A constructive approach to a class of ergodic HJB equations with nonsmooth cost

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Abstract

We consider a class of ergodic Hamilton-Jacobi-Bellman (HJB) equations, related to large time asymptotics of non-smooth multiplicative functional of diffusion processes. Under suitable ergodicity assumptions on the underlying diffusion, we show existence of these asymptotics, and that they solve the related HJB equation in the viscosity sense.

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1 Introduction

Let $(x_t)_{t\geq 0}$ be a continuous-time, homogeneous Markov process with infinitesimal generator L. To fix ideas, assume x_t is $\mathbb{R}^d$-valued. Given a function $c : \mathbb{R}^d \to \mathbb{R}$ and $\gamma > 0$, we are interested in obtaining long-time asymptotics of the functional

$$S(T,x) := \log E_x \left[\exp \left(\gamma \int_0^T c(x_t)dt \right) \right],$$

where E_x is the expectation conditioned to $x_0 = x$. Let $\varphi(T,x) = e^{S(T,x)}$. At least at the formal level, φ is a solution of the equation

$$\partial_t \varphi(t,x) = L\varphi(t,x) + \gamma c(x)\varphi(t,x).$$

If a Perron-Frobenius-type Theorem holds for the operator $L + \gamma c$, then for T large $\varphi(T,x)$ gets close to $e^{\lambda T} v(x)$, where λ is the largest eigenvalue of $L + \gamma c$, and v is the corresponding strictly positive eigenfunction. In other words, setting $V(x) := \log v(x)$, we obtain

$$S(T,x) = \lambda T + V(x) + o(T),$$

i.e.

$$\lambda = \lim_{T \to +\infty} \frac{1}{T} \log E_x \left[\exp \left(\gamma \int_0^T c(x_t)dt \right) \right] \tag{1.1}$$

and

$$V(x) = \lim_{T \to +\infty} \left\{ \log E_x \left[\exp \left(\gamma \int_0^T c(x_t)dt \right) \right] - \lambda T \right\}. \tag{1.2}$$

Note also that the pair (λ, V) is a solution of the nonlinear equation

$$\lambda = e^{-V} L(e^V) + \gamma c. \tag{1.3}$$

The actual proof of the existence of the limits (1.1) and (1.2) is, in general, not simple, and various assumptions are required. If the empirical measures

$$\mathfrak{L}_t := \frac{1}{t} \int_0^t \delta_{x_s} ds$$

of the Markov process obey a Large Deviation Principle with rate function $i(\mu)$ (which is known under fairly general conditions), and $c(\cdot)$ is continuous and bounded (but weaker conditions on $c(\cdot)$ may suffice), then the limit (1.1) exists, and

$$\lambda = \sup_{\mu} \left[\int c d\mu - i(\mu) \right] \tag{1.4}$$

where in (1.3) μ varies over probability measures on $\mathbb{R}^d$. The existence of the limit (1.2), i.e. the second-order asymptotics of $S(T,x)$, is a harder problem. For processes taking values in a compact space, where things are simpler, we refer to [9], Section 4. For $\mathbb{R}^d$-valued diffusions, conditions for existence of solutions of (1.3) are given in [8] and [14]. In [8] it is also shown that, under sufficient ergodicity of $(x_t)_{t\geq 0}$ and if $c(\cdot)$ is bounded and sufficiently smooth, then (1.3) has a solution, which needs not to be the unique one, for which (1.1) and (1.2) hold.

In order to relate the above problem to stochastic control, assume that $(x_t)_{t\geq 0}$ is a diffusion of the form

$$dx_t = b(x_t)dt + dB_t, \tag{1.5}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. We then consider the controlled diffusion

$$dx_t^u = [b(x_t^u) + u_t]dt + dB_t, \tag{1.6}$$

where $(u_t)_{t\geq 0}$ is a progressively measurable, square-integrable, $\mathbb{R}^d$-valued process. Define the performance functional

$$J(x,u) := \limsup_{T\to+\infty} \frac{1}{T} E_x \left[\int_0^T \left(\gamma c(x_t^u) - \frac{1}{2}|u_t|^2 \right) dt \right]. \tag{1.7}$$

The aim of the controller consists in maximizing the performance $J(x,u)$ over u. The fact that

$$\sup_u J(x,u) = \lambda$$

with λ as in (1.1), is a consequence of a well known duality principle in stochastic control (see e.g. [6]). Moreover, if the Hamilton-Jacobi-Bellman equation

$$\begin{aligned} \lambda &= \frac{1}{2}\Delta V(x) + \max_{u\in\mathbb{R}^d} \left[(b(x)+u)\cdot\nabla V(x) + \gamma c(x) - \frac{1}{2}|u|^2 \right] \\ &= \frac{1}{2}\Delta V(x) + b(x)\cdot\nabla V(x) + \frac{1}{2}|\nabla V(x)|^2 + \gamma c(x) \end{aligned} \tag{1.8}$$

has a sufficiently nice solution, then $u_t^* := \nabla V(x_t^{u^*})$ provides an optimal feedback control. Moreover, equation (1.8) coincides with equation (1.3).

A considerable improvement to the understanding of equation (1.3) (actually to a more general version of it) is due to Kaise & Sheu [12]. They showed, under reasonable conditions on $b(\cdot)$ and $c(\cdot)$, that (1.3) has indeed multiple solutions, even after identifying solutions that differ by a constant. It is shown in [12] that there exists $\lambda \in \mathbb{R}$ such that the equation

$$\mu = \frac{1}{2}\Delta V + b\cdot\nabla V + \frac{1}{2}|\nabla V|^2 + \gamma c$$

has a (smooth) solution if and only if $\mu \geq \lambda$. Moreover, for $\mu = \lambda$, this solution is unique up to additive constant. Kaise & Sheu also indicates that this λ should be as in (1.1). They do not address the possibility of interpreting *one* solution V as in (1.2).

We remark that all cited results require regularity on the cost $c(\cdot)$; in particular, the available proof of the existence of the limits (1.1) and (1.2) rely on boundedness of $c(\cdot)$ and $\nabla c(\cdot)$. The main object of this paper is to propose a totally different approach to the above problems, by tackling (1.1) and (1.2), for the diffusion (1.5), directly, without relying on properties of equation (1.3). Our approach seems to have the following advantages.

1. Besides some inevitable growth conditions, no smoothness of $c(\cdot)$, not even continuity, is required.
2. In (1.1) and (1.2) the integral

$$\int_0^T c(x_t)dt$$

could be replaced by

$$\int_{[0,T]} c(x_t)d\mu(t),$$

where μ could be of the following forms:

   i. μ is a σ-finite periodic measure, for instance $\mu(dt) = \sum_{k\geq 0} \delta_{k\Delta}(dt)$ for some $\Delta > 0$. In this last case the cost acts at discrete-time only.

   ii. μ is a random measure, independent of $(B_t)_{t\geq 0}$, translation invariant and sufficiently ergodic in law. For instance we could take $\mu(dt) = \sum_n \delta_{\tau_n}(dt)$, where $(\tau_n)_{n\geq 0}$ are the points of a Poisson process.

3. Jump processes, rather than diffusions, should also be treatable.

Not surprisingly, there is a price to pay. At the present stage our results hold for γ in some interval $[0, \overline{\gamma}]$ which is certainly not optimal. Note that however one could get an explicit expression for $\overline{\gamma}$ (carefully following the proofs), as a function of the constants c_b and K_b appearing in conditions (DC) and (CC) of section 3.

The paper is organized as follows. In section 2 we prove the existence of the limits (1.1) and (1.2) under some general conditions on the diffusion process. In section 3 we give sufficient conditions on the drift b only, for these conditions to be fulfilled. In section 4 we show that V and λ given by (1.2) and (1.1) respectively are linked to the equation (1.8), more precisely we show that V is a viscosity solution of (1.8).

We consider here only diffusions whose diffusion coefficient is the identity matrix. The uniformly elliptic case could be dealt with minor modifications. It is worth noticing that the whole content of section 2 is based on Assumptions A1-A6 below, which do not refer to any specific form of the Markov process. The fact that the process is a diffusion plays a role in sections 3 and 4.

2 Existence of the limits (λ, V).

We begin by stating our assumptions on the $\mathbb{R}^d$-valued diffusion

$$dx_t = b(x_t)dt + dB_t. \tag{2.1}$$

A1. Equation (2.1) has, for every deterministic initial condition, a unique strong solution.

A2. There is $C > 0$ such that
$$|c(x)| \le C(|x|^2 + 1).$$

A3. The process (2.1) has a unique invariant probability measure $m(dx)$ such that, for some $\beta > 0$,
$$\int e^{\beta|x|^2} m(dx) < +\infty.$$

A4. The transition probability of the process (2.1) admits a density $p_t(x, y)$ with respect to $m(dx)$. Furthermore there exist $K > 0$, $p > 2$ and $t_0 > 0$ such that for $t \ge t_0$,
$$\| \, p_t(.,.) \, \|_{\mathbb{L}^p(m \otimes m)} \le \; K \, .$$

A5. Let $P_t f(x) = E_x(f(x_t))$ which extends as a continuous semi-group on $\mathbb{L}^2(m)$. For all $f \in \mathbb{L}^2(m)$,
$$\int \left| P_t f(x) - \left(\int f dm \right) \right|^2 \, m(dx) \; \to \; 0 \qquad \text{as } t \to +\infty.$$

A6. For all $a > 0$ and all x there exists $\beta_{a,x} > 0$ such that
$$E_x \left[e^{\beta_{a,x} \int_0^a \, |x_s|^2 \, ds} \right] < +\infty \, .$$

We shall say that A6 is uniformly satisfied if for all $a > 0$ there exist $\beta_a > 0$ and a locally bounded function h_a such that for all x,
$$E_x \left[e^{\beta_a \int_0^a \, |x_s|^2 \, ds} \right] \le h_a(x) \, .$$

Notice that Assumption A4 implies that the semi-group $P_t f(x) = E_x(f(x_t))$ maps continuously $\mathbb{L}^2(m)$ into $\mathbb{L}^p(m), p > 2$, for $t \ge t_0$. Hence, according to Gross hypercontractivity theorem (see e.g. [1]), m satisfies a defective logarithmic Sobolev inequality. If m is absolutely continuous with respect to the Lebesgue measure, $m(dx) = e^{-V} dx$, and V is locally bounded, a result by Röckner and Wang says that m satisfies a so called "weak Poincaré inequality", hence thanks to a result by Aida, m will satisfy a tight log-Sobolev inequality (for all these results see the book of Wang [17]). In particular m will both satisfy a Poincaré inequality (or spectral gap inequality), so that Assumption A5 is satisfied, and a gaussian concentration inequality implying A3. Section 3 will be devoted to giving sufficient conditions for these hypotheses to hold. We begin to show that the limits (1.1) and (1.2) exist along suitable sequences.

Proposition 1 . *Under A1-A6, for every $a > 0$ large enough, there exists $\gamma(a)$ such that for all $\gamma < \gamma(a)$ and all x the limits*
$$\lambda_a = \lim_{n \to +\infty} \frac{1}{an} \log E_x \left[\exp \left(\gamma \int_0^{an} c(x_t) dt \right) \right] \tag{2.2}$$

and

$$V_a(x) = \lim_{n\to+\infty} \left\{ \log E_x \left[\exp\left(\gamma \int_0^{an} c(x_t)dt \right) \right] - \lambda an \right\}, \tag{2.3}$$

exist. If A6 is uniformly satisfied, then the limit in (2.3) *is uniform on compact sets.*

The proof of Proposition 1 is done via a cluster expansion technique. The convergence of the expansion requires some small parameter, which is obtained by choosing γ small and by a suitable choice of the time step a in the discretization. We will not try to give explicit bounds on the $\gamma(a)$ in Proposition 1, even though this would be possible.

In what follows, define

$$\begin{aligned} \psi_\gamma(t,x,y) &:= \log E_x \left[\exp\left(\gamma \int_0^t c(x_s)ds \right) \Big| x_t = y \right] \\ &= \log E_{xy} \left[\exp\left(\gamma \int_0^t c(x_s)ds \right) \right], \end{aligned} \tag{2.4}$$

where E_{xy} denotes the expectation under the law of the bridge of $(x_s)_{0\le s\le t}$ between x and y.

Proof of Proposition 1. Consider a time-step $a > 0$. Note that

$$\begin{aligned} e^{S(an,x)} &= E_x \left[\exp\left(\gamma \int_0^{an} c(x_t)dt \right) \right] \\ &= E_x \left[\exp\left(\sum_{k=0}^{n-1} \psi_\gamma(a, x_{ka}, x_{(k+1)a}) \right) \right] \\ &= \mathbb{E} \left[\exp\left(\sum_{k=0}^{n-1} \psi_\gamma(a, \xi_k, \xi_{k+1}) \right) \prod_{k=0}^{n-1} p_a(\xi_k, \xi_{k+1}) \right] \\ &= \mathbb{E} \left[\exp\left(\sum_{k=0}^{n-1} \phi_\gamma(a, \xi_k, \xi_{k+1}) \right) \right], \end{aligned} \tag{2.5}$$

where $\mathbb{E}$ is the expectation with respect to a probability $\mathbb{P}$, $\xi_0 = x$, $\xi_1, \ldots, \xi_n$ are random variables that, under $\mathbb{P}$, are i.i.d. with law $m(dx)$, and

$$\phi_\gamma(a,x,y) = \psi_\gamma(a,x,y) + \log p_a(x,y).$$

A cluster in this context is a subset of $\mathbb{Z}^+ := \{0,1,2,\ldots\}$ of the form $\{k, k+1, ..., k+l\}$. We say that two clusters are separated if their union is not a cluster or, equivalently, if there is an integer k which is strictly bigger than all elements of one cluster and strictly smaller that all elements of the other. We denote by $\mathcal{C}$ the set of all clusters, while $\mathcal{C}_n$ denotes the set of clusters contained in $\{0, 1, ..., n-1\}$. The usual cluster expansion procedure yields

$$\exp\left(\sum_{k=0}^{n-1}\phi_\gamma(a,\xi_k,\xi_{k+1})\right) = \prod_{k=0}^{n-1}\left[\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right)+1\right]$$

$$= \sum_{\tau\subseteq\{0,1,\dots,n-1\}}\prod_{k\in\tau}\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right) = \sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{\tau_1,\dots,\tau_p\in\mathcal{C}_n \\ \text{separated}}}\prod_{i=1}^{p}\prod_{k\in\tau_i}\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right)$$

$$= \sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{\tau_1,\dots,\tau_p\in\mathcal{C}_n \\ \text{separated}}} q_{\tau_1}q_{\tau_2}\cdots q_{\tau_p},$$

where

$$q_{\tau_i} := \prod_{k\in\tau_i}\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right),$$

and we have used the fact that any subset of $\{0,...,n-1\}$ is union of p separated clusters for some $p\geq 0$, and these clusters can be rearranged in $p!$ ways. The key remark is that if τ_i and τ_j are separated clusters, then $\mathbb{E}(q_{\tau_i}q_{\tau_j}) = \mathbb{E}(q_{\tau_i})\mathbb{E}(q_{\tau_j})$. Thus, by (2.5),

$$e^{S(an,x)} = \sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{\tau_1,\dots,\tau_p\in\mathcal{C}_n \\ \text{separated}}}\mathbb{E}(q_{\tau_1})\mathbb{E}(q_{\tau_2})\cdots\mathbb{E}(q_{\tau_p}).$$

Now the logarithm of the above expression can be rewritten as

$$S(an,x) = \sum_{\tau\in\mathcal{C}_n,\tau\neq\emptyset}\sum_{p\geq 0}\sum_{\substack{\tau_1,\dots,\tau_p\in\mathcal{C}_n \\ \tau_1\cup\cdots\cup\tau_p=\tau}} a_p(\tau_1,\tau_2,\dots,\tau_p)\mathbb{E}(q_{\tau_1})\mathbb{E}(q_{\tau_2})\cdots\mathbb{E}(q_{\tau_p}) := \sum_{\tau\in\mathcal{C}_n,\tau\neq\emptyset}\Gamma_\tau,$$

where the coefficients $a_p(\tau_1,\dots,\tau_p)$ come from the Taylor expansion of the logarithm (see [13] page 492). Now note that Γ_τ depends on x if and only if $0\in\tau$, i.e. $\tau = \{0,1,...,m\}$ for some m. In what follows we write Γ_m in place of $\Gamma_{\{0,1,\dots,m\}}$. Thus we can write

$$\sum_{\tau\in\mathcal{C}_n,\tau\neq\emptyset}\Gamma_\tau = \sum_{m=0}^{n-1}\Gamma_m + \sum_{i=1}^{n-1}\sum_{\substack{\tau\in\mathcal{C}_n,\tau\not\ni 0 \\ i\in\tau}}\frac{1}{|\tau|}\Gamma_\tau = \sum_{m=0}^{n-1}\Gamma_m + (n-1)\sum_{\substack{\tau\in\mathcal{C}_n,\tau\not\ni 0 \\ 1\in\tau}}\frac{1}{|\tau|}\Gamma_\tau, \tag{2.6}$$

where in the last step we used the fact that, for $0\notin\tau$, Γ_τ is invariant by translation and permutation of τ, property that is inherited from the measure $\mathbb{P}$. Thus, at a formal level, the limits (2.2) and (2.3) are given by

$$V_a(x) = \sum_{m=0}^{+\infty}\Gamma_m, \tag{2.7}$$

$$\lambda_a = \frac{1}{a}\sum_{\substack{\tau\in\mathcal{C},\tau\not\ni 0 \\ 1\in\tau}}\frac{1}{|\tau|}\Gamma_\tau, \tag{2.8}$$

provided the sums in (2.7) and (2.8). By the usual cluster expansion estimates (see e.g. [7]), the convergence of (2.8) follows from the strong cluster estimates: there exists $\rho \in (0,1)$ such that for $\tau \in \mathcal{C}$, $0 \notin \tau$,

$$|\mathbb{E}(q_\tau)| \leq \rho^{|\tau|}. \tag{2.9}$$

In the case of $\tau \ni 0$, we will prove an estimate of the type

$$|\mathbb{E}(q_\tau)| \leq C(x)\rho^{|\tau|}. \tag{2.10}$$

This is enough for the convergence of (2.8) and of (2.7) for each fixed x.

Thus, to complete the proof, we only have to prove the estimates (2.9) and (2.10), for γ sufficiently small. We begin by proving (2.9). Recall that

$$|\mathbb{E}(q_\tau)| = \left|\mathbb{E}\left[\prod_{k\in\tau}\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right)\right]\right|.$$

By the generalized Hölder inequality in [15], Lemma 5.2, we have

$$|\mathbb{E}(q_\tau)| \leq \prod_{k\in\tau}\left[\mathbb{E}\left[\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right)^2\right]\right]^{1/2} = \rho^{|\tau|} \tag{2.11}$$

for

$$\rho := \left[\mathbb{E}\left[\left(e^{\phi_\gamma(a,\xi_1,\xi_2)}-1\right)^2\right]\right]^{1/2}.$$

We now show how to make ρ strictly less than 1 by choosing a sufficiently large and γ small enough.

$$\begin{aligned}
\rho^2 &= \mathbb{E}\left(\left(e^{\psi_\gamma(a,\xi_1,\xi_2)}p_a(\xi_1,\xi_2)-1\right)^2\right)\\
&= \int_{\mathbb{R}^{2d}}\left[E_{xy}\left(e^{\gamma\int_0^a c(x_s)ds}\right)p_a(x,y)-1\right]^2 m(dx)m(dy)\\
&= \int_{\mathbb{R}^{2d}}\left[E_{xy}\left(e^{\gamma\int_0^a c(x_s)ds}-1\right)p_a(x,y)+(p_a(x,y)-1)\right]^2 m(dx)m(dy)\\
&\leq 2\int_{\mathbb{R}^{2d}}E_{xy}^2\left(e^{\gamma\int_0^a c(x_s)ds}-1\right)p_a^2(x,y)m(dx)m(dy)+2\int_{\mathbb{R}^{2d}}(p_a(x,y)-1)^2\, m(dx)m(dy)
\end{aligned}$$

The second integral term in the above right hand side goes to 0 as $a \to +\infty$ thanks to Lemma 1 below. We thus analyze the first integral term in the right hand side of the above inequality. For any $\varepsilon \in]0,1[$, by Hölder inequality,

$$\begin{aligned}
I(a,\gamma) &:= \int_{\mathbb{R}^{2d}}E_{xy}^2\left(e^{\gamma\int_0^a c(x_s)ds}-1\right)p_a^{\varepsilon}(x,y)p_a^{2-\varepsilon}(x,y)m(dx)m(dy)\\
&\leq \left[\int_{\mathbb{R}^{2d}}E_{xy}^{2/\varepsilon}\left(e^{\gamma\int_0^a c(x_s)ds}-1\right)p_a(x,y)m(dx)m(dy)\right]^{\varepsilon}\left[\int_{\mathbb{R}^{2d}}p_a^{\frac{2-\varepsilon}{1-\varepsilon}}(x,y)m(dx)m(dy)\right]^{1-\varepsilon}\\
&= \left[\mathbb{E}\left(|e^{\gamma\int_0^a c(x_s)ds}-1|^{2/\varepsilon}\right)\right]^{\varepsilon}\,\|\, p_a \,\|^{2-\varepsilon}_{\mathbb{L}^{\frac{2-\varepsilon}{1-\varepsilon}}(m\otimes m)}
\end{aligned}$$

Thanks to Assumption A4, for a large enough and ε small enough (such that $\frac{2-\varepsilon}{1-\varepsilon} \leq p$), $\| p_a \|^{2-\varepsilon}_{\mathbb{L}^{\frac{2-\varepsilon}{1-\varepsilon}}(m\otimes m)} < +\infty$.
To complete the proof, it remains to control $J(a,\gamma) := \mathbb{E}\left(|e^{\gamma\int_0^a c(x_s)ds} - 1|^{2/\varepsilon}\right)$.
Since $e^{\gamma\int_0^a c(x_s)ds} - 1 = \gamma\int_0^a c(x_s)ds \int_0^1 e^{u\gamma\int_0^a c(x_s)ds}du$,

$$\begin{aligned} J(a,\gamma) &= \gamma^{2/\varepsilon}\mathbb{E}\left(\left|\int_0^a c(x_s)ds\right|^{2/\varepsilon}\left(\int_0^1 e^{u\gamma\int_0^a c(x_s)ds}du\right)^{2/\varepsilon}\right) \\ &\leq \gamma^{2/\varepsilon}\mathbb{E}^{1/2}\left(\left|\int_0^a c(x_s)ds\right|^{4/\varepsilon}\right)\mathbb{E}^{1/2}\left(e^{\frac{4\gamma}{\varepsilon}\int_0^a c(x_s)ds}\right) \end{aligned}$$

By Jensen's inequality

$$\begin{aligned} J(a,\gamma) &\leq \gamma^{2/\varepsilon}a^{2/\varepsilon-1/2}\left[\mathbb{E}\left(\int_0^a |c(x_s)|^{4/\varepsilon}ds\right)\right]^{1/2}\left[\frac{1}{a}\int_0^a \mathbb{E}\left(e^{\frac{4\gamma}{\varepsilon}ac(x_s)}\right)ds\right]^{1/2} \\ &\leq (\gamma a)^{2/\varepsilon}\left[\int |c(x)|^{4/\varepsilon}m(dx)\right]^{1/2}\left[\int e^{\frac{4\gamma a}{\varepsilon}c(x)}m(dx)\right]^{1/2} \end{aligned}$$

The first integral term of the right hand side in the above inequality is finite due to Assumption A2 and A3. For the same reason, if $\gamma a < \frac{\varepsilon}{4C}\beta$, the last integral term of the right hand side is finite. Then, for γa small enough, $J(a,\gamma)$ is as small as we want and the cluster estimate ρ is smaller than 1, which completes the proof.

For the proof of (2.10), we proceed in the same way, just observing that the first factor in the right hand side of (2.11) is now dependent on x. The additional term to control is $E_x\left[e^{q\gamma\int_0^a c(x_s)\,ds}\right]$ for some large $q > 1$. This can be done using A2 and A6. ∎

It remains to state and prove

Lemma 1 *If A4 and A5 are satisfied,* $\lim_{a\to+\infty}\int_{\mathbb{R}^{2d}}(p_a(x,y)-1)^2\, m(dx)m(dy) = 0$.

Proof. Recall the semi-group $P_t f(x) = \int f(y)\, p_t(x,y)\, m(dy)$. By Assumption A5, P_t is a contraction semi-group on $\mathbb{L}^2(m)$, and

$$\int \left|P_t f(x) - \left(\int f dm\right)\right|^2 m(dx) \;\to\; 0 \quad \text{as } t\to+\infty.$$

Notice also that for $a > b > 0$ and m almost all y,

$$p_a(x,y) = P_{a-b}p_b(.,y)(x)$$

in $\mathbb{L}^2(m)$ for all rational times a and b and, by invariance of m, $\int p_b(x,y)m(dx) = 1$. Consider now an increasing sequence $(a_n)_{n\geq 0}$ such that $a_n \to +\infty$. We have to show that, for any such sequence,

$$\int_{\mathbb{R}^{2d}}(p_{a_n}(x,y)-1)^2\, m(dx)m(dy) \to 0 \tag{2.12}$$

as $n \to +\infty$. It is not restrictive to assume $a_1 > t_0$, where t_0 is the constant in Assumption A4. By Assumption A5, for (m almost all) fixed y,

$$\int (p_{a_n}(x,y) - 1)^2 \; m(dx) = \int (P_{a_n - a_1} p_{a_1}(.,y)(x) - 1)^2 \; m(dx) \to 0$$

as $n \to +\infty$. But thanks to Assumption A4, the sequence

$$y \mapsto \int (p_{a_n}(x,y) - 1)^2 \; m(dx)$$

is uniformly integrable. Thus the conclusion (2.12) follows by Vitali convergence theorem. ∎

Now we shall see why the limits (2.2) and (2.3) do not depend on the time step a, yielding the limits (1.1) and (1.2).

So we choose once for all some convenient a and consider the corresponding set of convenient γ's, yielding for each γ a λ obtained thanks to Proposition 1. For large T we choose n such that $a(n-1) \leq T < an$.

Remark that if $c = c^+ - c^-$ with non-negative c^+ and c^-,

$$\begin{aligned} E_x\left[e^{\gamma \int_0^{a(n-1)} c(x_s)ds} \, e^{-\gamma \int_{a(n-1)}^{an} c^-(x_s)ds}\right] &\leq E_x\left[e^{\gamma \int_0^T c(x_s)ds}\right] \\ &\leq E_x\left[e^{\gamma \int_0^{a(n-1)} c(x_s)ds} \, e^{\gamma \int_{a(n-1)}^{an} c^+(x_s)ds}\right] . \end{aligned}$$

We denote by $e^{S^-(an,x)}$ and $e^{S^+(an,x)}$ respectively the lower and upper bounds above. Both $S^-(an,x)$ and $S^+(an,x)$ can be calculated using the same cluster expansion except that we have to replace $\psi_\gamma(a, x_{a(n-1)}, x_{an})$ by a similar ψ^- (resp. ψ^+) obtained by replacing c by $-c^-$ (resp c^+). We thus obtain a similar decomposition $S^-(an,x) = \sum_{T \in \mathfrak{C}_n, T \neq \emptyset} \Gamma^-_T$ with $\Gamma^-_T = \Gamma_T$ if $n-1 \notin T$ and Γ^-_T obviously modified if $n-1 \in T$. In particular in the decomposition (2.6) we see that, in the first sum, the only modified term is Γ_{n-1}. But since $-c^-$ also satisfies A2, the same estimates as in the proof of the previous Theorem show that the modified term Γ^-_{n-1} goes to 0 as T (hence n) goes to infinity, by possibly choosing smaller γ. The second sum in (2.6) is a little bit more intricate to study. The only modified terms are those obtained for $(n-1) \in T$ (notice that we can no more use translation or permutation invariance for these clusters). But summing up all these modified terms gives a quantity which is smaller than $M(an) - M(a(n-1))$ where $M(an)$ is defined similarly as $S(an,x)$ replacing $\mathbb{E}(q_T)$ by $\rho^{|T|}$ for some ρ strictly less than 1 and larger than $\mathbb{E}(q_T)$ and the modified $\mathbb{E}(q^-_T)$. Since this difference goes to 0 as n goes to infinity, and since we can control $S^+(an,x)$ in exactly the same way, we have obtained

Theorem 1 . *Under A1-A6 there is $\overline{\gamma} > 0$ such that for every $\gamma < \overline{\gamma}$ the limits (1.1) and (1.2) exist.*

Furthermore if A6 is uniformly satisfied, the convergence in (1.2) *is uniform on compact sets.*

3 Some properties of diffusion processes and their invariant measures.

We consider the $\mathbb{R}^d$-valued diffusion process $(x_t)_t$ defined by the stochastic differential equation (2.1), with an unique invariant probability measure $m(dx)$. We denote by L its associated infinitesimal generator i.e. $L = \frac{1}{2}\Delta + b\nabla$.

First we discuss existence, uniqueness, non explosion, existence and uniqueness of an invariant probability measure, existence of kernels and time reversal.

If b is local Lipschitz, existence and strong uniqueness are ensured up to the explosion time, starting from any x. If there exists a Lyapunov function ψ, i.e. ψ is smooth, larger than 1, goes to $+\infty$ at infinity and such that $L\psi \le C\psi$ for some constant C, then the explosion time is a.s. infinite, just applying Ito's formula to ψ up to the exit time of the level sets of ψ (in the same spirit as in [16], Théorème 2.2.19).

Existence and uniqueness of an invariant probability measure m is ensured by the existence of some ψ as before, but satisfying the stronger condition $L\psi \le C\,\mathbb{I}_K$ for some compact subset K.
In particular if b satisfies for some $R \ge 0$,

$$\langle b(x), x\rangle \;\le\; -r \quad \text{ for some } r > d/2 \text{ and all } x \text{ s.t. } |x| \ge R, \tag{3.1}$$

we may choose $\psi(x) = 1 + |x|^2$, so that (2.1) has an unique solution starting from any x, and an unique invariant probability measure m.

Assume in addition that $b \in C^1$. Then Malliavin calculus shows that the law of x_t is absolutely continuous w.r.t. Lebesgue measure for all initial conditions x and all $t > 0$. Hence m is also absolutely continuous w.r.t. Lebesgue measure, and it can be shown that dm/dy is a.e. positive.

Hence the law of x_t with initial condition x denoted by $P_t\delta_x(dy)$ is absolutely continuous w.r.t. m, i.e. there exists some kernel p_t such that $P_t\delta_x(dy) = p_t(x, y)\, m(dy)$.

Another proof of this fact follows from [3] where additional (but unnecessary here) regularity is derived.

The semi-group P_t defined on the bounded Borel functions, extends to a contraction semi-group, denoted again by P_t, on $\mathbb{L}^2(\mu)$, whose generator coincides with L on smooth functions. We thus have a pair of generators (L, L^*) and of semi-groups (P_t, P_t^*) in duality w.r.t. m, i.e.

$$\int gP_t f dm = \int fP_t^* g dm \quad \text{ and } \quad P_t^* g(x) = \int p_t^*(x, y) m(dy)\,,$$

with $p_t^*(x, y) = p_t(y, x)$.

Remark 1 If in addition, $\int b^2 dm < +\infty$, we may write $b = -\nabla V + a$ (see [5]) where $-\nabla V$ is the orthogonal projection (in $\mathbb{L}^2(m)$) of b onto the closure of the gradients of test functions (it is not known whether ∇V is actually the gradient of some function V in $\mathbb{L}^2(m)$, so that this is an abuse of notation). Hence $\int a \cdot \nabla f dm = 0$ for all smooth f. Following [10] and [4], one knows that the time reversal of the stationary process is a drifted Brownian motion with drift $b^* = -\nabla V - a$. The associated semi-group is the dual semi-group.

We do not need this remark in the remaining of the paper. $\diamondsuit$

The aim of this section is to find sufficient conditions for the following three properties to hold

(1) m satisfies a logarithmic Sobolev inequality,

(2) $p_t(.,.) \in \mathbb{L}^p(m \otimes m)$ for some $p > 2$ and t large,

(3) there exists some $\beta > 0$ such that $\int e^{\beta |y|^2} m(dy) < +\infty$.

It is well known that (1) implies (3), but we shall try to obtain explicit estimates. Similarly in some situations (2) implies (1) (recall the discussion in the previous section), but again we shall look at both properties separately. Nevertheless if m satisfies a log-Sobolev inequality, it satisfies a spectral gap inequality and A5 is thus satisfied.

Since we do not have any information on m, our condition will be a "drift condition", namely we shall say that condition (DC) is satisfied if

$$\textbf{Condition (DC)} \quad \exists c_b > 0 \text{ and } \exists R \geq 0 \text{ s.t. for } |x| \geq R, \quad \langle b(x), x \rangle \leq -c_b |x|^2 . \tag{3.2}$$

Since (DC) is stronger than (3.1), all the previous discussion is available. In addition

Lemma 2 *If (DC) holds, then for all $\lambda < c_b$, $\int e^{\lambda |y|^2} m(dy) < +\infty$.*

Proof. Since (DC) holds there exists $D > 0$ such that $\langle b(x), x \rangle \leq -c_b |x|^2 + D$ for all x, and $c = c_b$ in the proof.

Let g_n be a smooth non-decreasing concave function defined on $\mathbb{R}^+$ such that $g_n(u) = u$ if $u \leq n-1$ and $g_n(u) = n$ if $u \geq n$ (such a function exists). Let $f_n(x) = \exp(\lambda \, g_n(|x|^2))$, for $\lambda < c$.

Then $\nabla f_n(x) = 2\lambda f_n(x) g_n'(|x|^2) x$ and

$$\Delta f_n(x) = 2\lambda f_n(x) \left(2g_n''(|x|^2)|x|^2 + 2\lambda (g_n')^2(|x|^2)|x|^2 + d g_n'(|x|^2) \right) ,$$

so that

$$\begin{aligned} L f_n(x) &= \lambda \, f_n(x) \, \left((2g_n''(|x|^2)|x|^2 + d g_n'(|x|^2)) + 2 g_n'(|x|^2)(\lambda g_n'(|x|^2)|x|^2 + \langle b(x), x \rangle) \right) \\ &\leq \lambda \, f_n(x)(d + 2D - 2(c - \lambda)|x|^2) \\ &\leq \lambda (d + 2D) e^{\lambda \frac{d+2D}{c-\lambda}} \; - \; \lambda (d + 2D) \, f_n(x) \, , \end{aligned}$$

since

$$d + 2D - 2(c-\lambda)|x|^2 \leq -(c-\lambda)|x|^2 \leq -(d+2D)$$

for $|x|^2 \geq \frac{d+2D}{c-\lambda}$.

For short, there exist c_1 and c_2 positive constants such that for all n, $Lf_n \leq c_1 - c_2 f_n$.

Define $h_n(s) = \mathbb{E}\left[e^{\lambda g_n(|X_s^x|^2)}\right]$. Ito's formula yields

$$h_n(t) \leq h_n(0) + c_1 t - c_2 \int_0^t h_n(s)ds\,,$$

hence applying Gronwall's lemma we obtain

$$\mathbb{E}\left[e^{\lambda g_n(|X_t^x|^2)}\right] \leq \frac{c_1}{c_2} + e^{-c_2 t}\, e^{\lambda g_n(|x|^2)}\,. \tag{3.3}$$

Integrating (3.3) with respect to the invariant measure m yields

$$(1 - e^{-c_2 t}) \int e^{\lambda g_n(|y|^2)} m(dy) \leq \frac{c_1}{c_2}\,.$$

We may thus choose t large enough for $e^{-c_2 t} \leq 1/2$ and then use monotone convergence theorem with $n \to +\infty$ in order to obtain $\int e^{\lambda |y|^2} m(dy) < +\infty$ for $\lambda < c_b$. ∎

The next step will be the study of hypercontractivity. To this end we introduce another condition called a "curvature condition" denoted by (CC). First recall the notation (used by Wang)

$$\langle \nabla_\xi b(x), \xi \rangle = \sum_{i,j} \xi_i\, \partial_i b_j(x)\, \xi_j\,. \tag{3.4}$$

The curvature condition is then

$$\textbf{Condition (CC)} \quad \exists K_b \in \mathbb{R} \text{ s.t. for all } x \text{ and all } \xi\ , \quad \langle \nabla_\xi b(x), \xi \rangle \leq K_b |\xi|^2\,. \tag{3.5}$$

Now we may use the deep results by Wang in [17] Theorem 5.7.3 and Corollary 5.7.2 and Theorem 5.7.1 in order to get

Lemma 3 *Let $b \in C^1$. Assume that conditions (DC) and (CC) are satisfied with $c_b > K_b$. Then*

(1) for all $1 < p < q < +\infty$ there exists $t_{p,q}$ such that for $t \geq t_{p,q}$, P_t is a bounded operator from $\mathbb{L}^p(m)$ into $\mathbb{L}^q(m)$ with norm equal to 1 (we shall say that the semi-group P_t is hypercontractive),

(2) m satisfies a logarithmic Sobolev inequality, i.e. there exists some C_G such that for all smooth f with compact support satisfying $\int f^2 dm = 1$, it holds

$$\int f^2 \log f^2\, dm \;\leq\; C_G \int |\nabla f|^2\, dm\,,$$

(3) *the semi-group P_t^* is hypercontractive too (with a family $t^*_{p,q}$ of contraction times).*

The final statement is obtained by duality, in particular if $t^*_{p,q} = t_{p',q'}$ where p' and q' are the conjugates of p and q.

Remark 2 The condition $c_b > K_b$ is a mild condition. Indeed, if we replace (DC) by a stronger (but more symmetric) condition namely

Condition (DCC) for all x and y it holds

$$\langle b(x) - b(y)\,,\, x-y\rangle \leq -c_b\,|x-y|^2$$

(if $b = -\nabla V$ this is a convexity assumption) then we may choose $K_b < 0$. One guess that (this can be rigorously done in the symmetric case, i.e $b = \nabla V$), if b can be written as $b_1 + b_2$ with $b_1 \in C^1$ satisfying (DCC) and b_2 a C^1 compactly supported function, then Lemma 3 is still true, with the only $c_b > 0$ condition.

It is worth noticing that if we reinforce (DC) assuming the following (SDC) condition

$$\lim_{|x|\to\infty} \langle b(x), \frac{x}{|x|^2}\rangle = -\infty\,,$$

then we may always choose $c_b > K_b$ (if K_b is finite of course). In this situation it can be shown (see [17] Corollary 5.7.7) that the semi-group is even superbounded. $\diamondsuit$

The proof of all these results lies on a beautiful Harnack inequality derived by Wang ([17] Theorem 2.5.2)

$$(P_t f(x))^\alpha \;\leq\; P_t f^\alpha(y)\, \exp\left(\frac{\alpha}{2(\alpha-1)}\, K_b(1-e^{-2K_b t})^{-1}\,|x-y|^2\right) \tag{3.6}$$

holding for $t > 0$, $\alpha > 1$, all (x, y) and all nonnegative continuous and bounded f, with the convention $K_b(1-e^{-2K_b t})^{-1} = 1/2t$ if $K_b = 0$ (see also [1] Lemma 7.5.4 if $\alpha = 2$).

This inequality is the key for the following lemma

Lemma 4 *Let $b \in C^1$. Assume that conditions (DC) and (CC) are satisfied. Then for all $p > 2$, $p_t(.,.) \in \mathbb{L}^p(m \otimes m)$ for all t such that*

$$c_b \;>\; \frac{K_b\, p\,(p-1)}{1-e^{-2K_b t}}\,.$$

In particular, if $K_b \leq 0$, then for all $p > 2$ there exists t_p such that $p_t(.,.) \in \mathbb{L}^p(m\otimes m)$ for $t \geq t_p$, while for $K_b > 0$ such a t_p exists provided $c_b > K_b\, p\,(p-1)$.

Proof. We shall first derive an upper bound for the density.

Let $\alpha > 1$, $D_t := \{x \in \mathbb{R}^d, |x| \leq \gamma(t)\}$ for some increasing function γ going to ∞ and f be nonnegative and bounded. Integrating Harnack inequality for P_t with respect to $m(dy)$ on D_t and denoting

$$\kappa(t) = \frac{\alpha}{2(\alpha-1)}\, K_b(1-e^{-2K_b t})^{-1}$$

we get

$$\begin{aligned} ((P_t f)(x))^\alpha &\leq \int_{D_t} (P_t f^\alpha)(y)\, e^{\kappa(t)|x-y|^2}\, m(dy)/m(D_t) \\ &\leq \int f^\alpha(y)\, (P_t^*(\mathbb{I}_{D_t}(.)\, e^{\kappa(t)|x-.|^2}))(y)\, m(dy)/m(D_t) \\ &\leq e^{2\kappa(t)(|x|^2+\gamma^2(t))} \int f^\alpha(y)\; m(dy)/m(D_t)\,, \end{aligned}$$

since

$$\| \, \mathbb{I}_{D_t}(.)\, e^{\kappa(t)|x-.|^2} \, \|_\infty \leq e^{2\kappa(t)(|x|^2+\gamma^2(t))}\,.$$

If

$$\theta(t,x) = \left(e^{2\kappa(t)\gamma^2(t)}/m(D_t)\right)\, e^{2\kappa(t)|x|^2}\,,$$

we thus have

$$((P_t f)(x))^\alpha \leq \theta(t) \int f^\alpha(y)\, m(dy)\,. \tag{3.7}$$

Applying the previous inequality with (a continuous approximation of, and then taking limits)

$$f_N(z) = p_t^\beta(x,z)\, \mathbb{I}_{\{p_t(x,z)\leq N\}}$$

yields

$$\left(\int p_t^{1+\beta}(x,z)\, \mathbb{I}_{\{p_t(x,z)\leq N\}}\, m(dz)\right)^\alpha \leq \theta(t,x) \int p_t^{\alpha\,\beta}(x,y)\, \mathbb{I}_{\{p_t(x,y)\leq N\}}\; m(dy)\,,$$

i.e. letting N go to ∞ and choosing $1+\beta = \alpha\,\beta$ hence $\beta = 1/(\alpha-1)$

$$\int p_t^{\frac{\alpha}{\alpha-1}}(x,y)\, m(dy) \;\leq\; \theta^{1/(\alpha-1)}(t,x)\,. \tag{3.8}$$

According to Lemma 2, the right hand side in (3.8) is in $\mathbb{L}^1(m)$ provided

$$c_b > \frac{2\kappa(t)}{\alpha-1} = \frac{\alpha K_b}{(\alpha-1)^2(1-e^{-2K_b t})}\,. \tag{3.9}$$

In particular if $p > 2$ define $1 < \alpha = p/(p-1) < 2$. Hence $p_t(.,.) \in \mathbb{L}^p(m\otimes m)$ provided

$$c_b > \frac{K_b\, p\,(p-1)}{1-e^{-2K_b t}}\,.$$

∎

We may thus state the main result of this section,

Theorem 2 *Let $b \in C^1$. If condition (DC) holds, Assumption A6 is uniformly satisfied.*

If conditions (DC) and (CC) are satisfied and $c_b > 2K_b$, then

- *for all $\lambda < c_b$, $\int e^{\lambda |x|^2} m(dx) < +\infty$,*
- *the semi-groups P_t and P_t^* are hypercontractive,*
- *there exist $p > 2$ and t_p such that for $t \geq t_p$, $p_t(.,.) \in \mathbb{L}^p(m \otimes m)$.*

In particular assumptions A1, A3, A4 and A5 are satisfied.

Proof. The only thing to prove is the first statement since all others statements are consequence of the three previous lemmata.

Using Ito's formula up to the exit time T_M of the ball of center 0 and radius M we have

$$E_x \left[e^{\theta \, |x_{t \wedge T_M}|^2} \right] = e^{\theta |x|^2} + E_x \left[\int_0^{t \wedge T_M} \left(2\theta \, \langle b(x_s), x_s \rangle + d\theta + 2\theta^2 |x_s|^2 \right) \, e^{\theta \, |x_s|^2} \, ds \right] .$$

In particular if condition (DC) holds with $c_b > \theta$, the integrand in the right hand side is non-positive for large values of $|x_s|$, hence we can let M go to infinity in order to show that there exists some constant κ (depending on (DC) and θ) such that

$$E_x \left[e^{\theta \, |x_t|^2} \right] < e^{\theta |x|^2} + \kappa t \, .$$

Accordingly using Jensen inequality

$$\begin{aligned} E_x \left[e^{\int_0^a \, \beta \, |x_s|^2 \, ds} \right] &= E_x \left[e^{\frac{1}{a} \, \int_0^a \, a \, \beta \, |x_s|^2 \, ds} \right] \\ &\leq \frac{1}{a} \, E_x \left[\int_0^a \, e^{a \, \beta \, |x_s|^2} \, ds \right] < +\infty \end{aligned}$$

as soon as $a\beta < c_b$. The proof is completed. ∎

Remark 3 It can be interesting to get explicit local bounds for the density. Here is a result in this direction. It will not be used, however, in this paper.

Assume that $b \in C^1$, that (DC) holds and that (CC) holds for some $K_b \leq 0$. Assume in addition that the drift b^* of the dual semi-group belongs to C^1.

Then there exist $\underline{a}(t)$ and $\bar{a}(t)$ going to 1 when t goes to ∞ , $\underline{\beta}(t)$ and $\bar{\beta}(t)$ going to 0 when t goes to ∞ and some $T_b > 0$ such that for all $t > T_b$ the following estimates hold

$$\underline{a}(t) \, e^{-\underline{\beta}(t)(|x|^2+|y|^2)} \leq p_t(x,y) \leq \bar{a}(t) \, e^{\bar{\beta}(t)(|x|^2+|y|^2)}$$

for all pair (x,y).

Let us prove this statement. First we derive the upper bound. We follow the proof of Lemma 4 up to (3.8). Similarly by duality we have

$$\int p_t^{\frac{\alpha}{\alpha-1}}(y,x)\, m(dy) \;\leq\; \theta^{1/(\alpha-1)}(t,x)\,. \tag{3.10}$$

But the semi-group property yields

$$\begin{aligned} p_{2t}(x,y) &= \int p_t(x,z)\, p_t(z,y)\, m(dz) \\ &\leq \left(\int p_t^2(x,z)\, m(dz)\right)^{1/2} \left(\int p_t^2(z,y)\, m(dz)\right)^{1/2} \\ &\leq \theta^{\frac{1}{2}}(t,x)\, \theta^{\frac{1}{2}}(t,y) \end{aligned}$$

where we have arbitrarily chosen $\alpha = 2$, i.e

$$p_t(x,y) \;\leq\; \frac{e^{2\kappa(t/2)\gamma^2(t/2)}}{m(D_{t/2})}\, e^{\kappa(t/2)(|x|^2+|y|^2)}\,. \tag{3.11}$$

Since $K_b \leq 0$, $\kappa(t) \to 0$ as $t \to +\infty$ so that we may choose now any γ such that $\gamma(t) \to +\infty$ and $\kappa(t)\gamma^2(t) \to 0$ as $t \to +\infty$.

We turn now to the lower bound.

We use Harnack's inequality with $f(u) = p_t(u,z)\, \mathbb{1}_{|u|\leq N}$ and $\alpha = 2$ again. It yields

$$\begin{aligned} \left(\int p_t(x,u)\, p_t(u,z)\, \mathbb{1}_{|u|\leq N}\, m(du)\right)^2 &\leq \left(\int p_t(y,u)\, p_t^2(u,z)\, \mathbb{1}_{|u|\leq N}\, m(du)\right) e^{\kappa(t)|x-y|^2} \\ &\leq \left(\int p_t(y,u)\, p_t^2(u,z)\, m(du)\right) e^{\kappa(t)|x-y|^2} \end{aligned}$$

hence, letting N go to ∞ we obtain for all x,y,z

$$p_{2t}^2(x,z) \;\leq\; \left(\int p_t(y,u)\, p_t^2(u,z)\, m(du)\right) e^{\kappa(t)|x-y|^2}\,. \tag{3.12}$$

Now we plug our upper bound into the right hand side of (3.12), bounding $p_t^{1/2}(y,u)$ and $p_t^{3/2}(u,z)$ respectively. It furnishes

$$\begin{aligned} p_{2t}^2(x,z) &\leq \left(\int (p_t(y,u)\, p_t(u,z))^{1/2}\; e^{2\bar\beta(t)|u|^2}\, m(du)\right) \bar a^2(t)\, e^{\frac{1}{2}\bar\beta(t)(|y|^2+3|z|^2)}\, e^{\kappa(t)|x-y|^2} \\ &\leq \left(\int e^{4\bar\beta(t)|u|^2} m(du)\right)^{1/2} p_{2t}^{1/2}(y,z)\, \bar a^2(t)\, e^{\frac{1}{2}\bar\beta(t)(|y|^2+3|z|^2)}\, e^{2\kappa(t)(|x|^2+|y|^2)}\,, \end{aligned}$$

where we used Cauchy-Schwarz inequality. We may take the square root of both sides and then integrate with respect to $m(dx)$. It yields

$$1 \;\leq\; p_{2t}^{1/4}(y,z)\, \bar a(t)\, e^{\frac{1}{4}\bar\beta(t)(|y|^2+3|z|^2)}\, e^{\kappa(t)|y|^2}\, \delta(t)$$

where

$$\delta(t) = \left(\int e^{4\bar{\beta}(t)|u|^2} m(du) \right)^{1/4} \left(\int e^{\kappa(t)|x|^2} m(dx) \right)$$

goes to 1 as t goes to $+\infty$, thanks to the concentration property of m (i.e. Lemma 2) and Lebesgue theorem. Notice that $\delta(t)$ is finite for t large enough, so that the lower bound we have obtained is true for $t > T_b$.

This scheme of proof is standard in the symmetric case where sharper bounds can be obtained. However the integral maximum principle (see e.g. Proposition 13 in [2]) does not hold in the non-symmetric case, so that we have to replace it, but obtain worse bounds. ♢

4 The limiting function as viscosity solution

We are considering the function

$$\varphi(t, x) := E_x \left[\exp \left(\gamma \int_0^t c(x_s) ds \right) \right]. \tag{4.1}$$

We have shown that (under some assumptions we shall assume to be in force below), for γ sufficiently small, the limits

$$\lambda := \lim_t \frac{1}{t} \log \varphi(t, x) \tag{4.2}$$

and

$$V(x) := \lim_t [\log \varphi(t, x) - \lambda t] \tag{4.3}$$

exist uniformly over compact sets. We want to show that V is a *viscosity solution* of the Hamilton-Jacobi-Bellman equation (1.8) or, equivalently, that $v(x) := e^{V(x)}$ is a viscosity solution of the linear equation

$$-\left[\frac{1}{2} \Delta v + b \cdot \nabla v + \gamma c v \right] + \lambda v = 0. \tag{4.4}$$

In the last few years A. Gulisashvili and J.A. Van Casteren have been writing a series of papers and a book [11] one the evolution operator for $\varphi(t, x)$, the so-called *Feynman-Kac propagator*. They work in a rather general context (locally compact, second countable Hausdorff space), but then they specialize their results in $\mathbb{R}^d$. They give conditions under which $\varphi(t, x)$ is continuous and is a viscosity solution of the corresponding parabolic equation, which are the following:
Condition 1. The semigroup of the process x_t transforms bounded measurable functions to bounded continuous function (the strong Feller property);
Condition 2. The function $x \mapsto \int_0^t E_x[|c(x_s)|] ds$ is bounded, and goes to zero uniformly as $t \to 0$.

While condition 1 is acceptable (it is implied by our assumptions on the diffusion in section 3), condition 2 cannot hold true for c unbounded. In what follows we assume condition 1, but only quadratic growth on c (i.e. assumption A2).

Proposition 2 . *Assume that conditions A1 to A5 are satisfied (hence the limits (4.2) and (4.3) exist) and that condition (DC) is satisfied, so that condition A6 is uniformly satisfied. Then $v(\cdot)$ is continuous, and it is a viscosity solution of equation (4.4).*

Proof. Step 1. Continuity of $\varphi(t,x)$. We first establish continuity in x.
First note that, according to the proofs in section 2, for $\gamma < \bar{\gamma}$, one can find some $\delta \in]1, \frac{\bar{\gamma}}{\gamma}[$ and some function $h_\gamma(t,x)$ which is bounded on compact sets such that

$$E_x\left[\exp\left(\gamma\delta\int_0^t |c(x_s)|ds\right)\right] \leq h_\gamma(t,x) \tag{4.5}$$

for all $t > 0$ and $x \in \mathbb{R}^d$.
Note that, for $0 < \epsilon < t$,

$$\begin{aligned} |\varphi(t,x) \; - \; & \varphi(t,y)| \\ = \; & \left| E_x\left[\varphi(t-\epsilon, x_\epsilon)\exp\left(\gamma\int_0^\epsilon c(x_s)ds\right)\right] - E_y\left[\varphi(t-\epsilon, x_\epsilon)\exp\left(\gamma\int_0^\epsilon c(x_s)ds\right)\right]\right| \\ \leq \; & E_x\left[\left|e^{\gamma\int_0^\epsilon c(x_s)ds} - 1\right|\varphi(t-\epsilon,x_\epsilon)\right] + E_y\left[\left|e^{\gamma\int_0^\epsilon c(x_s)ds} - 1\right|\varphi(t-\epsilon,x_\epsilon)\right] \\ & + \left|E_x[\varphi(t-\epsilon,x_\epsilon)] - E_y[\varphi(t-\epsilon,x_\epsilon)]\right|. \end{aligned} \tag{4.6}$$

We begin by estimating the first term in the r.h.s. of (4.6). By Hölder inequality

$$\begin{aligned} & E_x\left[\left|e^{\gamma\int_0^\epsilon c(x_s)ds} - 1\right|\varphi(t-\epsilon,x_\epsilon)\right] \\ & \qquad \leq \left\{E_x\left[\left|e^{\gamma\int_0^\epsilon c(x_s)ds} - 1\right|^p\right]\right\}^{1/p}\left\{E_x\left[\exp\left(\gamma\delta\int_0^t |c(x_s)|ds\right)\right]\right\}^{1/\delta}, \end{aligned} \tag{4.7}$$

where $p = \frac{\delta}{\delta-1}$. Our aim is to show that the l.h.s. of (4.7) goes to 0 as $\epsilon \to 0$, uniformly in x varying in a compact set. By (4.5), the second factor in the r.h.s. of (4.7) is locally bounded. Thus, it is enough to show that

$$E_x\left[\left|e^{\gamma\int_0^\epsilon c(x_s)ds} - 1\right|^p\right]$$

goes to zero uniformly in compact sets. By the inequality $|e^x - 1| \leq |x|e^{|x|}$, Cauchy-Schwartz inequality and Jensen's inequality

$$\begin{aligned} E_x\left[\left|e^{\gamma\int_0^\epsilon c(x_s)ds} - 1\right|^p\right]^2 \; & \leq \; \gamma^{2p}E_x\left[\left(\int_0^\epsilon c(x_s)ds\right)^p e^{\gamma p\int_0^\epsilon |c(x_s)|ds}\right]^2 \\ & \leq \; \gamma^{2p}E_x\left[\left(\int_0^\epsilon c(x_s)ds\right)^{2p}\right]E_x\left[e^{2\gamma p\int_0^\epsilon |c(x_s)|ds}\right] \\ & \leq \; \epsilon^{2p-1}\gamma^{2p}\int_0^\epsilon E_x\left[|c(x_s)|^{2p}\right]ds\; E_x\left[e^{2\gamma p\int_0^\epsilon |c(x_s)|ds}\right] \\ & \leq \; \epsilon^{2p-2}\gamma^{2p}\int_0^\epsilon E_x\left[|c(x_s)|^{2p}\right]ds\int_0^\epsilon E_x\left[e^{2\gamma p\epsilon|c(x_s)|}\right]ds \end{aligned} \tag{4.8}$$

Since $p > 1$, it is enough to show that the two integrals in (4.8) are locally bounded. This follows easily from the assumption that $c(\cdot)$ has quadratic growth (see A2 where the constant C is defined), and from the proof of the first part of theorem 2, as soon as $2\gamma p C\varepsilon < c_b$. Indeed we get some exponential integrability which is strong enough to control both terms. This establishes continuity in x.

To get joint continuity in (t,x) just observe that, by the integrability condition (4.5), we can differentiate in t $\varphi(t,x)$, and show that this derivative is locally bonded. Thus $\varphi(t,x)$ is locally Lipschitz in t, locally uniformly in x. This, together with continuity in x, implies joint continuity.

Step 2. Viscosity solution of the parabolic equation. In what follows we introduce the upper-semicontinuous (resp. lower-semicontinuous) extension c^* (resp. c_*) of $c(\cdot)$:

$$c^*(x) := \limsup_{y\to x} c(y) \quad c_*(x) := \liminf_{y\to x} c(y).$$

Moreover, let $v_T(t,x) := \varphi(T-t,x)$. We now show that v_T is a viscosity solution (in $[0,T]$) of the parabolic equation

$$-\left(\partial_t v_T + b\cdot\nabla v_T + \frac{1}{2}\Delta v_T + \gamma c v_T\right) = 0. \tag{4.9}$$

Since v_T is continuous, this amounts to show that the following two properties hold true.

i. (Supersolution property). Let $(t,x)\in[0,T)\times\mathbb{R}^d$ and let $\psi:[0,T)\times\mathbb{R}^d\to\mathbb{R}$ be a smooth function such that $\psi(t,x)=v_T(t,x)$, and $v_T-\psi$ has a local maximum at (t,x) (there may be no such function). Then

$$-\left(\partial_t\psi(t,x) + b(x)\cdot\nabla\psi(t,x) + \frac{1}{2}\Delta\psi(t,x) + \gamma c^*(x) v_T(t,x)\right) \le 0.$$

ii. (Subsolution property). Let $(t,x)\in[0,T)\times\mathbb{R}^d$ and let $\psi:[0,T)\times\mathbb{R}^d\to\mathbb{R}$ be a smooth function such that $\psi(t,x)=v_T(t,x)$, and $v_T-\psi$ has a local minimum at (t,x). Then

$$-\left(\partial_t\psi(t,x) + b(x)\cdot\nabla\psi(t,x) + \frac{1}{2}\Delta\psi(t,x) + \gamma c_*(x) v_T(t,x)\right) \ge 0.$$

$v_T-\psi$ has a *strict* local extreme in (t,x). Indeed, if $v_T-\psi$ has a local extreme at (t,x) and $\tilde\psi(s,y) := \psi(s,y)\pm[(s-t)^2+|x-y|^4]$ (where the sign depends on whether we are dealing with a maximum or a minimum), then $v_T-\tilde\psi$ has a *strict* local extreme in (t,x), and ψ and $\tilde\psi$ have the same first space and time derivatives and second space derivatives at (t,x). We now observe the following identities.

$$\begin{aligned}\varphi(t,x) = 1 - \int_0^t \frac{d}{ds} E_x\left[\exp\left(\int_s^t \gamma c(x_\tau)d\tau\right)\right]ds &= 1+\gamma\int_0^t E_x\left[c(x_s)\exp\left(\gamma\int_s^t c(x_\tau)d\tau\right)\right]ds \\ &= 1+\gamma\int_0^t E_x\left[c(x_s)\varphi(t-s,x_s)\right]ds,\end{aligned}$$

where all steps are justified by (4.5). It follows that, for $\epsilon > 0$,

$$\varphi(t,x) - E_x[\varphi(t-\epsilon, x_\epsilon)] = \gamma E_x\left[\int_0^\epsilon c(x_s)\varphi(t-s,x_s)ds\right].$$

By a change $t \mapsto T-t$ of the time variable, we get

$$v_T(t,x) - E_x[v_T(t+\epsilon, x_\epsilon)] = \gamma E_x\left[\int_0^\epsilon c(x_s)v_T(t+s,x_s)ds\right]. \tag{4.10}$$

Now we use (4.10) to prove that v_T has the subsolution property. The supersolution property is proved in the same way. Note that, both properties are local, so it is not restrictive to assume the test functions ψ to have compact support.
So let ψ be a smooth function with compact support such that $\psi(t,x) = v_T(t,x)$, and $v_T - \psi$ has a local minimum at (t,x). We first claim that

$$\limsup_{\epsilon\to 0} \frac{v_T(t,x) - E_x[v_T(t+\epsilon,x_\epsilon)]}{\epsilon} \le \limsup_{\epsilon\to 0} \frac{\psi(t,x) - E_x[\psi(t+\epsilon,x_\epsilon)]}{\epsilon}. \tag{4.11}$$

This is done by a simple localization. Let $\rho > 0$ be such that $v_T(s,y) \ge \psi(s,y)$ for $(s,y) \in [t-\rho, t+\rho] \times B(x,\rho)$. Then, for $|\epsilon| < \rho$,

$$\begin{aligned}
&\frac{v_T(t,x) - E_x[v_T(t+\epsilon,x_\epsilon)]}{\epsilon} \\
=\; & E_x\left[\frac{v_T(t,x) - v_T(t+\epsilon,x_\epsilon)}{\epsilon}\mathbb{1}_{|x_\epsilon - x|\le\rho}\right] + E_x\left[\frac{v_T(t,x) - v_T(t+\epsilon,x_\epsilon)}{\epsilon}\mathbb{1}_{|x_\epsilon - x|>\rho}\right] \\
\le\; & E_x\left[\frac{\psi(t,x) - \psi(t+\epsilon,x_\epsilon)}{\epsilon}\mathbb{1}_{|x_\epsilon - x|\le\rho}\right] + E_x\left[\frac{v_T(t,x) - v_T(t+\epsilon,x_\epsilon)}{\epsilon}\mathbb{1}_{|x_\epsilon - x|>\rho}\right] \\
=\; & E_x\left[\frac{\psi(t,x) - \psi(t+\epsilon,x_\epsilon)}{\epsilon}\right] \\
& -E_x\left[\frac{\psi(t,x) - \psi(t+\epsilon,x_\epsilon)}{\epsilon}\mathbb{1}_{|x_\epsilon - x|>\rho}\right] + E_x\left[\frac{v_T(t,x) - v_T(t+\epsilon,x_\epsilon)}{\epsilon}\mathbb{1}_{|x_\epsilon - x|>\rho}\right].
\end{aligned}$$

Thus, in order to obtain (4.11), it is enough to show that the last two terms go to zero as $\epsilon \to 0$. We only deal with the last, the other being easier.

$$\begin{aligned}
\left|E_x\left[\frac{v_T(t,x) - v_T(t+\epsilon,x_\epsilon)}{\epsilon}\mathbb{1}_{|x_\epsilon - x|>\rho}\right]\right| &\le \frac{2}{\epsilon}E_x\left[e^{\gamma\int_0^T |c(x_s)|ds}\mathbb{1}_{|x_\epsilon - x|>\rho}\right] \\
&\le \frac{2}{\epsilon}E_x\left[e^{\gamma\delta\int_0^T |c(x_s)|ds}\right] E_x(\mathbb{1}_{|x_\epsilon - x|>\rho})^{1-\frac{1}{\delta}},
\end{aligned}$$

that goes to zero as $\epsilon \to 0$ since, by small time estimates again, $E_x(\mathbb{1}_{|x_\epsilon - x|>\rho}) = o(\epsilon)$. This establishes (4.11). On the other hand, by Ito's rule,

$$\lim_{\epsilon\to 0} \frac{\psi(t,x) - E_x[\psi(t+\epsilon,x_\epsilon)]}{\epsilon} = -\left(\partial_t\psi(t,x) + b(t,x)\cdot\nabla\psi(t,x) + \frac{1}{2}\Delta\psi(t,x)\right). \tag{4.12}$$

Putting together (4.10), (4.11) and (4.12), the subsolution property follows from

$$\liminf_{\epsilon\to 0} \frac{1}{\epsilon} E_x \left[\int_0^\epsilon c(x_s) v_T(t+s, x_s) ds \right] \geq c_*(x) v_T(t,x), \tag{4.13}$$

where the above convergence is again controlled by small time estimates and the fact that v_T is continuous.

Step 3. Conclusion. Letting $\tilde{v}_T(t,x) := v_T(t,x)e^{-\lambda(T-t)}$, it is easily checked that $\tilde{v}_T$ is a viscosity solution of

$$-\left(\partial_t \tilde{v}_T + b \cdot \nabla \tilde{v}_T + \frac{1}{2} \Delta \tilde{v}_T + \gamma c v_T \right) + \lambda \tilde{v} = 0. \tag{4.14}$$

Moreover $\tilde{v}_T(t,x) \to v(x)$ as $T \to +\infty$ uniformly on compact sets. In particular v is continuous. Now, it is a standard argument, that I now sketch, to show that v is a viscosity solution of (4.4).

Let $x \in \mathbb{R}^d$, and let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $v(x) = \psi(x)$ and $v - \psi$ has a local minimum at x. Fix $t > 0$, and define $\tilde{\psi}(s,y) := \psi(y) - |y-x|^4 - (s-t)^2$. Note that $v - \tilde{\psi}$ has a *strict* local minimum at (t,x), and

$$\partial_t \tilde{\psi}(t,x) + b(t,x) \cdot \nabla \tilde{\psi}(t,x) + \frac{1}{2} \Delta \tilde{\psi}(t,x) = b(x) \cdot \nabla \psi(x) + \frac{1}{2} \Delta \psi(x). \tag{4.15}$$

A simple exercise in uniform convergence show that there is a sequence $(t_n, x_n) \to (t,x)$ as $n \to +\infty$, such that $\tilde{v}_n - \tilde{\psi}$ has a local minimum at (t_n, x_n). Therefore, since $\tilde{v}$ is a viscosity solution of (4.14)

$$-(\partial_t \tilde{\psi}(t_n,x_n) + b(x_n) \cdot \nabla \tilde{\psi}(t_n,x_n) + \frac{1}{2} \Delta \tilde{\psi}(t_n,x_n) + \gamma c_*(x_n) \tilde{v}_n(x_n)) + \lambda \tilde{v}_n(x_n) \geq 0. \tag{4.16}$$

Letting $n \to +\infty$, using (4.15) and lower-semicontinuity of c_* we obtain the subsolution property for equation (4.4). The supersolution property is obtained in the same way.

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