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Ternutator identities

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Abstract

The ternary commutator or ternutator, defined as the alternating sum of the product of three operators, has recently drawn much attention as an interesting structure generalizing the commutator. The ternutator satisfies cubic identities analogous to the quadratic Jacobi identity for the commutator. We present various forms of these identities and discuss the possibility of using them to define ternary algebras.

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1. Introduction

Some time ago Nambu [1] introduced antisymmetric brackets depending on three functions and a derivation, generalizing the Poisson bracket and satisfying the Nambu identity, which generalizes the Jacobi identity. These brackets have been discussed, for example, in [2–4]. Nambu also considered the possible quantization of his brackets: for three arbitrary operators A_1, A_2, A_3 (with an associative product) in an arbitrary vector space, he defined the related ternary commutator $[A_1, A_2, A_3]$, which we call the *ternutator*, by the alternating sum over the permutations of $(1, 2, 3)$,

$$\begin{aligned}[A_1, A_2, A_3] &:= A_1A_2A_3 + A_2A_3A_1 + A_3A_1A_2 - A_1A_3A_2 - A_2A_1A_3 - A_3A_2A_1 \\ &=: \sum_{\pi\in S_3} \mathrm{sgn}(\pi) A_{\pi(1)}A_{\pi(2)}A_{\pi(3)}. \qquad (1)\end{aligned}$$

This can be expressed in terms of the commutator as

$$[A_1, A_2, A_3] = A_1[A_2, A_3] + A_2[A_3, A_1] + A_3[A_1, A_2]$$
$$= [A_2, A_3]A_1 + [A_3, A_1]A_2 + [A_1, A_2]A_3. \tag{2}$$

The ternutator is trilinear in the operators and has the property of being totally skew-symmetric,

$$[A_1, A_2, A_3] = -[A_2, A_1, A_3] = -[A_3, A_2, A_1], \tag{3}$$

which leads to the cyclic property, $[A_1, A_2, A_3] = [A_2, A_3, A_1] = [A_3, A_1, A_2]$. This bracket, as a particular case of those introduced by Fillipov in [5], has recently attracted renewed interest in the physics literature (e.g. [6–11]). Earlier physical applications of ternary algebraic structures are reviewed in [12].

The ternutator generalizes the commutator, which is the binary operation underlying Lie algebras. The Jacobi identity plays a crucial role in the classification of Lie algebras. In this paper, we present identities for the ternutator, analogous to the Jacobi identity for the commutator. This problem has also been discussed by others, e.g. [13, 14]. All n-linear brackets defined by the alternating sum of the product of any odd number $n \geqslant 5$ of operators have recently been shown [15] to satisfy similar identities. Just as the Jacobi identity provides the defining relations for Lie algebras, we discuss the possibility of using the ternutator identities to define ternary algebras. Some specific examples of sets of structure constants satisfying the ternutator identity are given in section 4.

2. Ternutators and their identities

The Jacobi identity for commutators, i.e. the equality of the two ways of writing the ternutator in terms of the commutator in (2), can be expressed as an alternating sum over permutations of (1, 2, 3),

$$\sum_{\iota\in S_3} \epsilon_{\iota_1\iota_2\iota_3}[[A_{\iota_1}, A_{\iota_2}], A_{\iota_3}] = 2\sum_{\substack{\iota\in S_3\\ \iota_1<\iota_2}} \epsilon_{\iota_1\iota_2\iota_3}[[A_{\iota_1}, A_{\iota_2}], A_{\iota_3}] = 0, \tag{4}$$

where the second sum is over all permutations $(\iota_1, \iota_2, \iota_3)$ of (1, 2, 3) satisfying the condition $\iota_1 < \iota_2$ and $\epsilon_{\iota_1\iota_2\iota_3}$ is the completely antisymmetric Levi Civita symbol.

The following result for the ternutator (1) is well known (e.g. [13]).

Lemma 1. *There are no identities of second order in the ternutator.*

Proof. The most general second-order identity is

$$\sum_{\substack{\iota\in S_5\\ \iota_1<\iota_2<\iota_3,\iota_4<\iota_5}} c(\iota_1, \iota_2, \iota_3)[[A_{\iota_1}, A_{\iota_2}, A_{\iota_3}], A_{\iota_4}, A_{\iota_5}] = 0, \tag{5}$$

where the sum is over all permutations $(\iota_1, \ldots, \iota_5)$ of $(1, \ldots, 5)$ satisfying the conditions $\iota_1 < \iota_2 < \iota_3$ and $\iota_4 < \iota_5$. This has ten summands, multilinear in the five operators $A_i, i = 1, \ldots, 5$. It is easy to check that these ten terms are linearly independent. Hence the ten coefficients $c(\iota_1, \iota_2, \iota_3)$ must be zero. □

There are two types of monomials cubic in the ternutator, both involving seven operators. Label any set of seven operators (not necessarily linearly independent) $A_1, \ldots, A_7$ and define

$$T_1(A_1, \ldots, A_7) := [[[A_1, A_2, A_3], A_4, A_5], A_6, A_7] \tag{6}$$

$$T_2(A_1, \ldots, A_7) := [[A_1, A_2, A_3], [A_4, A_5, A_6], A_7]. \tag{7}$$

In virtue of the symmetry properties of the ternutator, there are 210 independent monomials of the form $T_1(A_{\iota_1}, A_{\iota_2}, A_{\iota_3}, A_{\iota_4}, A_{\iota_5}, A_{\iota_6}, A_{\iota_7})$ labelled by the permutations of $(\iota_1, \ldots, \iota_7)$ of $(1, \ldots, 7)$ satisfying $(\iota_1 < \iota_2 < \iota_3)$, $(\iota_4 < \iota_5)$ and $(\iota_6 < \iota_7)$, and 70 monomials of the form $T_2(A_{\iota_1}, A_{\iota_2}, A_{\iota_3}, A_{\iota_4}, A_{\iota_5}, A_{\iota_6}, A_{\iota_7})$, with conditions $(\iota_1 < \iota_2 < \iota_3)$, $(\iota_4 < \iota_5 < \iota_6)$, $(\iota_1 < \iota_4)$. An identity cubic in ternutators, if it exists, is a linear dependence amongst these 280 terms:

$$\sum_{\substack{\iota\in S_7 \\ \iota_1<\iota_2<\iota_3,\iota_4<\iota_5,\iota_4<\iota_6<\iota_7}} c_1(\iota_1,\iota_2,\iota_3,\iota_4,\iota_5)\; T_1(A_{\iota_1}, A_{\iota_2}, A_{\iota_3}, A_{\iota_4}, A_{\iota_5}, A_{\iota_6}, A_{\iota_7}) + \sum_{\substack{\iota\in S_7 \\ \iota_1<\iota_2<\iota_3,\iota_1<\iota_4<\iota_5<\iota_6}} c_2(\iota_1,\iota_2,\iota_3,\iota_7)\; T_2(A_{\iota_1}, A_{\iota_2}, A_{\iota_3}, A_{\iota_4}, A_{\iota_5}, A_{\iota_6}, A_{\iota_7}) = 0, \tag{8}$$

where the sums are over all permutations $\iota \in S_7$, $\iota : n \mapsto \iota_n$, $n \in \{1, \ldots, 7\}$, satisfying the given restrictions. The following result was also known to Bremner [13].

Theorem 1. *Among any seven operators, there exist precisely seven independent identities cubic in the ternutator.*

In order to prove this, we have used REDUCE and independently MAPLE to find that the 280 coefficients in (8) are constrained by 273 independent linear relations leaving a seven-parameter space of identities. Hence there is a basis of seven independent identities, which generalize the usual Jacobi identity for commutators.

Consider seven arbitrary operators and label them A_i, $i = 1, \ldots, 7$. Pick out one of them, say A_7, and consider the following alternating sums of T_1 and T_2, (skew) symmetric under permutations of the remaining six operators:

$$I_1(\iota_1, \ldots \iota_6; 7) = \sum_{\substack{\iota\in S_6 \\ \iota_1<\iota_2<\iota_3,\iota_5<\iota_6}} \epsilon_{\iota_1\iota_2\iota_3\iota_4\iota_5\iota_6}\, [[[A_{\iota_1}, A_{\iota_2}, A_{\iota_3}], A_{\iota_4}, A_7], A_{\iota_5}, A_{\iota_6}]$$

$$I_2(\iota_1, \ldots \iota_6; 7) = \sum_{\substack{\iota\in S_6 \\ \iota_1<\iota_2<\iota_3,\iota_4<\iota_5}} \epsilon_{\iota_1\iota_2\iota_3\iota_4\iota_5\iota_6}\, [[A_{\iota_1}, A_{\iota_2}, A_{\iota_3}], [A_{\iota_4}, A_{\iota_5}, A_7], A_{\iota_6}]. \tag{9}$$

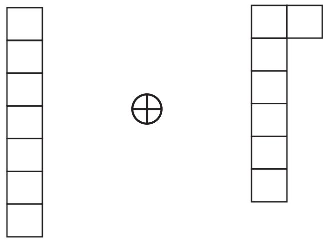
The identity is then

$$I = I_1 + I_2 \equiv 0. \tag{10}$$

An alternative way of writing the identity is as follows. Take seven operators $A_a, A_b, A_c, A_d, A_e, A_f, A_g$ and single out A_g. We can write the identity I as the alternating sum over all permutations of the six indices (a, b, c, d, e, f),

$$12\, I = \sum_{S(a,b,c,d,e,f)} \operatorname{sgn}(\pi)\, ([[[A_a, A_b, A_c], A_d, A_g], A_e, A_f] + [[A_a, A_b, A_c], [A_d, A_e, A_g], A_f]) \equiv 0, \tag{11}$$

where $\operatorname{sgn}(\pi)$ is the sign of the permutation. The factor 12 here is the number of times a given term is repeated in the sum over all permutations. The seven independent identities correspond to the seven possibilities of singling out any one, say A_g. These seven identities transform under S_7 as the direct sum of the representations associated with the Young tableaux



(12)

These identities were known to Bremner [13] in a slightly different form. They were independently rediscovered in this explicit form by one of us (JN). There exist further equivalent expressions for these identities. Again singling out A_7, we may write

$$\begin{aligned} 3 \sum_{\substack{\mu\in S_6\\ \mu_1<\mu_2<\mu_3,\ \mu_4<\mu_5<\mu_6,\ \mu_1<\mu_4}} & \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\,[[A_{\mu_1}, A_{\mu_2}, A_{\mu_3}], [A_{\mu_4}, A_{\mu_5}, A_{\mu_6}], A_7] \\ &= \sum_{\substack{\mu\in S_7\\ \mu_1<\mu_2<\mu_3,\ \mu_4<\mu_5<7,\ \mu_6<\mu_7}} \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7}\,[[[A_{\mu_1}, A_{\mu_2}, A_{\mu_3}], A_{\mu_4}, A_{\mu_5}], A_{\mu_6}, A_{\mu_7}] \\ &\quad - 2 \sum_{\substack{\mu\in S_7\\ \mu_1<\mu_2<\mu_3,\ \mu_5=7,\ \mu_6<\mu_7}} \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7}\,[[[A_{\mu_1}, A_{\mu_2}, A_{\mu_3}], A_{\mu_4}, A_{\mu_5}], A_{\mu_6}, A_{\mu_7}]. \end{aligned} \tag{13}$$

The left-hand side has ten summands. In the first term of the right-hand side there are 150 summands and in the second term 60 summands. This linear dependence amongst monomials cubic in the ternutator has alternative expression:

$$\begin{aligned} 3 \sum_{\substack{\mu\in S_6\\ \mu_1<\mu_2<\mu_3,\ \mu_4<\mu_5<\mu_6,\ \mu_1<\mu_4}} & \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\,[[A_{\mu_1}, A_{\mu_2}, A_{\mu_3}], [A_{\mu_4}, A_{\mu_5}, A_{\mu_6}], A_7] \\ &= \sum_{\substack{\mu\in S_6\\ \mu_1<\mu_2,\ \mu_3<\mu_4,\ \mu_5<\mu_6}} \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\,[[[A_{\mu_1}, A_{\mu_2}, A_7], A_{\mu_3}, A_{\mu_4}], A_{\mu_5}, A_{\mu_6}] \\ &\quad + \sum_{\substack{\mu\in S_6\\ \mu_1<\mu_2<\mu_3,\ \mu_4<\mu_5,}} \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\,[[[A_{\mu_1}, A_{\mu_2}, A_{\mu_3}], A_{\mu_4}, A_{\mu_5}], A_{\mu_6}, A_7] \\ &\quad - 2 \sum_{\substack{\mu\in S_6\\ \mu_1<\mu_2<\mu_3,\ \mu_5<\mu_6}} \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}[[[A_{\mu_1}, A_{\mu_2}, A_{\mu_3}], A_{\mu_4}, A_7], A_{\mu_5}, A_{\mu_6}]. \end{aligned} \tag{14}$$

Here, there are ten summands on the left-hand side, 90 terms in the first part of the right-hand side, 60 terms in the second part and 60 in the third part.

3. Ternary algebras

A ternary algebra is a vector space in which a ternary composition is given by the 'ternutation relations',

$$[e_i, e_j, e_k] = t_{ijk}{}^m e_m, \tag{15}$$

where e_i are basis elements and the structure constants $t_{ijk}{}^m$ are completely antisymmetric in i, j, k and transform as (3, 1)-tensors under a nonsingular change of basis, $e'_j = S^k_j e_k$,

$\det(S) \neq 0$. Inserting these basis elements in the polynomials I_1 and I_2 in (9) yields cubic polynomials in the structure constants,

$$I_1(t; a, b, c, d, e, f; g, q) = \sum_{S(a,b,c,d,e,f)} \operatorname{sgn}(\pi)\, t_{abc}{}^{p}\, t_{pdg}{}^{n}\, t_{nef}{}^{q}$$

$$I_2(t; a, b, c, d, e, f; g, q) = \sum_{S(a,b,c,d,e,f)} \operatorname{sgn}(\pi)\, t_{abc}{}^{m} t_{deg}{}^{n} t_{mnf}{}^{q}, \qquad (16)$$

where the sum is over all permutations of (a, b, c, d, e, f). The ternutator identities (11) then yield identities for the structure constants $t_{ijk}{}^{m}$,

$$I(t; a, b, c, d, e, f; g, q) := (I_1 + I_2)(t; a, b, c, d, e, f; g, q) = 0. \qquad (17)$$

Just as a set of structure constants satisfying the Jacobi identities converts a vector space into a Lie algebra, it would be interesting if the identities (17) similarly provide necessary and sufficient conditions for a set of structure constants $\{t_{ijk}{}^{m}\}$ to define a ternary algebra with relations (15).

The associativity of the multiplication implies the quadratic Jacobi identity for the commutator (4) as well as the cubic identities (11) for the ternutator. If we lift associativity, both these identities do not hold. For the alternative multiplication for the basis elements of the imaginary octonions, $e_i e_j = -\delta_{ij} + \psi_{ijk} e_k$, the left-hand side of the Jacobi identity (4) is proportional to the associator $(e_i, e_j, e_k) = (e_i e_j) e_k - e_i (e_j e_k) = \varphi_{ijkl} e_l$, where both ψ_{ijk} and φ_{ijkl} are completely antisymmetric. The ternutator $[e_i, e_j, e_k]$ is also proportional to the associator, and we have verified by direct calculation using REDUCE that the G_2-invariant structure constants of the associator algebra $\varphi_{ijkl} = \frac{1}{6}\epsilon_{ijklmnp}\psi_{mnp}$ with nonzero components,

$$\varphi_{1234} = \varphi_{1256} = \varphi_{1357} = \varphi_{1476} = \varphi_{2376} = \varphi_{2475} = \varphi_{3456} = 1, \qquad (18)$$

corresponding to the octonion structure constants $\{\psi_{127} = \psi_{163} = \psi_{154} = \psi_{253} = \psi_{246} = \psi_{347} = \psi_{567} = 1\}$ indeed do not satisfy the identities (17). Instead, the two polynomials in (16) evaluated for φ are equal.

Indeed, in dimensions 6 and 7, for antisymmetric t's, the values of the two polynomials in (16) are equal, so in these dimensions the identities (17) are equivalent to either $I_1 = 0$ or $I_2 = 0$. We have checked explicitly that the equality $I_1 = I_2 \neq 0$ also holds for the components of the Spin(7)-invariant 4-form [16, 17] defined by (18) together with the further nonzero components $\varphi_{ijk8} = \psi_{ijk}$, namely, $\varphi_{1278} = \varphi_{1386} = \varphi_{1485} = \varphi_{2385} = \varphi_{2468} = \varphi_{3478} = \varphi_{5678} = 1$, and hence this 4-form also does not satisfy the ternutator identity.

Let π be a permutation of the four indices (a, b, c, d). The following quadratic equations in the structure constants

$$\sum_{\pi \in S(a,b,c,d)} \operatorname{sgn}(\pi) t_{abc}{}^{m} t_{mdg}{}^{q} = 0 \qquad (19)$$

are related to the above-mentioned Nambu identity and have been advocated [11] as defining conditions for algebras (1). These equations are by no means necessary conditions for the existence of the ternary algebra since the ternutator does not satisfy a quadratic identity (see theorem 1). However, it is readily seen that if (19) holds then our cubic necessary conditions (11) are satisfied. Hence the relations (19) could be sufficient (but by no means necessary) to guarantee the existence of the corresponding ternary algebra.

The Cartan–Killing metric plays a crucial role in the classification of Lie algebras. There are several tensors, constructed from the structure constants $t_{ijk}{}^{m}$, which possibly could play a similar role for ternary algebras. For Lie algebras, in terms of the adjoint map of element e_j with respect to an element e_i defined by $\mathrm{ad}_{e_i} e_j ;= [e_i, e_j]$, the Cartan–Killing form is given by $g(e_i, e_j) := \mathrm{Tr}(\mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_j})$. In terms of the structure constants of the Lie algebra we have $g_{ij} = f_{im}{}^n f_{jn}{}^m$. Using the ternutation relations (15) to define an analogous endomorphism of the ternary algebra, $X_{e_i,e_j} e_k := [e_i, e_j, e_k]$, yields the most natural candidate for a 'metric', namely $\mathrm{Tr}(X_{e_i,e_j} \circ X_{e_k,e_l})$, which is clearly antisymmetric in ij, antisymmetric in kl and symmetric under interchange of these ordered pairs (in other words, it has the symmetries of the Riemann tensor). This object has components

$$g_{ij,kl} = t_{ijn}{}^m\, t_{klm}{}^n \tag{20}$$

and can serve as a metric on the $N(N-1)/2$-dimensional space of 2-forms indexed by $\{ij\}$, $i < j$. If this metric is invertible, its inverse $g^{ij,kl}$ then satisfies

$$\sum_{k<l} g_{ij,kl}\, g^{kl,mn} = \delta_i^m \delta_j^n \tag{21}$$

and affords the construction of the further symmetric tensor,

$$g^{pq} = \frac{1}{2}\left(t_{ijk}{}^p g^{ij,kq} + t_{ijk}{}^q g^{ij,kp}\right), \tag{22}$$

which is a candidate for a metric in the linear space of the operators.

If a Lie algebra is semi-simple, a particularly important subset of Lie algebras, it is well known that the metric $g_{ij} = f_{ia}{}^b f_{jb}{}^a$ formed from the structure constants is nonsingular and that $f_{ijk} = g_{ka} f_{ij}{}^a$ is completely antisymmetric in its indices. It would be remarkable if analogous statements hold for ternary algebras. In particular, we expect a special role for ternary algebras having the following two properties:

(a) there exists a basis in the space of operators such that the structure constants are completely antisymmetric,

$$[e_i, e_j, e_k] = t_{ijkm} e_m, \quad \text{with} \quad t_{ijkm} \quad \text{antisymmetric}, \tag{23}$$

(b) in this basis, a metric appears in the form

$$\sum_{a<b<c} t_{iabc} t_{jabc} = \lambda \delta_{ij}. \tag{24}$$

4. Examples

The conditions (17) are so overdetermined that a complete classification of solutions, without imposing further conditions, seems to be a rather difficult task. However, isolated solutions in lower dimensions can certainly be found. We display four solutions in six dimensions with the structure constants being given by components of a 4-form t_{ijkm}, which satisfy the identities (17). It is convenient to use the Hodge-dual 2-form $\widetilde{t} = * t$ with components $\widetilde{t}_{a_1a_2} = \frac{1}{4!}\epsilon_{a_1a_2a_3a_4a_5a_6} t_{a_3a_4a_5a_6}$.

Example 1

$$\widetilde{t}_{12} = -\widetilde{t}_{13} = -\widetilde{t}_{14} = \widetilde{t}_{15} = -\widetilde{t}_{23} = -\widetilde{t}_{24} = \widetilde{t}_{26} = \widetilde{t}_{35} = -\widetilde{t}_{36} = \widetilde{t}_{45} = -\widetilde{t}_{46} = \widetilde{t}_{56} = 1. \tag{25}$$

Example 2

$$-\widetilde{t}_{12} = \widetilde{t}_{13} = \widetilde{t}_{14} = -\widetilde{t}_{15} = \widetilde{t}_{26} = -\widetilde{t}_{36} = -\widetilde{t}_{46} = \widetilde{t}_{56} = 1. \tag{26}$$

Example 3

$$\widetilde{t}_{14} = \widetilde{t}_{15} = -\widetilde{t}_{16} = -\widetilde{t}_{24} - \widetilde{t}_{25} = \widetilde{t}_{26} = \widetilde{t}_{34} = \widetilde{t}_{35} = -\widetilde{t}_{36} = 1. \tag{27}$$

Example 4

$$\begin{aligned}&\widetilde{t}_{46} = -1, \quad \widetilde{t}_{56} = 1,\\ &\widetilde{t}_{36} = x_1, \quad \widetilde{t}_{26} = -x_3, \quad \widetilde{t}_{35} = -x_2, \quad \widetilde{t}_{34} = x_2,\\ &\widetilde{t}_{25} = x_4, \quad \widetilde{t}_{24} = -x_4, \quad \widetilde{t}_{23} = x_4 x_1 - x_2 x_3.\end{aligned} \tag{28}$$

In the first three examples, the structure constants satisfy the calibration criterion

$$t^3_{abcd} = t_{abcd} \quad \Longrightarrow \quad \widetilde{t}^3_{ab} = \widetilde{t}_{ab}. \tag{29}$$

Example 4 does not seem to be a calibration.

5. Conclusions

Ternutators and ternary algebras generalize the familiar commutators and Lie algebras and have recently appeared in various contexts in the literature (see e.g. [18]). In this paper, we have obtained explicit forms of the seven identities, cubic in ternutators, which are obeyed by any seven operators with an associative product. We have shown that these identities exhaust the space of identities for seven operators. It remains an open question whether higher order ($q \geqslant 4$) identities involving $2q + 1$ operators are trivial consequences of the seven basic identities. The seven independent identities for seven operators clearly imply identities (of fourth order in the ternutator) for nine operators. As is clear from (11), the former seven identities can be considered as operations $X(B_1, \ldots, B_6; B_7)$, depending on seven operators, skew-symmetric in the first six. Amongst the identities for any given choice of nine operators $A_i, i = 1 \ldots 9$ there exist those having one of the following structures: $A = [X(A_1, \ldots, A_6; A_7), A_8, A_9]$, $B = X(A_1, \ldots, A_6; [A_7, A_8, A_9])$, $C = X(A_1, \ldots, A_5, [A_6, A_8, A_9]; A_7)$. There are $7 \cdot \binom{9}{2} = 252$ identities of type A, and $\binom{9}{3} = 84$ identities of type B and type C. So altogether 420 identities amongst 9 operators have their origins in the identities for seven operators. Are there any further identities?

As a consequence of these identities, the structure constants of ternary algebras must obey cubic relations, which are highly overdetermined. We have given some examples of solutions of these identities in six dimensions.

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