nabla^i_\alpha,\nabla_{\beta\dot\beta}]=\epsilon_{\alpha\beta}(c_2 t^{ij}_{[2]}+c_{11}\theta^{ij}_{[11]})\nabla^j_{\dot\beta}, \tag{28}$$

$$[\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}]=\epsilon_{\dot\alpha\dot\beta}(\overline{c_2} t^{ij}_{[2]}+\overline{c_{11}}\theta^{ij}_{[11]})\nabla^j_{\beta}, \tag{29}$$

$$[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]=r(\epsilon_{\alpha\beta}M_{\dot\alpha\dot\beta}+\epsilon_{\dot\alpha\dot\beta}M_{\alpha\beta}). \tag{30}$$

Comments:

(a) The equations involving Y^i and $Z^{ij}=-Z^{ji}$ on the right-hand side have been written so as to exhibit manifestly the irreducibility of these operators. In particular, use of the partially trace-free invariant tensors as coefficients automatically yields Y^i satisfying (14) and Z^{ij} satisfying (16), since using these tensors guarantees that the relevant term vanishes when one replaces Z^{kl} by $\theta^k_{[1]}V^l$ and independently Y^k by $\theta^k S$.

(b) For the ∇^i_α and the $\nabla^i_{\dot\alpha}$, we have not separated the permutation-irreducible tensors explicitly. However, the tensors t from (17) have been chosen to correspond to the decomposition into the irreducible pieces.

(c) That the two terms on the right-hand side of (30) always have the same coefficient, can be easily deduced from the Jacobi identity for three $\nabla_{\alpha\dot\alpha}$'s. The parameter r distinguishes the two main classes of supersymmetry algebras we shall consider: The contraction to the $r=0$ case corresponds to the *algebras of super-Poincaré type* and for $r\neq 0$ we obtain *algebras of the super de Sitter type*. We shall not consider algebras of superconformal type, which have a second element transforming as a Lorentz vector, the generator of conformal transformations.

(d) The right-hand sides involve the most general Lorentz covariant terms. This guarantees that Jacobi identities involving one M are automatically satisfied.

(e) The fifteen complex parameters $\{a_2,a_{11}\}$, $\{b_2,b_{11},b_{21},b_{22}\}$, $\{\overline{b_2},\overline{b_{11}},\overline{b_{21}},\overline{b_{22}}\}$, $\{c_2,c_{11}\}$, $\{\overline{c_2},\overline{c_{11}}\}$, and $\{r\}$ are *a priori* independent. They are to be chosen so as to satisfy the super Jacobi identities, which we shall consider in the next section.

D. The action of $\mathfrak{h}$ on the superderivations

The most general commutation relations of the Lorentz scalar operators Y and Z with the superderivations compatible with Lorentz and permutation covariance, e.g.,

$$[Y^i,\nabla^j_\alpha]=(d_3\theta^{ijk}_{[3]}+d^a_{21}\theta^{ijk}_{[21]}+d^b_{21}\theta^{ikj}_{[21]}+d^c_{21}\theta^{kji}_{[21]}+d_{111}\theta^{ijk}_{[111]})\nabla^k_\alpha, \tag{31}$$

on imposition of the trace conditions, yield the following eight-parameter set of relations involving the partially trace-free tensors (17):

$$[Y^i,\nabla^j_\alpha]=(d_3 t^{ijk}_{[3]}+d^a_{21}t^{ijk}_{[21]}+d^b_{21}t^{ikj}_{[21]})\nabla^k_\alpha, \tag{32}$$

$$[Y^i,\nabla^j_{\dot\alpha}]=(\overline{d}_3 t^{ijk}_{[3]}+\overline{d}^a_{21}t^{ijk}_{[21]}+\overline{d}^b_{21}t^{ikj}_{[21]})\nabla^k_{\dot\alpha}, \tag{33}$$

$$[Y^i,\nabla_{\alpha\dot\alpha}]=0, \tag{34}$$

$$[Z^{ij},\nabla^k_\alpha]=f_{22}t^{jkil}_{[22]}\nabla^l_\alpha, \tag{35}$$

$$[Z^{ij},\nabla^{k}_{\dot\alpha}]=\overline{f}_{22}t^{jkil}_{[22]}\nabla^{l}_{\dot\alpha}, \tag{36}$$

$$[Z^{ij},\nabla_{\alpha\dot\alpha}]=0. \tag{37}$$

We note that the Coleman–Mandula-type relation, $[\mathfrak{h},\mathcal{D}_0]=0$, is an immediate consequence of the trace conditions. However, the internal symmetry can still act nontrivially on the odd derivations.

E. The commutators in $\mathfrak{h}$

The subalgebra of the Y's and Z's has the Lorentz and permutation covariant form satisfying the trace-conditions,

$$[Y^{i},Y^{j}]=g_{22}t^{imjn}_{[22]}Z^{mn}, \tag{38}$$

$$[Z^{ij},Y^{k}]=h_{22}t^{jkil}_{[22]}Y^{l}+h_{32}t^{jklim}_{[32]}Z^{lm}, \tag{39}$$

$$[Z^{ij},Z^{kl}]=k_{222}t^{jkmiln}_{[222]}Z^{mn}. \tag{40}$$

In fact the Jacobi identities always imply that $h_{32}=0$ (see below). This reduces the number of parameters to three, which are constrained by the Jacobi identities.

III. DEMOCRATIC LIE ALGEBRAS $\mathfrak{g}$

The *a priori* Lorentz covariant commutators of our N-extended democratic algebras must satisfy super Jacobi identities which guarantee that the products of the underlying operators are associative. We shall now consider the constraints imposed on the parameters in (25)–(30), (32)–(37), (38)–(40) by the super Jacobi identities. Let us first recall that, by construction, all the Jacobi identities involving at least one M are automatically satisfied. We begin with the subalgebra $\mathfrak{h}$.

A. Democratic Lie algebras $\mathfrak{h}$

To find all possible S_N democratic algebras containing the $N(N-1)/2$ generators Y and Z, the Jacobi identities for (38)–(40) need to be satisfied. These yield the following four conditions on the four parameters g_{22}, h_{22}, h_{32}, k_{222}:

$$\begin{gathered} h_{32}h_{22}=h_{32}k_{222}=0,\\ h_{22}(h_{22}-2k_{222})=0,\\ Ng_{22}(h_{22}-2k_{222})-2h_{32}^{2}=0. \end{gathered} \tag{41}$$

They generally imply that $h_{32}=0$, leaving the conditions

$$h_{22}(h_{22}-2k_{222})=0, \quad g_{22}(h_{22}-2k_{222})=0. \tag{42}$$

These equations lead to a classification in five distinct categories:
(**1a**) Abelian $\mathfrak{h}$: all the scalar operators commute

$$g_{22}=h_{22}=k_{222}=0 \tag{43}$$

and the Y and Z can still be renormalized freely.
(**1b**) The Z's commute, they commute with the Y's but the commutators of the Y's generate the Z's. By renormalization of the Z's or the Y's, we find

$$g_{22}=1, \quad h_{22}=k_{222}=0. \tag{44}$$

(**2**) The Z's form an so$(N-1)$ algebra, with $N-1$ commuting Y's which moreover are so$(N-1)$ scalars, i.e., do not transform under the Z. Using the normalization freedom, we may write

$$g_{22}=h_{22}=0, \quad k_{222}=1. \tag{45}$$

(**3a**) The inhomogeneous so$(N-1)$ case. By normalization of the Z's, the parameters can be brought to

$$g_{22}=0, \quad h_{22}=2, \quad k_{222}=1. \tag{46}$$

The Y's behave as a vector under so$(N-1)$ and commute. They behave as momenta with respect to so$(N-1)$ and hence this corresponds to an inhomogeneous so$(N-1)$ algebra. The normalizations of the Y's can still be adjusted freely.
(**3b**) The so(N) case. We clearly have as many Y and Z operators as there are generators of so(N), which is indeed a particular democratic Lie algebra $\mathfrak{h}$. In this case, by suitable renormalizations of the Y's and the Z's, the parameters can be brought to their so(N) values, which we normalize as

$$g_{22}=1, \quad h_{22}=2, \quad k_{222}=1. \tag{47}$$

That these values correspond to so(N) can be seen as follows. The commutation relations of the $N(N-1)/2$ generators $M^{ij}=M^{ji}$ of so(N) are usually written as,

$$[M^{ij},M^{kl}]=\theta^{jk}_{[2]}M^{il}-\theta^{ik}_{[2]}M^{jl}-\theta^{jl}_{[2]}M^{ik}+\theta^{il}_{[2]}M^{jk}.$$

Defining projections

$$V^{j}=\theta^{k}_{[1]}M^{kj}, \qquad T^{jk}=M^{jk}-\frac{1}{N}(\theta^{j}_{[1]}V^{k}\theta^{k}_{[1]}V^{j}), \tag{48}$$

we obtain that the subset of the T operators alone form a democratic so$(N-1)$ subalgebra [with $(N-1)(N-2)/2$ independent operators] of the so(N) algebra. The $N-1$ independent V operators transform as a vector under the so$(N-1)$ subalgebra. The V and the T satisfy precisely the commutation relations (38)–(40) satisfied by Y and Z, respectively, with

$$g_{22}=-\frac{N}{2}, \quad h_{22}=1, \quad k_{222}=\frac{1}{2}. \tag{49}$$

Since there are possible arbitrary democratic rescalings of V with respect to Y and of T with respect to Z, the algebra of the Y's and the Z's corresponds to an so(N) algebra provided (47) holds.

B. Supersymmetry algebras $\mathfrak{g}$

The full discussion for the rest of the super Jacobi identities is rather intricate. We discuss the full set of solutions in the Appendix, discussing the main features here.

We have chosen to discuss the general solution of the Jacobi identities in terms of two criteria:

(1) The first criterion is related to the appearance of the term $\theta^{ij}_{[2]}\nabla_{\alpha\dot{\alpha}}$ in the anticommutators of ∇^{i}_{α} with $\nabla^{j}_{\dot{\alpha}}$ (parameter a_2) and of the Y's in the anticommutator of two ∇_{a}'s (parameter b_{21}) or of two $\nabla_{\dot{\alpha}}$'s (parameter $\overline{b}_{21}$).
(2) The second criterion reveals the structure of the algebra $\mathfrak{h}$ of the Lorentz scalar elements as discussed in the preceding section.

We use the values of the parameters a_2, b_{21}, and $\overline{b}_{21}$ as the basis of our classification. It follows from (25), (26) and (27) that, if any of these three parameters is nonzero, it may be renormalized to one by rescaling the three superderivations democratically. Hence, using also the fact that we have a natural symmetry under the interchange of the dotted and undotted operators, we are led to six independent classes of superalgebras:

$$\begin{aligned}
&\text{Class A:}\quad a_2=1,\ b_{21}=1,\ \overline{b_{21}}=1,\\
&\text{Class B:}\quad a_2=1,\ b_{21}=1,\ \overline{b_{21}}=0,\\
&\text{Class C:}\quad a_2=0,\ b_{21}=1,\ \overline{b_{21}}=1,\\
&\text{Class D:}\quad a_2=1,\ b_{21}=0,\ \overline{b_{21}}=0,\\
&\text{Class E:}\quad a_2=0,\ b_{21}=1,\ \overline{b_{21}}=0,\\
&\text{Class F:}\quad a_2=0,\ b_{21}=0,\ \overline{b_{21}}=0,
\end{aligned} \tag{50}$$

which we discuss in detail in the Appendix. Classes B and E are *chiral*, not having the mirror symmetry under the chirality interchanges between dotted and undotted indices ($\alpha\leftrightarrow\dot{\alpha},\ldots$) and between the parameters $c\leftrightarrow\overline{c},\ldots$ (for existing unbarred-barred pairs). The two further classes,

$$\begin{aligned}
&\text{Class B}':\quad a_2=1,\ b_{21}=0,\ \overline{b_{21}}=1,\\
&\text{Class E}':\quad a_2=0,\ b_{21}=0,\ \overline{b_{21}}=1,
\end{aligned} \tag{51}$$

can clearly be obtained trivially from the B and E classes by performing the above chirality exchanges; and we do not explicitly discuss these.

Within the above classes, the discussion is subdivided according to the values of k_{222} and h_{22}, corresponding to the division in Sec. III A,

$$\begin{aligned}
&\text{Case 1:}\quad k_{222}=0,\ h_{22}=0,\\
&\text{Case 2:}\quad k_{222}=1,\ h_{22}=0,\\
&\text{Case 3:}\quad k_{222}=1,\ h_{22}=2.
\end{aligned} \tag{52}$$

C. Some solutions of the super Jacobi identities

In this section, we discuss the main noteworthy features revealed by our approach. Let us consider Case A3 from the Appendix:

$$\{\nabla^i_\alpha,\nabla^j_{\dot{\alpha}}\}=(t^{ij}_{[2]}+a_{11}\theta^{ij}_{[11]})\nabla_{\alpha\dot{\alpha}}, \tag{53}$$

$$\{\nabla^i_\alpha,\nabla^j_\beta\}=4(b_{22}t^{ij}_{[2]}+a_{11}\overline{b_{22}}\theta^{ij}_{[11]})M_{\alpha\beta}+\epsilon_{\alpha\beta}((t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+b_{22}t^{ikjl}_{[22]}Z^{kl}), \tag{54}$$

$$\{\nabla^i_{\dot{\alpha}},\nabla^j_{\dot{\beta}}\}=4(\overline{b_{22}}t^{ij}_{[2]}+a_{11}b_{22}\theta^{ij}_{[11]})M_{\dot{\alpha}\dot{\beta}}+\epsilon_{\dot{\alpha}\dot{\beta}}((t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+\overline{b_{22}}t^{ikjl}_{[22]}Z^{kl}), \tag{55}$$

$$[\nabla^i_\alpha,\nabla_{\beta\dot{\beta}}]=4\epsilon_{\alpha\beta}\left(b_{22}t^{ij}_{[2]}+\frac{\overline{b_{22}}}{N}\theta^{ij}_{[11]}\right)\nabla^j_{\dot{\beta}}, \tag{56}$$

$$[\nabla^i_{\dot{\alpha}},\nabla_{\beta\dot{\beta}}]=4\epsilon_{\dot{\alpha}\dot{\beta}}\left(\overline{b_{22}}t^{ij}_{[2]}+\frac{b_{22}}{N}\theta^{ij}_{[11]}\right)\nabla^j_\beta, \tag{57}$$

$$[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]=16b_{22}\overline{b_{22}}(\epsilon_{\alpha\beta}M_{\dot{\alpha}\dot{\beta}}+\epsilon_{\dot{\alpha}\dot{\beta}}M_{\alpha\beta}), \tag{58}$$

$$[Y^i,\nabla_{\alpha\dot{\alpha}}]=0, \tag{59}$$

$$[Z^{ij},\nabla_{\alpha\dot{\alpha}}]=0, \tag{60}$$

$$[Y^i,\nabla^j_\alpha]=4\left(\overline{b}_{22}a_{11}t^{ikj}_{[21]}-\frac{b_{22}}{N}t^{ijk}_{[21]}\right)\nabla^k_\alpha, \tag{61}$$

$$[Y^i,\nabla^j_{\dot{\alpha}}]=4\left(b_{22}a_{11}t^{ikj}_{[21]}-\frac{\overline{b}_{22}}{N}t^{ijk}_{[21]}\right)\nabla^k_{\dot{\alpha}}, \tag{62}$$

$$[Z^{ij},\nabla^k_\alpha]=2t^{jkil}_{[22]}\nabla^l_\alpha, \tag{63}$$

$$[Z^{ij},\nabla^k_{\dot{\alpha}}]=2t^{jkil}_{[22]}\nabla^l_{\dot{\alpha}}, \tag{64}$$

$$[Y^i,Y^j]=-4a_{11}b_{22}\overline{b}_{22}t^{imjn}_{[22]}Z^{mn}, \tag{65}$$

$$[Z^{ij},Y^k]=2t^{jkil}_{[22]}Y^l, \tag{66}$$

$$[Z^{ij},Z^{kl}]=t^{jkmiln}_{[222]}Z^{mn}. \tag{67}$$

The main unusual features displayed by this algebra are:

(1) Nontrivial action of the subalgebra $\mathfrak{h}$ on the vector space of superderivations $\mathcal{D}$;
(2) Non-Abelian subalgebra of the Lorentz scalar generators;
(3) Occurrence of the a_{11} term in (53).

The above example is of super de Sitter-type. A chiral super Poincaré-type example, also displaying these interesting features, is given by Case B3:

$$\{\nabla^i_\alpha,\nabla^j_{\dot{\alpha}}\}=(t^{ij}_{[2]}+a_{11}\theta^{ij}_{[11]})\nabla_{\alpha\dot{\alpha}}, \tag{68}$$

$$\{\nabla^i_\alpha,\nabla^j_\beta\}=(4b_{22}t^{ij}_{[2]}+Nc_{11}a_{11}\theta^{ij}_{[11]})M_{\alpha\beta}+\epsilon_{\alpha\beta}((t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+b_{22}t^{ikjl}_{[22]}Z^{kl}), \tag{69}$$

$$\{\nabla^i_{\dot{\alpha}},\nabla^j_{\dot{\beta}}\}=0, \tag{70}$$

$$[\nabla^i_\alpha,\nabla_{\beta\dot{\beta}}]=\epsilon_{\alpha\beta}(4b_{22}t^{ij}_{[2]}+c_{11}\theta^{ij}_{[11]})\nabla^j_{\dot{\beta}}, \tag{71}$$

$$[\nabla^i_{\dot{\alpha}},\nabla_{\beta\dot{\beta}}]=0, \tag{72}$$

$$[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]=0, \tag{73}$$

$$[Y^i,\nabla_{\alpha\dot{\alpha}}]=0, \tag{74}$$

$$[Z^{ij},\nabla_{\alpha\dot{\alpha}}]=0, \tag{75}$$

$$[Y^i,\nabla^j_\alpha]=\left(-\frac{4}{N}b_{22}t^{ijk}_{[21]}+Na_{11}c_{11}t^{ikj}_{[21]}\right)\nabla^k_\alpha, \tag{76}$$

$$[Y^i,\nabla^j_{\dot{\alpha}}]=(-c_{11}t^{ijk}_{[21]}+4a_{11}b_{22}t^{ikj}_{[21]})\nabla^k_{\dot{\alpha}}, \tag{77}$$

$$[Z^{ij},\nabla^k_\alpha]=2t^{jkil}_{[22]}\nabla^l_\alpha, \tag{78}$$

$$[Z^{ij},\nabla^k_{\dot\alpha}]=2t^{jkil}_{[22]}\nabla^l_{\dot\alpha}, \tag{79}$$

$$[Y^i,Y^j]=-Na_{11}c_{11}b_{22}t^{imjn}_{[22]}Z^{mn}, \tag{80}$$

$$[Z^{ij},Y^k]=2t^{jkil}_{[22]}Y^l, \tag{81}$$

$$[Z^{ij},Z^{kl}]=t^{jkmiln}_{[222]}Z^{mn}. \tag{82}$$

IV. CONCLUSION

The inclusion of multiplicities in our program,[8] extending in a Lorentz covariant way the algebra of coordinates and derivatives, has been shown to exhibit interesting new features and a rather rich structure of solutions for the super Jacobi identities. In order to obtain explicit solutions, we have chosen to restrict ourselves in this article to a set of operators of spin less than or equal to one and to impose *democracy*. Within these restricted hypotheses, we have been able to classify fully the allowed superalgebras of derivations and superderivations. Apart from the well-known examples,[1,2] new and potentially interesting cases have been uncovered.

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APPENDIX

With classes (A–F) defined in (50) and subcases (1–3) defined by (52) the full classification of the democratic supersymmetry algebras is given below.

1. Class A

Imposing the super Jacobi identities together with the class A constraints, $a_2=1$, $b_{21}=1$, $\overline{b}_{21}=1$, yields the relations,

$$\{\nabla^i_\alpha,\nabla^j_{\dot\alpha}\}=(t^{ij}_{[2]}+a_{11}\theta^{ij}_{[11]})\nabla_{\alpha\dot\alpha}, \tag{A1}$$

$$\{\nabla^i_\alpha,\nabla^j_\beta\}=4k_{222}(b_{22}t^{ij}_{[2]}+a_{11}\overline{b}_{22}\theta^{ij}_{[11]})M_{\alpha\beta}+\epsilon_{\alpha\beta}((t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+b_{22}t^{ikjl}_{[22]}Z^{kl}), \tag{A2}$$

$$\{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\}=4k_{222}(\overline{b}_{22}t^{ij}_{[2]}+a_{11}b_{22}\theta^{ij}_{[11]})M_{\dot\alpha\dot\beta}+\epsilon_{\dot\alpha\dot\beta}((t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+\overline{b}_{22}t^{ikjl}_{[22]}Z^{kl}), \tag{A3}$$

$$[\nabla^i_\alpha,\nabla_{\beta\dot\beta}]=4\epsilon_{\alpha\beta}k_{222}\left(b_{22}t^{ij}_{[2]}+\frac{\overline{b}_{22}}{N}\theta^{ij}_{[11]}\right)\nabla^j_{\dot\beta}, \tag{A4}$$

$$[\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}]=4\epsilon_{\dot\alpha\dot\beta}k_{222}\left(\overline{b}_{22}t^{ij}_{[2]}+\frac{b_{22}}{N}\theta^{ij}_{[11]}\right)\nabla^j_{\beta}, \tag{A5}$$

$$[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]=16b_{22}\overline{b}_{22}k^2_{222}(\epsilon_{\alpha\beta}M_{\dot\alpha\dot\beta}+\epsilon_{\dot\alpha\dot\beta}M_{\alpha\beta}), \tag{A6}$$

$$[Y^i,\nabla_{\alpha\dot\alpha}]=0, \tag{A7}$$

$$[Z^{ij},\nabla_{\alpha\dot\alpha}]=0, \tag{A8}$$

$$[Y^i,\nabla^j_\alpha]=4k_{222}\left(\overline{b_{22}}a_{11}t^{ikj}_{[21]}-\frac{b_{22}}{N}t^{ijk}_{[21]}\right)\nabla^k_\alpha, \tag{A9}$$

$$[Y^i,\nabla^j_{\dot\alpha}]=4k_{222}\left(b_{22}a_{11}t^{ikj}_{[21]}-\frac{\overline{b_{22}}}{N}t^{ijk}_{[21]}\right)\nabla^k_{\dot\alpha}, \tag{A10}$$

$$[Z^{ij},\nabla^k_\alpha]=h_{22}t^{jkil}_{[22]}\nabla^l_\alpha, \tag{A11}$$

$$[Z^{ij},\nabla^k_{\dot\alpha}]=h_{22}t^{jkil}_{[22]}\nabla^l_{\dot\alpha}, \tag{A12}$$

$$[Y^i,Y^j]=-4a_{11}b_{22}\overline{b_{22}}k_{222}t^{imjn}_{[22]}Z^{mn}, \tag{A13}$$

$$[Z^{ij},Y^k]=h_{22}t^{jkil}_{[22]}Y^l, \tag{A14}$$

$$[Z^{ij},Z^{kl}]=k_{222}t^{jkmiln}_{[222]}Z^{mn}, \tag{A15}$$

with the space of class A superalgebras defined by solutions of the system of quadratic equations,

$$\begin{aligned} &b_{22}(2k_{222}-h_{22})=0,\\ &\overline{b_{22}}(2k_{222}-h_{22})=0,\\ &h_{22}(2k_{222}-h_{22})=0. \end{aligned} \tag{A16}$$

We find three subcases [see (52)]

Case A1: Since $h_{22}=k_{222}=0$, the parameters a_{11}, b_{22}, and $\overline{b_{22}}$ are free. This includes the standard super Poincaré algebra with Abelian algebra $\mathfrak{h}$ of central charges.

Case A2: Here a_{11} is free, $k_{222}=1$ and all other parameters are zero. There is an $\mathrm{so}(N-1)$ subalgebra [see (45)] of the Z's which decouples.

Case A3: This is a much less trivial case (see Sec. III C) and the full $\mathrm{so}(N)$ algebra (49) is included in the algebra. The independent parameters are a_{11}, b_{22}, $\overline{b_{22}}$ while $h_{22}=2$, $k_{222}=1$.

2. Class B

This class is *chiral* of super Poincaré-type: $a_2=1$, $b_{21}=1$, $\overline{b_{21}}=0$. It has relations

$$\{\nabla^i_\alpha,\nabla^j_{\dot\alpha}\}=(t^{ij}_{[2]}+a_{11}\theta^{ij}_{[11]})\nabla_{\alpha\dot\alpha}, \tag{B1}$$

$$\{\nabla^i_\alpha,\nabla^j_\beta\}=(4b_{22}k_{222}t^{ij}_{[2]}+Nc_{11}a_{11}\theta^{ij}_{[11]})M_{\alpha\beta}+\epsilon_{\alpha\beta}((t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+b_{22}t^{ikjl}_{[22]}Z^{kl}), \tag{B2}$$

$$\{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\}=\epsilon_{\dot\alpha\dot\beta}\overline{b_{22}}t^{ikjl}_{[22]}Z^{kl}, \tag{B3}$$

$$[\nabla^i_\alpha,\nabla_{\beta\dot\beta}]=\epsilon_{\alpha\beta}(4b_{22}k_{222}t^{ij}_{[2]}+c_{11}\theta^{ij}_{[11]})\nabla^j_{\dot\beta}, \tag{B4}$$

$$[\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}]=0, \tag{B5}$$

$$[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]=0, \tag{B6}$$

$$[Y^i,\nabla_{\alpha\dot\alpha}]=0, \tag{B7}$$

$$[Z^{ij},\nabla_{\alpha\dot\alpha}]=0, \tag{B8}$$

$$[Y^i,\nabla^j_\alpha]=\left(-\frac{4}{N}b_{22}k_{222}t^{ijk}_{[21]}+Na_{11}c_{11}t^{ikj}_{[21]}\right)\nabla^k_\alpha, \tag{B9}$$

$$[Y^i,\nabla^j_{\dot\alpha}]=(-c_{11}t^{ijk}_{[21]}+4a_{11}b_{22}k_{222}t^{ikj}_{[21]})\nabla^k_{\dot\alpha}, \tag{B10}$$

$$[Z^{ij},\nabla^k_\alpha]=h_{22}t^{jkil}_{[22]}\nabla^l_\alpha, \tag{B11}$$

$$[Z^{ij},\nabla^k_{\dot\alpha}]=h_{22}t^{jkil}_{[22]}\nabla^l_{\dot\alpha}, \tag{B12}$$

$$[Y^i,Y^j]=-Na_{11}c_{11}b_{22}t^{imjn}_{[22]}Z^{mn}, \tag{B13}$$

$$[Z^{ij},Y^k]=h_{22}t^{jkil}_{[22]}Y^l, \tag{B14}$$

$$[Z^{ij},Z^{kl}]=k_{222}t^{jkmiln}_{[222]}Z^{mn}. \tag{B15}$$

Here the parameters are constrained by the system of equations,

$$\begin{aligned}\overline{b_{22}}h_{22}=\overline{b_{22}}k_{222}=0,\\ b_{22}(2k_{222}-h_{22})=0,\\ h_{22}(2k_{222}-h_{22})=0,\end{aligned} \tag{B16}$$

defining the space of class B superalgebras. They are all of chiral super Poincaré-type. There are three subcases of solutions [see (52)]:

Case B1: The parameters a_{11}, b_{22}, $\overline{b_{22}}$, and c_{11} are free, $h_{22}=k_{222}=0$. The Z's are central, not the Y's.

Case B2: The parameters a_{11}, and c_{11} are free, $k_{222}=1$ and the remaining are zero. The subalgebra $\mathfrak{h}$ contains the $\mathrm{so}(N-1)$ of the Z's which decouples. The subalgebra of the Y's is Abelian.

Case B3: The parameters a_{11}, b_{22}, and c_{11} are free, $h_{22}=2$, $k_{222}=1$, $\overline{b_{22}}=0$ (see Sec. III C).

3. Class C

This class contains super algebras of the de Sitter-type. They allow contractions to super Poincaré-type algebras by setting c_2 and/or $\overline{c_2}$ to zero. The relations $a_2=0$, $b_{21}=1$, $\overline{b_{21}}=1$ yield the superbrackets,

$$\{\nabla^i_\alpha,\nabla^j_{\dot\alpha}\}=a_{11}\theta^{ij}_{[11]}\nabla_{\alpha\dot\alpha}, \tag{C1}$$

$$\{\nabla^i_\alpha,\nabla^j_\beta\}=a_{11}\overline{c_2}\,\theta^{ij}_{[11]}M_{\alpha\beta}+\epsilon_{\alpha\beta}((t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+b_{22}t^{ikjl}_{[22]}Z^{kl}), \tag{C2}$$

$$\{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\}=a_{11}c_2\,\theta^{ij}_{[11]}M_{\dot\alpha\dot\beta}+\epsilon_{\dot\alpha\dot\beta}((t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+\overline{b_{22}}t^{ikjl}_{[22]}Z^{kl}), \tag{C3}$$

$$[\nabla^i_\alpha,\nabla_{\beta\dot\beta}]=\epsilon_{\alpha\beta}\left(c_2t^{ij}_{[2]}+\frac{\overline{c_2}}{N}\theta^{ij}_{[11]}\right)\nabla^j_{\dot\beta}, \tag{C4}$$

$$[\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}]=\epsilon_{\dot\alpha\dot\beta}\left(\overline{c_2}t^{ij}_{[2]}+\frac{c_2}{N}\theta^{ij}_{[11]}\right)\nabla^j_\beta, \tag{C5}$$

$$[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]=c_2\overline{c_2}(\epsilon_{\alpha\beta}M_{\dot\alpha\dot\beta}+\epsilon_{\dot\alpha\dot\beta}M_{\alpha\beta}), \tag{C6}$$

$$[Y^i,\nabla_{\alpha\dot\alpha}]=0, \tag{C7}$$

$$[Z^{ij},\nabla_{\alpha\dot{\alpha}}]=0, \tag{C8}$$

$$[Y^i,\nabla^j_\alpha]=a_{11}\overline{c_2}t^{ikj}_{[21]}\nabla^k_\alpha, \tag{C9}$$

$$[Y^i,\nabla^j_{\dot{\alpha}}]=a_{11}c_2t^{ikj}_{[21]}\nabla^k_{\dot{\alpha}}, \tag{C10}$$

$$[Z^{ij},\nabla^k_\alpha]=h_{22}t^{jkil}_{[22]}\nabla^l_\alpha, \tag{C11}$$

$$[Z^{ij},\nabla^k_{\dot{\alpha}}]=h_{22}t^{jkil}_{[22]}\nabla^l_{\dot{\alpha}}, \tag{C12}$$

$$[Y^i,Y^j]=-a_{11}b_{22}\overline{c_2}t^{imjn}_{[22]}Z^{mn}, \tag{C13}$$

$$[Z^{ij},Y^k]=h_{22}t^{jkil}_{[22]}Y^l, \tag{C14}$$

$$[Z^{ij},Z^{kl}]=k_{222}t^{jkmiln}_{[222]}Z^{mn}. \tag{C15}$$

Here the parameters are constrained by the system of equations

$$\begin{gathered}\overline{b_{22}}c_2-b_{22}\overline{c_2}=0,\\ b_{22}h_{22}=b_{22}k_{222}=0,\\ \overline{b_{22}}h_{22}=\overline{b_{22}}k_{222}=0,\\ a_{11}\overline{c_2}b_{22}(2k_{222}-h_{22})=0,\\ h_{22}(2k_{222}-h_{22})=0.\end{gathered} \tag{C16}$$

We find three subcases [see (52)]:

Case C1: We have $h_{22}=k_{222}=0$ while a_{11} is free and b_{22}, $\overline{b_{22}}$, c_2, $\overline{c_2}$, are constrained by the condition

$$\overline{b_{22}}c_2=b_{22}\overline{c_2}. \tag{C17}$$

The Z's are central charges.

Case C2: The parameters a_{11}, c_2, $\overline{c_2}$ are free, $k_{222}=1$ and the remaining are zero. The subalgebra $\mathrm{so}(N-1)\subset\mathfrak{h}$ of the Z's decouples.

Case C3: The parameters a_{11}, c_2, $\overline{c_2}$ are free, $h_{22}=2$, $k_{222}=1$, and $b_{22}=\overline{b_{22}}=0$.

4. Class D

This has $a_2=1$, $b_{21}=0$, $\overline{b_{21}}=0$, yielding

$$\{\nabla^i_\alpha,\nabla^j_{\dot{\alpha}}\}=(t^{ij}_{[2]}+a_{11}\theta^{ij}_{[11]})\nabla_{\alpha\dot{\alpha}}, \tag{D1}$$

$$\{\nabla^i_\alpha,\nabla^j_\beta\}=\epsilon_{\alpha\beta}b_{22}t^{ikjl}_{[22]}Z^{kl}, \tag{D2}$$

$$\{\nabla^i_{\dot{\alpha}},\nabla^j_{\dot{\beta}}\}=\epsilon_{\dot{\alpha}\dot{\beta}}\overline{b_{22}}t^{ikjl}_{[22]}Z^{kl}, \tag{D3}$$

$$[\nabla^i_\alpha,\nabla_{\beta\dot{\beta}}]=0, \tag{D4}$$

$$[\nabla^i_{\dot{\alpha}},\nabla_{\beta\dot{\beta}}]=0, \tag{D5}$$

$$[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]=0, \tag{D6}$$

$$[Y^i,\nabla_{\alpha\dot{\alpha}}]=0, \tag{D7}$$

$$[Z^{ij},\nabla_{\alpha\dot{\alpha}}]=0, \tag{D8}$$

$$[Y^i,\nabla^j_\alpha]=(d_3t^{ijk}_{[3]}+d^a_{21}t^{ijk}_{[21]}-N\overline{d^a_{21}}a_{11}t^{ikj}_{[21]})\nabla^k_\alpha, \tag{D9}$$

$$[Y^i,\nabla^j_{\dot{\alpha}}]=(-d_3t^{ijk}_{[3]}+\overline{d^a_{21}}t^{ijk}_{[21]}-Na_{11}d^a_{21}t^{ikj}_{[21]})\nabla^k_{\dot{\alpha}}, \tag{D10}$$

$$[Z^{ij},\nabla^k_\alpha]=f_{22}t^{jkil}_{[22]}\nabla^l_\alpha, \tag{D11}$$

$$[Z^{ij},\nabla^k_{\dot{\alpha}}]=f_{22}t^{jkil}_{[22]}\nabla^l_{\dot{\alpha}}, \tag{D12}$$

$$[Y^i,Y^j]=g_{22}t^{imjn}_{[22]}Z^{mn}, \tag{D13}$$

$$[Z^{ij},Y^k]=h_{22}t^{jkil}_{[22]}Y^l, \tag{D14}$$

$$[Z^{ij},Z^{kl}]=k_{222}t^{jkmiln}_{[222]}Z^{mn}. \tag{D15}$$

The remaining parameters must satisfy the 18 equations,

$$d_3b_{22}=0,\ \ d_3\overline{b_{22}}=0,\ \ d_3f_{22}=0,\ \ d_3h_{22}=0, \tag{D16}$$

$$b_{22}f_{22}=0,\ \ b_{22}h_{22}=0,\ \ b_{22}k_{222}=0,\ \ b_{22}a_{11}\overline{d^a_{21}}=0, \tag{D17}$$

$$\overline{b_{22}}f_{22}=0,\ \ \overline{b_{22}}h_{22}=0,\ \ \overline{b_{22}}k_{222}=0,\ \ \overline{b_{22}}a_{11}d^a_{21}=0, \tag{D18}$$

$$h_{22}(h_{22}-2k_{222})=0,\ \ g_{22}(h_{22}-2k_{222})=0,\ \ f_{22}(f_{22}-2k_{222})=0, \tag{D19}$$

$$d^a_{21}(f_{22}-h_{22})=0,\ \ \overline{d^a_{21}}(f_{22}-h_{22})=0,\ \ d_3^2+2Nf_{22}g_{22}+N^3a_{11}d^a_{21}\overline{d^a_{21}}=0. \tag{D20}$$

We find seven essentially different subcases [see (52)]:

Case D1a: The parameter g_{22} is free while $h_{22}=k_{222}=b_{22}=\overline{b_{22}}=f_{22}=0$ and a_{11}, d_3, d^a_{21}, $\overline{d^a_{21}}$, satisfy the condition

$$a_{11}d^a_{21}\overline{d^a_{21}}+\frac{d_3^2}{N^3}=0. \tag{D21}$$

Case D1b: The parameters $b_{22}\neq 0$, d^a_{21} and g_{22} are free, a_{11} and $\overline{d^a_{21}}$ are constrained by

$$\overline{d^a_{21}}a_{11}=0 \tag{D22}$$

and the remaining parameters are zero.

Case D1c: The parameters $b_{22}\neq 0$, $\overline{b_{22}}\neq 0$, g_{22} are free, a_{11}, d^a_{21}, $\overline{d^a_{21}}$ satisfy the conditions

$$d^a_{21}a_{11}=0,\quad \overline{d^a_{21}}a_{11}=0, \tag{D23}$$

and the remaining parameters are zero.

Case D2a: Here $k_{222}=1$, the parameters a_{11}, d_3, d^a_{21}, $\overline{d^a_{21}}$ satisfy the condition (D21) and the remaining parameters are zero.

Case D2b: All the parameters are zero except $k_{222}=1$, $f_{22}=2$ and a_{11} which is free.

Case D3a: All the other parameters are zero except $k_{222}=1$, $h_{22}=2$ and a_{11}, g_{22} which are free.

Case D3b: All the parameters are zero except $k_{222}=1$, $h_{22}=2$, $f_{22}=2$ and g_{22}, a_{11}, d^a_{21}, $\overline{d^a_{21}}$ satisfy

$$4g_{22}+N^2a_{11}d^a_{21}\overline{d^a_{21}}=0.$$

5. Class E

Imposing $a_2=0$, $b_{21}=1$, $\overline{b_{21}}=0$, we obtain the chiral superalgebra,

$$\{\nabla^i_\alpha,\nabla^j_{\dot\alpha}\}=a_{11}\theta^{ij}_{[11]}\nabla_{\alpha\dot\alpha}, \tag{E1}$$

$$\{\nabla^i_\alpha,\nabla^j_\beta\}=Na_{11}c_{11}\theta^{ij}_{[11]}M_{\alpha\beta}+\epsilon_{\alpha\beta}((t^{ikj}_{[21]}-t^{jki}_{[21]})Y^k+b_{22}t^{ikjl}_{[22]}Z^{kl}), \tag{E2}$$

$$\{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\}=\epsilon_{\dot\alpha\dot\beta}\overline{b_{22}}t^{ikjl}_{[22]}Z^{kl}, \tag{E3}$$

$$[\nabla^i_\alpha,\nabla_{\beta\dot\beta}]=\epsilon_{\alpha\beta}(c_2t^{ij}_{[2]}+c_{11}\theta^{ij}_{[11]})\nabla^j_{\dot\beta}, \tag{E4}$$

$$[\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}]=0, \tag{E5}$$

$$[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]=0, \tag{E6}$$

$$[Y^i,\nabla_{\alpha\dot\alpha}]=0, \tag{E7}$$

$$[Z^{ij},\nabla_{\alpha\dot\alpha}]=0, \tag{E8}$$

$$[Y^i,\nabla^j_\alpha]=Na_{11}c_{11}t^{ikj}_{[21]}\nabla^k_\alpha, \tag{E9}$$

$$[Y^i,\nabla^j_{\dot\alpha}]=a_{11}c_2t^{ikj}_{[21]}\nabla^k_{\dot\alpha}, \tag{E10}$$

$$[Z^{ij},\nabla^k_\alpha]=h_{22}t^{jkil}_{[22]}\nabla^l_\alpha, \tag{E11}$$

$$[Z^{ij},\nabla^k_{\dot\alpha}]=\overline{f_{22}}t^{jkil}_{[22]}\nabla^l_{\dot\alpha}, \tag{E12}$$

$$[Y^i,Y^j]=-Na_{11}c_{11}b_{22}t^{imjn}_{[22]}Z^{mn}, \tag{E13}$$

$$[Z^{ij},Y^k]=h_{22}t^{jkil}_{[22]}Y^l, \tag{E14}$$

$$[Z^{ij},Z^{kl}]=k_{222}t^{jkmiln}_{[222]}Z^{mn}. \tag{E15}$$

Here a_{11} and c_{11} are free and the remaining parameters satisfy the constraints,

$$b_{22}\overline{f_{22}}=b_{22}h_{22}=b_{22}k_{222}=0, \tag{E16}$$

$$\overline{b_{22}}c_2=\overline{b_{22}}\overline{f_{22}}=\overline{b_{22}}h_{22}=\overline{b_{22}}k_{222}=0, \tag{E17}$$

$$h_{22}(h_{22}-2k_{222})=\overline{f_{22}}(\overline{f_{22}}-2k_{222})=c_2(\overline{f_{22}}-h_{22})=0. \tag{E18}$$

We find six essentially different subcases:

Case E1a: The parameters $h_{22}=k_{222}=\overline{f_{22}}=c_2=0$, and a_{11}, b_{22}, $\overline{b_{22}}$, c_{11} are free.

Case E1b: The parameters $h_{22}=k_{222}=\overline{f_{22}}=\overline{b_{22}}=0$, and a_{11}, b_{22}, c_2, c_{11} are free.

Case E2a: Here $k_{222}=1$, a_{11}, c_{11}, and c_2 are free and the remaining parameters are zero.

Case E2b: Here $k_{222}=1$, $\overline{f}_{22}=2$, a_{11}, c_{11} are free and the remaining parameters are zero.

Case E3a: Here $k_{222}=1$, $h_{22}=2$, a_{11}, c_{11} are free and the remaining parameters are zero.

Case E3b: Here $k_{222}=1$, $h_{22}=2$, $\overline{f}_{22}=2$, a_{11}, c_{11}, c_2 are free and the remaining parameters are zero.

6. Class F

Class F has the following basic relations, $a_2=0$, $b_{21}=0$, $\overline{b}_{21}=0$, which yield the superalgebra

$$\{\nabla^i_\alpha,\nabla^j_{\dot\alpha}\}=a_{11}\theta^{ij}_{[11]}\nabla_{\alpha\dot\alpha}, \tag{F1}$$

$$\{\nabla^i_\alpha,\nabla^j_\beta\}=\epsilon_{\alpha\beta}b_{22}t^{ikjl}_{[22]}Z^{kl}, \tag{F2}$$

$$\{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\}=\epsilon_{\dot\alpha\dot\beta}\overline{b}_{22}t^{ikjl}_{[22]}Z^{kl}, \tag{F3}$$

$$[\nabla^i_\alpha,\nabla_{\beta\dot\beta}]=\epsilon_{\alpha\beta}(c_2t^{ij}_{[2]}+c_{11}\theta^{ij}_{[11]})\nabla^j_{\dot\beta}, \tag{F4}$$

$$[\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}]=\epsilon_{\dot\alpha\dot\beta}(\overline{c}_2t^{ij}_{[2]}+\overline{c}_{11}\theta^{ij}_{[11]})\nabla^j_\beta, \tag{F5}$$

$$[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]=c_2\overline{c}_2(\epsilon_{\alpha\beta}M_{\dot\alpha\dot\beta}+\epsilon_{\dot\alpha\dot\beta}M_{\alpha\beta}), \tag{F6}$$

$$[Y^i,\nabla_{\alpha\dot\alpha}]=0, \tag{F7}$$

$$[Z^{ij},\nabla_{\alpha\dot\alpha}]=0, \tag{F8}$$

$$[Y^i,\nabla^j_\alpha]=(d_3t^{ijk}_{[3]}+d^a_{21}t^{ijk}_{[21]}+d^b_{21}t^{ikj}_{[21]})\nabla^k_\alpha, \tag{F9}$$

$$[Y^i,\nabla^j_{\dot\alpha}]=(\overline{d}_3t^{ijk}_{[3]}+\overline{d}^a_{21}t^{ijk}_{[21]}+\overline{d}^b_{21}t^{ikj}_{[21]})\nabla^k_{\dot\alpha}, \tag{F10}$$

$$[Z^{ij},\nabla^k_\alpha]=f_{22}t^{jkil}_{[22]}\nabla^l_\alpha, \tag{F11}$$

$$[Z^{ij},\nabla^k_{\dot\alpha}]=\overline{f}_{22}t^{jkil}_{[22]}\nabla^l_{\dot\alpha}, \tag{F12}$$

$$[Y^i,Y^j]=g_{22}t^{imjn}_{[22]}Z^{mn}, \tag{F13}$$

$$[Z^{ij},Y^k]=h_{22}t^{jkil}_{[22]}Y^l, \tag{F14}$$

$$[Z^{ij},Z^{kl}]=k_{222}t^{jkmiln}_{[222]}Z^{mn}. \tag{F15}$$

For $k_{222}=h_{22}=0$, i.e., case F1, we will limit ourselves to giving the conditions which have to be fulfilled. In the cases F2, F3 where $k_{222}=1$, we give a more precise discussion. There are many subcases which we have classified as follows:

$$\begin{aligned}&\text{subcase a:}\quad a_{11}\neq 0,\\&\text{subcase b:}\quad a_{11}=0,\ f_{22}=0,\ \overline{f}_{22}=0,\\&\text{subcase c:}\quad a_{11}=0,\ f_{22}=2,\ \overline{f}_{22}=0,\\&\text{subcase d:}\quad a_{11}=0,\ f_{22}=2,\ \overline{f}_{22}=2.\end{aligned} \tag{F16}$$

With this in mind, we find nine essentially different subcases.

Case F1: With $h_{22}=k_{222}=0$, which implies $f_{22}=\overline{f}_{22}=0$, the 20 conditions to be fulfilled are

$$\begin{gathered}
b_{22}d_{21}^{b}=0, \quad b_{22}d_3=0,\\
\overline{b}_{22}\overline{d}_{21}^{b}=0, \ \overline{b}_{22}\overline{d}_3=0,\\
a_{11}c_2=0, \ a_{11}c_{11}=0,\\
a_{11}\overline{c}_2=0, \ a_{11}\overline{c}_{11}=0,\\
a_{11}d_{21}^{a}=0, \ a_{11}\overline{d}_{21}^{a}=0,\\
c_2\overline{c}_2-N^2c_{11}\overline{c}_{11}=0, \ \overline{b}_{22}c_2-b_{22}\overline{c}_2=0,\\
c_2(\overline{d}_3-d_3)=0, \ \overline{c}_2(\overline{d}_3-d_3)=0,\\
c_2\overline{d}_{21}^{a}-Nc_{11}d_{21}^{a}=0, \ \overline{c}_2\overline{d}_{21}^{b}-N\overline{c}_{11}d_{21}^{b}=0,\\
c_2d_{21}^{b}-Nc_{11}\overline{d}_{21}^{b}=0, \ \overline{c}_2d_{21}^{a}-N\overline{c}_{11}\overline{d}_{21}^{a}=0,\\
d_3^2-N^2d_{21}^{a}d_{21}^{b}=0, \ \overline{d}_3^2-N^2\overline{d}_{21}^{a}\overline{d}_{21}^{b}=0.
\end{gathered} \tag{F17}$$

This leads to a rather long, easy but uninteresting discussion which we will not give.

Case F2a: Here $k_{222}=1$, $a_{11}\neq 0$, d_{21}^{b}, $\overline{d}_{21}^{b}$, f_{22}, and $\overline{f}_{22}$ satisfy the following conditions:

$$\begin{gathered}
f_{22}(f_{22}-2)=0, \quad \overline{f}_{22}(\overline{f}_{22}-2)=0,\\
f_{22}d_{21}^{b}=0, \quad \overline{f}_{22}\overline{d}_{21}^{b}=0,
\end{gathered} \tag{F18}$$

and the remaining parameters are zero.

Case F2b: Here $k_{222}=1$, c_2, c_{11}, $\overline{c}_2$, $\overline{c}_{11}$, d_{21}^{a}, d_{21}^{b}, d_3, $\overline{d}_{21}^{a}$, $\overline{d}_{21}^{b}$, $\overline{d}_3$ satisfy the conditions,

$$\begin{gathered}
c_2\overline{c}_2-N^2c_{11}\overline{c}_{11}=0,\\
c_2(\overline{d}_3-d_3)=0, \ \overline{c}_2(\overline{d}_3-d_3)=0,\\
c_2\overline{d}_{21}^{a}-Nc_{11}d_{21}^{a}=0, \ \overline{c}_2\overline{d}_{21}^{b}-N\overline{c}_{11}d_{21}^{b}=0,\\
c_2d_{21}^{b}-Nc_{11}\overline{d}_{21}^{b}=0, \ \overline{c}_2d_{21}^{a}-N\overline{c}_{11}\overline{d}_{21}^{a}=0,\\
d_3^2-N^2d_{21}^{a}d_{21}^{b}=0, \ \overline{d}_3^2-N^2\overline{d}_{21}^{a}\overline{d}_{21}^{b}=0,
\end{gathered} \tag{F19}$$

and the remaining parameters are zero.

Case F2c: All the parameters are zero except $k_{222}=1$, $f_{22}=2$ and c_{11}, $\overline{c}_{11}$, $\overline{d}_3$, $\overline{d}_{21}^{a}$, and $\overline{d}_{21}^{b}$ which satisfy

$$\begin{gathered}
c_{11}\overline{c}_{11}=0, \ \overline{d}_3^2-N^2\overline{d}_{21}^{a}\overline{d}_{21}^{b}=0,\\
c_{11}\overline{d}_{21}^{b}=0, \ \overline{c}_{11}\overline{d}_{21}^{a}=0.
\end{gathered} \tag{F20}$$

Case F2d: All the parameters are zero except $k_{222}=1$, $f_{22}=\overline{f}_{22}=2$ and c_2, c_{11}, $\overline{c}_2$, $\overline{c}_{11}$ which satisfy the condition,

$$c_2\overline{c_2}-N^2c_{11}\overline{c_{11}}=0. \tag{F21}$$

Case F3a: All the parameters are zero except $k_{222}=1$, $h_{22}=2$, $a_{11}\neq 0$, and d_{21}^b, $\overline{d_{21}^b}$, f_{22}, $\overline{f_{22}}$, and g_{22} which satisfy the conditions,

$$\begin{gathered} f_{22}(f_{22}-2)=0,\ \ \overline{f_{22}}(\overline{f_{22}}-2)=0,\\ d_{21}^b(f_{22}-2)=0,\ \ \overline{d_{21}^b}(\overline{f_{22}}-2)=0,\\ f_{22}g_{22}=0,\ \ \overline{f_{22}}g_{22}=0. \end{gathered} \tag{F22}$$

Case F3b: All the parameters are zero except for $k_{222}=1$, $h_{22}=2$, g_{22}, and c_2, c_{11}, $\overline{c_2}$, and $\overline{c_{11}}$ which satisfy the condition,

$$c_2\overline{c_2}-N^2c_{11}\overline{c_{11}}=0. \tag{F23}$$

Case F3c: All the parameters are zero except $k_{222}=1$, $h_{22}=2$, $f_{22}=2$, and c_{11}, $\overline{c_{11}}$, d_{21}^a, and d_{21}^b which satisfy the conditions,

$$c_{11}\overline{c_{11}}=0,\quad c_{11}d_{21}^a=0,\quad \overline{c_{11}}d_{21}^b=0, \tag{F24}$$

and the dependent parameter,

$$g_{22}=\frac{N}{4}d_{21}^a d_{21}^b. \tag{F25}$$

Case F3d: All the parameters are zero except for $k_{222}=1$, $h_{22}=2$, $f_{22}=\overline{f_{22}}=2$, and c_{11}, c_2, $\overline{c_{11}}$, $\overline{c_2}$, d_{21}^a, d_{21}^b $\overline{d_{21}^a}$, and $\overline{d_{21}^b}$ which satisfy the conditions,

$$\begin{gathered} c_2\overline{d_{21}^a}-Nc_{11}d_{21}^a=0,\ \ \overline{c_2}\overline{d_{21}^b}-N\overline{c_{11}}d_{21}^b=0,\\ c_2d_{21}^b-Nc_{11}\overline{d_{21}^b}=0,\ \ \overline{c_2}d_{21}^a-N\overline{c_{11}}\overline{d_{21}^a}=0,\\ c_2\overline{c_2}-N^2c_{11}\overline{c_{11}}=0,\ \ d_{21}^a d_{21}^b-\overline{d_{21}^a}\overline{d_{21}^b}=0, \end{gathered} \tag{F26}$$

and the dependent parameter,

$$g_{22}=\frac{N}{4}d_{21}^a d_{21}^b. \tag{F27}$$

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