Self-duality for eight-dimensional gauge theories

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Using a technique based essentially on group theory and an ansatz inspired by the old instanton analysis, we obtain a simplified set of differential equations expressing self-duality for eight-dimensional gauge theories with an SO(8) gauge group. We obtain explicit solutions, some of which, with spherical symmetry, have finite action for a theory defined by a Lagrangian of higher order.

I. INTRODUCTION

Recently, increased attention has been focused on gauge theories in higher-dimensional spaces. First[1] natural extensions of the four-dimensional case were sought for the Lagrangian ($\mu=1,\ldots,d$)

$$L=\tfrac{1}{4}\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) \tag{1.1}$$

with

$$D_\mu=\partial_\mu+A_\mu\ , \tag{1.2a}$$

$$F_{\mu\nu}=[D_\mu,D_\nu]\ . \tag{1.2b}$$

In Eq. (1.1) the trace is understood for the gauge indices. As usual the equations of motion are

$$[D_\mu,F_{\mu\nu}]=0 \tag{1.3}$$

and the Bianchi identities are

$$\sum_{\mu\nu\rho}^{c}[D_\mu,F_{\nu\rho}]=0\ . \tag{1.4}$$

In (1.4) the sum extends on the cyclic (c) permutations of the three indices.

It was immediately shown that the four-dimensional self-duality ($\eta=\pm1$)

$$F_{\mu\nu}=\frac{\eta}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{1.5}$$

could be extended in the secular equation

$$\lambda F_{\mu\nu}=T_{\mu\nu\rho\sigma}F_{\rho\sigma}\ , \tag{1.6}$$

where the numerical tensor T is completely antisymmetrical and λ is an eigenvalue. Equations (1.5) and (1.6), which are linear relations among the F's, guarantee that the equations of motion are satisfied as a consequence of the Bianchi identities [see Eqs. (A1)—(A7) in the Appendix as an example].

It was realized that the case of eight dimensions ($d=8$) is of particular interest. Spinors and vectors have the same dimension and T can be invariant under an SO(7) subgroup with the vector of SO(8) behaving as an eight-dimensional spinor of SO(7). Unfortunately, as expected, the solutions of (1.6), and in particular the spherically symmetric solutions[2] tend to produce F's which behave as the inverse of r^2 at infinity and hence lead to infinite action.

Another generalization of self-duality can be proposed[3,4] for spaces of dimension $4n$. Introduce

$$K_{\alpha_1\cdots\alpha_{2n}}=\sum_P \mathrm{sgn}(P)D_{\alpha_1}\cdots D_{\alpha_{2n}}\ , \tag{1.7}$$

where the sum extends over all the permutations of the α's and sgn(P) is the sign of the permutation. Clearly K is a function of degree n in the fields F in (1.2b). Defining the dual of K by

$$\widetilde{K}_{\alpha_1\cdots\alpha_{2n}}=\frac{1}{2n!}\epsilon_{\alpha_1\cdots\alpha_{2n}\alpha_{2n+1}\cdots\alpha_{4n}}K_{\alpha_{2n+1}\cdots\alpha_{4n}} \tag{1.8}$$

self-duality now reads

$$K_{\alpha_1\cdots\alpha_{2n}}=\eta\widetilde{K}_{\alpha_1\cdots\alpha_{2n}}\ . \tag{1.9}$$

If the Lagrangian is taken to be[4]

$$L=\frac{1}{2(2n)!}\mathrm{Tr}(K_{\alpha_1\cdots\alpha_{2n}}K_{\alpha_1\cdots\alpha_{2n}})\ , \tag{1.10}$$

and F behaves in its normal inverse r^2 law one may expect the action to be marginally convergent.

The equations of motion which follow from (1.10) are quite interesting. Indeed defining

$$R_{\alpha\beta}=\sum_{k=0}^{n-1}F_{\alpha_1\alpha_2}F_{\alpha_3\alpha_4}\cdots F_{\alpha_{2k-1}\alpha_{2k}}K_{\alpha\beta\alpha_1\alpha_2\cdots\alpha_{2n-2}}\times F_{\alpha_{2k+1}\alpha_{2k+2}}\cdots F_{\alpha_{2n-3}\alpha_{2n-2}} \tag{1.11}$$

the equations of motion are equivalent to

$$[D_\alpha, R_{\alpha\beta}]=0 \ . \quad (1.12)$$

These equations are physically acceptable since the starting Lagrangian is at most quadratic in any derivative. Because of the complete antisymmetry of K and the Bianchi identities (1.4), the equations of motion are also

$$\sum_{k=0}^{n-1} F_{\alpha_1\alpha_2} \cdots F_{\alpha_{2k-1}\alpha_{2k}}[D_\alpha, K_{\alpha\beta\alpha_1\cdots\alpha_{2n-2}}] \times F_{\alpha_{2k+1}\alpha_{2k+2}} \cdots F_{\alpha_{2n-3}\alpha_{2n-2}}=0 \ . \quad (1.13)$$

Hence

$$[D_{\alpha_1}, K_{\alpha_1\cdots\alpha_{2n}}]=0 \quad (1.14)$$

are sufficient conditions (but not necessary) to fulfill the equations of motion. Finally it is clear that self-duality (1.9) is even a stronger sufficient condition.

A weaker condition can be obtained as follows. Let $\widetilde{R}$ be defined by a formula analogous to (1.11) with K replaced by $\widetilde{K}$. The sufficient condition is

$$\lambda \widetilde{R}_{\alpha\beta}=R_{\alpha\beta} \ , \quad (1.15)$$

where λ is an arbitrary c number.

In what follows we will analyze the self-duality equations (1.9) in eight dimensions ($n=2$) and for a gauge group SO(8) with an ansatz inspired by the old four-dimensional instanton analysis, the so-called Corrigan-Fairlie-Wilzcek ansatz. In particular we will obtain spherically symmetric solutions with finite action (1.10) and simple generalizations thereof. A few comments will be made on the seemingly less restrictive sufficient conditions (1.15).

II. SO(8) GAUGE THEORY IN EIGHT DIMENSIONS

In order to study the formally spherically symmetric equations for an SO(8) gauge theory in eight dimensions, let us recall that there are three nonequivalent eight-dimensional representations of SO(8) labeled, respectively, by V, W, and X, namely, the vector representation ($i,j,\alpha\beta=1,\ldots,8$)

$$V^{ij}_{\alpha\beta}=\delta^i_\alpha\delta^j_\beta-\delta^j_\alpha\delta^i_\beta \quad (2.1)$$

and two spinor representations conveniently written in terms of the Λ and Ω matrices defined in the Appendix and which were introduced before[1,2] ($A,B=1,\ldots,7$):

$$W^{ij}_{AB}=\Omega^{AB}_{ij} \ , \qquad W^{ij}_{A8}=\Lambda^A_{ij} \quad (2.2a)$$

or

$$X^{ij}_{AB}=\Omega^{AB}_{ij} \ , \qquad X^{ij}_{A8}=-\Lambda^A_{ij} \ . \quad (2.2b)$$

These three representations satisfy evidently the commutation relations

$$[V_{\alpha\beta}, V_{\gamma\delta}]=g_{\beta\gamma}V_{\alpha\delta}+g_{\alpha\delta}V_{\beta\gamma}-g_{\alpha\gamma}V_{\beta\delta}-g_{\beta\delta}V_{\alpha\gamma} \ . \quad (2.3)$$

Formally the potentials A for an SO(8) gauge theory in eight dimensions, if spherical symmetry is imposed, behave as

$$A^{ij}_\mu : \mathbf{28}\times\mathbf{8}_V=\mathbf{8}_V+\mathbf{56}_V+\mathbf{160}_V \ , \quad (2.4)$$

where by definition μ ($\mu=1,\ldots,8$) transforms with the vector representation and ij ($i,j=1,\ldots,8$) are the SO(8) antisymmetric gauge indices (the adjoint 28-dimensional representation).

In analogy with the Corrigan-Fairlie-Wilzcek ansatz, it is tempting to use, in (2.4), only the vector part and write

$$A^{ij}_\mu=W^{ij}_{\mu\nu}\partial_\nu \ln\rho \ . \quad (2.5)$$

Let us remark that we have used a spinor-type connection between the SO(8) i,j indices and the vector indices μ,ν. It can be checked easily that using V in (2.1) instead of W in (2.2a) leads to nothing as interesting. The roles of X in (2.2b) and W in (2.2a) can be exchanged with essentially no new result.

From the ansatz (2.5), the F's can be computed to be

$$F^{ij}_{\mu\nu}=F^1W^{ij}_{\mu\nu}+W^{ij}_{\mu\rho}F^{35}_{\nu\rho}-W^{ij}_{\nu\rho}F^{35}_{\mu\rho} \ , \quad (2.6)$$

where F^1 is a singlet and F^{35} a $\mathbf{35}_V$ symmetrical tensor of zero trace:

$$F^1=-\tfrac{1}{4}[\partial_\mu\partial_\mu\ln\rho+3(\partial_\mu\ln\rho)(\partial_\mu\ln\rho)] \ , \quad (2.7)$$

$$F^{35}_{\alpha\beta}=(\partial_\alpha\ln\rho)(\partial_\beta\ln\rho)-\partial_\alpha\partial_\beta\ln\rho -\tfrac{1}{8}\delta_{\alpha\beta}[(\partial_\mu\ln\rho)(\partial_\mu\ln\rho)-\partial_\mu\partial_\mu\ln\rho] \ . \quad (2.8)$$

Let us remark that in general a spherically symmetric F behaves as

$$F^{ij}_{\alpha\beta} : \mathbf{28}\times\mathbf{28}=\mathbf{1}+\mathbf{28}+\mathbf{35}_V+\mathbf{35}_W+\mathbf{35}_X+\mathbf{300}+\mathbf{350} \quad (2.9)$$

and the ansatz (2.5) has restricted the allowed terms to a singlet and a $\mathbf{35}_V$.

Analogously, K in (1.7) which here can be written

$$K^{ij}_{\alpha\beta\gamma\delta}=\sum_{\beta\gamma\delta}^{c}[F_{\alpha\beta},F_{\gamma\delta}]^{ij}_+ \quad (2.10)$$

behaves as a $\mathbf{1}+\mathbf{35}_W$ in ij and as a $\mathbf{35}_W+\mathbf{35}_X$ in $\alpha\beta\gamma\delta$, i.e., the decomposition of the reducible 70-dimensional completely antisymmetric representation in its self-dual $\mathbf{35}_W$ and anti-self-dual pieces $\mathbf{35}_X$. Hence for the self-dual part

$$K_{\rm SD}: (\mathbf{1}+\mathbf{35}_W)\times\mathbf{35}_W =\mathbf{35}_W+\mathbf{1}+\mathbf{28}+\mathbf{35}_W+\mathbf{294}_W+\mathbf{300}+\mathbf{567}_W \quad (2.11a)$$

and for the anti-self-dual part

$$K_{\rm ASD}: (\mathbf{1}+\mathbf{35}_W)\times\mathbf{35}_X=\mathbf{35}_X+\mathbf{35}_V+\mathbf{350}+\mathbf{840}_V \ . \quad (2.11b)$$

Upon inspection we can make the crucial remark that all representations belonging to the self-dual and anti-self-dual parts are different.

Finally before solving the equation for our ansatz let us note for completeness that the R tensor (1.11) becomes here

$$\tilde{R}^{ij}_{\alpha\beta}=[F_{\gamma\delta},K_{\alpha\beta\gamma\delta}]^{ij}_{+} \tag{2.12}$$

with a spherically symmetric behavior identical to that of F in (2.9). The index content of $\tilde{R}$ is identical to that of R.

III. SPHERICALLY SYMMETRIC SOLUTIONS

We will now study self-duality (1.9) for our ansatz (2.5) in general and then apply our results to full spherical symmetry, i.e., when ρ is restricted to depend only on

$$r^2=\sum_\mu x_\mu{}^2 \,. \tag{3.1}$$

Before turning to the explicit equation let us note that, since K is constructed out of the symmetrized product of two F's in (2.10), which themselves are built out of a singlet (F^1) and a **35** (F^{35}), there can appear only the following representations in K:

$$\begin{aligned}&(F^1)^2 : \mathbf{1}\,,\\ K &: F^1\times F^{35} : \mathbf{35}_V\,,\\ &(F^{35}\times F^{35})_S : \mathbf{1}+\mathbf{35}_V+\mathbf{294}_V+\mathbf{300}\,.\end{aligned} \tag{3.2}$$

Comparing with the allowed content (2.11) we see that $\mathbf{294}_V$ cannot be present and that the **1**'s and the **300** are self-dual while the $\mathbf{35}_V$'s are anti-self-dual.

Solutions of the self-duality equations will thus be obtained by equating to zero either the **1** and the **300** (leading to an anti-self-dual solution) or the $\mathbf{35}_V$ (leading to a self-dual solution). After some algebra one obtains the equations which follow.

Let us denote by $F^>$ the tensors obtained in squaring F^{35}. The relevant pieces behaving as a singlet, a $\mathbf{35}_V$ and a **300** are as follows (the allowed $\mathbf{294}_V$ has not been explicitly written down since it does not play any role in the analysis):

$$\begin{aligned}&F^{>1}=F^{35}_{\alpha\beta}F^{35}_{\alpha\beta}\,,\quad F^{>35}_{\alpha\beta}=F^{35}_{\alpha\gamma}F^{35}_{\beta\gamma}-\tfrac{1}{8}\delta_{\alpha\beta}F^{>1}\,,\\ &F^{>300}_{\alpha\beta\gamma\delta}=2F^{35}_{\alpha\beta}F^{35}_{\gamma\delta}-F^{35}_{\alpha\delta}F^{35}_{\beta\gamma}-F^{35}_{\alpha\gamma}F^{35}_{\beta\delta}+\tfrac{1}{6}(2\delta_{\alpha\beta}F^{>35}_{\gamma\delta}+2\delta_{\gamma\delta}F^{>35}_{\alpha\beta}-\delta_{\alpha\delta}F^{>35}_{\beta\gamma}-\delta_{\gamma\beta}F^{>35}_{\alpha\delta}-\delta_{\alpha\gamma}F^{>35}_{\beta\delta}-\delta_{\beta\delta}F^{>35}_{\alpha\gamma})\\ &\qquad+\tfrac{1}{56}(2\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\gamma}\delta_{\beta\delta})F^{>1}\,.\end{aligned} \tag{3.3}$$

With the use of the Clebsch-Gordan coefficients given in the Appendix, K can be written [see (A9)—(A12)]

$$K^{ij}_{\alpha\beta\gamma\delta}=S^{ij}_{\alpha\beta\gamma\delta}[(F^1)^2-\tfrac{1}{14}F^{>1}]+Q^{ij}_{\alpha\beta\gamma\delta,\rho\sigma}(F^1F^{35}_{\rho\sigma}-\tfrac{1}{3}F^{>35}_{\rho\sigma})+T^{ij}_{\alpha\beta\gamma\delta,\rho\sigma\mu\nu}F^{>300}_{\rho\sigma\mu\nu}\,. \tag{3.4}$$

Recalling the definitions (2.7), (2.8), and (3.3) the equation of self-duality now becomes

$$F^1F^{35}_{\alpha\beta}-\tfrac{1}{3}F^{>35}_{\alpha\beta}=0 \tag{3.5}$$

leading to self-dual solutions or

$$(F^1)^2-\tfrac{1}{14}F^{>1}=0\,,\quad F^{>300}_{\alpha\beta\gamma\delta}=0\,, \tag{3.6}$$

which together lead to anti-self-dual solutions.

Instead of trying to solve these equations in general let us restrict ourselves to the truly spherically symmetric case where ρ depends only on r^2. Let the prime be the derivative with respect to r^2. Then from (2.7)

$$F^1=-(r^2\rho\rho''+4\rho\rho'+2\rho'^2r^2)\rho^{-2}\,, \tag{3.7}$$

$$F^{35}_{\alpha\beta}=4(-\rho\rho''+2\rho'^2)\rho^{-2}x_{\alpha\beta} \tag{3.8}$$

with

$$x_{\alpha\beta}=x_\alpha x_\beta-\tfrac{1}{8}r^2\delta_{\alpha\beta}\,. \tag{3.9}$$

From these, $F^>$ can be computed easily. Since the representation **300** has the symmetry of the Young tableau (2.2), it is identically zero (no antisymmetrical tensor can be obtained from the commuting x's). One obtains

$$\begin{aligned}F^{>1}&=14(-\rho\rho''+2\rho'^2)^2\rho^{-4}r^4\,,\\ F^{>35}_{\alpha\beta}&=12(-\rho\rho''+2\rho'^2)^2\rho^{-4}r^2x_{\alpha\beta}\,.\end{aligned} \tag{3.10}$$

The self-duality equation (3.5), proportional to the tensor (3.9), leads (up to a factor $-\frac{1}{16}$) to

$$(-\rho\rho''+2\rho'^2)(\rho+\rho'r^2)\rho'\rho^{-4}=0 \tag{3.11}$$

while the anti-self-duality (3.6) gives (up to a factor $\frac{1}{8}$)

$$(\rho''r^2+2\rho')(\rho+\rho'r^2)\rho'\rho^{-3}=0\,. \tag{3.12}$$

In (3.11) and (3.12) there are two trivial solutions: first when $\rho'=0$ the A potentials (2.5) are identically zero and the solution is trivial, then when the second factor vanishes

$$\rho+\rho'r^2=0\,. \tag{3.13}$$

K is identically zero, the fields F are also zero but A is nonzero

$$A^{ij}_\mu=-W^{ij}_{\mu\nu}2x_\nu r^{-2}\,. \tag{3.14}$$

The interesting cases are obtained when the first factor in (3.11) or in (3.12) vanishes. In the case of self-duality the solution for ρ is

$$\rho=(cr^2+d)^{-1} \tag{3.15}$$

which leads to (with $\theta=d/c$)

$$A^{ij}_\mu=-W^{ij}_{\mu\nu}2x_\nu(r^2+\theta)^{-1}\,, \tag{3.16a}$$

$$F^{ij}_{\mu\nu}=W^{ij}_{\mu\nu}4\theta(r^2+\theta)^{-2}\,, \tag{3.16b}$$

$$K^{ij}_{\mu\nu\rho\sigma}=16S^{ij}_{\mu\nu\rho\sigma}\theta^2(r^2+\theta)^{-4}\,. \tag{3.16c}$$

In the case of anti-self-duality the solution for ρ is

$$\rho=(c+d\,r^2)r^{-2} \tag{3.17}$$

which leads to (with $\theta = d/c$)

$$A^{ij}_{\alpha} = -W^{ij}_{\mu\nu} 2x_{\nu}(\theta r^2+1)r^{-2} , \tag{3.18a}$$

$$F^{ij}_{\mu\nu} = [r^2 W^{ij}_{\mu\nu} - 4(W^{ij}_{\mu\sigma}x_{\nu\sigma} - W^{ij}_{\nu\sigma}x_{\mu\sigma})] \times 2\theta(1+\theta r^2)^{-2} r^{-2} , \tag{3.18b}$$

$$K^{ij}_{\mu\nu\rho\sigma} = -32 Q^{ij}_{\mu\nu\rho\sigma,\alpha\beta} x_{\alpha\beta} \theta^2 (1+\theta r^2)^{-4} r^{-2} . \tag{3.18c}$$

IV. CONCLUSION

In the previous section, we have obtained two (spherically symmetric) essentially different solutions of the theory defined in Eq. (1.10) and for an eight-dimensional Euclidean space-time and a gauge group SO(8). Let us stress that the way we have produced the solutions is based essentially on group-theory arguments, decomposition of irreducible representations, etc.

The solutions (3.15) and (3.17) are very like the so-called (four-dimensional) instanton;[5] indeed, both are expressed in terms of rational functions, they are invariant under the full rotation group of the space where they are living (spherically symmetric) and both describe some localized object with most of the action concentrated around a single point of space-time.

As expected, trying to relax the hypothesis of spherical symmetry in the above section makes the equations much more complicated. One more solution can be obtained, however, by equating the tensor F^{35} in (2.8) to zero; it takes the form

$$\rho = (a + b_{\mu}x_{\mu})^{-1} , \tag{4.1}$$

where a and b are constant. The function ρ is constant on the family of hyperplanes parallel to

$$b_{\mu}x_{\mu} = 0 \tag{4.2}$$

and singular when

$$b_{\mu}x_{\mu} + a = 0 . \tag{4.3}$$

More interesting would be the discovery of a superposition principle like the multi-instanton solutions[6] (in four-dimensions). This is under investigation; however, the following observation could be helpful. It is the fact that our solutions are still solutions after the substitution

$$r^2 \to (x_{\mu} - f_{\mu})(x_{\mu} - g_{\mu}) , \quad x_{\alpha\beta} \to \tfrac{1}{4}(2x_{\alpha} - f_{\alpha} - g_{\alpha})(2x_{\beta} - f_{\beta} - g_{\beta}) . \tag{4.4}$$

For the self-dual solution (3.15) the above substitution is an obvious redefinition of the integration constants; for the anti-self-dual ones it is more subtle and it enlarges somehow the space of the known solutions.

Finally we would like to stress that the weaker sufficient conditions (1.15) are less amenable to group theory as the representations appearing in R and $\tilde{R}$ do not obviously split with respect to λ.

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APPENDIX

In this appendix, we collect some definition of tensors appearing in the text. First, the seven antisymmetric matrices Λ can be defined by the equations

$$F_{81} + F_{72} + F_{45} + F_{36} = 0 , \tag{A1}$$

$$F_{82} + F_{17} + F_{35} + F_{64} = 0 , \tag{A2}$$

$$F_{83} + F_{74} + F_{52} + F_{61} = 0 , \tag{A3}$$

$$F_{84} + F_{37} + F_{51} + F_{26} = 0 , \tag{A4}$$

$$F_{85} + F_{76} + F_{14} + F_{23} = 0 , \tag{A5}$$

$$F_{86} + F_{57} + F_{13} + F_{42} = 0 , \tag{A6}$$

$$F_{87} + F_{65} + F_{43} + F_{21} = 0 , \tag{A7}$$

introduced in Refs. 1 and 2 and equivalent to ($B = 1, \ldots, 7$)

$$\Lambda^{B}_{\mu\nu} F_{\mu\nu} = 0 . \tag{A8}$$

Then, the 21 antisymmetric Ω matrices are defined by

$$\Omega^{AB}_{\mu\nu} = \tfrac{1}{2}(\Lambda^{A}_{\mu\alpha}\Lambda^{B}_{\nu\alpha} - \Lambda^{B}_{\mu\alpha}\Lambda^{A}_{\nu\alpha}) \tag{A9}$$

completing the full basis of the algebra of SO(8) [see 2.2a)]. The definition of K in Eq. (3.4) involves several tensors entering as Clebsch-Gordan coefficients. The symbol c denotes the sum over the cyclic permutations of the indices written under the summation sign

$$S^{ij}_{\alpha\beta\gamma\delta} = \sum^{c}_{\beta\gamma\delta} [W_{\alpha\beta}, W_{\gamma\delta}]^{ij}_{+} , \tag{A10}$$

$$Q_{\alpha\beta\gamma\delta,\rho\sigma} = \tfrac{1}{2}\sum^{c}_{\beta\gamma\delta} [[[W_{\alpha\beta}, W_{\gamma\rho}]_{+}\delta_{\delta\sigma} + [W_{\gamma\delta}, W_{\alpha\rho}]_{+}\delta_{\beta\sigma} - [W_{\alpha\beta}, W_{\delta\rho}]_{+}\delta_{\gamma\sigma} - [W_{\gamma\delta}, W_{\beta\rho}]_{+}\delta_{\alpha\sigma}] + (\rho,\sigma)] - \mathrm{Tr}(\rho,\sigma) , \tag{A11}$$

$$T_{\alpha\beta\gamma\delta,\mu\nu\rho\sigma}=\frac{1}{24}\Bigg\{\Bigg\{\Bigg\{\Bigg\{\Bigg\{\Bigg\{\sum_{\beta\gamma\delta}^{c}(2[W_{\alpha\mu},W_{\gamma\rho}]_{+}\delta_{\beta\nu}\delta_{\delta\sigma} - [W_{\alpha\mu},W_{\gamma\nu}]\delta_{\beta\rho}\delta_{\delta\sigma}-[W_{\alpha\mu},W_{\gamma\nu}]_{+}\delta_{\beta\sigma}\delta_{\delta\rho})\Bigg\}-(\alpha,\beta)\Bigg\}-(\gamma,\delta)\Bigg\} + (\mu,\nu)\Bigg\}+(\rho,\sigma)\Bigg\}+(\mu\nu,\rho\sigma)\Bigg\}-\mathrm{Tr}(\mu,\nu,\rho,\sigma)\ . \tag{A12}$$

The gauge indices have not been written explicitly. In (A11) and (A12), terms like $+(\alpha,\beta)$ $[-(\alpha,\beta)]$ mean symmetrization [antisymmetrization] with respect to the interchange of α and β. The symbol $-\mathrm{Tr}$ means that the usual trace terms have to be subtracted in such a way that all relevant traces of the tensor are zero.

It is easy to check that the S and T tensors are self-dual and that Q is anti-self-dual with respect to their indices $\alpha\beta\gamma\delta$.

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