Yang–Mills connections over manifolds with Grassmann structure

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Let M be a manifold with Grassmann structure, i.e., with an isomorphism of the cotangent bundle $T^*M \cong E \otimes H$ with the tensor product of two vector bundles E and H. We define the notion of a half-flat connection ∇^W in a vector bundle $W \to M$ as a connection whose curvature $F \in S^2E \otimes \Lambda^2 H \otimes \mathrm{End}\ W \subset \Lambda^2 T^*M \otimes \mathrm{End}\ W$. Under appropriate assumptions, for example, when the Grassmann structure is associated with a quaternionic Kähler structure on M, half-flatness implies the Yang–Mills equations. Inspired by the harmonic space approach, we develop a local construction of (holomorphic) half-flat connections ∇^W over a complex manifold with (holomorphic) Grassmann structure equipped with a suitable linear connection. Any such connection ∇^W can be obtained from a prepotential by solving a system of linear first order ODEs. The construction can be applied, for instance, to the complexification of hyper-Kähler manifolds or more generally to hyper-Kähler manifolds with admissible torsion and to their higher-spin analogs. It yields solutions of the Yang–Mills equations.
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I. INTRODUCTION

The Yang–Mills self-duality equations have played an important role in field theory and in differential geometry. They are the main source of examples of solutions of the Yang–Mills equations on four-dimensional manifolds.[1] The self-duality equations $*F^\nabla = F^\nabla$ mean that the curvature F^∇ of a connection ∇ over a Riemannian four-fold M is an eigenvector of the Hodge star operator, associated with the volume four-form, which acts on two-forms. This apparently four-dimensional construction has an analog in Riemannian manifolds M of arbitrary dimensions. Any four-form Ω on M defines an endomorphism B_Ω of the space of two-forms and one can define (Ω,λ)-self-duality as the condition, $B_\Omega F^\nabla = \lambda F^\nabla$, that the curvature is an eigenvector of B_Ω with eigenvalue $\lambda = \mathrm{const} \neq 0$. Under appropriate assumptions on Ω (for example, if it is co-closed) this implies the Yang–Mills equations, just as in four dimensions. For instance, this works for a constant Ω in flat space[2,3] and for a parallel four-form on a Riemannian manifold with special holonomy (some examples are discussed in Refs. 4–8). If Ω is, for example, the canonical parallel four-form associated to a quaternionic Kähler manifold M of dimension $4m$, then the eigenspaces of B_Ω are the irreducible $\mathrm{Sp}(m)\cdot \mathrm{Sp}(1)$-submodules of the space of two-forms. In terms of the

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associated locally defined Grassmann structure $T^{*\mathbb{C}}M = E\otimes H$, i.e., the identification of the complexified cotangent bundle $T^{*\mathbb{C}}M$ with a tensor product of two vector bundles E and H of rank $2m$ and 2, respectively, the eigenspace decomposition is given by

$$\Lambda^2 T^{*\mathbb{C}}M = S^2E\otimes\Lambda^2H \oplus \Lambda_0^2E\otimes S^2H \oplus \omega_E\otimes S^2H,$$

with corresponding B_Ω-eigenvalues $\lambda_1=1, \lambda_2=-1/3, \lambda_3=-(2m+1)/3$.[3,9] Here ω_E and ω_H are two-forms on E^* and H^* such that the complex metric on $T^{\mathbb{C}}M$ is given by $\omega_E\otimes\omega_H$ and Λ_0^2E denotes the traceless part of Λ^2E with respect to ω_E. The eigenspaces of B_Ω can thus be described in terms of the Grassmann structure, which is a natural generalization of the well-known spinor decomposition of a vector in four dimensions. A two-form on any manifold with Grassmann structure is called half-flat if it belongs to the eigenspace $S^2E\otimes\Lambda^2H$ and a connection ∇ with half-flat curvature is called half-flat. If the Grassmann structure is associated with the quaternionic Kähler structure, then a half-flat connection is the same as an (Ω,λ_1)-self-dual connection and hence satisfies the Yang–Mills equations. Inspired by the harmonic space approach,[10] we develop a construction of locally defined holomorphic half-flat connections on a manifold M with holomorphic admissible half-flat Grassmann structure, namely, a holomorphic Grassmann structure $T^*M=E\otimes H$ with holomorphic connections ∇^E and ∇^H in the bundles E and H, respectively, such that ∇^H is flat and the torsion of the linear connection $\nabla=\nabla^E\otimes \mathrm{Id}+\mathrm{Id}\otimes\nabla^H$ has no component in $S^3H\otimes E^*\otimes\Lambda^2E$. The construction associates to a holomorphic prepotential a half-flat connection through the solution of a system of linear first order ODEs. The construction can be applied, for instance, to the complexification of hyper-Kähler manifolds or, more generally, to hyper-Kähler manifolds with admissible torsion. Our construction of gauge fields on such curved backgrounds extends that of Ref. 10, where flat torsion-free backgrounds were considered. Moreover, we provide a geometric description of the harmonic space method of Ref. 10.

We note that using analytic continuation any real analytic connection ∇ over a real analytic Grassmann manifold allows extension to a holomorphic connection $\nabla^{\mathbb{C}}$ over a holomorphic Grassmann manifold and ∇ can be reconstructed from $\nabla^{\mathbb{C}}$ in terms of some antiholomorphic involution.

The main idea of our construction is to pull-back a half-flat connection ∇ in a holomorphic vector bundle $\nu: W\to M$ to the harmonic space S_H. The latter is the space of all symplectic frames $h=(h_+,h_-)$ in the vector bundle H^*. The group $\mathrm{Sp}(1,\mathbb{C})$ acts freely on S_H, with the orbit space $S_H/\mathrm{Sp}(1,\mathbb{C})=M$. Hence, the projection $\pi: S_H\to M$ is an $\mathrm{Sp}(1,\mathbb{C})$-principal bundle. Choosing a (local) trivialization, $M\ni x\mapsto(h_1(x),h_2(x))\in S_H$, of π we can make the identification $S_H=\mathrm{Sp}(1,\mathbb{C})\times M$. There exists a canonical decomposition,

$$TS_H = T^vS_H\oplus\mathcal{D}_+\oplus\mathcal{D}_-,$$

of the (holomorphic) tangent bundle into the vertical subbundle T^vS_H and two (holomorphic) distributions $\mathcal{D}_+$ and $\mathcal{D}_-$ spanned, respectively, by vector fields X_+^e and X_-^e canonically associated with sections e of the bundle E^*. If the Grassmann structure is admissible and half-flat, the distributions $\mathcal{D}_+$ and $\mathcal{D}_-$ are integrable. The vertical distribution T^vS_H is spanned by vector fields $\partial_0, \partial_{++}, \partial_{--}$, which correspond to the standard generators of the Lie algebra $\mathfrak{sp}(1,\mathbb{C})$. A half-flat connection ∇ in the bundle $\nu: W\to M$ induces the pull-back connection $\pi^*\nabla$ in the pull-back bundle $\pi^*\nu: \pi^*W\to S_H$. Since ∇ is half-flat, the curvature F of $\pi^*\nabla$ satisfies certain equations (see Definitions 6 and 7). A connection in $\pi^*\nu$ satisfying these equations is called a half-flat connection over S_H and is gauge equivalent to the pull-back of a half-flat connection over M. Any half-flat connection over S_H is flat along the leaves of the integrable distribution $\langle\mathcal{D}_+,\partial_0\rangle$ spanned by ∂_0 and $\mathcal{D}_+$. We can therefore choose a frame of the vector bundle $\pi^*\nu$ which is parallel along its leaves. Such a frame is called an analytic frame. With respect to such a frame a half-flat connection has no potentials in the directions of the distribution $\langle\mathcal{D}_+,\partial_0\rangle$. Starting from a matrix-valued function (prepotential) A_{++} on S_H, which is constant along the leaves of the distribution $\mathcal{D}_+$ and satisfies the homogeneity condition $\partial_0A_{++}=2A_{++}$, we construct a connection which satisfies almost all the conditions of half-flatness. We call such a connection an almost half-flat connection. It is half-flat if and only if its curvature satisfies the equation $F(\partial_{--},\mathcal{D}_-)=0$. The construction of an almost half-flat connection reduces to the solution of first order linear ODEs. Assuming that the almost half-flat connection ∇ is defined globally along the fibers (over $\pi^{-1}U$, where $U \subset M$ is a domain in M) we can modify ∇ to a half-flat connection over S_H which is the pull-back of a half-flat connection over M. In order to do this, we rewrite ∇ with respect to a "central frame," namely, a frame parallel along the fibers of π. The transformation from the analytic to the central frame reduces to the solution of the system of equations

$$\partial_{++}\Phi = -A_{++}\Phi, \quad \partial_0 \Phi = 0.$$

With respect to the central frame the potential $C(X_+^e)$ of the connection ∇ in the direction of the vector field $X_+^e \in \mathcal{D}_+$ has the form $C(X_+^e) = u_+^\alpha C_\alpha^e$, where C_α^e are matrix-valued functions on $M = M \times \{\mathrm{Id}\} \subset M \times \mathrm{Sp}(1,\mathbb{C})$ and $u_\pm^\alpha, \alpha = 1,2$, are matrix coefficients of $\mathrm{Sp}(1,\mathbb{C})$. The matrix-valued functions C_1^e, C_2^e define the desired half-flat connection on M given by

$$\nabla^M_{e\otimes h_1} = e \otimes h_1 + C_1^e, \quad \nabla^M_{e\otimes h_2} = e\otimes h_2 + C_2^e.$$

Moreover, any half-flat connection may be obtained in this way.

The above construction allows generalization to manifolds with spin $m/2$ Grassmann structure. This means that the cotangent bundle is identified as $T^*M = E \otimes F = E \otimes S^m H$, where E and H are (holomorphic) vector bundles of rank p and 2, respectively. If a connection ∇^E on E and a flat connection ∇^H on H are given, then the Grassmann structure is called half-flat. The connection ∇^H defines a flat connection ∇^F on $F = S^m H$ and the linear connection $\nabla = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^F$. The associated harmonic space $\pi: S_H \to M$ is defined as above, as the space of all symplectic frames $h = (h_+, h_-)$ in H^*. Its tangent space has decomposition

$$TS_H = T^v S_H \oplus \bigoplus_{k=0}^{m} \mathcal{D}_{k+} \oplus \bigoplus_{k=1}^{m} \mathcal{D}_{k-}.$$

Under certain conditions on the torsion of ∇ the distribution $\mathcal{D}_{(+)}^k := \oplus_{i=0}^k \mathcal{D}_{(m-2i)+}$, $k \leq m/2$, is integrable. Such a half-flat Grassmann structure is called k-admissible. Generalizing the notion of a half-flat connection, we may define a k-partially flat connection ∇ over a manifold with half-flat spin $m/2$ Grassmann structure such that the pull-back connection $\pi^*\nabla$ has no curvature in the directions of $\mathcal{D}_{(+)}^k$. The harmonic space method can be applied to construct k-partially flat connections over k-admissible half-flat spin $m/2$ Grassmann manifolds. In the final section we consider the case of $m = 3$ and sketch the construction of zero- and one-partially flat connections. The latter are Yang–Mills connections.

II. GENERALIZED SELF-DUALITY FOR MANIFOLDS OF DIMENSION GREATER THAN FOUR

A. Yang–Mills data

Let $\nu: W \to M$ be a real vector bundle over M and ∇ a connection in ν, that is a bilinear map

$$\nabla: \mathfrak{X}(M) \times \Gamma(\nu) \to \Gamma(\nu),$$

$$(X, \sigma) \mapsto \nabla_X \sigma,$$

which is $C^\infty(M)$-linear in the vector field $X \in \mathfrak{X}(M)$ and satisfies the Leibniz rule $\nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X \sigma$, for any function $f \in C^\infty(M)$ and any section, $\sigma \in \Gamma(\nu)$, of ν. The map ∇ can be extended to a complex bilinear map,

$$\nabla: \mathfrak{X}^{\mathbb{C}}(M) \times \Gamma(W^{\mathbb{C}} \to M) \to \Gamma(W^{\mathbb{C}} \to M),$$

$$(X, \sigma) \mapsto \nabla_X \sigma, \tag{1}$$

where $\mathfrak{X}^{\mathbb{C}}(M)$ is the space of complex vector fields $X+iY$; $X,Y\in\mathfrak{X}(M)$ and $W^{\mathbb{C}}\to M$ is the complexification of the vector bundle ν. Note that ∇ satisfies the reality condition

$$\nabla_{\bar{X}}\bar{\sigma}=\overline{\nabla_X\sigma},\quad X\in\mathfrak{X}^{\mathbb{C}}(M),\quad \sigma\in\Gamma(W^{\mathbb{C}}\to M), \tag{2}$$

where the bar denotes complex conjugation. Conversely, any $\mathbb{C}$-bilinear map (1) which is $\mathfrak{X}^{\mathbb{C}}(M)$-linear and satisfies the Leibniz rule and the reality condition (2) defines a connection ∇ in the real vector bundle ν. If the reality condition (2) is dropped, then (1) defines a connection in the complex vector bundle $W^{\mathbb{C}}\to M$.

Let $\varphi=(\varphi_1,\ldots,\varphi_r)$ denote a local frame of ν such that for any section $\sigma\in\Gamma(\nu)$, $\sigma=\Sigma s^i\varphi_i=:\varphi\cdot s$, where s^i are the coordinates of σ with respect to the frame φ and $s=(s^1,\ldots,s^r)^t$. Then the connection ∇ in ν has local expression

$$\nabla_X\sigma=\nabla_X(s^i\varphi_i)=\varphi\cdot\nabla_X s:=\left(Xs^i+\sum_j A^i_j(X)s^j\right)\varphi_i,$$

where $A^i_j(X)=(\nabla_X\varphi_j,\varphi^i)$ and $\varphi^*=(\varphi^1,\ldots,\varphi^r)$ denotes the dual frame. The locally defined matrix-valued one-form $A=(A^i_j):M\to\mathfrak{gl}(r,\mathbb{R})$ is called the Yang–Mills potential with respect to the frame φ. If the vector bundle ν has structure group G, i.e., if it is a bundle associated with a principal G-bundle $P\to M$ and a representation $\rho:G\to GL(r,\mathbb{R})$, such that $W=P\times_G\mathbb{R}^r$, then we may always choose a frame φ for which the potential takes values in the Lie algebra $\mathfrak{g}=\mathrm{Lie}\,\rho(G)\subset\mathfrak{gl}(r,\mathbb{R})$. We will symbolically write $\nabla_X=X+A$, $A=A^{\varphi}$. A change of frame (gauge transformation) $\varphi'=\varphi U$ induces changes $s'=U^{-1}s$ and $\varphi'(X+A'(X))s'=\varphi'\nabla_X s'=\varphi\nabla_X s=\varphi(X+A(X))s=\varphi' U^{-1}(X+A(X))Us'$, yielding the transformation rule for the potential,

$$A'=U^{-1}(XU)+U^{-1}A(X)U=U^{-1}\nabla_X U. \tag{3}$$

The curvature of the connection ∇, $F=F^{\nabla}\in\Omega^2_M(\mathrm{End}\,W)=\Gamma(\Lambda^2T^*M\otimes\mathrm{End}\,W)$, is given by

$$F(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}=XA(Y)-YA(X)+[A(X),A(Y)]-A([X,Y]).$$

The Jacobi identity for ∇_X is equivalent to the Bianchi identity, $d^{\nabla}F^{\nabla}=0$. Here the covariant derivative $d^{\nabla}:\Omega^p(\mathrm{End}\,W)\to\Omega^{p+1}(\mathrm{End}\,W)$ is defined by

$$d^{\nabla}(\omega\otimes C)=d\omega\otimes C+(-1)^p\omega\wedge\nabla C,$$

where ω is a p-form and C is a section of End W. (The connection ∇ on W induces a connection on End W denoted by the same symbol.)

On any n-dimensional oriented pseudo-Riemannian (or complex Riemannian) manifold, (M,g) using the canonical volume form $\mathrm{vol}^g\in\Lambda^nT^*M$, we define the Hodge $*$ operator which interchanges forms of complementary degree, $*:\Lambda^pT^*M\to\Lambda^{n-p}T^*M$, by the relation $\langle\alpha,\beta\rangle\mathrm{vol}^g=\alpha\wedge*\beta$, where $\alpha,\beta\in\Lambda^pT^*M$ and $\langle.,.\rangle$ is the natural scalar product on Λ^pT^*M induced by the metric g. We define $*:\Lambda^pT^*M\otimes\mathrm{End}\,W\to\Lambda^{n-p}T^*M\otimes\mathrm{End}\,W$ by $*(\omega\otimes C):=(*\omega\otimes C)$.

Definition 1: Let $\nu:W\to M$ be a real vector bundle over a pseudo-Riemannian manifold (M,g). A **YM connection** *∇ in ν is one which satisfies the Yang–Mills equation*

$$d^{\nabla}*F^{\nabla}=0.$$

On a closed manifold this is the Euler–Lagrange equation for the YM functional

$$\|F^{\nabla}\|^2=\int_M |F^{\nabla}|^2\ \mathrm{vol}^g, \tag{4}$$

where the norm on $\Lambda^2 T^*M \otimes \mathrm{End}\, W$ is induced by the pseudo-Riemannian metric on M and the natural metric on End W.

B. Self-duality conditions

On a Riemannian four-manifold, the $*$ operator maps two-forms to two-forms and has eigenvalues ± 1. The curvature tensor therefore has decomposition into the eigenspaces of the $*$ operator,

$$F^\nabla = F^\nabla_{+1} \oplus F^\nabla_{-1} \in \Omega^+_M(\mathrm{End}\, W) \oplus \Omega^-_M(\mathrm{End}\, W).$$

This splitting corresponds to the decomposition of the SO(4)-module $\Lambda^2\mathbb{R}^4 = \Lambda^2_+ \oplus \Lambda^2_- \cong \mathfrak{so}(4) = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ into its irreducible submodules. We call ∇ and F^∇ self-dual or anti-self-dual if $F^\nabla_{-1} := \frac{1}{2}(F^\nabla - *F^\nabla) = 0$ or $F^\nabla_{+1} := \frac{1}{2}(F^\nabla + *F^\nabla) = 0$, respectively. For (anti-)self-dual connections, the YM equation, $d^\nabla * F^\nabla = 0$, is an immediate consequence of the Bianchi identity, $d^\nabla F^\nabla = 0$. On closed manifolds (anti-) self-dual connections in fact minimize the YM functional (4), since the inequality

$$\|F^\nabla\|^2 = \|F^\nabla_{+1}\|^2 + \|F^\nabla_{-1}\|^2 \geqslant |\|F^\nabla_{+1}\|^2 - \|F^\nabla_{-1}\|^2| = 8\pi^2 |c_2(W)[M]|$$

is saturated. Here $c_2(W)[M] = (1/8\pi^2)\int_M \mathrm{tr}\quad F^\nabla \wedge F^\nabla$ is the evaluation of the second Chern class of the bundle W on the fundamental cycle.

The apparently four-dimensional notion of self-duality has an analog in higher dimensions. The construction originally given in Ref. 2 for flat spaces extends to arbitrary manifolds (M,g), of dimension greater than four, as follows.

For $\Omega \in \Omega^4(M)$ we define a symmetric tracefree endomorphism field $B_\Omega : \Lambda^2 T^*M \to \Lambda^2 T^*M$ by

$$B_\Omega \omega := *(*\Omega \wedge \omega), \tag{5}$$

where $\omega \in \Lambda^2 T^*M$. This endomorphism is zero if and only if the four-form Ω is zero. Moreover, we have the following.

Lemma 1: Let

$$\Omega = \sum \Omega_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l, \quad \omega = \sum \omega_{ij} e^i \wedge e^j$$

be the expressions for Ω and ω with respect to a frame e^i of T^*M. Then B_Ω is given as the contraction

$$B_\Omega \omega = 12 \sum g^{ii'} g^{jj'} \; \Omega_{ijkl} \; \omega_{i'j'} \; e^k \wedge e^l.$$

Proof: It is sufficient to check the above formula for decomposable forms $\Omega = e^i \wedge e^j \wedge e^k \wedge e^l$ and $\omega = e^m \wedge e^n$, where the e^i form an orthonormal basis of T^*M. □

Definition 2: A four-form $\Omega \in \Omega^4(M)$ on a pseudo-Riemannian manifold M is called **appropriate** *if there exists a nonzero real constant eigenvalue λ of the endomorphism field B_Ω.*

We note that on a Riemannian manifold the eigenvalues of B_Ω are real for any four-form Ω. A generalization of the four-dimensional notion of self-duality may now be defined:

Definition 3: Let Ω be an appropriate four-form on a pseudo-Riemannian manifold (M,g) and $\lambda \neq 0 \in \mathbb{R}$. A connection ∇ in a vector bundle $\nu: W \to M$ is **(Ω,λ)-self-dual** *if its curvature F^∇ satisfies the linear algebraic system*

$$B_\Omega F^\nabla = \lambda F^\nabla, \tag{6}$$

$$(d*\Omega)\wedge F^{\nabla}=0\,. \tag{7}$$

Theorem 1: *Let* (M,g) *be a pseudo-Riemannian manifold with an appropriate four-form* Ω. *Then any* (Ω,λ)*-self-dual connection* ∇ *is a YM connection.*

Proof: Using (6) and (5) we obtain

$$d^{\nabla}*F^{\nabla}=\frac{1}{\lambda}d^{\nabla}*B_{\Omega}F^{\nabla}=\pm\frac{1}{\lambda}d^{\nabla}(*\Omega\wedge F^{\nabla})=\pm\frac{1}{\lambda}((d*\Omega)\wedge F^{\nabla}+*\Omega\wedge d^{\nabla}F^{\nabla})=0,$$

in virtue of (7) and the Bianchi identity $d^{\nabla}F^{\nabla}=0$. □

Examples of manifolds admitting appropriate four-forms are easily obtained. Let V be a pseudo-Euclidean vector space and $G\subset \mathrm{SO}(V)$ be a linear group preserving a nonzero element $\Omega_0\in\Lambda^4V$. Denote by Ω_{ijkl} the components of Ω_0 with respect to an orthonormal basis of V. Given a manifold M with a G-structure, $\pi:P\rightarrow M$, i.e., a principal G-subbundle of the bundle of frames on M, we can define a four-form $\Omega:=\Sigma\Omega_{ijkl}e^i\wedge e^j\wedge e^k\wedge e^l$, where $(e^1,\ldots,e^n)$ is a coframe dual to a G-frame $p=(e_1,\ldots,e_n)\in P$. Since $G\subset\mathrm{SO}(V)$, M has the structure of an oriented pseudo-Riemannian manifold and we can define the operator B_{Ω}. The matrix components of $B_{\Omega}=\Sigma B_{ij}^{kl}e^i\wedge e^j\otimes e_k\wedge e_l$ are constant for any G-frame and so are its eigenvalues. Hence Ω is appropriate if the endomorphism $B_{\Omega_0}\in\Lambda^4V$ has a nonzero real eigenvalue λ. This is automatic in the Riemannian case.

There exist many examples of subgroups $G\subset\mathrm{SO}(V)$ admitting nonzero G-invariant four-forms, as shown by the following construction. Let $G\subset\mathrm{SO}(V)$ be a closed subgroup of the pseudo-orthogonal group $\mathrm{SO}(V)$ and $\mathfrak{g}\subset\mathfrak{so}(V)\cong\Lambda^2V^*$ its Lie algebra. Assume that $\mathfrak{g}$ admits a G-invariant symmetric nondegenerate bilinear form $B\in S^2(\mathfrak{g}^*)^G$, where W^G denotes the space of G-invariant elements of a G-module W. We can then identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ via B and consider B as an element of $(S^2(\mathfrak{g}))^G\subset(S^2\Lambda^2V^*)^G$. A G-invariant four-form is then defined by $\Omega_0^G:=\mathrm{alt}B\in(\Lambda^4V^*)^G$, where $\mathrm{alt}:S^2\Lambda^2V^*\rightarrow\Lambda^4V^*$ denotes alternation. We denote the corresponding four-form on a manifold with G-structure by Ω^G. The following variant of a theorem by Kostant[11] provides a wealth of examples of nonzero Ω_0^G's.

Theorem 2: *Let* $G\subset\mathrm{SO}(V)$ *be a closed subgroup whose Lie algebra* $\mathfrak{g}$ *admits a nondegenerate* G*-invariant bilinear form* $B\in(S^2\mathfrak{g})^G$. *If the* G*-module* V *is not equivalent to the isotropy module of a pseudo-Riemannian symmetric space, then the four-form* $\Omega_0^G:=\mathrm{alt}B\in(\Lambda^4V)^G$ *is nonzero.*

Proof: Recall that the $\mathrm{SO}(V)$-module $S^2\Lambda^2V$ decomposes according to $S^2\Lambda^2V=\mathcal{R}(\mathfrak{so}(V))+\Lambda^4V$, where $\mathcal{R}(\mathfrak{so}(V))$ denotes the space of curvature tensors of type $\mathfrak{so}(V)$, i.e., the space of two-forms fulfilling the first Bianchi identity or the kernel of the map $\mathrm{alt}:S^2\Lambda^2V\rightarrow\Lambda^4V$. If $\Omega_0^G=\mathrm{alt}B=0$, then B is a nonzero element of $\mathcal{R}(\mathfrak{so}(V))\cap S^2(\mathfrak{g})^G=\mathcal{R}(\mathfrak{g})^G$. Since B is a G-invariant two-form on V with values in $\mathfrak{g}$ it can be used to define a Lie bracket $[\,\cdot\,,\cdot\,]$ on the vector space $\mathfrak{l}=\mathfrak{g}\oplus V$ thus,

(i) $\mathfrak{g}$ is a subalgebra of $\mathfrak{l}$,
(ii) V is a $\mathfrak{g}$-submodule with action defined by the inclusion $\mathfrak{g}\subset\mathfrak{so}(V)$, and
(iii) $[u,v]:=B(u,v)\in\mathfrak{g}$ if $u,v\in V$.

The Jacobi identity follows from the Bianchi identity and the G-invariance. Let L be the simply connected Lie group with Lie algebra $\mathfrak{l}$. Then L/G_0 is a Riemannian symmetric space with V as its isotropy module, where $G_0\subset L$ is the connected Lie subgroup with $\mathrm{Lie}G_0=\mathfrak{g}$. □

Clearly, (7) is automatic if the four-form Ω is co-closed, $d*\Omega=0$. This is the case, for example, if Ω is parallel. In the Riemannian case the Berger list of irreducible holonomy groups[12] and a theorem of Kostant[11] yield the following result.

Theorem 3: *Let* M *be a complete simply connected irreducible Riemannian manifold of dimension* $n\geqslant 4$ *with holonomy group* $\mathrm{Hol}\subset\mathrm{SO}(n)$, $\mathrm{Hol}\neq\mathrm{SO}(n)$. *Then* M *admits a nontrivial parallel four-form if one of the following holds: (i)* M *is not a symmetric space or (ii)* M *is a symmetric space and has a nonsimple holonomy or, equivalently, isotropy group.*

Proof: By Berger's theorem on Riemannian irreducible holonomy groups,[12] we have

(a) M is not a symmetric space and its holonomy group is one of $\mathrm{U}(n/2)$, $\mathrm{SU}(n/2)$, $\mathrm{Sp}(n/4)\mathrm{Sp}(1)$, $\mathrm{Sp}(n/4)$, G_2, $\mathrm{Spin}(7)$, or
(b) M is a symmetric space.

All the groups in (a) admit invariant four-forms. These are given below. A theorem of Kostant[11] states that a simply connected irreducible Riemannian symmetric space G/K has no nonzero parallel four-form if and only if the isotropy group K is simple. □

In the following examples we explicitly describe parallel (hence appropriate) four-forms Ω on Riemannian n-manifolds with holonomy groups $\mathrm{Hol}\neq \mathrm{SO}(n)$ from Berger's list.

(1) Kähler manifolds, $\mathrm{Hol}\subset \mathrm{U}(m)\subset \mathrm{SO}(2m)$, $n=2m$: $\Omega=\omega\wedge\omega$, where ω is the Kähler form. One can check that this is proportional to $\Omega^{\mathrm{SU}(m)}$ and that any parallel four-form is proportional to $\omega\wedge\omega$ if the holonomy group is $\mathrm{SU}(m)$ or $\mathrm{U}(m)$. If $\mathrm{Hol}\subset \mathrm{Sp}(k)\subset \mathrm{SU}(2k)\subset \mathrm{SO}(4k)$, $n=4k>4$, i.e., if the manifold is hyper-Kähler, there exist three skewsymmetric parallel complex structures $J_\alpha, \alpha=1,2,3$. Then there exist six independent parallel four-forms $\omega_\alpha\wedge\omega_\beta$, $\alpha,\beta=1,2,3$, where ω_α is the Kähler form associated to J_α. For low dimensional examples, eigenvalues and eigenspaces of B_Ω are given in Ref. 2.

(2) Quaternionic Kähler manifolds, $\mathrm{Hol}\subset \mathrm{Sp}(m)\mathrm{Sp}(1)\subset \mathrm{SO}(4m)$, $n=4m$. In this case there exist three locally defined almost complex structures J_α, with corresponding Kähler forms ω_α, such that the four-form $\Omega:=\Sigma_\alpha\omega_\alpha\wedge\omega_\alpha$ is globally defined and parallel. This will be discussed in more detail in Sec. II C.

(3) $\mathrm{Hol}\subset G_2\subset \mathrm{SO}(7)$. Let $V=\mathbb{O}=\mathbb{R}1+\mathrm{Im}\,\mathbb{O}=\mathbb{R}\oplus\mathbb{R}^7=\mathbb{R}^8$ be the algebra of octonions. Recall that $G_2=\mathrm{Aut}(\mathbb{O})$ is the group of automorphisms of the octonions. We can decompose the product of two octonions a,b into its real and imaginary parts as follows:

$$ab=a\cdot b=\langle a,b\rangle 1+\tfrac{1}{2}[a,b],$$

where $\langle a,b\rangle$ is the scalar product and $[a,b]=ab-ba$ is the commutator. We define a three-form φ and a four-form ψ on $\mathrm{Im}\,\mathbb{O}=\mathbb{R}^7$ by the formulas

$$\varphi(x,y,z):=\langle x\cdot y,z\rangle=\tfrac{1}{2}\langle [x,y],z\rangle$$

$$\psi(x,y,z,w):=\langle [x,y,z],w\rangle,$$

where $[x,y,z]=(xy)z-x(yz)$ is the associator. It is known that $\psi=*\varphi$. Notice that G_2 is the group of isometries of $\mathbb{O}=\mathbb{R}^8$ which fix the identity element 1 and preserve the three-form φ (or equivalently the four-form ψ) on $\mathrm{Im}\,\mathbb{O}$. The four-form ψ defines a parallel four-form on any Riemannian seven-fold with holonomy $G_2\subset \mathrm{SO}(7)$. It is known[13] that $\Lambda^4\mathbb{R}^7=\mathbb{R}\psi\oplus V^7(\pi_1)\oplus V^{27}(2\pi_1)$, where $V^d(\pi)$ is the d-dimensional real irreducible representation of G_2 with highest weight π and π_i denotes the ith fundamental weight of G_2. From this it follows that the four-form $\Omega_0^{G_2}$ coincides with ψ up to scaling. The corresponding endomorphism B_ψ of $\Lambda^2\mathbb{R}^7=\mathfrak{g}_2\oplus\mathbb{R}^7$ has two distinct eigenvalues which correspond to the two irreducible G_2-submodules $\mathfrak{g}_2$ and $\mathbb{R}^7\subset\Lambda^2\mathbb{R}^7$ (see Ref. 2).

(4) $\mathrm{Hol}\subset \mathrm{Spin}(7)\subset \mathrm{SO}(8)$. Using the three- and four-forms φ and ψ on $\mathbb{R}^7$ introduced in the G_2-case, we construct the four-form

$$\Omega=dt\wedge\varphi+\psi,$$

where t is the first coordinate on $\mathbb{R}^8=\mathbb{R}1+\mathbb{R}^7$. In particular,

$$\Omega(1,x,y,z)=\varphi(x,y,z),\quad \Omega(x,y,z,w)=\psi(x,y,z,w),\quad x,y,z,w\in\mathbb{R}^7.$$

This four-form Ω defines a parallel four-form on any Riemannian eight-fold with holonomy $\mathrm{Spin}(7)\subset \mathrm{SO}(8)$. It is known that $\Lambda^4\mathbb{R}^8=\mathbb{R}\Omega\oplus V^7(\pi_1)\oplus V^{27}(2\pi_1)\oplus\Lambda^4\mathbb{R}^7$. From this it follows that the four-form $\Omega_0^{\mathrm{Spin}(7)}$ coincides with Ω up to scaling. The corresponding endomorphism B_Ω of $\Lambda^2\mathbb{R}^8=\mathfrak{spin}_7\oplus\mathbb{R}^7$ has two distinct eigenvalues which correspond to the two irreducible Spin(7)-submodules $\mathfrak{spin}_7$ and $\mathbb{R}^7\subset\Lambda^2\mathbb{R}^8$ (see Ref. 2).

C. Quaternionic Kähler case

Now we discuss in more detail the case of quaternionic Kähler manifolds (Example 2 above). Riemannian manifolds (M,g) with holonomy group $\mathrm{Hol}\subset\mathrm{Sp}(m)\mathrm{Sp}(1)$ are called quaternionic Kähler manifolds. A quaternionic Kähler manifold with holonomy group $\mathrm{Hol}\subset\mathrm{Sp}(m)$ is called hyper-Kähler. On any quaternionic Kähler manifold M, there exists a rank 3 vector subbundle $Q\subset\mathrm{End}\,TM$, invariant under parallel transport, which is locally spanned by three almost complex structures $(J_\alpha)=(J_1,J_2,J_3=J_1J_2=-J_2J_1)$. The latter are in general only locally defined. The (globally defined) vector bundle Q is called the **quaternionic structure** of M. A local frame (J_α) as above is called a **standard frame** for Q. Similarly, a **standard basis** of Q at $m\in M$ is a triple $I,J,K=IJ=-JI\in Q_m$ of complex structures on T_mM. A quaternionic Kähler manifold is hyper-Kähler if and only if there exists a globally defined parallel standard frame $(J_\alpha)=(J_1,J_2,J_3=J_1J_2=-J_2J_1)$.

Given a standard frame, we may locally define three nondegenerate two-forms $\omega_\alpha:=g(J_\alpha\cdot,\cdot)$. The four-form

$$\Omega:=\sum_\alpha \omega_\alpha\wedge\omega_\alpha$$

is independent of the choice of standard frame and defines a global parallel four-form.

To describe the eigenspace decomposition of Ω it is convenient to use the Grassmann structure (i.e., generalized spinor decomposition) of a quaternionic Kähler manifold. Recall that a **Grassmann structure** on a (real) manifold M is defined as an isomorphism $T^{*\mathbb{C}}M\cong E\otimes H$ of the complexified cotangent bundle with the tensor product of two complex vector bundles E and H over M. Any quaternionic Kähler manifold admits a (locally defined) Grassmann structure $T^{*\mathbb{C}}M=E\otimes H$, where H has rank 2, such that the holonomy group $\mathrm{Hol}\subset\mathrm{Sp}(E)\otimes\mathrm{Sp}(H)$. This follows from the fact that any complex irreducible representation of the group $\mathrm{Sp}(m)\times\mathrm{Sp}(1)$ is a tensor product of irreducible representations of its factors.

The complex extension $g^{\mathbb{C}}$ of the Riemannian metric defines a complex bilinear metric on $T^{\mathbb{C}}M$, which locally factorizes as $g^{\mathbb{C}}=\omega_E\otimes\omega_H$, where ω_E and ω_H are sections of Λ^2E and Λ^2H, defining complex symplectic forms on the fibers of E^* and H^*, respectively. We call ω_E and ω_H the symplectic forms of the symplectic vector bundles E^* and H^*.

In terms of the Grassmann structure the eigenspaces V_λ of the endomorphism B_Ω on $\Lambda^2T^{*\mathbb{C}}M$ are given by[3,9]

$$V_{\lambda_1}=S^2E\otimes\omega_H,\quad V_{\lambda_2}=\Lambda_0^2E\otimes S^2H,\quad V_{\lambda_3}=\omega_E\otimes S^2H,$$

where Λ_0^2E is the space of ω_E-traceless two-forms and the eigenvalues are $\lambda_1=1$, $\lambda_2=-\frac{1}{3}$ and $\lambda_3=-(2m+1)/3$. In particular the λ_1-self-duality condition takes the form

$$F^\nabla\in S^2E\otimes\omega_H\otimes\mathrm{End}\,W. \tag{8}$$

Note that since Ω is parallel it is appropriate and co-closed and hence the (Ω,λ)-self-duality equations (Definition 3) reduce to (6), which implies the Yang–Mills equation. It is known (see Theorem 1 of Ref. 4) that λ_1- and λ_3-self-dual connections correspond to absolute minima of the Yang–Mills functional on compact quaternionic Kähler manifolds.

D. Self-duality as half-flatness

The λ_1-self-duality equation (8) in fact depends only on the existence of the factorization $T^{*\mathbb{C}}M \cong E \otimes H$ and the symplectic structure in H^*. A connection ∇ in a vector bundle W over a manifold M with a Grassmann structure is called **half-flat** if its curvature satisfies the condition

$$F^\nabla \in S^2 E \otimes \Lambda^2 H \otimes \mathrm{End}\ W. \tag{9}$$

In general such half-flat connections are *not* YM connections (with respect to some metric), but it is possible to impose further conditions on F^∇ in order to enforce the YM equation. In fact, it is the half-flatness of the connection, rather than the YM property, which is crucial for our construction of solutions.

Proposition 1: A connection ∇ in a vector bundle $W \to M$ over a quaternionic Kähler manifold is half-flat if and only if it is λ_1-self-dual. Hence any such connection is a Yang–Mills connection.

Proof: The result follows from (8) and (9) since $\Lambda^2 H$ is the line bundle generated by ω_H.□

The Levi-Civita connection on a hyper-Kähler manifold is an example of a half-flat linear connection. Its complexification gives an example of what we call an admissible half-flat Grassmann structure in the next section.

III. MANIFOLDS WITH HALF-FLAT HOLOMORPHIC GRASSMANN STRUCTURE

Our goal is to give a construction of half-flat connections in a vector bundle $\nu: W \to M$ over a manifold M. If all objects are real analytic, using analytic continuation we may obtain corresponding complex analytic objects. Specifically, assume that the manifold M and the bundle ν are real analytic. Then M is defined by an atlas of charts with analytic transition functions. Extending these functions to complex holomorphic functions, we may extend M to a complex manifold $M^{\mathbb{C}}$ with antiholomorphic involution τ such that $M = (M^{\mathbb{C}})^\tau$, the fixed point set of τ. Similarly, a real analytic vector bundle $\nu: W \to M$ can be extended to a holomorphic vector bundle $\nu^{\mathbb{C}}: W^{\mathbb{C}} \to M^{\mathbb{C}}$. Moreover, an analytic connection ∇ in ν can be extended to a holomorphic connection ∇ in $\nu^{\mathbb{C}}$. A holomorphic extension of a Yang–Mills connection is also a Yang–Mills connection. In the rest of this article, we shall assume that all objects (manifolds, bundles and connections) are holomorphic. In Sec. IV we shall give a construction of half-flat connections in a holomorphic bundle $W \to M$ over a complex manifold M with holomorphic Grassmann structure. Now we describe the required geometrical notions. In particular, we provide a description of the harmonic spaces of Ref. 10 in geometric language. Our description affords application to the construction of half-flat connections over more general manifolds than the flat torsion-free backgrounds previously considered in the harmonic space literature (see, e.g., Ref. 10).

A. Grassmann structure

Let M be a complex manifold with holomorphic Grassmann structure $T^*M = E \otimes H$, the isomorphism of the holomorphic cotangent bundle over M with the tensor product of holomorphic vector bundles E and H over M of rank p and q, respectively. Then $TM = E^* \otimes H^*$. A holomorphic linear connection ∇ on M is called a **holomorphic Grassmann connection** if it preserves the holomorphic Grassmann structure. This means that for any vector field X on M and local sections $e \in \Gamma(E)$ and $h \in \Gamma(H)$,

$$\nabla_X(e \otimes h) = \nabla_X^E e \otimes h + e \otimes \nabla_X^H h,$$

where ∇^E, ∇^H are connections in the bundles E, H, respectively.

Definition 4: A holomorphic Grassmann structure, $T^*M = E \otimes H$, on a complex manifold M with a holomorphic Grassmann connection $\nabla = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^H$ is called **half-flat** *if the connection ∇^H in the holomorphic vector bundle $H \to M$ is flat. A manifold with such a half-flat holomorphic Grassmann structure is called a* **half-flat Grassmann manifold**.

Assumption: In this section we assume that M is a manifold with a half-flat holomorphic Grassmann structure $(T^*M = E \otimes H, \nabla = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^H)$, such that H has rank 2 and that a

∇^H-parallel nondegenerate fiber-wise two-form $\omega_H \in \Gamma(\Lambda^2 H)$ in the bundle H^* is fixed. If, in addition, a ∇^E-parallel nondegenerate two-form $\omega_E \in \Gamma(\Lambda^2 E)$ is fixed, then we can define a ∇-parallel complex Riemannian metric $g=\omega_E \otimes \omega_H$ on M. We do not assume, in general, that the linear connection ∇ is torsion-free.

The torsion of a linear connection belongs to $TM \otimes \Lambda^2 T^*M$. Since $T^*M = E \otimes H$, we have the decomposition

$$\begin{aligned} TM\otimes\Lambda^2 T^*M &= TM\otimes(\Lambda^2 E\otimes S^2 H \oplus S^2 E\otimes \Lambda^2 H)\\ &= E^*H^*(\Lambda^2 E S^2 H \oplus S^2 E\,\omega_H)\\ &\cong E^*\Lambda^2 E(S^3 H \oplus \omega_H H)\oplus E^* S^2 E\,\omega_H H, \end{aligned} \tag{10}$$

where we omit the $\otimes$'s and we identify H^* with H using ω_H.

Definition 5: A half-flat connection is called **admissible** *if its torsion tensor has no component in $E^*\otimes\Lambda^2 E\otimes S^3 H$. A half-flat Grassmann manifold (M,∇) is called* **admissible** *if ∇ is* **admissible**.

We remark that if the torsion of a half-flat connection is E-symmetric, i.e., if it belongs to $TM\otimes S^2 E\otimes \Lambda^2 H = TM\otimes S^2 E\otimes \omega_H$, then the connection is admissible. It follows from the above decomposition that the torsion tensor of any admissible connection can be written as

$$T(e\otimes h, e'\otimes h') = T_1(e,e')\otimes\omega_H(h,h')h_1 + T_2(e,e')\otimes\omega_H(h,h_2)h' + T_2(e,e')\otimes\omega_H(h',h_2)h,$$

where e,e' are sections of E^*, h_1,h_2 are fixed sections of $H\cong H^*$, $T_1\in\Gamma(E^*\otimes S^2 E)$ and $T_2 \in\Gamma(E^*\otimes\Lambda^2 E)$. This shows that admissibility of the connection means that the torsion can be represented as the sum of two tensors linear in ω_H.

B. Harmonic space

Let M be a half-flat Grassmann manifold. We denote by S_H the $\mathrm{Sp}(1,\mathbb{C})$-principal holomorphic bundle over M consisting of symplectic bases of $H_m^* \cong H_m \cong \mathbb{C}^2$, $m\in M$,

$$S_H = \{ s=(h_+,h_-) \ | \ \omega_H(h_+,h_-)=1 \}.$$

The bundle $S_H \to M$ is called **harmonic space**.[10] A parallel (local) section

$$m \mapsto s_m = (h_1(m), h_2(m)) \in S_H$$

defines a trivialization

$$M\times \mathrm{Sp}(1,\mathbb{C}) \cong S_H,$$

given by

$$(m,\mathcal{U})\mapsto s_m\mathcal{U} = \left(h_+ = \sum_{\alpha=1}^{2} h_\alpha u_+^\alpha \ , \ h_- = \sum_{\alpha=1}^{2} h_\alpha u_-^\alpha \right), \quad \mathcal{U} = \begin{pmatrix} u_+^1 & u_-^1 \\ u_+^2 & u_-^2 \end{pmatrix}; \det\ \mathcal{U}=1.$$

We denote by $\partial_{++}, \partial_{--}, \partial_0$ the fundamental vector fields on S_H generated by the standard generators of $\mathrm{Sp}(1,\mathbb{C})$,

$$\partial_{++} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \partial_{--} \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \partial_0 \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the relations

$$[\partial_{++}, \partial_{--}] = \partial_0, \quad [\partial_0, \partial_{++}] = 2\partial_{++}, \quad [\partial_0, \partial_{--}] = -2\partial_{--}.$$

Consider $\mathrm{Mat}(2,\mathbb{C})$, the vector space of two by two matrices. The matrix coefficients $u_{\pm}^{\alpha}$ are coordinates on this vector space. One can easily check that the vector fields $u_{+}^{\alpha}\partial/\partial u_{-}^{\alpha}$, $u_{-}^{\alpha}\partial/\partial u_{+}^{\alpha}$ and $u_{+}^{\alpha}\partial/\partial u_{+}^{\alpha}-u_{-}^{\alpha}\partial/\partial u_{-}^{\alpha}$ annihilate the function $\det\mathcal{U}=\epsilon_{\beta\gamma}u_{+}^{\beta}u_{-}^{\gamma}$, where $\epsilon_{\beta\gamma}$ are the matrix coefficients of the standard symplectic form of $\mathbb{C}^2$. Therefore these vector fields are tangent to the submanifold $\mathrm{Sp}(1,\mathbb{C})=\{\det\mathcal{U}=1\}\subset\mathrm{Mat}(2,\mathbb{C})$. One can easily prove the following lemma.

Lemma 2: In terms of the identification, $S_H\cong M\times\mathrm{Sp}(1,\mathbb{C})$, the fundamental vector fields on S_H generated by the standard generators of $\mathrm{Sp}(1,\mathbb{C})$ *may be written*

$$\partial_{++}=u_{+}^{\alpha}\frac{\partial}{\partial u_{-}^{\alpha}},\quad \partial_{--}=u_{-}^{\alpha}\frac{\partial}{\partial u_{+}^{\alpha}},\quad \partial_0=u_{+}^{\alpha}\frac{\partial}{\partial u_{+}^{\alpha}}-u_{-}^{\alpha}\frac{\partial}{\partial u_{-}^{\alpha}}.$$

We say that a function f on S_H has **charge** c if $\partial_0 f=cf$. The charge measures the difference in the degrees of homogeneity in u_+ and u_-.

Note that any frame $(h_+,h_-)\in S_H$ defines an isomorphism $\mathbb{C}^2\xrightarrow{\sim}H_m^*$ given by $(z^1,z^2)\mapsto z^1h_++z^2h_-$. This induces an isomorphism

$$\mathfrak{sp}(1,\mathbb{C})=\mathfrak{sp}(\mathbb{C}^2)\cong S^2\mathbb{C}^2\xrightarrow{\sim}S^2H_m^*=\mathrm{span}_{\mathbb{C}}\{h_+^2,h_-^2,h_+\vee h_-\},$$

where we have identified $\mathfrak{sp}(\mathbb{C}^2)$ with $S^2\mathbb{C}^2$ using the symplectic form of $\mathbb{C}^2$. The generators of $\mathfrak{sp}(1,\mathbb{C})$ corresponding to h_+^2, $-h_-^2$, $-h_+\vee h_-$ under this identification are precisely ∂_{++}, ∂_{--}, ∂_0 respectively.

C. Canonical distributions on harmonic space

Let $S_H=\{(h_+,h_-)|h_{\pm}=u_{\pm}^{\alpha}h_{\alpha},(u_{\pm}^{\alpha})\in\mathrm{Sp}(1,\mathbb{C})\}$ be the harmonic space associated to a half-flat Grassmann manifold M. Here we have fixed a parallel symplectic frame (h_1,h_2) of H^* which defines the trivialization $S_H=M\times\mathrm{Sp}(1,\mathbb{C})$ of the holomorphic bundle S_H. In particular, the matrix coefficients $u_{\pm}^{\alpha}$ of $\mathrm{Sp}(1,\mathbb{C})$ will be considered as holomorphic functions on S_H. Together with local coordinates (x^i) of M, we obtain a system $(x^i,u_{\pm}^{\alpha})$ of local (nonhomogeneous–homogeneous) coordinates on S_H.

For any section $e\in\Gamma(E^*)$ we define vector fields $X_{\pm}^{e}\in\mathfrak{X}(S_H)$ by the formula

$$X_{\pm}^{e}|_{(h_+,h_-)}=\widetilde{e\otimes h_{\pm}},$$

where $\tilde{Y}$ stands for the horizontal lift of a tangent vector Y on M with respect to the connection ∇^H. Since the frame h_α is parallel, this horizontal lift coincides with the horizontal lift with respect to the splitting $S_H=M\times\mathrm{Sp}(1,\mathbb{C})$. This shows that the vector fields $X_{\pm}^{e}$ are tangent to $M\times\{(h_+,h_-)\}$ and hence annihilate $u_{\pm}^{\alpha}$. If $h_{\pm}=u_{\pm}^{\alpha}h_{\alpha}$, then $X_{\pm}^{e}=u_{\pm}^{\alpha}\widetilde{X_{\alpha}^{e}}$, where $X_{\alpha}^{e}:=e\otimes h_{\alpha}$.

There exists a canonical decomposition

$$TS_H=T^vS_H\oplus\mathcal{D}_+\oplus\mathcal{D}_-$$

of the (holomorphic) tangent bundle into the vertical subbundle T^vS_H and two (holomorphic) distributions $\mathcal{D}_+$ and $\mathcal{D}_-$ spanned, respectively, by vector fields X_+^e and X_-^e associated with sections e of the bundle E^*. The vertical distribution T^vS_H is spanned by the vector fields $\partial_0,\partial_{++},\partial_{--}$, which correspond to the standard generators of the Lie algebra $\mathfrak{sp}(1,\mathbb{C})$.

Lemma 3: The vector fields $X_{\pm}^{e}\in\mathfrak{X}(S_H)$ satisfy the following commutation relations:

$$[\partial_0,X_{\pm}^{e}]=\pm X_{\pm}^{e},\quad[\partial_{\pm\pm},X_{\pm}^{e}]=0,\quad[\partial_{\pm\pm},X_{\mp}^{e}]=X_{\pm}^{e},$$

$$[X_+^e,X_-^{e'}]=X_-^{\nabla_{\pi_*X_+^e}e'}-X_+^{\nabla_{\pi_*X_-^{e'}}e}-\tilde{T}(\pi_*X_+^e,\pi_*X_-^{e'}),\tag{11}$$

$$[X_{\pm}^{e}, X_{\pm}^{e'}] = X_{\pm}^{\nabla_{\pi_* X_{\pm}^{e}} e'} - X_{\pm}^{\nabla_{\pi_* X_{\pm}^{e'}} e} - \tilde{T}(\pi_* X_{\pm}^{e}, \pi_* X_{\pm}^{e'}),$$

where T is the torsion of the Grassmann connection, $\tilde{T}(X,Y) := \widetilde{T(X,Y)}$ denotes the horizontal lift of the vector $T(X,Y)$ and we have used the abbreviation $\nabla_X e := \nabla_X^E e$.

Proof: The first three equations follow from $X_{\pm}^{e} = u_{\pm}^{\alpha} \widetilde{e \otimes h_{\alpha}}$ and the expression for the fundamental vector fields given in Lemma 2. To prove the last equation we first compute the Lie bracket of two vector fields $X = e \otimes h$ and $X' = e' \otimes h$ on M, where h is parallel:

$$[X,X'] = \nabla_X X' - \nabla_{X'} X - T(X,X') = (\nabla_X e' - \nabla_{X'} e) \otimes h - T(X,X'). \tag{12}$$

Using this we calculate the commutator

$$\begin{aligned}[X_{\pm}^{e},\ X_{\pm}^{e'}] &= u_{\pm}^{\alpha} u_{\pm}^{\beta} (\widetilde{\nabla_{X_{\alpha}^{e}} X_{\beta}^{e'}} - \widetilde{\nabla_{X_{\beta}^{e'}} X_{\alpha}^{e}} - \tilde{T}(X_{\alpha}^{e}, X_{\beta}^{e'})) \\ &= (\nabla_{\pi_* X_{\pm}^{e}} e' \otimes h_{\pm})^{\sim} - (\nabla_{\pi_* X_{\pm}^{e'}} e \otimes h_{\pm})^{\sim} - u_{\pm}^{\alpha} u_{\pm}^{\beta} \tilde{T}(X_{\alpha}^{e}, X_{\beta}^{e'}) \\ &= X_{\pm}^{\nabla_{\pi_* X_{\pm}^{e}} e' - \nabla_{\pi_* X_{\pm}^{e'}} e} - \tilde{T}(\pi_* X_{\pm}^{e}, \pi_* X_{\pm}^{e'}).\end{aligned}$$

The expression for $[X_{+}^{e}, X_{-}^{e'}]$ follows similarly. □

We shall use the abbreviation $T(X_{\pm}^{e}, X_{\pm}^{e'}) := \tilde{T}(\pi_* X_{\pm}^{e}, \pi_* X_{\pm}^{e'})$.

Proposition 2: The following conditions are equivalent:

(i) *For any parallel section $h \in \Gamma(H^*)$ the distribution $E^* \otimes h$ on M is integrable.*
(ii) *The distribution $\mathcal{D}_+$ [associated to any parallel frame (h_1, h_2)] on S_H is integrable.*
(iii) *The distribution $\mathcal{D}_-$ on S_H is integrable.*
(iv) *The holomorphic Grassmann structure is admissible, i.e., it has admissible connection.*

Proof: The formula (12), where h is parallel, shows that the distribution $E^* \otimes h$ is integrable if and only if

$$T(E^* \otimes h, E^* \otimes h) \subset E^* \otimes h. \tag{13}$$

Using the decomposition (10), one can check that this condition is satisfied for all parallel sections h if and only if the connection is admissible. This proves the equivalence of (i) and (iv). Since $\pi_*(X_+^{e}|_{(h_+, h_-)}) = e \otimes h_+$, the last equation in (11) shows that the distribution $\mathcal{D}_+$ is integrable if and only if (13) holds for all h. Thus (i) is equivalent to (ii). The equivalence of (i) and (iii) is proved similarly. □

IV. CONSTRUCTION OF HALF-FLAT CONNECTIONS OVER HALF-FLAT GRASSMANN MANIFOLDS

A. Half-flat connections over half-flat Grassmann manifolds

In this section we describe the *harmonic space method*[10] for constructing half-flat connections ∇ (Definition 6 below) in a holomorphic vector bundle $\nu: W \to M$ over a complex manifold M with admissible half-flat holomorphic Grassmann structure. The basic ingredient of the construction is the lift of geometric data from M to S_H via $\pi: S_H \to M$. Let ∇ be a holomorphic connection in a holomorphic vector bundle $\nu: W \to M$. Its curvature

$$F(e \otimes h_{\alpha}, e' \otimes h_{\beta}) = \omega_H(h_{\alpha}, h_{\beta}) F^{(ee')} + F_{\alpha\beta}^{[ee']}, \tag{14}$$

where (h_1,h_2) is the fixed parallel local frame of H^* and e,e' are local sections of E^*. The curvature component $F^{(ee')}$ is symmetric in e,e' and $F^{[ee']}_{\alpha\beta}$ is skew in e,e' and symmetric in α,β. Lifting (14) to S_H we obtain the curvature of the pull-back connection $\pi^*\nabla$ in $\pi^*\nu:\pi^*W\to S_H$ with components, $F(v,\cdot)=0$, $\forall v\in T^vS_H$, together with

$$F(X^e_\pm,X^{e'}_\pm)=F^{[ee']}_{\pm\pm}:=u^\alpha_\pm u^\beta_\pm F^{[ee']}_{\alpha\beta},$$

$$F(X^e_+,X^{e'}_-)=F^{(ee')}+F^{[ee']}_{+-}:=F^{(ee')}+u^\alpha_+ u^\beta_- F^{[ee']}_{\alpha\beta}.$$

Definition 6: A holomorphic connection ∇ *in a holomorphic vector bundle* $\nu:W\to M$ *over a complex manifold* M *with holomorphic Grassmann structure,* $T^*M=E\otimes H$, *is called* **half-flat** *if its curvature* F *satisfies the equation*

$$F(e\otimes h_\alpha,\, e'\otimes h_\beta)=\omega_H(h_\alpha,h_\beta)F^{(ee')}, \tag{15}$$

where (h_1,h_2) *is a parallel local frame of* H^* *and* $F^{(ee')}$ *is symmetric in the local sections* e,e' *of* E^*.

Note that (15) is equivalent to (9). From this definition it follows that for any $h\in H^*$ we have $F(e\otimes h,\, e'\otimes h)=0$.

Definition 7: A connection in a holomorphic vector bundle $W\to S_H$ *over harmonic space* S_H *is called* **half-flat** *if its curvature* F *satisfies the equations*

$$\begin{aligned} F(X^e_+,X^{e'}_+)&=0,\\ F(X^e_+,X^{e'}_-)&=F^{(ee')},\\ F(X^e_-,X^{e'}_-)&=0,\\ F(v,\cdot)&=0,\quad \forall v\in T^vS_H, \end{aligned} \tag{16}$$

where $F^{(ee')}$ *is symmetric in the local sections* e,e' *of* E^*.

Definition 8: Let $\nu:W\to M$ *be a holomorphic vector bundle and* ∇ *a connection in* $\pi^*\nu:\pi^*W\to S_H$, *where* $\pi:S_H\to M$. *A local frame of* $\pi^*\nu$ *defined on* $\pi^{-1}(U)$, *where* U *is an open subset of* M, *is called a* **central frame** *with respect to* ∇ *if it is parallel along the fibers of the bundle* $\pi:S_H\to M$.

Remark: If $\chi=(\chi_1,\ldots,\chi_r)$ is a local frame of ν, then $\pi^*\chi$ will be a central frame with respect to the pull-back $\pi^*\nabla$ of any connection ∇ in ν. The connection one-form A of $\pi^*\nabla$ with respect to the frame $\pi^*\chi$ satisfies $A(v)=0$, $A(X^e_\pm)=u^\alpha_\pm A^e_\alpha$, where v is any vertical vector and $A^e_\alpha=A(\widetilde{X^e_\alpha})$ is a matrix-valued function on M. Conversely, any connection satisfying these conditions is the pull-back of the connection over M with potential $A(X^e_\alpha)=A^e_\alpha$.

Proposition 3: Let $\pi:S\to M$ *be any fiber bundle with simply connected fibers over a simply connected manifold* M. *There is a natural one-to-one correspondence between gauge equivalence classes of connections* ∇^M *in the trivial bundle* $\mathbb{C}^r\times M$ *and gauge equivalence classes of connections* ∇^S *in* $\mathbb{C}^r\times S$ *satisfying the curvature constraint* $F(v,\cdot)=0$ *for all vertical vectors* v.

Proof: It is clear that the pull-back $\nabla^S=\pi^*\nabla^M$ to S of a connection ∇^M defined over M satisfies the curvature constraint. To prove the converse, we will apply the following elementary lemma to the connection one-form A of a connection ∇ over $N=S$.

Lemma 4: Let $\pi:N\to M$ *be a submersion with connected fibers and* α *a* p*-form on* N. *Then* α *is the pull-back* $\pi^*\beta$ *of a* p*-form* β *on* M *if and only if the inner products* $\iota_v\alpha=\iota_v d\alpha=0$ *for all vertical tangent vectors* v.

Since the connection ∇^S is flat along the (simply connected) fibers of π there exists a central frame $\psi=(\psi_1,\ldots,\psi_r)$ for ∇^S. Let A be the connection one-form of ∇^S with respect to this central frame. We then have $A(v)=0$ for any vertical vector v and the curvature condition $F(v,\cdot)=0$ implies $dA(v,\cdot)=0$. Now the above lemma shows that A is the pull-back of a one-form B on M, which defines a connection ∇^M in the trivial bundle $\mathbb{C}^r\times M$. Since any two central frames differ by a gauge transformation which is a matrix-valued function on M the connection ∇^M is well defined up to a gauge transformation. The pull-back $\pi^*\nabla^M$ is gauge equivalent to ∇^S since it has the same expression with respect to the standard frame of $\mathbb{C}^r\times S$ (which is the pull-back of the standard frame of $\mathbb{C}^r\times M$) as ∇^S with respect to the central frame ψ. It is clear that the pull-backs of gauge equivalent connections over M are gauge equivalent connections over S. Applying a gauge transformation to a connection ∇^S which has connection one-form A with respect to a central frame ψ we get a new connection $(\nabla^S)'$, which has the same connection form A with respect to the transformed frame ψ'. The frame ψ' is therefore central with respect to $(\nabla^S)'$ and the two connections ∇^S and $(\nabla^S)'$ define the same gauge equivalence class of connections over M. □

Proposition 4: Let $\nu: W=\mathbb{C}^r\times M\to M$ be a trivial vector bundle over a complex manifold M with admissible half-flat holomorphic Grassmann structure and $\pi^*\nu:\pi^*W=\mathbb{C}^r\times S_H\to S_H$ its pull-back to S_H. Then any half-flat connection over S_H is gauge equivalent to the pull-back of a half-flat connection over M.

Proof: It is clear that the pull-back of a half-flat connection is half-flat. To prove the converse, we apply Proposition 3, by which a half-flat connection ∇^S over S_H is gauge equivalent to a pull-back connection $\pi^*\nabla^M$, which is necessarily half-flat. This implies that ∇^M is half-flat. In fact, if the connection ∇^M were not half-flat, then it would have a nontrivial curvature component $F_{\alpha\beta}^{[ee']}$ which would imply that its pull-back $\pi^*\nabla^M$ has, for instance, a nonzero curvature component $F_{++}^{[ee']}$. But this is impossible since $\pi^*\nabla^M$ is half-flat. □

Corollary 1: The connection one-form A of a half-flat connection over S_H with respect to a central frame ψ has the form

$$A(v)=0,\quad A(X_\pm^e)=u_\pm^\alpha A_\alpha^e,$$

where v is any vertical vector and $A_\alpha^e=A(\widetilde{X_\alpha^e})$ is a matrix-valued function on M.

Remark: This shows that the half-flat connection is completely determined by the potential in the $\mathcal{D}_+$-direction, $A(X_+^e)=u_+^\alpha A_\alpha^e$, with respect to a central frame.

Proof: This follows from Proposition 4 and the remark following Definition 8. □

B. The construction

In this section we construct half-flat connections in a bundle $\nu: W\to M$ over a manifold M with a half-flat admissible Grassmann structure. First we define the weaker notion of an almost half-flat connection over S_H and show how to construct all such connections from appropriate prepotentials. Then we show that any almost half-flat connection over S_H may be used to construct a half-flat connection on M. Since our construction is local in M, we shall assume that the bundles π, ν and $\pi^*\nu$ are trivial, i.e., $\pi: M\times \mathrm{Sp}(1,\mathbb{C})\to M$, $\nu: M\times\mathbb{C}^r\to M$ and $\pi^*\nu: S_H\times\mathbb{C}^r\to S_H$.

1. Construction of almost half-flat connections

The restriction of a half-flat connection to a leaf of the integrable distribution $\langle\mathcal{D}_+,\partial_0\rangle$ is clearly flat.

Definition 9: A frame $\varphi_1,\ldots,\varphi_r$ in the holomorphic vector bundle $\pi^*\nu:\mathbb{C}^r\times S_H$ which is parallel along leaves of the integrable distribution $\langle\mathcal{D}_+,\partial_0\rangle$ is called an **analytic frame**.

With respect to an analytic frame a connection in the vector bundle $\pi^*\nu$ has components

$$\nabla_{\partial_0}^S=\partial_0,$$

$$\nabla_{X_+^e}^S=X_+^e,$$

$$\nabla_{\partial_{++}}^S=\partial_{++}+A_{++}\ :=\partial_{++}+A(\partial_{++}),$$

$$\nabla^S_{\partial_{--}} = \partial_{--} + A_{--} \; := \partial_{--} + A(\partial_{--}),$$

$$\nabla^S_{X^e_-} = X^e_- + A(X^e_-).$$

Definition 10: A connection ∇^S *over* S_H *is called* **almost half-flat** *if its curvature satisfies the following equations:*

$$F(X^e_+, X^{e'}_+) = F(X^e_+, v) = 0, \quad \forall v \in T^v S_H,$$
$$F(\partial_{++}, \cdot) = F(\partial_0, \cdot) = 0\,. \tag{17}$$

In fact, these equations are not independent; for instance the Bianchi identity with arguments $(X_+, \partial_{++}, \partial_{--})$ together with $F(\partial_{++}, \partial_{--}) = F(\partial_{\pm\pm}, X^e_+) = 0$ implies the equation $F(\partial_{++}, X^e_-) = 0$.

Proposition 5: Any almost half-flat connection satisfies the following equation:

$$F(X^e_+, X^{e'}_-) = F(X^{e'}_+, X^e_-).$$

Proof: Using the integrability of $\mathcal{D}_+$ and $F(X^e_+, X^{e'}_+) = F(\partial_{--}, X^e_+) = 0$, we obtain

$$\begin{aligned} 0 &= [\nabla^S_{\partial_{--}}, F(X^e_+, X^{e'}_+)] \\ &= [\nabla^S_{\partial_{--}}, [\nabla^S_{X^e_+}, \nabla^S_{X^{e'}_+}]] - [\nabla^S_{\partial_{--}}, \nabla^S_{[X^e_+, X^{e'}_+]}] \\ &= [\nabla^S_{X^e_-}, \nabla^S_{X^{e'}_+}] + [\nabla^S_{X^e_+}, \nabla^S_{X^{e'}_-}] - \nabla^S_{[X^e_-, X^{e'}_+]} - \nabla^S_{[X^e_+, X^{e'}_-]} \\ &= F(X^e_-, X^{e'}_+) - F(X^{e'}_-, X^e_+)\;. \end{aligned}$$

□

It follows that an almost half-flat connection is a generalization of a half-flat connection, satisfying only those equations in (16), that involve curvatures with $\partial_0, \partial_{++}$ or X^e_+ in one of the arguments.

Proposition 6: An almost half-flat connection is half-flat if and only if it satisfies $F(\partial_{--}, X^e_-) = 0$.

Proof: By Proposition 5 an almost half-flat connection is required to satisfy all the half-flatness equations (16) with the exception of

$$F(\partial_{--}, X^e_-) = 0 \quad \text{and} \quad F(X^e_-, X^{e'}_-) = 0. \tag{18}$$

The second equation here follows from the first by virtue of the Bianchi identity with arguments $(X^e_+, X^{e'}_-, \partial_{--})$. □

The following proposition shows that an almost half-flat connection is completely determined by the potentials A_{++} and A_{--} with respect to an analytic frame.

Proposition 7: Let ∇^S *be an almost half-flat connection in the vector bundle* $\pi^*\nu: \mathbb{C}^r \times S_H \to S_H$ *with potentials* A_{++}, A_{--} *and* $A(X^e_-)$ *in an analytic frame. Then we have following.*

(i) *The potential* A_{++} *is analytic and has charge* $+2$, *i.e.,*

$$X^e_+ A_{++} = 0, \quad \partial_0 A_{++} = 2A_{++}\,. \tag{19}$$

(ii) *The potential* A_{--} *satisfies*

$$\partial_{++}A_{--} - \partial_{--}A_{++} + [A_{++}, A_{--}] = 0, \quad \partial_0 A_{--} = -2A_{--}\,. \tag{20}$$

(iii) *The potential* $A(X^e_-)$ *is determined by* A_{--} *and has charge* -1:

$$A(X_-^e) = -X_+^e A_{--} , \quad \partial_0 A(X_-^e) = -A(X_-^e) \ . \tag{21}$$

Conversely, any matrix-valued potentials A_{++}, A_{--} *and* $A(X_-^e)$ *satisfying* (19)–(21) *define an almost half-flat connection.*

Proof: (i) The curvature constraints $F(X_+^e, \partial_{++}) = 0$, $F(\partial_0, \partial_{++}) = 0$, in an analytic frame, take the form (19).

(ii) The further almost half-flatness conditions, $F(\partial_{++}, \partial_{--}) = F(\partial_0, \partial_{--}) = 0$, give Eqs. (20) for the potential A_{--}.

(iii) Having obtained A_{--}, we can find $A(X_-^e)$ from the equations $F(X_+^e, \partial_{--}) = F(\partial_0, X_-^e) = 0$, which take the form

$$X_+^e A_{--} = A([X_+^e, \partial_{--}]) = -A(X_-^e), \quad \partial_0 A(X_-^e) = -A(X_-^e) \ . \tag{22}$$

The second equation follows from the first. □

We can now write an algorithm for the construction of all almost half-flat connections:

Theorem 4: *Let* A_{++} *be an analytic prepotential, i.e., a matrix-valued function on a domain* $U = \pi^{-1}(V) \subset S_H$, *where* $V \subset M$ *is a simply connected domain, satisfying (19). Let* Φ *be an invertible matrix-valued function on U which satisfies the equations*

$$\partial_{++}\Phi = -A_{++}\Phi , \quad \partial_0 \Phi = 0 \ . \tag{23}$$

It always exists. The pair (A_{++}, Φ) *determines an almost half-flat connection* $\nabla^S = \nabla^{(A_{++}, \Phi)}$. *Its potentials with respect to an analytic frame are given by* A_{++}, $A_{--} = -(\partial_{--}\Phi)\Phi^{-1}$ *and* $A(X_-^e) = -X_+^e A_{--}$. *Conversely, any almost half-flat connection is of this form.*

Proof: We consider the connection defined by A_{++} and $A(\partial_0) = 0$ along an orbit sB of the Borel subgroup of $SL(2,\mathbb{C})$,

$$B = \left\{ \begin{pmatrix} t_0 & t_1 \\ 0 & t_0^{-1} \end{pmatrix} \middle| \, t_0 \in \mathbb{C}^*, t_1 \in \mathbb{C} \right\} \cong \mathbb{C}^* \times \mathbb{C} \quad \text{(diffeomorphic)}.$$

It is flat since the second equation of (19) is equivalent to $F(\partial_0, \partial_{++}) = 0$ (vanishing of the curvature along sB). Moreover, it has trivial holonomy since the fundamental group of $B \cong \mathbb{C}^* \times \mathbb{C}$ coincides with the fundamental group of the $\mathbb{C}^*$-factor and the potential is zero in the direction of ∂_0 which is tangent to $\mathbb{C}^*$. An invertible solution to the system (23) exists and defines a parallel frame Φ with respect to the flat connection with trivial holonomy defined along each orbit of the Borel group. Since the space of Borel orbits in U is diffeomorphic to $V \times \mathbb{C}P^1$ and is therefore simply connected, a solution Φ exists on the domain U. Now, given any such solution of (23), we define $A_{--} := -(\partial_{--}\Phi)\Phi^{-1}$. This solves (20), since $F(\partial_{\pm\pm}, \partial_0) = F(\partial_{++}, \partial_{--}) = 0$ is the integrability condition for the system $\partial_{\pm\pm}\Phi = -A_{\pm\pm}\Phi$, $\partial_0\Phi = 0$. Finally, we define $A(X_-^e) := -X_+^e A_{--}$, obtaining an almost half-flat connection by Proposition 7. Now the converse statement follows also from Proposition 7. □

2. Transformation to the central frame

Since an almost half-flat connection $\nabla = \nabla^S$ is flat in vertical directions, it admits a central frame ψ. The following lemma shows that the solution Φ of Eq. (23) gives a gauge transformation from an analytic frame φ to a central frame $\psi = \varphi\Phi$ for the almost half-flat connection $\nabla^{(A_{++}, \Phi)}$.

Lemma 5: Let $\nabla = \nabla^{(A_{++}, \Phi)}$ *be the almost half-flat connection associated to the analytic prepotential* A_{++} *with respect to the analytic frame* φ *and an invertible solution* Φ *of (23). Then the frame* $\psi := \varphi\Phi$ *is a central frame for the connection* ∇, *i.e., the potentials* $C(\partial_{\pm\pm})$ *and* $C(\partial_0)$ *with respect to that frame vanish.*

Proof: The result follows from the pure gauge form of $A(\partial_{\pm\pm})$ and $A(\partial_0)$ and the transformation law (3) for potentials. □

With respect to the central frame ψ, the almost half-flat connection constructed above takes the form

$$\nabla^S_{X^e_+} = X^e_+ + C(X^e_+) = X^e_+ + \Phi^{-1} X^e_+ \Phi,$$

$$\nabla^S_{X^e_-} = X^e_- + C(X^e_-) = X^e_- + \Phi^{-1} X^e_- \Phi + \Phi^{-1} X^e_+ (\partial_{--}\Phi\Phi^{-1})\Phi,$$

$$\nabla^S_{\partial_{++}} = \partial_{++}, \quad \nabla^S_{\partial_{--}} = \partial_{--}, \quad \nabla^S_{\partial_0} = \partial_0 .$$

Moreover, the equations $F(\partial_{++}, X^e_+) = F(\partial_0, X^e_+) = 0$ imply that the potential $C(X^e_+)$ satisfies the equations

$$\partial_{++} C(X^e_+) = 0, \quad \partial_0 C(X^e_+) = C(X^e_+). \tag{24}$$

3. Construction of half-flat connections

We assume now that the analytic prepotential A_{++} is defined globally along the fibers of $\pi: S_H \to M$. Then, restricting M to an appropriate domain, we may assume that A_{++} is defined globally on S_H. The previous construction then provides an almost half-flat connection over S_H. Using this connection, we may construct a half-flat connection on M. The crucial point is the following:

Proposition 8: The potential $C(X^e_+)$ of an almost half-flat connection ∇ with respect to a central frame is linear in u^α_+, namely,

$$C(X^e_+) = u^\alpha_+ C(\widetilde{X^e_\alpha}) =: u^\alpha_+ C^e_\alpha , \tag{25}$$

where $(x^i, u^\alpha_\pm)$ are the local coordinates associated with the trivialization $S_H = M \times \mathrm{Sp}(1,\mathbb{C})$ and $C^e_\alpha = C^e_\alpha(x^i)$ is a matrix-valued function on M.

Proof: Due to Eqs. (24), the result follows from Lemma 6.

Lemma 6: (i) If a holomorphic function f_+, defined on some domain

$$U \subset \{u^2_+ \neq 0\} \subset \mathrm{Sp}(1,\mathbb{C}) = \left\{ \mathcal{U} = \begin{pmatrix} u^1_+ & u^1_- \\ u^2_+ & u^2_- \end{pmatrix} ; \det \mathcal{U} = 1 \right\},$$

satisfies

$$\partial_{++} f_+ = 0, \quad \partial_0 f_+ = f_+, \tag{26}$$

then $f_+ = u^\alpha_+ f_\alpha(u^1_+/u^2_+)$. Here $f_\alpha(u^1_+/u^2_+)$ are holomorphic functions on U invariant under the right action of the Lie algebra of upper-triangular matrices.

(ii) Moreover, if the function f_+ is globally defined, then it is linear in u^α_+, i.e., $f_+ = u^\alpha_+ f_\alpha, f_\alpha = const$.

Proof: (i) One can immediately check that $f_+ = u^\alpha_+ f_\alpha(u^1_+/u^2_+)$ is a solution of (26). We note that the quotient of any two solutions of (26) is a solution of the corresponding homogeneous system,

$$\partial_{++} f = 0, \quad \partial_0 f = 0. \tag{27}$$

It is sufficient to check that any solution of (27) is a function of u^1_+/u^2_+. To prove this we use the local factorization of $\mathrm{Sp}(1,\mathbb{C})$ into the product of a Borel subgroup $\mathcal{B}$ and a nilpotent subgroup as follows:

$$\begin{pmatrix} u_+^1 & u_-^1 \\ u_+^2 & u_-^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \quad \mathcal{B} = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}.$$

Then $c = u_+^1 / u_+^2$ and $\partial_0, \partial_{++}$ are generators of the right action of $\mathcal{B}$. This implies that the solutions of (27) are precisely the local functions on $\mathrm{Sp}(1,\mathbb{C})$ invariant under the right action of $\mathcal{B}$. In terms of the local coordinate system (a,b,c) on $\mathrm{Sp}(1,\mathbb{C})$ such functions are functions of $c = u_+^1/u_+^2$ alone.

(ii) The restriction $V|_{\mathrm{Sp}(1)}$ to $\mathrm{Sp}(1)$ of any irreducible $\mathrm{Sp}(1,\mathbb{C})$-module V of holomorphic functions is a (finite dimensional) irreducible $\mathrm{Sp}(1)$-module of smooth functions on $\mathrm{Sp}(1)$. The condition (26) shows that f_+ is a highest weight vector with weight $+1$. Hence f_+ generates a two-dimensional submodule $\langle f_+ \rangle = \mathrm{span}\{f_+, f_- := \partial_{--} f_+\}$ of holomorphic functions. It remains to show that any two-dimensional module of holomorphic functions on $\mathrm{Sp}(1,\mathbb{C})$ is spanned by linear functions. We know two such modules, generated by the highest weight vectors u_+^1 and u_+^2 respectively. On the other hand, by the Peter–Weyl theorem the multiplicity of the two-dimensional irreducible representation of $\mathrm{Sp}(1)$ in $L^2(\mathrm{Sp}(1))$ is 2. □

Using Proposition 8, with respect to a central frame, we can write $\nabla_{X_+^e} = X_+^e + u_+^\alpha C_\alpha^e$ where the coefficients $C_\alpha^e = C_\alpha^e(x^i)$ are matrix valued functions of coordinates x^i on M. Using them we define a new connection in the trivial bundle $\mathbb{C}^r \times S_H$ over S_H by

$$\hat{\nabla}_{X_\pm^e} = X_\pm^e + u_\pm^\alpha C_\alpha^e,$$

$$\hat{\nabla}_{\partial_{\pm\pm}} = \partial_{\pm\pm}, \quad \hat{\nabla}_{\partial_0} = \partial_0.$$

Our main result now follows:

Theorem 5: *Let M be a complex manifold with a half-flat admissible Grassmann structure. Let A_{++} be an analytic prepotential, i.e., a solution of (19), and Φ an invertible solution of (23). Then the connection $\hat{\nabla} = \hat{\nabla}^{(A_{++},\Phi)}$ constructed from the data (A_{++},Φ) is a half-flat connection in the trivial vector bundle $\mathbb{C}^r \times S_H \to S_H$ and it is the pull-back of the following half-flat connection ∇^M in the bundle $\mathbb{C}^r \times M \to M$:*

$$\nabla^M_{X_\alpha^e} = X_\alpha^e + C_\alpha^e. \tag{28}$$

Conversely, any half-flat connection over S (or M) is gauge equivalent to one obtained from the above construction.

Proof: The remark after Definition 8 shows that the connection $\hat{\nabla}$ is the pull-back of the connection ∇^M. It suffices now to show that ∇^M is half-flat. Note that the connections ∇ and $\hat{\nabla}$ coincide in the direction of X_+^e. Hence, using $C_+^e := u_+^\alpha C_\alpha^e$, we have

$$\begin{aligned} 0 = F^\nabla(X_+^e, X_+^{e'}) = F^{\hat{\nabla}}(X_+^e, X_+^{e'}) &= X_+^e C_+^{e'} - X_+^{e'} C_+^e + [C_+^e, C_+^{e'}] - C([X_+^e, X_+^{e'}]) \\ &= u_+^\alpha u_+^\beta (X_\alpha^e C_\beta^{e'} - X_\beta^{e'} C_\alpha^e + [C_\alpha^e, C_\beta^{e'}] - C([X_\alpha^e, X_\beta^{e'}])) \\ &= u_+^\alpha u_+^\beta F^{\nabla^M}(X_\alpha^e, X_\beta^{e'}), \end{aligned}$$

since $X_+^e u_+^\beta = 0$. This shows that the curvature $F^{\nabla^M}(X_\alpha^e, X_\beta^{e'})$ is skew-symmetric in α,β, i.e., it belongs to $\Lambda^2 H \otimes S^2 E \otimes \mathrm{End}\, W$. In other words, the connection ∇^M is half-flat.

Conversely, let ∇^S be a half-flat connection over S_H. By Proposition 4 we may assume that it is a pull-back of a half-flat connection ∇^M over M. Since the restriction of ∇^S to the leaves of $\langle \mathcal{D}_+, \partial_0 \rangle$ is flat, there exists an analytic frame [i.e., a frame such that $A(X_+^e) = A(\partial_0) = 0$, in which the potential $A(\partial_{++})$ satisfies the equations (19)]. Since ∇^S is flat along the (simply-connected) fibers, there exists an invertible solution Φ to the system

$$\partial_{++}\Phi + A_{++}\Phi = \partial_{--}\Phi + A_{--}\Phi = \partial_0\Phi = 0.$$

This shows that ∇^S is gauge-equivalent to the almost half-flat connection $\nabla^{(A_{++},\Phi)} = \hat{\nabla}^{(A_{++},\Phi)}$. □

C. Application to hyper-Kähler manifolds with admissible torsion

The above construction can be applied to the complexification of hyper-Kähler manifolds. Recall that any hyper-Kähler manifold admits a (locally defined) Grassmann structure $T^{*\mathbb{C}}M = E\otimes H$, such that the Levi–Civita connection on the cotangent bundle $\nabla = \nabla^E\otimes \mathrm{Id} + \mathrm{Id}\otimes\nabla^H$ is half-flat, i.e., the connection ∇^H is flat. Since the hyper-Kähler metric is Ricci flat, hence analytic, we may, using analytic continuation, extend the manifold M to a complex manifold $M^{\mathbb{C}}$ with holomorphic extension of the hyper-Kähler structure, in particular, we have a holomorphic Ricci flat metric on $M^{\mathbb{C}}$ with holonomy in $\mathrm{Sp}(n,\mathbb{C})$ and half-flat Grassmann structure. This Grassmann structure is admissible since the Levi–Civita connection on $M^{\mathbb{C}}$ has no torsion. Hence we can apply the harmonic space method to construct half-flat connections on holomorphic vector bundles $W\to M^{\mathbb{C}}$. The complex version of Proposition 1 shows that such connections are Yang–Mills connections. More generally, the method of construction of half-flat connections extends to real analytic (possibly indefinite) hyper-Kähler manifolds with admissible torsion, i.e., with torsion which has zero component in $S^3H\otimes E^*\otimes\Lambda^2E$. A **hyper-Kähler manifold with admissible torsion** is defined as a pseudo-Riemannian manifold (M,g) with a linear metric connection ∇ with holonomy in $\mathrm{Sp}(k,l)$ which has admissible torsion. As in the (torsion-free) hyper-Kähler case there exists a parallel four-form given by $\Omega = \Sigma_\alpha \omega_\alpha\wedge\omega_\alpha$, $\omega_\alpha := g(J_\alpha\cdot,\cdot)$, and half-flat connections are characterized as connections with curvature in $V_{\lambda_1}\otimes \mathrm{End}\, W$, where V_{λ_1} is the λ_1-eigenspace of the endomorphism B_Ω associated to Ω. If the form Ω is co-closed, then any half-flat connection will be (Ω,λ_1)-self-dual and thus a Yang–Mills connection. We remark that co-closedness of Ω is equivalent to a linear $\mathrm{Sp}(k,l)$-invariant condition on the torsion.

V. GENERALIZATION TO HIGHER-SPIN GRASSMANN MANIFOLDS

A. Higher-spin Grassmann structures

The construction discussed in the previous section is in fact the $m=1$ specialization of a more general construction of connections on *spin m/2 Grassmann manifolds*, which we discuss in this section. These manifolds were considered in Ref. 14.

Definition 11: A **spin** $m/2$ **Grassmann structure** *on a (complex) manifold M is a holomorphic Grassmann structure of the form* $T^*M\cong E\otimes F = E\otimes S^mH$, *with a holomorphic Grassmann connection* $\nabla = \nabla^E\otimes\mathrm{Id} + \mathrm{Id}\otimes\nabla^F$, *where H is a rank 2 holomorphic vector bundle over M with holomorphic symplectic connection* ∇^H *and symplectic form* $\omega_H\in\Gamma(\Lambda^2H)$, *and* ∇^F *is the connection in* $F=S^mH$ *induced by* ∇^H. *M is called* **half-flat** *if the connection* ∇^F *is flat.*

The bundle S^mH is associated with the spin $m/2$ representation of the group $\mathrm{Sp}(1,\mathbb{C})$. Any frame (h_1,h_2) for H^* defines a frame for S^mH^* $(h_A := h_{\alpha_1}h_{\alpha_2}\cdots h_{\alpha_m})$, where the multi-index $A := \alpha_1\alpha_2\cdots\alpha_m$, $\alpha_i = 1,2$. The ∇^H-parallel symplectic form ω_H on H^* induces a bilinear form ω_H^m on $F^* = S^mH^*$ given by

$$\omega_H^m(h_A,h_B) := \underset{A}{\mathfrak{S}}\,\underset{B}{\mathfrak{S}}\;\omega_H(h_{\alpha_1},h_{\beta_1})\,\omega_H(h_{\alpha_2},h_{\beta_2})\cdots\omega_H(h_{\alpha_m},h_{\beta_m}),$$

where $\mathfrak{S}_A$ denotes the sum over all permutations of the α's. This form is skew-symmetric if m is odd and symmetric if m is even. To any section $e\in\Gamma(E^*)$ and multi-index A we associate the vector field $X_A^e := e\otimes h_A$ on M.

The construction of half-flat connections described in Sec. IV B may be adapted to obtain certain "partially flat" connections in vector bundles $W\to M$, provided that the torsion of ∇ obeys certain admissibility conditions.

Definition 12: Let (M,∇) *be a half-flat spin m/2 Grassmann manifold. For any section* $e\in\Gamma(E^*)$ *we define vector fields*

$$X^e_{(m-2i)+} := u_-^{\alpha_1}\cdots u_-^{\alpha_i} u_+^{\alpha_{i+1}}\cdots u_+^{\alpha_m} X^e_A \quad \text{if} \quad m-2i \geqslant 0,$$

$$X^e_{(2i-m)-} := u_-^{\alpha_1}\cdots u_-^{\alpha_i} u_+^{\alpha_{i+1}}\cdots u_+^{\alpha_m} X^e_A \quad \text{if} \quad m-2i < 0,$$

on the principal bundle S_H of symplectic frames in H. The distribution spanned by these vector fields is denoted by $\mathcal{D}_{k+} := \langle X^E_{k+}\rangle$ for $k \geqslant 0$, $k \equiv m \bmod 2$ and $\mathcal{D}_{k-} := \langle X^E_{k-}\rangle$ for $k>0$, $k \equiv m \bmod 2$. We define also

$$\mathcal{D}^k_{(\pm)} := \bigoplus_{i=0}^{k} \mathcal{D}_{(m-2i)\pm}.$$

The Grassmann connection ∇ is called **k-admissible** *if it preserves the distribution $\mathcal{D}^k_{(\pm)}$, i.e.,*

$$T(\mathcal{D}^k_{(\pm)}, \mathcal{D}^k_{(\pm)}) \subset \mathcal{D}^k_{(\pm)}. \tag{29}$$

The Grassmann manifold (M,∇) is called **k-admissible** *if the connection ∇ is k-admissible.*

For small m we shall write X^e_0, X^e_+, X^e_-, X^e_{++}, etc. instead of X^e_{0+}, X^e_{1+}, X^e_{1-}, X^e_{2+}, etc.

Proposition 9: Let (M,∇) be a half-flat spin $m/2$ Grassmann manifold. Then the distribution $\mathcal{D}^k_{(\pm)}$ is integrable if and only if the torsion of the Grassmann connection ∇ satisfies Eq. (29).

The proof is similar to that of Proposition 2.

B. Partially flat connections over higher-spin Grassmann manifolds

Let (M,∇) be a half-flat spin $m/2$ Grassmann manifold and $\nu: W \to M$ a holomorphic vector bundle. Since our constructions are local we will assume that W is trivial. In the higher spin ($m > 1$) case, there exists, as a natural generalization of the notion of a half-flat connection, the more refined notion of a k-partially flat connection in ν. The space of two-forms $\Lambda^2 T^*M$ has the following decomposition into $GL(E)\otimes \mathrm{Sp}(1,\mathbb{C})$-submodules:

$$\Lambda^2 T^*M = \Lambda^2(E\otimes S^m H) = \Lambda^2 E \otimes S^2 S^m H \oplus S^2 E \otimes \Lambda^2 S^m H,$$

where

$$S^2 S^m H = S^{2m} H \oplus \omega_H^2 S^{2m-4} H \oplus \cdots \oplus \omega_H^{2[m/2]} S^{2m-4[m/2]} H,$$

$$\Lambda^2 S^m H = \omega_H S^{2m-2} H \oplus \omega_H^3 S^{2m-6} H \oplus \cdots \oplus \omega_H^{2[m/2]+1} S^{2m-4[m/2]-2} H.$$

Here we use the convention that $S^l H = 0$ if $l < 0$.

Let ∇ be a connection in the vector bundle $W \to M$. Its curvature has the following decomposition, corresponding to the above decomposition of $\Lambda^2 T^*M$ into irreducible $GL(E)\cdot \mathrm{Sp}(1,\mathbb{C})$-submodules:

$$F(X^e_A, X^{e'}_B) = \underset{A}{\mathfrak{S}}\underset{B}{\mathfrak{S}} \sum_{k=0}^{[m/2]} \Big(\omega_H(h_{\alpha_1},h_{\beta_1})\cdots\omega_H(h_{\alpha_{2k}},h_{\beta_{2k}}) \overset{(2k)}{F}{}^{[ee']}_{\alpha_{2k+1}\cdots\alpha_m\beta_{2k+1}\cdots\beta_m}$$

$$+ \omega_H(h_{\alpha_1},h_{\beta_1})\cdots\omega_H(h_{\alpha_{2k+1}},h_{\beta_{2k+1}}) \overset{(2k+1)}{F}{}^{(ee')}_{\alpha_{2k+2}\cdots\alpha_m\beta_{2k+2}\cdots\beta_m}\Big), \tag{30}$$

where the tensors $F^{(2k)} \in \Gamma(\Lambda^2 E \otimes S^{2m-4k} H)$ and $F^{(2k+1)} \in \Gamma(S^2 E \otimes S^{2m-4k-2} H)$.

We note that half-flat connections are those which satisfy the conditions

$$\overset{(2i)}{F} = 0, \quad \text{for all } i \in \mathbb{N}. \tag{31}$$

For $m>1$ these conditions are not suitable for application of the harmonic space method. However, the following more refined restrictions on the curvature are amenable to the method (cf. Ref. 15).

Definition 13: A connection ∇ in the vector bundle $\nu:W\rightarrow M$ is called **k-partially flat** *if $F^{(i)}=0$ for all $i\leqslant 2k$. Here $0\leqslant k\leqslant[(m+2)/2]$.*

Clearly, $[(m+2)/2]$-partially flat connections are simply flat connections. We note that for $m=1$, zero-partially flat connections are precisely half-flat connections. For general odd $m=2p+1$, zero-partially flat connections in a vector bundle ν over flat spaces with spin $m/2$ Grassmann structure were considered by Ward.[3] He chose E to be a rank 2 flat bundle and showed that zero-partially flat connections, for $m>1$, do not correspond to Yang–Mills connections. Therefore, in our more general setting, we clearly cannot expect zero-partially flat connections to satisfy the Yang–Mills equations for $m>1$. On the other hand, the penultimate case, $k=[m/2]$, is particularly interesting for odd m:

Theorem 6: *Let M be a half-flat spin $m/2$ Grassmann manifold M. If m is odd and the vector bundle $E^*\rightarrow M$ admits a ∇^E-parallel symplectic form ω_E, then M has canonical $\mathrm{Sp}(E)\cdot\mathrm{Sp}(H)$-invariant metric $g=\omega_E\otimes\omega_H^m$ and four-form $\Omega\neq 0$. If Ω is co-closed with respect to the metric g, then any $(m-1)/2$-partially flat connection ∇ in a vector bundle W over M is (Ω,λ)-self-dual and hence it is a Yang–Mills connection.*

Proof: To describe Ω we use the following notation: e_a is a basis of E^*, h_α is a basis of H^*, h_A is the corresponding basis of S^mH^* and $X_{aA}:=e_a\otimes h_A$ is the corresponding basis of $TM=E^*\otimes S^mH^*$. With respect to these bases, the skew symmetric forms ω_E, ω_H and ω_H^m are represented by the matrices ω_{ab}, $\omega_{\alpha\beta}$ and ω_{AB}, respectively. We define Ω by

$$\Omega:=\sum \omega_{ab}\,\omega_{cd}\,\omega_{AC}\,\omega_{BD}\,X^{aA}\wedge X^{bB}\wedge X^{cC}\wedge X^{dD},$$

where X^{aA} is the basis dual to X_{aA}. This form is obviously $\mathrm{Sp}(E)\cdot\mathrm{Sp}(H)$-invariant since we used only ω_E and ω_H in the definition. One can easily check that $\Omega\neq 0$. The connection ∇ is $(m-1)/2$-partially flat if and only if its curvature F belongs to the space

$$S^2E\otimes\omega_H^m\otimes\mathrm{End}\ W\subset\ S^2E\otimes\Lambda^2S^mH\otimes\mathrm{End}\ W\subset\ \Lambda^2(E\otimes S^mH)\otimes\mathrm{End}\ W.$$

Here we use the decomposition

$$\Lambda^2S^mH=\omega_HS^{2m-2}H\oplus\omega_H^3S^{2m-6}H\oplus\cdots\oplus\mathbb{C}\omega_H^m.$$

The $\mathrm{Sp}(E)\cdot\mathrm{Sp}(H)$-submodule $S^2E\otimes\omega_H^m\subset\Lambda^2T^*M$ is irreducible. Therefore it is contained in an eigenspace V_λ of the $\mathrm{Sp}(E)\cdot\mathrm{Sp}(H)$-invariant operator $B_\Omega:\Lambda^2T^*M\rightarrow\wedge^2T^*M$. It remains to check that $\lambda\neq 0$. By Lemma 1 it suffices to compute the contraction $K=K_{cCdD}X^{cC}X^{dD}$ of a tensor $S=S_{ab}\omega_{AB}e^ae^b\otimes h^Ah^B$ in $S^2E\otimes\omega_H^m$ with Ω:

$$\begin{aligned}-K_{cCdD}=S^{ab}\omega^{AB}(&\omega_{ab}\omega_{cd}\omega_{AC}\omega_{BD}+\omega_{ac}\omega_{db}\omega_{AD}\omega_{CB}+\omega_{ad}\omega_{bc}\omega_{AB}\omega_{DC}-\omega_{ba}\omega_{cd}\omega_{BC}\omega_{AD}\\&-\omega_{bc}\omega_{da}\omega_{BD}\omega_{CA}-\omega_{bd}\omega_{ac}\omega_{BA}\omega_{DC}-\omega_{ca}\omega_{bd}\omega_{CB}\omega_{AD}-\omega_{cb}\omega_{da}\omega_{CD}\omega_{BA}\\&-\omega_{cd}\omega_{ab}\omega_{CA}\omega_{DB}+\omega_{da}\omega_{bc}\omega_{DB}\omega_{AC}+\omega_{db}\omega_{ca}\omega_{DC}\omega_{BA}+\omega_{dc}\omega_{ab}\omega_{DA}\omega_{CB})\\=4(m+1)&S_{cd}\omega_{CD}.\end{aligned}$$

Here ω^{ab} and ω^{AB} denote the inverses of ω_{ab} and ω_{AB} and $S^{ab}=\omega^{aa'}\omega^{bb'}S_{a'b'}$. We have used that $S^{ab}\omega^{AB}$ is skew-symmetric under interchange of aA with bB and that $\omega^{AB}\omega_{AB}=-(m+1)$. The above calculation shows that $\lambda=-4(m+1)\neq 0$ and hence any $(m-1)/2$-partially flat connection is (Ω,λ)-self-dual and is a Yang–Mills connection by Theorem 1. □

The analogous result does not hold if m is even.

Proposition 10: If m is even and the vector bundle $E^* \to M$ admits a ∇^E-parallel metric γ_E, then M has canonical $\mathrm{SO}(E)\cdot\mathrm{Sp}(H)$*-invariant metric* $g=\gamma_E \otimes \omega_H^m$ *and four-form* $\Omega \neq 0$.

Proof: Analogously to the case of m odd, we can define Ω by

$$\Omega := \sum \gamma_{ab}\,\gamma_{cd}\,\omega_{AC}\,\omega_{BD}\,X^{aA}\wedge X^{bB}\wedge X^{cC}\wedge X^{dD}.$$

Here $\gamma_{ab}=\gamma_E(e_a,e_b)$ and we recall that for even m the bilinear form ω_H^m is symmetric: $\omega_{AB} = \omega_{BA}$. □

For even m, a connection ∇ in a vector bundle W over M is $m/2$-partially flat if and only if its curvature F belongs to the space

$$(\Lambda^2 E\otimes \omega_H^m \oplus S^2E\otimes S^2H\otimes \omega_H^{m-1})\otimes \mathrm{End}\, W \subset (\Lambda^2E\otimes S^2S^mH \oplus S^2E\otimes \Lambda^2 S^mH)\otimes \mathrm{End}\, W$$
$$=\Lambda^2(E\otimes S^mH)\otimes \mathrm{End}\, W.$$

The $\mathrm{SO}(E)\cdot\mathrm{Sp}(H)$-submodule $\Lambda^2E\otimes\omega_H^m\oplus S^2E\otimes S^2H\otimes\omega_H^{m-1}\subset\Lambda^2T^*M$ is not irreducible, so unlike the odd m case we cannot conclude that it is contained in an eigenspace V_λ of the $\mathrm{SO}(E)\cdot\mathrm{Sp}(H)$-invariant operator $B_\Omega:\Lambda^2T^*M\to\Lambda^2T^*M$. In fact, examples are known (see Appendix B of Ref. 16) where B_Ω has different eigenvalues on each irreducible summand of Λ^2T^*M. Therefore, in the case of even m we cannot expect that $m/2$-partial flatness implies the Yang–Mills equations.

C. Construction of partially flat connections over higher spin Grassmann manifolds

Now we generalize the construction of half-flat connections over admissible half-flat Grassmann manifolds to the case of k-partially flat connections over k-admissible higher spin Grassmann manifolds M. The natural extension of the harmonic construction given in Sec. IV B yields k-partially flat connections in the vector bundle ν over the k-admissible spin $m/2$ Grassmann manifold M. Again, we lift the geometric data from M to S_H via the projection $\pi: S_H\to M$. The pull back $\pi^*\nabla$ of a k-partially flat connection ∇ in the trivial vector bundle $\nu: W=\mathbb{C}^r\times M\to M$ is a connection in the vector bundle $\pi^*\nu:\pi^*W\to S_H$ which satisfies equations defining the notion of a k-partially flat gauge connection on S_H. One can also define the weaker notion of an almost k-partially flat connection in $\pi^*\nu:\pi^*W\to S_H$. The latter may be constructed from a prepotential and it affords the construction of a k-partially flat connection in the bundle $W\to M$. To simplify our exposition we explain the construction in the $m=3$ case. Here the decomposition (30) of the curvature tensor takes the form

$$F(X^e_{\alpha_1\alpha_2\alpha_3},X^{e'}_{\beta_1\beta_2\beta_3}) = \mathop{\mathfrak{S}}_{A}\mathop{\mathfrak{S}}_{B}\Big(\overset{(0)}{F}{}^{[ee']}_{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3} + \omega_H(h_{\alpha_1},h_{\beta_1})\overset{(1)}{F}{}^{(ee')}_{\alpha_2\alpha_3\beta_2\beta_3}$$
$$+\omega_H(h_{\alpha_1},h_{\beta_1})\,\omega_H(h_{\alpha_2},h_{\beta_2})\overset{(2)}{F}{}^{[ee']}_{\alpha_3\beta_3}$$
$$+\omega_H(h_{\alpha_1},h_{\beta_1})\,\omega_H(h_{\alpha_2},h_{\beta_2})\,\omega_H(h_{\alpha_3},h_{\beta_3})\overset{(3)}{F}{}^{(ee')}\Big). \tag{32}$$

In this case we have two nontrivial notions of partial flatness:

$$\text{zero-partial flatness:}\quad \overset{(0)}{F}=0, \tag{33}$$

$$\text{one-partial flatness:}\quad \overset{(0)}{F}=\overset{(1)}{F}=\overset{(2)}{F}=0. \tag{34}$$

Clearly, two-partial flatness is tantamount to flatness. By Theorem 6, a one-partially flat connection is a Yang–Mills connection.

1. Construction of zero-partially flat connections

Let M be a zero-admissible spin 3/2 Grassmann manifold with a zero-partially flat connection ∇ [satisfying (33)] in a holomorphic vector bundle $W \to M$. The pull-back of such a connection ∇ to a connection in the bundle $\pi^* W \to S_H$, where $\pi: S_H \to M$, has curvature F with components given by

$$F(X^{e}_{\pm\pm\pm}, X^{e'}_{\pm\pm\pm}) = 0,$$

$$F(X^{e'}_{\pm\pm\pm}, X^{e}_{\pm}) = u_{\pm}^{\alpha_1} u_{\pm}^{\alpha_2} u_{\pm}^{\alpha_3} u_{\pm}^{\beta_1} u_{\pm}^{\beta_2} u_{\mp}^{\beta_3} F(X^{e}_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3})$$
$$= \pm 12 \overset{(1)}{F}{}^{(ee')}_{\alpha_1\alpha_2\beta_1\beta_2} u_{\pm}^{\alpha_1} u_{\pm}^{\alpha_2} u_{\pm}^{\beta_1} u_{\pm}^{\beta_2} =: \pm 12 \overset{(1)}{F}{}^{(ee')}_{\pm\pm\pm\pm},$$

$$F(X^{e}_{\pm}, X^{e'}_{\pm}) = u_{\pm}^{\alpha_1} u_{\pm}^{\alpha_2} u_{\mp}^{\alpha_3} u_{\pm}^{\beta_1} u_{\pm}^{\beta_2} u_{\mp}^{\beta_3} F(X^{e}_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3}) = -8 \overset{(2)}{F}{}^{[ee']}_{\alpha_1\beta_1} u_{\pm}^{\alpha_1} u_{\pm}^{\beta_1} =: -8 \overset{(2)}{F}{}^{[ee']}_{\pm\pm},$$

$$F(X^{e}_{+++}, X^{e'}_{---}) = u_{+}^{\alpha_1} u_{+}^{\alpha_2} u_{+}^{\alpha_3} u_{-}^{\beta_1} u_{-}^{\beta_2} u_{-}^{\beta_3} F(X^{e}_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3})$$
$$= 36\big(\overset{(1)}{F}{}^{(ee')}_{\alpha_1\alpha_2\beta_1\beta_2} u_{+}^{\alpha_1} u_{+}^{\alpha_2} u_{-}^{\beta_1} u_{-}^{\beta_2} + \overset{(2)}{F}{}^{[ee']}_{\alpha_1\beta_1} u_{+}^{\alpha_1} u_{-}^{\beta_1} + \overset{(3)}{F}{}^{(ee')} \big)$$
$$=: 36\big(\overset{(1)}{F}{}^{(ee')}_{0} + \overset{(2)}{F}{}^{[ee']}_{0} + \overset{(3)}{F}{}^{(ee')}_{0} \big), \tag{35}$$

$$F(X^{e}_{\pm\pm\pm}, X^{e'}_{\mp}) = u_{\pm}^{\alpha_1} u_{\pm}^{\alpha_2} u_{\pm}^{\alpha_3} u_{\pm}^{\beta_1} u_{\mp}^{\beta_2} u_{\mp}^{\beta_3} F(X^{e}_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3}) = \pm 24 \overset{(1)}{F}{}^{(ee')}_{\pm\pm} + 12 \overset{(2)}{F}{}^{[ee']}_{\pm\pm},$$

$$F(X^{e}_{+}, X^{e'}_{-}) = u_{+}^{\alpha_1} u_{+}^{\alpha_2} u_{-}^{\alpha_3} u_{+}^{\beta_1} u_{-}^{\beta_2} u_{-}^{\beta_3} F(X^{e}_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3}) = 12 \overset{(1)}{F}{}^{(ee')}_{0} - 4 \overset{(2)}{F}{}^{[ee']}_{0} - 12 \overset{(3)}{F}{}^{(ee')}_{0},$$

$$F(v, \cdot) = 0,$$

where v, v' are vertical vector fields on S_H. The form of these components lead us to the following definition.

Definition 14: A connection in a holomorphic vector bundle $W \to S_H$ is **zero-partially flat** *if its curvature satisfies the equations*

$$F(X^{e}_{\pm\pm\pm}, X^{e'}_{\pm\pm\pm}) = 0,$$

$$F(X^{e}_{\pm\pm\pm}, X^{e'}_{\pm}) = F(X^{e'}_{\pm\pm\pm}, X^{e}_{\pm}),$$

$$F(X^{e}_{\pm}, X^{e'}_{\pm}) = -F(X^{e'}_{\pm}, X^{e}_{\pm}),$$

$$F(v, \cdot) = 0, \quad \forall v \in T^{v} S_H.$$

The restriction of a zero-partially flat connection to a leaf of the integrable distribution $\langle \mathcal{D}_{3+}, \partial_0 \rangle$ is clearly flat. In this case, an **analytic frame** in the holomorphic vector bundle $\pi^* \nu: \mathbb{C}^r \times S_H \to S_H$ is a frame which is parallel along leaves of this integrable distribution. With respect to such a frame, a connection in the vector bundle $\pi^* \nu$ can be written as

$$\nabla^S_{\partial_0} = \partial_0 ,$$

$$\nabla^S_{X^e_{+++}} = X^e_{+++} ,$$

$$\nabla^S_{X^e_{\pm}} = X^e_{\pm} + A(X^e_{\pm}),$$

$$\nabla^S_{\partial_{\pm\pm}} = \partial_{\pm\pm} + A(\partial_{\pm\pm}),$$

$$\nabla^S_{X^e_{---}} = X^e_{---} + A(X^e_{---}).$$

Definition 15: A connection ∇^S over S_H is called **almost zero-partially flat** *if its curvature satisfies the following equations:*

$$F(X^e_{+++}, X^{e'}_{+++}) = F(X^e_{+++}, v) = F(X^e_{\pm}, v) = 0, \quad \forall v \in T^v S_H, \qquad F(\partial_{++}, \cdot\,) = F(\partial_0, \cdot\,) = 0 . \tag{36}$$

Following the construction of almost half-flat connections, we may construct almost zero-partially flat connections, which allow deformation to a zero-partially flat connection. As in the case of a half-flat connection (cf. Proposition 7), an almost zero-partially flat connection is completely determined by the potentials $A_{\pm\pm} =: A(\partial_{\pm\pm})$ with respect to an analytic frame.

Proposition 11: Let ∇ be an almost zero-partially flat connection in the vector bundle $\pi^* \nu : \mathbb{C}^r \times S_H \to S_H$ with potentials A_{++}, A_{--}, $A(X^e_{\pm})$ and $A(X^e_{---})$ in an analytic frame. Then we have the following.

(i) The potential A_{++} is analytic and has charge 2, i.e.,

$$X^e_{+++} A_{++} = 0, \quad \partial_0 A_{++} = 2A_{++} . \tag{37}$$

(ii) The potential A_{--} satisfies

$$\partial_{++} A_{--} - \partial_{--} A_{++} + [A_{++}, A_{--}] = 0, \quad \partial_0 A_{--} = -2A_{--} . \tag{38}$$

(iii) The potentials $A(X^e_{\pm})$ and $A(X^e_{---})$ are then recursively determined as follows:

$$\begin{aligned} A(X^e_{+}) &= -\tfrac{1}{3} X^e_{+++} A_{--} , \\ A(X^e_{-}) &= \tfrac{1}{2}(\partial_{--} A(X^e_{+}) - X^e_{+} A_{--} + [A_{--}, A(X^e_{+})]), \\ A(X^e_{---}) &= \partial_{--} A(X^e_{-}) - X^e_{-} A_{--} + [A_{--}, A(X^e_{-})], \end{aligned} \tag{39}$$

and they have charges $+1, -1$, and -3, respectively, i.e.,

$$\partial_0 A(X^e_{\pm}) = \pm A(X^e_{\pm}), \quad \partial_0 A(X^e_{---}) = -3A(X^e_{---}). \tag{40}$$

Conversely, any set of matrix-valued potentials A_{++}, A_{--}, $A(X^e_{\pm})$ and $A(X^e_{---})$ satisfying (37)–(40) define an almost zero-partially flat connection.

Proof: (i) The curvature constraints $F(X^e_{+}, \partial_{++}) = 0$, $F(\partial_0, \partial_{++}) = 0$, in an analytic frame, take the form (37).

(ii) The further almost zero-partial-flatness conditions, $F(\partial_{++}, \partial_{--}) = F(\partial_0, \partial_{--}) = 0$, give Eqs. (38) for the potential A_{--}.

(iii) Having obtained A_{--}, we can find $A(X^e_{\pm})$ and $A(X^e_{---})$ from the equations

$$F(\partial_{--},X^e_{+++})=0\Leftrightarrow -X^e_{+++}A_{--}=A([\partial_{--},X^e_{+++}])=3A(X^e_{+}),$$

$$F(\partial_{--},X^e_{+})=0\Leftrightarrow \partial_{--}A(X^e_{+})-X^e_{+}A_{--}+[A_{--},A(X^e_{+})]=A([\partial_{--},X^e_{+}])=2A(X^e_{-}),$$

$$F(\partial_{--},X^e_{-})=0\Leftrightarrow \partial_{--}A(X^e_{-})-X^e_{-}A_{--}+[A_{--},A(X^e_{-})]=A([\partial_{--},X^e_{-}])=A(X^e_{---}).$$

Equations (40) follow from (39). □

Now, using this proposition, a modification of Theorem 4 gives an algorithm for the construction of all almost zero-partially flat connections.

Theorem 7: *Let A_{++} be an analytic prepotential, i.e., a matrix-valued function on a domain $U=\pi^{-1}(V)\subset S_H$, $V\subset M$ a simply connected domain, satisfying (37), and Φ an invertible matrix-valued function on U which satisfies the equations*

$$\partial_{++}\Phi=-A_{++}\Phi\,,\quad \partial_0\Phi=0\,. \tag{41}$$

Such a function Φ always exists. Then the pair (A_{++},Φ) determines an almost zero-partially flat connection $\nabla^S=\nabla^{(A_{++},\Phi)}$. Its potentials with respect to an analytic frame are given by A_{++}, $A_{--}=-(\partial_{--}\Phi)\Phi^{-1}$ and (39). Conversely, any almost zero-partially flat connection is of this form.

The proof follows that for Theorem 4 and uses Proposition 1.1.

To deform an almost zero-partially flat connection into a zero-partially flat connection, we need to find a transformation from the above analytic frame to a central frame. Analogously to Lemma 5 we may prove the following.

Lemma 7: Let $\nabla=\nabla^{(A_{++},\Phi)}$ be the almost zero-partially flat connection associated to the analytic prepotential A_{++} with respect to the analytic frame φ and an invertible solution Φ of (41). Then the frame $\psi:=\varphi\Phi$ is a central frame for the connection ∇, i.e., the potentials $C(\partial_{\pm\pm})$ and $C(\partial_0)$ with respect to that frame vanish.

With respect to the central frame ψ, the above almost zero-partially flat connection then takes the form

$$\nabla^S_{X^e_{+++}}=X^e_{+++}+C(X^e_{+++})=X^e_{+++}+\Phi\; X^e_{+++}\;\Phi^{-1},$$

$$\nabla^S_{X^e_{\pm}}=X^e_{\pm}+C(X^e_{\pm}),$$

$$\nabla^S_{X^e_{---}}=X^e_{---}+C(X^e_{---}),$$

$$\nabla^S_{\partial_{++}}=\partial_{++},\quad \nabla^S_{\partial_{--}}=\partial_{--},\quad \nabla^S_{\partial_0}=\partial_0\,,$$

where in terms of the analytic frame potentials $A(X)$, the central frame potentials $C(X)$ are given by $C(X)=\Phi^{-1}A(X)\Phi+\Phi^{-1}(X\Phi)$. Moreover, the equations $F(\partial_{++},X^e_{+++})=F(\partial_0,X^e_{+++})=0$ imply that the potential $C(X^e_{+++})$ satisfies the equations

$$\partial_{++}C(X^e_{+++})=0,\quad \partial_0 C(X^e_{+++})=3C(X^e_{+++})\,. \tag{42}$$

The following proposition is analogous to Proposition 8 in the half-flat case.

Proposition 12: The potential $C(X^e_{+++})$ of an almost zero-partially flat connection ∇ with respect to a central frame is cubic in u^α_+,

$$C(X^e_{+++})=u^\alpha_+u^\beta_+u^\gamma_+C(\widetilde{X^e_{\alpha\beta\gamma}})=u^\alpha_+u^\beta_+u^\gamma_+C^e_{\alpha\beta\gamma}, \tag{43}$$

where the coefficients $C^e_{\alpha\beta\gamma} = C^e_{\alpha\beta\gamma}(x^i)$, *symmetric in* α,β,γ, *are matrix valued functions of coordinates* x^i *on* M *and* $(x^i, u^\alpha_\pm)$ *are the local coordinates associated with the trivialization* $S_H = M \times \mathrm{Sp}(1,\mathbb{C})$.

With respect to a central frame, we can therefore write $\nabla_{X^e_{+++}} = X^e_{+++} + u^\alpha_+ u^\beta_+ u^\gamma_+ C^e_{\alpha\beta\gamma}$. Using $C^e_{\alpha\beta\gamma}$, we now define a new connection in $\pi^*\nu$ over S_H by

$$\hat{\nabla}_{X^e_{+++}} = X^e_{+++} + u^\alpha_+ u^\beta_+ u^\gamma_+ C^e_{\alpha\beta\gamma},$$

$$\hat{\nabla}_{X^e_{+}} = X^e_{+} + u^\alpha_+ u^\beta_+ u^\gamma_- C^e_{\alpha\beta\gamma},$$

$$\hat{\nabla}_{X^e_{-}} = X^e_{-} + u^\alpha_- u^\beta_- u^\gamma_+ C^e_{\alpha\beta\gamma},$$

$$\hat{\nabla}_{X^e_{---}} = X^e_{---} + u^\alpha_- u^\beta_- u^\gamma_- C^e_{\alpha\beta\gamma},$$

$$\hat{\nabla}_{\partial_{++}} = \partial_{++}, \quad \hat{\nabla}_{\partial_{--}} = \partial_{--}, \quad \hat{\nabla}_{\partial_0} = \partial_0 .$$

The following theorem is the analog of Theorem 5 in the half-flat case.

Theorem 8: *The constructed connection* $\hat{\nabla}$ *is a 0-partially flat connection in* $\pi^*\nu$ *over* S_H *and it is the pull-back of the following 0-partially flat connection* ∇^M *in* ν *over* M:

$$\nabla^M_{X^e_{\alpha\beta\gamma}} = X^e_{\alpha\beta\gamma} + C^e_{\alpha\beta\gamma}.$$

Proof: As in Lemma 1 we may show that the connection $\hat{\nabla}$ is the pull-back of the connection ∇^M. It then suffices to show that ∇^M is zero-partially flat. The connections ∇ and $\hat{\nabla}$ coincide in the direction of X^e_{+++}. Hence, using $C^e_{+++} := u^{\alpha_1}_+ u^{\alpha_2}_+ u^{\alpha_3}_+ C^e_{\alpha_1\alpha_2\alpha_3}$, we have

$$\begin{aligned}
0 = F^\nabla(X^e_{+++}, X^{e'}_{+++}) = F^{\hat{\nabla}}(X^e_{+++}, X^{e'}_{+++})& \\
= X^e_{+++} C^{e'}_{+++} - X^{e'}_{+++} C^e_{+++} + [C^e_{+++}, C^{e'}_{+++}] - C([X^e_{+++}, X^{e'}_{+++}])& \\
= u^{\alpha_1}_+ u^{\alpha_2}_+ u^{\alpha_3}_+ u^{\beta_1}_+ u^{\beta_2}_+ u^{\beta_3}_+ (X^e_{\alpha_1\alpha_2\alpha_3} C^{e'}_{\beta_1\beta_2\beta_3} - X^{e'}_{\beta_1\beta_2\beta_3} C^e_{\alpha_1\alpha_2\alpha_3}& \\
+ [C^e_{\alpha_1\alpha_2\alpha_3}, C^{e'}_{\beta_1\beta_2\beta_3}] - C([X^e_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3}]))& \\
= u^{\alpha_1}_+ u^{\alpha_2}_+ u^{\alpha_3}_+ u^{\beta_1}_+ u^{\beta_2}_+ u^{\beta_3}_+ \; F^{\nabla^M}(X^e_{\alpha_1\alpha_2\alpha_3}, X^{e'}_{\beta_1\beta_2\beta_3}),&
\end{aligned}$$

since $X^e_{+++} u^\beta_+ = 0$. This shows that the component $F^{(0)} = 0$ in the decomposition (32), i.e., the connection ∇^M is zero-partially flat. □

2. Construction of one-partially flat connections

Let M be a one-admissible spin 3/2 Grassmann manifold with a one-partially flat connection ∇ [satisfying (34)] in a holomorphic vector bundle $W \to M$. The pull-back of such a connection to a connection in the bundle $\pi^* W \to S_H$, where $\pi: S_H \to M$, has curvature F with components given by

$$F(X^e_{\pm\pm\pm}, X^{e'}_{\pm\pm\pm}) = 0,$$

$$F(X^{e'}_{\pm\pm\pm}, X^e_{\pm}) = 0,$$

$$F(X^e_{\pm},X^{e'}_{\pm})=0,$$

$$F(X^e_{+++},X^{e'}_{---})=36\overset{(3)}{F}{}^{(ee')},\tag{44}$$

$$F(X^e_{\pm\pm\pm},X^{e'}_{\mp})=0,$$

$$F(X^e_{+},X^{e'}_{-})=-12\overset{(3)}{F}{}^{(ee')},$$

$$F(v,\cdot)=0,$$

where v is any vertical vector field on S_H. A connection in a holomorphic vector bundle $W\to S_H$ is **one-partially flat** if its curvature satisfies the above equations. The restriction of a one-partially flat connection to a leaf of the integrable distribution $\langle\mathcal{D}_{3+},\mathcal{D}_+,\partial_0\rangle$ is clearly flat. In this case, an **analytic frame** in the holomorphic vector bundle $\pi^*\nu:\mathbb{C}^r\times S_H\to S_H$ is a frame which is parallel along leaves of this distribution. With respect to such a frame, a connection in the vector bundle $\pi^*\nu$ can be written as

$$\nabla^S_{\partial_0}=\partial_0,$$

$$\nabla^S_{X^e_{+++}}=X^e_{+++},$$

$$\nabla^S_{X^e_{+}}=X^e_{+},$$

$$\nabla^S_{\partial_{++}}=\partial_{++}+A_{++},$$

$$\nabla^S_{\partial_{--}}=\partial_{--}+A_{--},$$

$$\nabla^S_{X^e_{---}}=X^e_{---}+A(X^e_{---}),$$

$$\nabla^S_{X^e_{-}}=X^e_{-}+A(X^e_{-}),$$

with potentials $A(X^e_{+++})=A(X^e_{+})=A(\partial_0)=0$. We look for solutions of the system (44) in this analytic gauge.

Definition 16: A connection ∇^S *over* S_H *is called* **almost one-partially flat** *if its curvature satisfies the equations*

$$F(X^e_{+++},X^{e'}_{+++})=F(X^e_{+++},X^{e'}_{+})=F(X^e_{+},X^{e'}_{+})=0,$$
$$F(X^e_{+++},v)=F(X^e_{\pm},v)=F(\partial_{++},\cdot)=F(\partial_0,\cdot)=0,\quad \forall v\in T^vS_H.\tag{45}$$

In virtue of these equations, the potentials $A_{\pm\pm}=:A(\partial_{\pm\pm})$ determine all other potentials:

Proposition 13: Let ∇ *be an almost one-partially flat connection in the vector bundle* $\pi^*\nu:\mathbb{C}^r\times S_H\to S_H$ *with potentials* A_{++}, A_{--}, $A(X^e_{-})$ *and* $A(X^e_{---})$ *in an analytic frame. Then we have the following.*

(i) The potential A_{++} *is analytic and has charge* 2, *i.e.,*

$$X^e_{+++}A_{++}=0,\quad X^e_{+}A_{++}=0,\quad \partial_0A_{++}=2A_{++}.\tag{46}$$

(ii) The potential A_{--} *satisfies*

$$\partial_{++}A_{--}-\partial_{--}A_{++}+[A_{++},A_{--}]=0,\quad \partial_0 A_{--}=-2A_{--}. \tag{47}$$

(iii) The potentials $A(X^e_-)$ *and* $A(X^e_{---})$ *are then recursively determined as follows:*

$$A(X^e_-)=-\tfrac{1}{2}X^e_+A_{--},$$
$$A(X^e_{---})=\partial_{--}A(X^e_-)-X^e_-A_{--}+[A_{--},A(X^e_-)], \tag{48}$$

and they have charges -1 *and* -3*, respectively; i.e.,*

$$\partial_0 A(X^e_-)=\pm A(X^e_-),\quad \partial_0 A(X^e_{---})=-3A(X^e_{---}). \tag{49}$$

Conversely, any set of matrix-valued potentials A_{++}*,* A_{--}*,* $A(X^e_-)$ *and* $A(X^e_{---})$ *satisfying (46)–(49) define an almost one-partially flat connection.*

Proof: (i) Equations (46) are equivalent to $F(\partial_{++},X^e_{+++})=F(\partial_{++},X^e_+)=F(\partial_0,\partial_{++})=0$.

(ii) The further almost one-partial-flatness conditions, $F(\partial_{++},\partial_{--})=F(\partial_0,\partial_{--})=0$, give Eqs. (47).

(iii) Having obtained A_{--}, we can find $A(X^e_-)$ and $A(X^e_{---})$ from the equations

$$F(\partial_{--},X^e_+)=0\Leftrightarrow -X^e_+A_{--}=A([\partial_{--},X^e_+])=2A(X^e_-),$$

$$F(\partial_{--},X^e_-)=0\Leftrightarrow \partial_{--}A(X^e_-)-X^e_-A_{--}+[A_{--},A(X^e_-)]=A([\partial_{--},X^e_-])=A(X^e_{---}).$$

The equations (49) follow from (48). □

Now, starting from a prepotential A_{++}, which solves (46), we may construct an almost one-partially flat connection. The potential $A_{--}=-(\partial_{--}\Phi)\Phi^{-1}$ is determined, as before, from a solution Φ of (41). Then, with the remaining potentials in an analytic frame being given by (48) and satisfying (49), all the other equations in (45) follow. This shows that an almost one-partially flat connection is determined by an arbitrary analytic prepotential A_{++} and an invertible solution Φ of (41). As before, Φ is a transition function from an analytic frame to a central frame, in which the above almost one-partially flat connection takes the form

$$\nabla^S_{X^e_{+++}}=X^e_{+++}+C(X^e_{+++})=X^e_{+++}+\Phi\ X^e_{+++}\ \Phi^{-1},$$

$$\nabla^S_{X^e_+}=X^e_++C(X^e_+)=X^e_++\Phi\ X^e_+\ \Phi^{-1},$$

$$\nabla^S_{X^e_-}=X^e_-+C(X^e_-),$$

$$\nabla^S_{X^e_{---}}=X^e_{---}+C(X^e_{---}),$$

$$\nabla^S_{\partial_{++}}=\partial_{++},\quad \nabla^S_{\partial_{--}}=\partial_{--},\quad \nabla^S_{\partial_0}=\partial_0.$$

Moreover, the equations $F(\partial_{++},X^e_{+++})=F(\partial_0,X^e_{+++})=0$ imply that the potential $C(X^e_{+++})$ satisfies the equations

$$\partial_{++}C(X^e_{+++})=0,\quad \partial_0 C(X^e_{+++})=3C(X^e_{+++}).$$

Proposition 14: The potentials $C(X^e_{+++})$ *and* $C(X^e_+)$ *of an almost one-partially flat connection* ∇ *with respect to a central frame have the form*

$$C(X^e_{+++})=u^\alpha_+u^\beta_+u^\gamma_+C^e_{\alpha\beta\gamma},\quad C(X^e_+)=u^\alpha_+u^\beta_+u^\gamma_-C^e_{\alpha\beta\gamma},$$

where $C^e_{\alpha\beta\gamma}$ is a function on M, symmetric in α,β,γ.

With respect to a central frame, we can therefore write

$$\nabla_{X^e_{+++}} = X^e_{+++} + u^\alpha_+ u^\beta_+ u^\gamma_+ C^e_{\alpha\beta\gamma}, \quad \nabla_{X^e_+} = X^e_+ + u^\alpha_+ u^\beta_+ u^\gamma_- C^e_{\alpha\beta\gamma}.$$

We define a modified connection in the bundle $\pi^*\nu$ over S_H by

$$\begin{aligned}
\hat{\nabla}_{X^e_{+++}} &= X^e_{+++} + u^\alpha_+ u^\beta_+ u^\gamma_+ C^e_{\alpha\beta\gamma},\\
\hat{\nabla}_{X^e_+} &= X^e_+ + u^\alpha_+ u^\beta_+ u^\gamma_- C^e_{\alpha\beta\gamma},\\
\hat{\nabla}_{X^e_-} &= X^e_- + u^\alpha_- u^\beta_- u^\gamma_+ C^e_{\alpha\beta\gamma},\\
\hat{\nabla}_{X^e_{---}} &= X^e_{---} + u^\alpha_- u^\beta_- u^\gamma_- C^e_{\alpha\beta\gamma},\\
\hat{\nabla}_{\partial_{++}} = \partial_{++}, \quad &\hat{\nabla}_{\partial_{--}} = \partial_{--}, \quad \hat{\nabla}_{\partial_0} = \partial_0 .
\end{aligned} \tag{50}$$

As in the zero-partially flat case, we have the following.

Theorem 9: *The constructed connection $\hat{\nabla}$ is a one-partially flat connection in $\pi^*\nu$ over S_H and it is the pull-back of the following one-partially flat connection ∇^M in ν over M:*

$$\nabla^M_{X^e_{\alpha\beta\gamma}} = X^e_{\alpha\beta\gamma} + C^e_{\alpha\beta\gamma}. \tag{51}$$

Proof: As before one shows that the connection $\hat{\nabla}$ is the pull-back of the connection ∇^M. It remains to show that ∇^M is one-partially flat. Since any almost one-partially flat connection is almost zero-partially flat, we have $F^{(0)}=0$ by Theorem 8. Next we show that $F^{(1)}=0$. The connections ∇ and $\hat{\nabla}$ coincide in the direction of X^e_{+++} and X^e_+. Hence, using Eq. (35), which holds for zero-partially flat connections, we have

$$0 = F^{\nabla}(X^e_{+++}, X^{e'}_+) = F^{\hat{\nabla}}(X^e_{+++}, X^{e'}_+) = 12 \overset{(1)}{F}{}^{(ee')}_{\alpha_1\alpha_2\beta_1\beta_2} u^{\alpha_1}_+ u^{\alpha_2}_+ u^{\beta_1}_+ u^{\beta_2}_+ .$$

This shows that the component $F^{(1)}$ in the decomposition (32) vanishes. Similarly,

$$0 = F^{\nabla}(X^e_+, X^{e'}_+) = F^{\hat{\nabla}}(X^e_+, X^{e'}_+) = -8 \overset{(2)}{F}{}^{[ee']}_{\alpha_1\beta_1} u^{\alpha_1}_+ u^{\beta_1}_+$$

implies $F^{(2)[ee']}=0$, and hence that $\hat{\nabla}$ is one-partially flat. □

By Theorem 6, the one-partially flat connection $\nabla^M_{X^e_{\alpha\beta\gamma}}$ in (51) is a Yang–Mills connection.

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