$N=3$ extended supersymmetric gauge theories and an explicit construction of higher conservation laws

C. Devchand
Fakultät für Physik der Universität, D-7800 Freiburg, Federal Republic of Germany

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A Lagrangian for the superfield equations of motion for supersymmetric gauge theories with $N=3$ extended supersymmetry is presented. A novel formulation of the previously constructed infinitely many spinorial continuity equations considerably clarifies their structure.

I. INTRODUCTION

The maximally supersymmetric gauge theories in four dimensions[1,2] with $N=4$ (or equivalently $N=3$) supersymmetry have, among their many remarkable properties, the feature of ultraviolet finiteness to all orders in perturbation theory.[3–5] The conjunction of conformal invariance and solubility is a familiar feature of many quantum field theories and the strongly constrained dynamics implied by the ultraviolet finiteness suggests integrability of the field equations as a possible underlying classical precursor. The finiteness of maximal super-Yang–Mills therefore lends further promise to the notion that these are the appropriate theories for the realization of duality conjectures[6] which generalize the duality between the Thirring model and the sine–Gordon model[7] to four-dimensional gauge theories with monopole solutions. It is, however, clear that the equations of motion for the $N=4$ theory with Lagrangian[2]

$$\begin{aligned} L = (& -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}D_\mu \Phi_{ij} D^\mu \Phi^{ij} \\ & - i\lambda^{i} D\lambda_i + \lambda^{i}[\lambda^{cj},\Phi_{ij}] \\ & - \lambda^{c}_{i}[\lambda_j,\Phi^{ij}] + \tfrac{1}{4}[\Phi_{ij},\Phi_{kl}][\Phi^{ij},\Phi^{kl}]) \end{aligned} \tag{1}$$

[where all fields are gauge algebra valued; i,j run from 1 to 4; $\Phi^{ij} = \frac{1}{2}\epsilon^{ijkl}\Phi_{kl}$ and the spinors λ_i are chiral, $\gamma_5\lambda = -i\lambda$ and $\lambda^{ci} \equiv C\lambda^{i}$ yielding $\gamma_5\lambda^{c} = i\lambda^{c}$] are *not* completely integrable in any conventional sense; nor is the S matrix of the theory trivial, as would be expected of a four-dimensional theory with higher local conserved currents.[8] Nevertheless, in the superspace formulation,[9] the classical equations of motion for this theory, similarly to the Yang–Mills self-duality equations,[10] may be formulated in a geometric way as integrability conditions for a set of linear (superfield) equations based on the Witten–Manin supertwistor correspondence.[11–13] This structural similarity to completely integrable systems with solitons has led to many (hitherto largely futile) attempts (e.g., Refs. 14–19) to obtain meaningful explicit results on classical solutions and higher conservation laws and symmetries using methods analogous to those developed to study self-dual Yang–Mills or soliton equations of the Zakharov–Shabat type. This paper is a further contribution in this direction. We present, in Sec. II, a novel Lagrangian for a reformulated set of superfield equations of motion. This Lagrangian is rather similar to the Lagrangian for self-dual Yang–Mills discussed by Leznov and Saveliev.[20] In Sec. III we discuss the infinite number of continuity equations, first introduced in Ref. 15, which in the new formulation of this paper are considerably clarified. They acquire a more explicit form that is perhaps more amenable to interpretation. In Sec. IV we comment on the infinitesimal symmetry transformations of the equations of motion and we conclude (Sec. V) with some remarks on the reduction of these equations to the supersymmetric self-duality conditions.

II. $N=3$ SUPER-YANG–MILLS EQUATIONS AND A SUPERFIELD LAGRANGIAN

N-extended complexified superspace, a supermanifold of complex dimension $(4/4N)$ with complexified space-time coordinates $x^{\alpha\dot\beta} = x^\mu \sigma_\mu^{\alpha\dot\beta}$ and the anticommuting coordinates ϑ^{α}_{i}, $\bar\vartheta^{\dot\alpha j}$, where α, $\dot\alpha$ are two-component spinor indices and $i,j, = 1,\ldots,N$ is the internal SU(N) index, the upper and lower indices referring to fundamental and conjugate representations, respectively. The supertranslation vector fields $\partial_A = (\partial_{\alpha\dot\alpha}, \mathrm{D}^i_\alpha, \mathrm{D}_{\dot\alpha j})$,

$$\mathrm{D}^i_\alpha = \frac{\partial}{\partial\vartheta^{\alpha}_{i}} + \bar\vartheta^{\dot\beta i}\partial_{\alpha\dot\beta}, \quad \mathrm{D}_{\dot\beta j} = \frac{\partial}{\partial\bar\vartheta^{\dot\beta j}} + \vartheta^{\alpha}_{j}\partial_{\alpha\dot\beta},$$
$$\partial_{\alpha\dot\beta} = \frac{\partial}{\partial x^{\alpha\dot\beta}}, \tag{2}$$

provide a nonholonomic frame for superspace and realize the superalgebra

$$\begin{aligned} &\{\mathrm{D}^i_\alpha, \mathrm{D}^j_\beta\} = 0 = \{\mathrm{D}_{\dot\alpha i}, \mathrm{D}_{\dot\beta j}\}, \\ &\{\mathrm{D}^i_\alpha, \mathrm{D}_{\dot\beta j}\} = 2\delta^i_j \partial_{\alpha\dot\beta}, \\ &[\partial_{\alpha\dot\beta}, \mathrm{D}^i_\beta] = 0 = [\partial_{\alpha\dot\beta}, \mathrm{D}_{\dot\alpha j}] = [\partial_{\alpha\dot\alpha}, \partial_{\beta\dot\beta}]. \end{aligned} \tag{3}$$

The Lie-algebra valued components of the gauge superconnection $A_A = (A_{\alpha\dot\beta}, A^{i}_{\alpha}, A_{\dot\beta j})$ transform as usual under gauge transformations

$$A_A \to e^{\Lambda} A_A e^{-\Lambda} + e^{\Lambda}\partial_A e^{-\Lambda} \tag{4}$$

a covariant superfield transforming as $\Phi \to e^{\Lambda}\Phi e^{-\Lambda}$, where the gauge parameter Λ is a Lie algebra valued superfield, $\Lambda = T^a \Lambda^a(x,\vartheta,\bar\vartheta)$, T^a being the Lie algebra generators acting on the gauge indices of A_A and Φ. Introducing gauge-covariant derivatives $D_A = (D_{\alpha\dot\beta}, D^{i}_{\alpha}, D_{\dot\beta j})$,

$$\begin{aligned} D^{i}_{\alpha}\varphi &= \mathrm{D}^i_\alpha \varphi + [A^{i}_{\alpha}, \varphi], \\ D_{\dot\alpha j}\varphi &= \mathrm{D}_{\dot\alpha j}\varphi + [A_{\dot\alpha j}, \varphi], \\ D_{\alpha\dot\beta}\varphi &= \partial_{\alpha\dot\beta}\varphi + [A_{\alpha\dot\beta}, \varphi], \end{aligned} \tag{5}$$

transforming as $D_A \to e^{\Lambda}(D_A \varphi)e^{-\Lambda}$, the superfield curvatures F_{AB} may be formed by considering the graded commu-

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tator,

$$\{D^i_\alpha, D^j_\beta\} = F^{ij}_{\alpha\beta}, \quad \{D_{\dot\alpha i}, D_{\dot\beta j}\} = F_{\dot\alpha i,\dot\beta j},$$
$$[D_{\alpha\dot\beta}, D^i_\beta] = F^i_{\alpha\dot\beta,\beta}, \quad [D_{\alpha\dot\beta}, D_{\dot\alpha i}] = F_{\alpha\dot\beta,\dot\alpha i},$$
$$[D_{\alpha\dot\beta}, D_{\gamma\dot\delta}] = \epsilon_{\alpha\gamma} F_{\dot\beta\dot\delta} + \epsilon_{\dot\beta\dot\delta} F_{\alpha\gamma}, \tag{6}$$
$$\{D^i_\alpha, D_{\dot\beta j}\} = F^i_{\alpha,\dot\beta j} + 2\delta^i_j D_{\alpha\dot\beta},$$

where

$$F^{ij}_{\alpha\beta} = \mathrm{D}^i_\alpha A^j_\beta + \mathrm{D}^j_\beta A^i_\alpha + \{A^i_\alpha, A^j_\beta\},$$
$$F_{\dot\alpha i,\dot\beta j} = \mathrm{D}_{\dot\alpha i} A_{\dot\beta j} + \mathrm{D}_{\dot\beta j} A_{\dot\alpha i} + \{A_{\dot\alpha i}, A_{\dot\beta j}\},$$
$$F^i_{\alpha,\dot\beta j} = \mathrm{D}^i_\alpha A_{\dot\beta j} + \mathrm{D}_{\dot\beta j} A^i_\alpha + \{A^i_\alpha, A_{\dot\beta j}\} - 2\delta^i_j A_{\alpha\dot\beta}, \tag{7}$$
$$F^i_{\mu\alpha} = \partial_\mu A^i_\alpha - \mathrm{D}^i_\alpha A_\mu + [A_\mu, A^i_\alpha],$$
$$F_{\mu,\dot\alpha i} = \partial_\mu A_{\dot\alpha i} - \mathrm{D}_{\dot\alpha i} A_\mu + [A_\mu, A_{\dot\alpha i}],$$

and $F_{\alpha\gamma}(F_{\dot\beta\dot\delta})$ are the (anti-) self-dual parts of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

All the above F_{AB}'s are Lie algebra valued superfields transforming covariantly under gauge transformations. They are clearly not independent of each other, since the generalized Jacobi identity imposes relations amongst them, e.g.,

$$\begin{aligned} 0 \equiv {} & [D^i_\alpha, \{D_{\dot\alpha j}, D_{\dot\beta k}\}] + [D_{\dot\alpha j}, \{D_{\dot\beta k}, D^i_\alpha\}] \\ & + [D_{\dot\beta k}, \{D^i_\alpha, D_{\dot\alpha j}\}] \\ = {} & D^i_\alpha F_{\dot\alpha j,\dot\beta k} + D_{\dot\alpha j} F^i_{\alpha,\dot\beta k} + D_{\dot\beta k} F^i_{\alpha,\dot\alpha j} \\ & - 2\delta^i_k F_{\alpha\dot\beta,\dot\alpha j} - 2\delta^i_j F_{\alpha\dot\alpha,\dot\beta k} \end{aligned} \tag{8}$$

or

$$\begin{aligned} 0 \equiv {} & [D_{\alpha\dot\beta} + \{D^i_\beta, D_{\dot\alpha j}\}] + \{D^i_\beta, [D_{\dot\alpha j}, D_{\alpha\dot\beta}]\} \\ & + \{D_{\dot\alpha j}, [D^i_\beta, D_{\alpha\dot\beta}]\} \\ = {} & D_{\alpha\dot\beta} F^i_{\beta,\dot\alpha j} - D^i_\beta F_{\alpha\dot\beta,\dot\alpha j} - D_{\dot\alpha j} F^i_{\alpha\dot\beta,\beta} \\ & + 2\delta^i_j(\epsilon_{\alpha\beta} F_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} F_{\alpha\beta}). \end{aligned} \tag{9}$$

Nevertheless, the set of super Yang–Mills fields F_{AB} defined by Eq. (7) clearly has an enormous number of degrees of freedom; and in order to minimize the number of component fields to just those required for the construction of an irreducible representation of the supersymmetry algebra, one usually imposes the covariant (under both supersymmetry and gauge transformations) constraint equations,[9]

$$F^{(ij)}_{\alpha\beta} = 0 = F_{\dot\alpha(i,\dot\beta j)}, \tag{10a}$$
$$F^i_{\alpha,\dot\beta j} = 0. \tag{10b}$$

For $N<3$, these equations do not imply any equations in x space. They are therefore a suitable representation condition for a minimal multiplet of component fields. For $N\geqslant 3$, on the other hand, the constraints (10) do have dynamical content and do not yield an off-shell representation of the theory with only the minimal physical fields (viz., one spin 1, four spin $\frac{1}{2}$, and six spin 0). Remarkably, for the case of $N=3$, the constraints (10) are precisely equivalent to the equations of motion for the component fields.[21]

Equations (10a) have the solution

$$F^{ij}_{\alpha\beta} = \epsilon_{\alpha\beta} W^{ij}, \quad W^{ij} = \epsilon^{ijk} W_k, \tag{11a}$$
$$F_{\dot\alpha i,\dot\beta j} = \epsilon_{\dot\alpha\dot\beta} W_{ij}, \quad W_{ij} = \epsilon_{ijk} W^k, \tag{11b}$$

whereas the diagonal parts of (10b) are the conventional equations expressing the vector potential $A_{\alpha\dot\beta}$ entirely in terms of A^i_α and $A_{\dot\alpha i}$ in virtue of (7),

$$A_{\alpha\dot\beta} = \tfrac{1}{6}(\mathrm{D}^i_\alpha A_{\dot\beta i} + \mathrm{D}_{\dot\beta i} A^i_\alpha + \{A^i_\alpha, A_{\dot\beta i}\}). \tag{12}$$

The Bianchi identities can now be used to express the theory entirely in terms of the superfields $F_{\alpha\beta}$, $F_{\dot\alpha\dot\beta}$, W^k, W_k, $D^i_\alpha W_i$, $D_{\dot\alpha i} W^i$, $\epsilon_{ijk} D^i_\alpha W^j$, $\epsilon^{ijk} D_{\dot\alpha i} W_j$, and covariant derivatives thereof. The leading components (in a power series expansion in ϑ, $\bar\vartheta$) of $F_{\alpha\beta}$, $F_{\dot\alpha\dot\beta}$ yield the field strength components of the component vector field $A_{\alpha\dot\beta}(x)$, whereas the leading components of the other superfields yield the remaining component fields of the theory: two SU(3)-triplets of scalar fields (W^i, W_i), an SU(3)-scalar Majorana spinor $(\lambda_\alpha, \lambda_{\dot\alpha})$, and an SU(3)-triplet of spinors $(\chi_{\alpha i}, \chi^i_{\dot\alpha})$. These component fields have the following equations of motion [corresponding to the $N=3$ version of the theory (1)]:

The Dirac equations

$$\epsilon^{\alpha\beta} D_{\alpha\dot\beta} \lambda_\beta + [\chi^i_{\dot\beta}, W_i] = 0, \tag{13a}$$
$$\epsilon^{\dot\alpha\dot\beta} D_{\alpha\dot\alpha} \lambda_{\dot\beta} + [\chi_{i\alpha}, W^i] = 0, \tag{13b}$$
$$\epsilon^{\alpha\beta} D_{\alpha\dot\beta} \chi_{i\beta} + [\chi^j_{\dot\beta}, W^k]\epsilon_{ijk} - [\lambda_{\dot\beta}, W_i] = 0, \tag{13c}$$
$$\epsilon^{\dot\alpha\dot\beta} D_{\alpha\dot\alpha} \chi^j_{\dot\beta} + [\chi_{i\alpha}, W_k]\epsilon^{ijk} - [\lambda_\alpha, W^j] = 0, \tag{13d}$$

the scalar field equations

$$\begin{aligned} & \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} D_{\alpha\dot\alpha} D_{\beta\dot\beta} W_j + 2[[W^i, W_j], W_i] \\ & \quad - [[W^i, W_i], W_j] + \epsilon^{\alpha\beta}\{\chi_{\alpha j}, \lambda_\beta\} \\ & \quad - \tfrac{1}{2}\epsilon_{ijk}\epsilon^{\dot\alpha\dot\beta}\{\chi^i_{\dot\alpha}, \chi^k_{\dot\beta}\} = 0 \end{aligned} \tag{13e}$$

(similarly for W_i), and the Yang–Mills equation

$$\begin{aligned} & \epsilon^{\alpha\beta} D_{\alpha\dot\beta} F_{\gamma\beta} + \epsilon^{\dot\alpha\dot\gamma} D_{\gamma\dot\alpha} F_{\dot\gamma\dot\beta} + \{\chi_{\gamma k}, \chi^k_{\dot\beta}\} + \{\lambda_\gamma, \lambda_{\dot\beta}\} \\ & \quad + [W_i, D_{\gamma\dot\beta} W^i] + [W^i, D_{\gamma\dot\beta} W_i] = 0. \end{aligned} \tag{13f}$$

The equivalence proof of Ref. 21 eliminates the ϑ dependence of the gauge transformations (4) by subjecting the superconnection to the "transverse" gauge condition,

$$\vartheta^\alpha_i A^i_\alpha + \bar\vartheta^{\dot\alpha j} A_{\dot\alpha j} = 0, \tag{14}$$

which effectively eliminates all ϑ-dependent gauge transformations, while posing no restriction on x-dependent gauge transformations of the component fields. This allows the construction of a unique correspondence between a superconnection constrained by (10) and a component multiplet solving (13). For the $N=3$ theory, therefore, the constraint equations (10) are just a compact way of writing the field equations (13). These constraints equations, remarkably, also correspond to the vanishing of the supercurvature along supernull lines.[11] Consider the equations for the covariant constancy of sections along the direction in superspace given by $\xi^A = (\mu^\alpha \lambda^{\dot\alpha}, \mu^\alpha, \lambda_{\dot\alpha})$,

$$\mu^\alpha D^i_\alpha \Phi = 0 = \lambda^{\dot\alpha} D_{\dot\alpha j} \Phi, \tag{15a}$$
$$\mu^\alpha \lambda^{\dot\alpha} D_{\alpha\dot\alpha} \Phi = 0, \tag{15b}$$

where $v^{\alpha\dot\alpha} = \mu^\alpha \lambda^{\dot\alpha}$ is a null vector in x space and ξ^A is defined to be a supernull vector. Integrability of (15) (i.e., the path independence of Φ) requires $\xi^A A_A = (\mu^\alpha \lambda^{\dot\alpha} A_{\alpha\dot\alpha}, \mu^\alpha A^i_\alpha, \lambda^{\dot\alpha} A_{\dot\alpha i})$ to be a flat connection. This is tantamount to requiring that the supercurvature components satisfy (10), and allows the identification[11] of bundles with connections satisfying constraints (10) over super-Minkowski space with bundles over the supertwistor space of supernull lines with triviality conditions over certain $CP^1 \times CP^1$ submanifolds. It is the tantalizing similarity of (15) to Lax-type linear systems for soliton equations which led to previous attempts[14–19] to understand these theories using solitonic methods.

Following our previous approach,[15] we note that the following subset of constraints (10):

$$F^{ij}_{11} = 0 = F^{ij}_{22}, \tag{16}$$

$$F_{1i,1j} = 0 = F_{\dot{2}i,\dot{2}j}, \tag{17}$$

$$F^{i}_{1,1j} = 0 = F^{i}_{2,\dot{2}j}, \tag{18}$$

are equivalent to writing the spinor potentials and two components of the vector potential in pure gauge form,

$$A^{i}_{1} = g^{-1}\mathrm{D}^{i}_{1}g, \quad A_{1i} = g^{-1}\mathrm{D}_{1i}g, \quad A_{1\dot{1}} = g^{-1}\partial_{1\dot{1}}g, \tag{19}$$

$$A^{i}_{2} = h^{-1}\mathrm{D}^{i}_{2}h, \quad A_{\dot{2}i} = h^{-1}\mathrm{D}_{\dot{2}i}h, \quad A_{2\dot{2}} = h^{-1}\partial_{2\dot{2}}h. \tag{20}$$

Introducing the matrix

$$B = gh^{-1}, \tag{21}$$

the remaining components of the vector potential then take the form

$$A_{1\dot{2}} = g^{-1}\partial_{1\dot{2}}g + \tfrac{1}{6}g^{-1}\mathrm{D}^{i}_{1}(B\,\mathrm{D}_{\dot{2}i}B^{-1})g \tag{22a}$$

$$= h^{-1}\partial_{1\dot{2}}h + \tfrac{1}{6}h^{-1}\mathrm{D}_{\dot{2}i}(B^{-1}\mathrm{D}^{i}_{1}B)h, \tag{22b}$$

$$A_{2\dot{1}} = g^{-1}\partial_{2\dot{1}}g + \tfrac{1}{6}g^{-1}\mathrm{D}_{\dot{1}i}(B\,\mathrm{D}^{i}_{2}B^{-1})g \tag{23a}$$

$$= h^{-1}\partial_{2\dot{1}}h + \tfrac{1}{6}h^{-1}\mathrm{D}^{i}_{2}(B^{-1}\mathrm{D}_{\dot{1}i}B)h \tag{23b}$$

in virtue of (10b).

A gauge transformation (4) of these potentials (19)–(23) corresponds to the transformation

$$g \to Uge^{-\Lambda}, \quad h \to Vhe^{-\Lambda}, \tag{24}$$

where Λ is an arbitrary matrix superfield in the (complexified) gauge algebra, which we take to be $gl(n,\mathbb{C})$, and the matrices U and V are $GL(n,\mathbb{C})$ matrices satisfying

$$\mathrm{D}^{i}_{1}U = 0 = \mathrm{D}_{\dot{1}i}U, \quad \partial_{1\dot{1}}U = 0, \tag{25a}$$

$$\mathrm{D}^{i}_{2}V = 0 = \mathrm{D}_{\dot{2}i}V, \quad \partial_{2\dot{2}}V = 0. \tag{25b}$$

The remaining constraints in (10) may now be multiplied by g on the left and g^{-1} on the right to yield the equivalent equations,

$$\mathrm{D}^{(i}_{1}B\,\mathrm{D}^{j)}_{2}B^{-1}) = 0, \tag{26a}$$

$$\mathrm{D}_{\dot{1}(i}(B\,\mathrm{D}_{\dot{2}j)}B^{-1}) = 0, \tag{26b}$$

$$\mathrm{D}^{i}_{1}(B\,\mathrm{D}_{\dot{2}j}B^{-1}) - 2\delta^{i}_{j}gD_{1\dot{2}}g^{-1} = 0, \tag{26c}$$

$$\mathrm{D}_{\dot{1}i}(B\,\mathrm{D}^{j}_{2}B^{-1}) - 2\delta^{j}_{i}gD_{2\dot{1}}g^{-1} = 0. \tag{26d}$$

These equations transform covariantly under the U transformation in (24), being manifestly invariant under the V- and Λ-dependent parts of (24). Equations (26) may be solved by writing

$$B\,\mathrm{D}^{i}_{2}B^{-1} = \mathrm{D}^{i}_{1}x, \tag{27a}$$

$$B\,\mathrm{D}_{\dot{2}j}B^{-1} = \mathrm{D}_{\dot{1}j}y, \tag{27b}$$

where x and y are matrix superfields transforming under the gauge transformation (24) as

$$x \to UxU^{-1}, \quad y \to UyU^{-1}. \tag{28}$$

These superfields satisfy the equations

$$\mathrm{D}_{\dot{1}j}x = 0 = \mathrm{D}^{i}_{1}y \tag{29}$$

in order to satisfy the nondiagonal parts of (26c), (26d) yielding the following expressions for the potentials $A_{1\dot{2}}, A_{2\dot{1}}$ of (22), (23):

$$A_{1\dot{2}} = g^{-1}(\partial_{1\dot{2}} + \partial_{1\dot{1}}y)g, \tag{30a}$$

$$A_{2\dot{1}} = g^{-1}(\partial_{2\dot{1}} + \partial_{1\dot{1}}x)g. \tag{30b}$$

Similarly multiplying the curvature components in (16)–(18) by g on the left and g^{-1} on the right, we see that the first equality in (16)–(18) is identically satisfied, leaving the following forms of the right-hand side equations:

$$\mathrm{D}^{(i}_{2}(B\,\mathrm{D}^{j)}_{2}B^{-1}) + \{B\,\mathrm{D}^{i}_{2}B^{-1}, B\,\mathrm{D}^{j}_{2}B^{-1}\} = 0, \tag{31a}$$

$$\mathrm{D}_{\dot{2}(i}(B\,\mathrm{D}_{\dot{2}j)}B^{-1}) + \{B\,\mathrm{D}_{\dot{2}i}B^{-1}, B\,\mathrm{D}_{\dot{2}j}B^{-1}\} = 0, \tag{31b}$$

$$\mathrm{D}^{i}_{2}(B\,\mathrm{D}_{\dot{2}j}B^{-1}) + \mathrm{D}_{\dot{2}j}(B\,\mathrm{D}^{i}_{2}B^{-1}) + \{B\,\mathrm{D}^{i}_{2}B^{-1}, B\,\mathrm{D}_{\dot{2}j}B^{-1}\} = 2\delta^{i}_{j}gD_{2\dot{2}}g^{-1}. \tag{31c}$$

Inserting the solution (27) into these equations yields the equations

$$\mathrm{D}^{(i}_{2}\mathrm{D}^{j)}_{1}x + \{\mathrm{D}^{i}_{1}x, \mathrm{D}^{j}_{1}x\} = 0, \tag{32a}$$

$$\mathrm{D}_{\dot{2}(i}\mathrm{D}_{\dot{1}j)}y + \{\mathrm{D}_{\dot{1}i}y, \mathrm{D}_{\dot{1}j}y\} = 0, \tag{32b}$$

$$\mathrm{D}^{i}_{2}\mathrm{D}_{\dot{1}j}y + \mathrm{D}_{\dot{2}j}\mathrm{D}^{i}_{1}x + \{\mathrm{D}^{i}_{1}x, \mathrm{D}_{\dot{1}j}y\} = 2\delta^{i}_{j}gD_{2\dot{2}}g^{-1}. \tag{32c}$$

The last of these may be written, using (29),

$$\mathrm{D}^{i}_{1}\mathrm{D}_{\dot{2}j}x + \mathrm{D}_{\dot{1}j}\mathrm{D}^{i}_{2}y + \mathrm{D}_{\dot{1}j}[\mathrm{D}^{i}_{1}x, y] = 2\delta^{i}_{j}(\partial_{1\dot{2}}x + \partial_{2\dot{1}}y + [\partial_{1\dot{1}}x, y] - gD_{2\dot{2}}g^{-1}). \tag{33}$$

The nondiagonal parts of this equation are satisfied if the superfields x and y, in addition to (29), satisfy

$$\mathrm{D}_{\dot{2}j}x = 0, \tag{34a}$$

$$\mathrm{D}^{i}_{2}y + [\mathrm{D}^{i}_{1}x, y] = 0, \tag{34b}$$

yielding

$$A_{2\dot{2}} = g^{-1}(\partial_{2\dot{2}} + \partial_{1\dot{2}}x + \partial_{2\dot{1}}y + [\partial_{1\dot{1}}x, y])g. \tag{35}$$

In terms of the matrix superfields x and y satisfying (29), (34), the equations of motion for the theory therefore take the remarkably simple form of Eq. (32a), (32b).

Introducing the symmetric products of pairs of derivatives (2),

$$\mathrm{D}^{ij} \equiv \mathrm{D}^{\alpha i}\mathrm{D}^{j}_{\alpha} = \mathrm{D}^{ji},$$
$$\overline{\mathrm{D}}_{ij} \equiv \mathrm{D}_{\dot{\alpha}i}\mathrm{D}^{\dot{\alpha}}_{j} = \overline{\mathrm{D}}_{ji}, \tag{36}$$

Eqs. (32a), (32b) may be obtained by varying the Lagrangian density

$$L = \mathrm{tr}\{\overline{\mathrm{D}}_{ij}(\tfrac{1}{2}\mathrm{D}^{i}_{1}x\mathrm{D}^{j}_{2}x + x\mathrm{D}^{i}_{1}x\mathrm{D}^{j}_{1}x) + \mathrm{D}^{ij}(\tfrac{1}{2}\mathrm{D}_{\dot{1}i}y\mathrm{D}_{\dot{2}j}y + y\mathrm{D}_{\dot{1}i}y\mathrm{D}_{\dot{1}j}y)\}. \tag{37}$$

This Lagrangian can be considered to be an integral over a subspace of the odd part of superspace, since up to a total x derivative, a supercovariant derivative D^{i}_{α} is equivalent to an an ordinary spinor derivative $\partial/\partial\vartheta^{\alpha}_{i}$, which in turn is equivalent to a ϑ integration (since $\int d\vartheta\,\vartheta = 1$). The Lagrangian (37) is therefore a sum of integrals over a two-dimensional ϑ subspace and a two-dimensional $\bar{\vartheta}$ subspace. The functional (37) is rather reminiscent of the superinvariants constucted in Ref. 22 by integrating over even-dimensional submanifolds of extended superspace. The constraints (29), (34) on x and y may be incorported using Lagrange multipliers.

III. A CONSTRUCTION OF CONSERVED CURRENTS

Equations (32) are the consistency conditions for the following set of equations:

$$N^{i}\psi \equiv (\mathrm{D}^{i}_{1} + \mu \mathrm{D}^{i}_{2} + \mu \mathrm{D}^{i}_{1} x)\psi = 0, \tag{38a}$$

$$M_{i}\psi \equiv (\mathrm{D}_{\dot{1}i} + \lambda\,\mathrm{D}_{\dot{2}i} + \lambda\,\mathrm{D}_{\dot{1}i} y)\psi = 0, \tag{38b}$$

$$Z\psi \equiv (\partial_{1\dot{1}} + \lambda(\partial_{1\dot{2}} + \partial_{1\dot{1}} y) + \mu(\partial_{2\dot{1}} + \partial_{1\dot{1}} x) + \lambda\mu(\partial_{2\dot{2}} + \partial_{1\dot{2}} x + \partial_{2\dot{1}} y + [\partial_{1\dot{1}} x, y]))\psi = 0, \tag{38c}$$

where ψ is a matrix superfield in the gauge group depending on both parameters μ and λ. Consistency of the above equations is tantamount to requiring that (N^{i}, M_{i}, Z) satisfy the quantum-mechanical supersymmetry algebra,

$$\{N^{i}, N^{j}\} = 0 = \{M_{i}, M_{j}\}, \tag{39a}$$

$$\{N^{i}, M_{j}\} = 2\delta^{i}_{j} Z, \tag{39b}$$

$$[N^{i}, Z] = 0 = [M_{i}, Z]. \tag{39c,d}$$

The brackets (39a) yield relations (32a), (32b) and the relation (39b) corresponds to Eqs. (29), (34). The system (38) may be obtained from (15) by multiplying the latter by g^{-1} on the right and g on the left, inserting the solution (27), and denoting by parameters μ and λ the ratios of the components of the spinors μ^{α} and $\lambda^{\dot{\alpha}}$, $\mu^{2}/\mu^{1} = \mu$, $\lambda^{\dot{2}}/\lambda^{\dot{1}} = \lambda$.

Now we introduce the generating function for superfields $x^{(n)}, y^{(n)}$ $(n \geqslant 0, x^{(0)} = x, y^{(0)} = y)$, $\ln\psi$, where

$$\psi = \lim_{N\to\infty} \psi_{N};$$

$$\psi_{N} = e^{-\lambda^{N} y^{(N-1)}} e^{-\mu^{N} x^{(N-1)}} e^{-\lambda^{N-1} y^{(N-2)}} \times \cdots e^{-\mu^{2} x^{(1)}} e^{-\lambda y} e^{-\mu x}. \tag{40}$$

Requiring that this ψ satisfies (38) immediately yields an infinite number of conserved supercurrents. From (38) we obtain

$$\psi(\mathrm{D}^{i}_{1+\mu 2})\psi^{-1} \equiv \psi(\mathrm{D}^{i}_{1} + \mu\mathrm{D}^{i}_{2})\psi^{-1} = \mu \mathrm{D}^{i}_{1} x, \tag{41a}$$

$$\psi(\mathrm{D}_{\dot{1}+\lambda\dot{2},i})\psi^{-1} \equiv \psi(\mathrm{D}_{\dot{1}i} + \lambda\,\mathrm{D}_{\dot{2}i})\psi^{-1} = \lambda\,\mathrm{D}_{\dot{1}i} y, \tag{41b}$$

$$\psi(\partial_{1\dot{1}} + \mu\partial_{2\dot{1}} + \lambda\partial_{1\dot{2}} + \mu\lambda\partial_{2\dot{2}})\psi^{-1} = \mu\partial_{1\dot{1}} x + \lambda\partial_{1\dot{1}} y + \mu\lambda(\partial_{1\dot{2}} x + \partial_{2\dot{1}} y + [\partial_{1\dot{1}} x, y]). \tag{41c}$$

Equations (41a)–(41b) immediately yield

$$\mathrm{D}^{(i}_{1}(\psi \mathrm{D}^{j)}_{1+\mu 2}\,\psi^{-1}) = 0, \tag{42a}$$

$$\mathrm{D}_{\dot{1}(i}(\psi \mathrm{D}_{\dot{1}+\lambda\dot{2},j)}\,\psi^{-1}) = 0. \tag{42b}$$

Expanding (42) in μ,λ yields the Nth continuity equation as the trace [over the SU(3) indices] of the coefficient of μ^{N+1} and λ^{N+1} in (42a) and (42b), respectively. We consider

$$\psi_{3} = e^{-\lambda^{3} y^{(2)}} e^{-\mu^{3} x^{(2)}} e^{-\lambda^{2} y^{(1)}} e^{-\mu^{2} x^{(1)}} e^{-\lambda y} e^{-\mu x}.$$

Expanding $\psi_{3}\,\mathrm{D}^{i}_{1+\mu 2}\psi_{3}^{-1}$ up to order $\mu^{3}\lambda^{3}$ yields

$$\begin{aligned}
&\mathrm{D}^{i}_{1+\mu 2} + \lambda\,\mathrm{D}^{i}_{1+\mu 2} + (\lambda^{2}/2)[\mathrm{D}^{i}_{1+\mu 2}y,y] + (\lambda^{3}/6)[[\mathrm{D}^{i}_{1+\mu 2}y,y],y] + \mu\mathrm{D}^{i}_{1+\mu 2}x + \lambda\mu[\mathrm{D}^{i}_{1+\mu 2}x,y]\\
&\quad + (\lambda^{2}\mu/2)[[\mathrm{D}^{i}_{1+\mu 2}x,y],y] + (\lambda^{3}\mu/6)[[[\mathrm{D}^{i}_{1+\mu 2}x,y],y],y] + (\mu^{2}/2)[\mathrm{D}^{i}_{1+\mu 2}x,x] + (\lambda\mu^{2}/2)[[\mathrm{D}^{i}_{1+\mu 2}x,x],y]\\
&\quad + (\lambda^{2}\mu^{2}/4)[[[\mathrm{D}^{i}_{1+\mu 2}x,x]y],y] + (\lambda^{3}\mu^{2}/12)[[[[\mathrm{D}^{i}_{1+\mu 2}x,x],y],y],y] + (\mu^{3}/6)[[\mathrm{D}^{i}_{1}x,x],x]\\
&\quad + (\lambda\mu^{3}/6)[[[\mathrm{D}^{i}_{1}x,x],x],y] + (\lambda^{2}\mu^{3}/12)[[[[\mathrm{D}^{i}_{1}x,x],x],y],y] + (\lambda^{3}\mu^{3}/36)[\ldots[\mathrm{D}^{i}_{1}x,x],x],y],y],y]\\
&\quad + \mu^{2}\mathrm{D}^{i}_{1+\mu 2}x^{(1)} + \lambda\mu^{2}[\mathrm{D}^{i}_{1+\mu 2}y,x^{(1)}] + (\lambda^{2}\mu^{2}/2)[[\mathrm{D}^{i}_{1+\mu 2}y,y],x^{(1)}] + (\lambda^{3}\mu^{2}/6)[[[\mathrm{D}^{i}_{1+\mu 2}y,y],y],x^{(1)}]\\
&\quad + \mu^{3}[\mathrm{D}^{i}_{1+\mu 2}x,x^{(1)}] + \lambda^{2}\mathrm{D}^{i}_{1+\mu 2}y^{(1)} + \lambda^{3}[\mathrm{D}^{i}_{1+\mu 2}y,y^{(1)}] + \mu\lambda^{2}[\mathrm{D}^{i}_{1+\mu 2}x,y^{(1)}] + \mu\lambda^{3}[[\mathrm{D}^{i}_{1+\mu 2}x,y],y^{(1)}]\\
&\quad + (\lambda^{2}\mu^{2}/2)[[\mathrm{D}^{i}_{1+\mu 2}x,x],y^{(1)}] + (\lambda^{3}\mu^{2}/2)[[[\mathrm{D}^{i}_{1+\mu 2}x,x],y],y^{(1)}] + (\lambda^{2}\mu^{3}/6)[[[\mathrm{D}^{i}_{1}x,x],x],y^{(1)}]\\
&\quad + (\lambda^{3}\mu^{3}/6)[[[[\mathrm{D}^{i}_{1}x,x],x],y],y^{(1)}] + \lambda^{2}\mu^{2}[\mathrm{D}^{i}_{1+\mu 2}x^{(1)},y^{(1)}] + \lambda^{3}\mu^{2}[[\mathrm{D}^{i}_{1+\mu 2}y,x^{(1)}],y^{(1)}]\\
&\quad + \mu^{3}\mathrm{D}^{i}_{1}x^{(2)} + \mu^{3}\lambda\,[\mathrm{D}^{i}_{1}y,x^{(2)}] + (\mu^{3}\lambda^{2}/2)[[\mathrm{D}^{i}_{1}y,y],x^{(2)}] + (\mu^{3}\lambda^{3}/6)[[[\mathrm{D}^{i}_{1}y,y],y],x^{(2)}]\\
&\quad + \mu^{3}\lambda^{2}[\mathrm{D}^{i}_{1}y^{(1)},x^{(2)}] + \mu^{3}\lambda^{3}[\mathrm{D}^{i}_{1}y,y^{(1)}] + \lambda^{3}\mathrm{D}^{i}_{1+\mu 2}y^{(2)} + \lambda^{3}\mu[\mathrm{D}^{i}_{1+\mu 2}x,y^{(2)}] + (\lambda^{3}\mu^{2}/2)[[\mathrm{D}^{i}_{1+\mu 2}x,x],y^{(2)}]\\
&\quad + (\lambda^{3}\mu^{3}/6)[[[\mathrm{D}^{i}_{1}x,x]x],y^{(2)}] + \lambda^{3}\mu^{2}[\mathrm{D}^{i}_{1+\mu 2}x^{(1)},y^{(2)}] + \mu^{3}\lambda^{3}[\mathrm{D}^{i}_{1}x^{(2)},y^{(2)}].
\end{aligned} \tag{43a}$$

Similarly expanding $\psi_{3}\,\mathrm{D}_{\dot{1}+\lambda\dot{2},i}\psi_{3}^{-1}$ yields [we suppress the SU(3) index i in this formula]

$$\begin{aligned}
&\mathrm{D}_{\dot{1}+\lambda\dot{2}} + \lambda\,\mathrm{D}_{\dot{1}+\lambda\dot{2}} + (\lambda^{2}/2)[\mathrm{D}_{\dot{1}+\lambda\dot{2}}y,y] + (\lambda^{3}/6)[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}y,y],y] + \mu\mathrm{D}_{\dot{1}+\lambda\dot{2}}x + \mu\lambda\,[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,y]\\
&\quad + (\mu\lambda^{2}/2)[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,y],y] + (\mu\lambda^{3}/6)[[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,y],y],y] + (\mu^{2}/2)[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,x] + (\lambda\mu^{2}/2)[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,x],y]\\
&\quad + (\lambda^{2}\mu^{2}/4)[[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,x],y],y] + (\lambda^{3}\mu^{2}/12)[[[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,x],y],y],y] + (\mu^{3}/6)[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,x],x]\\
&\quad + (\lambda\mu^{3}/6)[[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,x],x],y] + (\lambda^{2}\mu^{3}/12)[[[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,x],x],y],y] + (\lambda^{3}\mu^{3}/36)[[[[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,x]x]y]y]y]\\
&\quad + \mu^{2}\mathrm{D}_{\dot{1}+\lambda\dot{2}}x^{(1)} + \mu^{2}\lambda\,[\mathrm{D}_{\dot{1}+\lambda\dot{2}}y,x^{(1)}] + (\mu^{2}\lambda^{2}/2)[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}y,y],x^{(1)}] + (\mu^{2}\lambda^{3}/6)[[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}y,y],y],x^{(1)}]\\
&\quad + \mu^{3}[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,x^{(1)}] + \lambda\mu^{3}[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,y],x^{(1)}] + (\mu^{3}\lambda^{2}/2)[[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,y],y],x^{(1)}]\\
&\quad + (\mu^{3}\lambda^{3}/6)[[[[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,y],y],y],x^{(1)}] + \lambda^{2}\mathrm{D}_{\dot{1}+\lambda\dot{2}}y^{(1)} + \lambda^{3}[\mathrm{D}_{\dot{1}}y,y^{(1)}] + \lambda^{2}\mu[\mathrm{D}_{\dot{1}+\lambda\dot{2}}x,y^{(1)}]
\end{aligned}$$

$$+\lambda^3\mu[[D_{\dot 1+\lambda\dot 2}x,y],y^{(1)}] + (\mu^2\lambda^2/2)[[D_{\dot 1+\lambda\dot 2}x,x],y^{(1)}] + (\mu^2\lambda^3/2)[[[D_{\dot 1+\lambda\dot 2}x,x],y],y^{(1)}]$$
$$+ (\mu^3\lambda^2/6)[[[D_{\dot 1+\lambda\dot 2}x,x],x],y^{(1)}] + (\mu^3\lambda^3/6)[[[[D_{\dot 1+\lambda\dot 2}x,x],x],y],y^{(1)}] + \lambda^2\mu^2[D_{\dot 1+\lambda\dot 2}x^{(1)},y^{(1)}]$$
$$+\mu^2\lambda^3[[D_{\dot 1+\lambda\dot 2}y,x^{(1)}],y^{(1)}] + \lambda^2\mu^3[[D_{\dot 1+\lambda\dot 2}x,x^{(1)}],y^{(1)}] + \lambda^3\mu^3[[[D_{\dot 1}x,y]x^{(1)}],y^{(1)}] + \mu^3 D_{\dot 1+\lambda\dot 2}x^{(2)}$$
$$+\mu^3\lambda[D_{\dot 1+\lambda\dot 2}y,x^{(2)}] + (\mu^3\lambda^2/2)[[D_{\dot 1+\lambda\dot 2}y,y],x^{(2)}] + (\mu^3\lambda^3/6)[[[D_{\dot 1}y,y],y],x^{(2)}]$$
$$+\mu^3\lambda^2[D_{\dot 1+\lambda\dot 2}y^{(1)},x^{(2)}] + \mu^3\lambda^3[[D_{\dot 1}y,y^{(1)}],x^{(2)}] + \lambda^3 D_{\dot 1+\lambda\dot 2}y^{(2)} + \lambda^3\mu[D_{\dot 1}x,y^{(2)}]$$
$$+ (\lambda^3\mu^2/2)[[D_{\dot 1}x,x],y^{(2)}] + (\mu^3\lambda^3/6)[[[D_{\dot 1}x,x],x],y^{(2)}] + \lambda^3\mu^2[D_{\dot 1}x^{(1)},y^{(2)}]$$
$$+\mu^3\lambda^3[[D_{\dot 1}x,x^{(1)}],y^{(2)}] + \lambda^3\mu^3[D_{\dot 1}x^{(2)},y^{(2)}] \quad (43b)$$

and

$$\psi_2(\partial_{1\dot 1} + \mu\partial_{2\dot 1} + \lambda\partial_{1\dot 2} + \mu\lambda\partial_{2\dot 2})\psi_2^{-1}$$
$$= \mu\partial_{1\dot 1}x + \lambda\partial_{1\dot 1}y + \mu\lambda(\partial_{1\dot 2}x + \partial_{2\dot 1}y + [\partial_{1\dot 1}x,y])$$
$$+\mu^2(\partial_{2\dot 1}x + \tfrac{1}{2}[\partial_{1\dot 1}x,x] + \partial_{1\dot 1}x^{(1)}) + \lambda^2(\partial_{1\dot 2}y + \tfrac{1}{2}[\partial_{1\dot 1}y,y] + \partial_{1\dot 1}y^{(1)}) + \mu\lambda^2(\partial_{2\dot 2}y + \tfrac{1}{2}[\partial_{2\dot 1}y,y] + [\partial_{1\dot 2}x,y]$$
$$+\tfrac{1}{2}[[\partial_{1\dot 1}x,y],y] + \partial_{2\dot 1}y^{(1)} + [\partial_{1\dot 1}x,y^{(1)}]) + \lambda\mu^2(\partial_{2\dot 2}x + \tfrac{1}{2}[\partial_{1\dot 2}x,x] + [\partial_{2\dot 1}x,y]$$
$$+\tfrac{1}{2}[[\partial_{1\dot 1}x,x],y] + \partial_{2\dot 1}x^{(1)} + [\partial_{1\dot 1}y,x^{(1)}]) + O(\mu^2\lambda^2). \quad (43c)$$

Therefore, Eqs. (41) contain Eqs. (29), (34),

$$D_{\dot\alpha j}x = 0, \quad (29a), (34a),$$

$$D^i_1 y = 0, \quad (29b),$$

$$D^i_2 y + [D^i_1 x,y] = 0 \quad , \quad (34b),$$

as well as the relations

$$D^i_1 x^{(1)} + D^i_2 x + \tfrac{1}{2}[D^i_1 x,x] = 0, \quad (44a)$$

$$D^i_1 x^{(2)} + D^i_2 x^{(1)} + [D^i_1 x,x^{(1)}] + \tfrac{1}{2}[D^i_2 x,x] + \tfrac{1}{6}[[D^i_1 x,x],x] = 0, \quad (44b)$$

$$D^i_1 y^{(n)} = 0, \quad n\geqslant 0, \quad y^{(0)} = y, \quad (44c)$$

$$D^i_2 y^{(n)} + [D^i_1 x,y^{(n)}] = 0, \quad n\geqslant 0, \quad (44d)$$

$$D^i_1 [x^{(1)},y] = 0, \quad (44e)$$

as coefficients of μ^2, μ^3, λ^{n+1}, $\mu\lambda^{n+1}$, $\mu^2\lambda$, respectively, in (41a), and

$$D_{\dot 1 i}y^{(1)} + D_{\dot 2 i}y + \tfrac{1}{2}[D_{\dot 1 i}y,y] = 0, \quad (45a)$$

$$D_{\dot 1 i}y^{(2)} + D_{\dot 2 i}y^{(1)} + [D_{\dot 1 i}y,y^{(1)}] + \tfrac{1}{2}[D_{\dot 2 i}y,y] + \tfrac{1}{6}[[D_{\dot 1 i}y,y],y] = 0, \quad (45b)$$

$$D_{\dot 1 i}x^{(n)} = 0, \quad n\geqslant 0, \quad x^{(0)} = x, \quad (45c)$$

$$D_{\dot 2 i}x^{(n)} + [D_{\dot 1 i}y,x^{(n)}] = 0, \quad n\geqslant 0, \quad (45d)$$

as coefficients of λ^2, λ^3, μ^{n+1}, $\mu^{n+1}\lambda$, respectively, in (41b). Equation (41c) yields the relations

$$\partial_{2\dot 1}x + \tfrac{1}{2}[\partial_{1\dot 1}x,x] + \partial_{1\dot 1}x^{(1)} = 0, \quad (46a)$$

$$\partial_{2\dot 2}x + \tfrac{1}{2}[\partial_{1\dot 2}x,x] + \partial_{1\dot 2}x^{(1)} + \partial_{1\dot 1}[y,x^{(1)}] = 0, \quad (46b)$$

which together with (29), (34), (44a), and (44e) may be used to verify the integrability condition (39c), as well as the relations

$$\partial_{1\dot 2}y + \tfrac{1}{2}[\partial_{1\dot 1}y,y] + \partial_{1\dot 1}y^{(1)} = 0, \quad (46a')$$

$$\partial_{2\dot 2}y + \tfrac{1}{2}[\partial_{2\dot 1}y,y] + \partial_{2\dot 1}y^{(1)} + [\partial_{1\dot 2}x,y] + \tfrac{1}{2}[[\partial_{1\dot 1}x,y],y] + [\partial_{1\dot 1}x,y^{(1)}] = 0, \quad (46b')$$

which together with (29), (34), and (45a) may similarly be used to verify the integrability condition (39d).

Now defining

$$J_2^{(n)i} = -D^i_1 x^{(n)}, \quad (47a)$$

$$J_{\dot 2 i}^{(n)} = -D_{\dot 1 i}y^{(n)}, \quad (47b)$$

we clearly have

$$D_1^{(i}J_2^{(n)j)} = 0 = D_{\dot 1(i}J_{\dot 2 j)}^{(n)}, \quad (48)$$

and, tracing over the SU (3) indices, yields infinitely many solutions of the conservation equation

$$D^i_1 J_{\dot 2 i} - D_{\dot 1 i}J^i_2 = 0. \quad (49)$$

Thus the coefficient of μ^{N+1} (λ^{N+1}) in (41a) [(41b)] yields an expression for $J_2^{(N)i}$ ($J_{\dot 2 i}^{(N)}$) in terms of $x^{(n)}$ ($y^{(n)}$), $n<N$, and its derivatives. Therefore, given superfields x,y solving Eqs. (32), (29), (34), we need to perform no further integrations in order to successively construct all the $J_2^{(n)i}$'s and $J_{\dot 2 i}^{(n)}$'s explicitly just by taking spinorial derivatives and commutators. From (44), (45), the components of the first two currents, for instance, are

$$J_2^{(1)i} = D^i_2 x + \tfrac{1}{2}[D^i_1 x,x], \quad (50a)$$

$$J_2^{(2)i} = D^i_2\, x^{(1)} + [D^i_1 x,x^{(1)}] + \tfrac{1}{2}[D^i_2 x,x] + \tfrac{1}{6}[[D^i_1 x,x]x], \quad (50b)$$

$$J_{\dot 2 i}^{(1)} = D_{\dot 2 i}x + \tfrac{1}{2}[D_{\dot 1 i}y,y], \quad (50c)$$

$$J_{\dot 2 i}^{(2)} = D_{\dot 2 i}y^{(1)} + [D_{\dot 1 i}y,y^{(1)}] + \tfrac{1}{2}[D_{\dot 2 i}y,y] + \tfrac{1}{6}[[D_{\dot 1 i}y,y]y]. \quad (50d)$$

IV. INFINITESIMAL SYMMETRY TRANSFORMATIONS

Analogously to the infinitesimal symmetry transformations obtained previously in Ref. 15, the equations of motion (32), (29), (34) are left invariant, to first order in the variation, under the transformations

$$\delta x = -\mu^{-1}(\psi T\psi^{-1}) + \xi, \quad (51a)$$

$$\delta y = -\lambda^{-1}(\psi T\psi^{-1}) + \eta, \quad (51b)$$

where ψ satisfies (38), and T is a matrix in the gauge algebra satisfying

$$(D_1^i + \mu D_2^i) T = 0, \quad (52a)$$

$$(D_{1i} + \lambda D_{2i}) T = 0, \quad (52b)$$

$$(\partial_{1\dot{1}} + \lambda \partial_{1\dot{2}} + \mu \partial_{2\dot{1}} + \lambda\mu \partial_{2\dot{2}}) T = 0, \quad (52c)$$

and the matrix superfields ξ and η are defined to be solutions of the following equations:

$$\begin{aligned} &D_1^i\, \xi = 0 = D_{1i} \eta, \\ &D_{\dot{\alpha} j} \xi = \mu^{-1} D_{\dot{\alpha} j} S, \\ &D_1^i\, \eta = \lambda^{-1} D_1^i\, S, \\ &D_2^i\, \eta = \mu^{-1} [D_1^i\, S, y] - \lambda^{-1} D_1^i\, S - [D_1^i\, x, \eta], \end{aligned} \quad (53)$$

where

$$S \equiv \psi T \psi^{-1}. \quad (54)$$

It is clear that (29), (34) are left invariant under the variations (51), whereas the invariance of Eq. (32a) requires the vanishing of

$$D_2^{(i} D_1^{j)} S + \{D_1^{(i} x, D_1^{j)} S\}. \quad (55)$$

Taking the second term in (55)

$$\begin{aligned} \{D_1^{(j} x, D_1^{i)} S\} &= - D_1^{(i} [D_1^{j)} x, S] \\ &= \mu^{-1} D_1^{(i} D_1^{j)} S + D_1^{(i} D_2^{j)} S, \end{aligned}$$

using (38a).

Thus (55) vanishes. The invariance of (32b) may similarly be verified using Eq. (38b).

The algebra of these infinitesimal transformations as well as their integrability to finite (Bäcklund) transformations will be discussed elsewhere.

V. SUPERSYMMETRIC SELF-DUALITY CONDITIONS

The supersymmetric self-duality equations arise as integrability conditions if, in addition to (38), the matrix superfield ψ is required to be μ independent,

$$\frac{\partial}{\partial \mu} \psi = 0. \quad (56)$$

This is tantamount to all the $x^{(n)}$'s, $n \geqslant 0$, in (40) being set to zero. Then Eqs. (38) reduce to

$$D_1^i\, \psi = 0 = (D_2^i + D_1^i x) \psi, \quad (57a)$$

$$(D_{1i} + \lambda D_{2i} + \lambda D_{1i} y) \psi = 0, \quad (57b)$$

$$[\partial_{1\dot{1}} + \lambda(\partial_{1\dot{2}} + \partial_{1\dot{1}} y)] \psi = 0, \quad (57c)$$

$$[\partial_{2\dot{1}} + \lambda(\partial_{2\dot{2}} + \partial_{2\dot{1}} y)] \psi = 0, \quad (57d)$$

a supersymmetrization of the linear system (57c), (57d) for the form of the self-duality equation

$$y_{y\bar{y}} + y_{z\bar{z}} = [y_y, y_z], \quad (58)$$

[where we use Yang's variables $x_{\alpha\beta} = \left[\begin{smallmatrix} y & -\bar{z} \\ z & \bar{y} \end{smallmatrix}\right]$, used by Leznov and Saveliev[20] in their reformulation of the conservation laws for which the generating function $\psi|_{\mu=0}$ in (40) is identical to that found in Ref. 23. Similarly, imposing λ independence on ψ yields the anti-self-duality equations.

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