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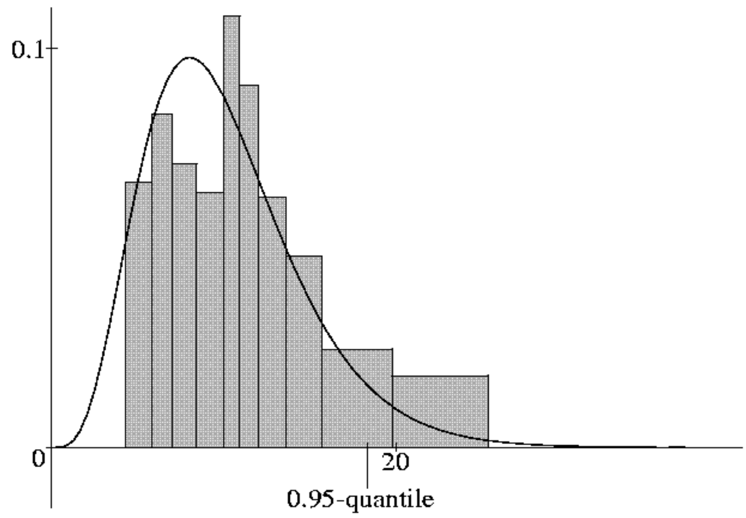

Mathematische Statistik und
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Abstract

We consider an infinite system of hard balls in $\mathbb{R}^d$ undergoing Brownian motions and submitted to a smooth pair potential. It is modelized by an infinite-dimensional Stochastic Differential Equation with an infinite-dimensional local time term. Existence and uniqueness of a strong solution is proven for such an equation with fixed deterministic initial condition. We also show that Gibbs measures are reversible measures.

AMS Classifications: 60H10, 60K35.
KEY-WORDS: Stochastic Differential Equation, local time, hard core potential, Gibbs measure, reversible measure.

1 Introduction

The aim of this paper is to construct and analyze an infinite system of interacting hard balls undergoing Brownian motions in $\mathbb{R}^d$ and starting from a fixed initial condition.

R. Lang ([Lan77]) constructed in a pioneer paper the reversible solution of an infinite gradient system of Brownian particles (balls with radius 0, i.e. points) submitted to a smooth pair interaction. It is a so-called *equilibrium dynamics* in Statistical Physics, since this process has a time-stationary distribution. J. Fritz solved some years later in [Fri87] the non-reversible case, which occurs when the initial distribution is no more Gibbsian. For this type of systems, the main difficulty comes from a possible explosion (i.e. an infinite number of particles can enter in a finite volume after a finite time).

On another side, a reversible system of infinitely many Brownian hard balls (without external potential) was studied by H. Tanemura [Tan96]. He constructs a unique solution to an infinite-dimensional Skohorod type equation where the hard core situation – balls can not overlap – appears as a local time term in addition to the basic Brownian motion. The (reversible) initial condition is ditributed like a Gibbs measure associated to the hard core potential.

In the present paper, the model is a mixture of both Lang's and Tanemura's models. We deal with Brownian motions submitted to the sum of a hard core potential and a smooth finite range pair potential. In [FR00] we proved existence and uniqueness of a reversible solution of the corresponding stochastic differential equation $(\mathcal{E})$, under the condition that the initial distribution is Gibbs with a small activity. We propose here the construction of a strong non-reversible solution of $(\mathcal{E})$, in the sense that the initial condition can be any deterministic configuration in a set of allowed configurations which is clearly identified.
Allthough some techniques in the proof of the main results are similar to those in [FR00], we adopt a new pathwise approach for the construction of the solution of $(\mathcal{E})$ which is much finer than in [FR00], where the time-stationarity of the solution was used at several places. Moreover the set of allowed initial configurations is explicitly given in Theorem 3.2, and we prove that any Gibbs measure associated with the dynamical interaction carries a.s. this set.

After a second section where notations are introduced, in section 3 we present the infinite dimensional equation $(\mathcal{E})$ and we state the results. The sequence of approximating solutions is built in section 4. In section 5 we prove technical estimates needed in section 6, for the convergence of the approximations. Finally, section 7 is devoted to complete the proof of the main results.

2 Configuration spaces and notations

The particles we deal with in the present paper evolve in $\mathbb{R}^d$, for a fixed $d \geqslant 1$, endowed with the euclidian norm denoted by $|\ |$. $B(y, \rho)$ will denote the closed ball centered in $y \in \mathbb{R}^d$ with radius ρ and more generally, for any $A \subset \mathbb{R}^d$, we define

$$B(A, \rho) = \{y \in \mathbb{R}^d \text{ such that } d(y, A) \leqslant \rho\}$$

where $d(y, A)$ denotes the (euclidian) distance between y and A. The volume of a subset A in $\mathbb{R}^d$ is also denoted by $|A|$.

The modelization of point configurations may be done at least in two ways.

The first possibility (used in Mathematical Physics) is to represent an n point configuration in $\mathbb{R}^d$ as a subset (with multiplicity) of cardinal n in $\mathbb{R}^d$, that is as an equivalence class on $(\mathbb{R}^d)^n$ under the action of the permutation group S_n on $\{1, \dots, n\}$. This modelization is especially useful for integral calculus.

The second and equivalent (more probabilistic) modelization uses point measures. An n point configuration in $\mathbb{R}^d$ is also a point measure $\sum_{i=1}^n \delta_{\xi_i}$ on $\mathbb{R}^d$, and, more generally, the set of all point configurations in $\mathbb{R}^d$ will be the set $\mathcal{M}$ of all point Radon measures on $\mathbb{R}^d$:

$$\mathcal{M} = \left\{ \xi = \sum_{i \in I} \delta_{\xi_i} \text{ such that } I \subset \mathbb{N},\ \xi_i \in \mathbb{R}^d \text{ and } \forall \Lambda \text{ compact in } \mathbb{R}^d\ , \xi(\Lambda) < +\infty \right\}.$$

Following the first representation of configurations, $\mathcal{M}$ will also be viewed as the set of all finite or countable subsets (with multiplicity) of $\mathbb{R}^d$ whose intersection with any compact subset of $\mathbb{R}^d$ is finite. $\mathcal{M}$ is endowed with the topology of vague convergence. We introduce the following notations :

- For $A \subset \mathbb{R}^d$, N_A is the counting variable on $\mathcal{M}$: $\quad N_A(\xi) = \text{Card}\{i \in \mathbb{N}, \xi_i \in A\}$.
- For $A \subset \mathbb{R}^d$, $\mathcal{B}_A$ is the σ-algebra on $\mathcal{M}$ generated by the sets $\{N_B = n\}$, $n \in \mathbb{N}$, $B \subset A$, B bounded.
- For $z > 0$, π^z (resp. π^z_A) is the Poisson process on $\mathbb{R}^d$ (resp. on A) with activity z, that is with intensity measure $z\ dy$ (resp. $z\ dy|_A$).
- π (resp. π_A) is the Poisson process on $\mathbb{R}^d$ (resp. on A) with intensity measure the Lebesgue measure dy (resp. $dy|_A$), i.e. $\pi = \pi^1$ and $\pi_A = \pi^1_A$.

The particles we deal with in this paper are not reduced to points but are hard spheres (or balls) of diameter r, for a fixed $r > 0$. Since balls may not overlap, the set of "allowed configurations" is the following subset of $\mathcal{M}$:

$$\mathcal{A} = \{\xi \in \mathcal{M} \text{ such that } \forall i \neq j\ |\xi_i - \xi_j| \geqslant r\}\ .$$

As long as it does not produce confusion, we will identify the point measure on $\mathbb{R}^d$, the subset of $\mathbb{R}^d$ corresponding to its support, and the representants of this subset in $(\mathbb{R}^d)^{\mathbb{N}}$, writing for example $\xi_\Lambda = \xi \cap \Lambda$ for the restriction of the configuration ξ to some subset Λ of $\mathbb{R}^d$, $\xi\eta$ for the concatenation of both configurations ξ and η, or $\mathcal{A} \cap (\mathbb{R}^d)^n$ for the set of all allowed n points configurations.

Let us denote $\mathcal{C}(\mathbb{R}^+, \mathcal{M})$ the set of continuous $\mathcal{M}$-valued paths on $\mathbb{R}^+$, endowed with the topology of uniform convergence on each compact time interval. $\mathcal{C}(\mathbb{R}^+, \mathcal{M})$ is the set of all possible paths, and the subset of all allowed paths is

$$\mathcal{C}(\mathbb{R}^+, \mathcal{A}) = \left\{ X \in \mathcal{C}(\mathbb{R}^+, \mathcal{M}) \text{ such that } \forall t \geqslant 0\ X(t) \in \mathcal{A} \right\}.$$

Sets $\mathcal{C}([0,T], \mathcal{M})$ and $\mathcal{C}([0,T], \mathcal{A})$ are defined similarly for any positive final time T.

Remark 2.1 : We study here the evolution of a particles configuration under the influence of an interaction potential with finite range R. Then a fixed particle can interact with at most a finite number $\overline{N}$ of particles. $\overline{N}$ only depends on d and R/r and is clearly bounded by $\frac{(R+r/2)^d}{(r/2)^d} - 1 = (1 + 2R/r)^d - 1$. See figure 1.

3 Statement of the results

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a right continuous filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ such that each $\mathcal{F}_t$ contains all P-negligible sets and let $(W_i(t), t \geqslant 0)_{i \in \mathbb{N}}$ be a family of $\mathcal{F}_t$-adapted independent d-dimensional Brownian motions.

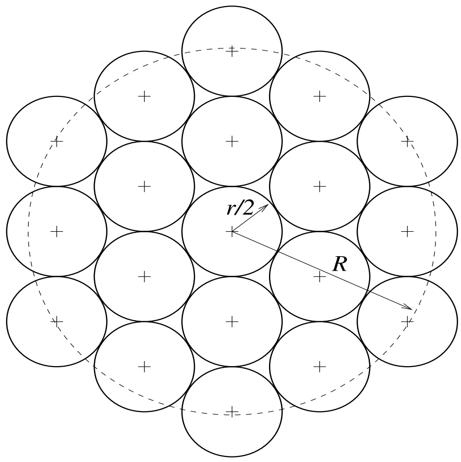

Figure 1: Example : Why $\overline{N} = 18$ if $d = 2$ and $R = 2r$.

We consider the following infinite system of stochastic equations :

$$(\mathcal{E}) \begin{cases} \text{For } i \in \mathbb{N}, t \in \mathbb{R}^+, \\ X_i(t) = X_i(0) + W_i(t) - \frac{1}{2} \sum_{j\in\mathbb{N}} \int_0^t \nabla\varphi(X_i(s) - X_j(s))ds \\ \qquad\qquad + \sum_{j\in\mathbb{N}} \int_0^t (X_i(s) - X_j(s))dL_{ij}(s) \end{cases}$$

where

- $(X_i(t), t\geqslant 0)_{i\in\mathbb{N}}$ is a family of continuous $\mathcal{A}$-valued processes, i.e. satisfying

$$|X_i(t) - X_j(t)| \geqslant r \quad \text{for } t\geqslant 0 \text{ and } i \neq j$$

- φ is a smooth stable pair potential with finite range R;
- $(L_{ij}(t), t\geqslant 0)_{i,j\in\mathbb{N}}$ is a family of non-decreasing $\mathbb{R}^+$-valued continuous processes satisfying :

$$L_{ij}(0) = 0, \quad L_{ij} \equiv L_{ji} \quad \text{and} \quad L_{ij}(t) = \int_0^t \mathbb{1}_{|X_i(s)-X_j(s)|=r}\, dL_{ij}(s) \ .$$

By convention, we will always take $L_{ii} \equiv 0$.

A solution of the system $(\mathcal{E})$ is a family $(X_i(t), L_{ij}(t), t\geqslant 0, i, j \in \mathbb{N})$ (or simply $(X_i(t), t\geqslant 0)_{i\in\mathbb{N}}$) of processes such that the equation $(\mathcal{E})$ and the above conditions are satisfied.

The process $(X_i(t), t\geqslant 0)_{i\in\mathbb{N}}$ evolves under a dynamics which contains a Brownian part as diffusion term and a drift modelizing a pair interaction which derives from the action of two potentials :

φ a pair potential, function on $\mathbb{R}^d$ of class $\mathcal{C}^2$ with finite range $R > r$, i.e. satisfying $\varphi(x) = 0$ if $|x|\geqslant R$ and $\varphi(x) = \varphi(-x)$ (then $\nabla\varphi(0) = 0$).

ψ a r-diameter hard core pair potential defined by $\psi(x) = +\infty$ if $|x| < r$ and $\psi(x) = 0$ otherwise.

Since the dynamics only depends on the sum $\varphi + \psi$, which is infinite for $|x| < r$, the values of $\varphi(x)$ may be chosen arbitrary for $|x| < r$. In particular, we may assume without restriction that φ vanishes in a neighborhood from 0. For the same reason, the smallest value of interaction between two particles is given by

$$\underline{\varphi} = \inf_{|x| \geqslant r} \varphi(x) \quad \leqslant 0.$$

This quantity $\underline{\varphi}$ is important in our study, in particular if it vanishes (repulsive potential) or if it is negative (partly attractive potential).

We now define the set $\mathcal{G}(z)$ of Gibbs states associated to the potential $\varphi + \psi$ with activity parameter $z \in \mathbb{R}^+$ (see e.g. [Geo79]). They are measures on $\mathcal{M}$ which are locally absolutely continuous with respect to π^z in the following sense :

For each compact subset Λ of $\mathbb{R}^d$, let us define a local density function by :

$$\begin{aligned} f_\Lambda^z(\xi|\eta) &= \frac{1}{Z_z^{\Lambda,\eta}} \exp\Big(-\frac{1}{2} \sum_{\substack{\xi_i,\xi_j \in \Lambda \\ i \neq j}} (\varphi+\psi)(\xi_i - \xi_j) - \sum_{\substack{\xi_i \in \Lambda \\ \eta_j \in \Lambda^c}} (\varphi+\psi)(\xi_i - \eta_j) \Big) \\ &= \frac{1}{Z_z^{\Lambda,\eta}} \, 1\!\!1_{\mathcal{A}}(\xi_\Lambda \eta_{\Lambda^c}) \, \exp\Big(-\frac{1}{2} \sum_{\substack{\xi_i,\xi_j \in \Lambda \\ i \neq j}} \varphi(\xi_i - \xi_j) - \sum_{\substack{\xi_i \in \Lambda \\ \eta_j \in \Lambda^c}} \varphi(\xi_i - \eta_j) \Big) \end{aligned} \tag{1}$$

where $Z_z^{\Lambda,\eta}$ is the renormalizing constant determined by $\int f_\Lambda^z(\xi|\eta) \, d\pi_\Lambda^z(\xi) = 1$, i.e.

$$Z_z^{\Lambda,\eta} = e^{-z|\Lambda|} \left(1 + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{\Lambda^n} 1\!\!1_{\mathcal{A}}(y_1 \cdots y_n \, \eta_{\Lambda^c}) \, \exp\Big(- \sum_{1 \leqslant i < j \leqslant n} \varphi(y_i - y_j) - \sum_{\substack{1 \leqslant i \leqslant n \\ \eta_j \in \Lambda^c}} \varphi(y_i - \eta_j) \Big) \, dy_1 \cdots dy_n \right)$$

Definition 3.1 *A Probability measure μ on $\mathcal{M}$ belongs to the set $\mathcal{G}(z)$ of Gibbs states with activity z and associated potential $\varphi + \psi$ if and only if, for each compact subset $\Lambda \subset \mathbb{R}^d$,*

$$d\mu(\xi|\mathcal{B}_{\Lambda^c})(\eta) = f_\Lambda^z(\xi|\eta) \, d\pi_\Lambda^z(\xi) \quad \text{for } \mu\text{-a.e. } \eta$$

that is if and only if, for each bounded measurable function F on $\mathcal{M}$

$$\begin{aligned} \int_{\mathcal{M}} F(\eta) \, d\mu(\eta) &= \int_{\mathcal{M}} \int_{\mathcal{M}_\Lambda} F(\xi \eta_{\Lambda^c}) \, f_\Lambda^z(\xi|\eta) \, d\pi_\Lambda^z(\xi) \, d\mu(\eta) \\ &= \int_{\mathcal{M}} \frac{e^{-z|\Lambda|}}{Z_z^{\Lambda,\eta}} \Bigg(F(\eta_{\Lambda^c}) + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{\Lambda^n} F(y_1 \cdots y_n \eta_{\Lambda^c}) \, 1\!\!1_{\mathcal{A}}(y_1 \cdots y_n \eta_{\Lambda^c}) \\ &\qquad \exp\Big(- \sum_{1 \leqslant i < j \leqslant n} \varphi(y_i - y_j) - \sum_{\substack{1 \leqslant i \leqslant n \\ \eta_j \in \Lambda^c}} \varphi(y_i - \eta_j) \Big) dy_1 \cdots dy_n \Bigg) d\mu(\eta) \ . \end{aligned}$$

Remark that any Gibbs measure in $\mathcal{G}(z)$ has its support included in $\mathcal{A}$. Dobrushin proved in [Dob69], using compactness argument, that there exists at least one element in $\mathcal{G}(z)$ when the potential contains a hard core component. Furthermore the set $\mathcal{G}(z)$ is convex and compact. About the cardinality of $\mathcal{G}(z)$, remarking that the sum of the hard core and the smooth potential φ is superstable and lower regular in the sense of Ruelle [Rue70], we do the following remarks :
- If z is small enough Ruelle proved that uniqueness holds. In our case, a sufficient condition would be : $z \leq e^{\overline{N}\underline{\varphi}-1} (|B(0,r)| + \int 1\!\!1_{r<|y|<R} |1 - e^{-\varphi(y)}| dy)^{-1}$.
- For z large enough it is conjectured (see [Geo79]) but still not proved that phase transition occurs : Card $\mathcal{G}(z) > 1$.

The main results of this paper are the following theorems, proved in the next sections.

Theorem 3.2 *The stochastic equation* $(\mathcal{E})$ *admits a solution with values in* $\mathcal{A}$ *for any deterministic initial configuration which belongs to the set* $\underline{\underline{\mathcal{A}}} \subset \mathcal{A}$ *defined by* $\underline{\underline{\mathcal{A}}} = \{x \in \mathcal{A} \ : P(\Omega_0^x \cap \Omega_1^x) = 1\}$ *(sets* Ω_0^x *and* Ω_1^x *are given in (14) and (22)). This solution is unique as element of* $\mathcal{C} \subset \mathcal{C}(\mathbb{R}^+, \mathcal{A})$*, a subset of paths with some regularity defined in section 7.*

Theorem 3.3 *If the initial configuration of the stochastic equation* $(\mathcal{E})$ *is random with distribution* $\mu \in \mathcal{G}(z)$ *for some* $z > 0$ *and* $\mu(\underline{\underline{\mathcal{A}}}) = 1$*, then this solution is time-reversible, that is its law is invariant with respect to the time reversal.*

Proposition 3.4 *Let* z_c *be a critical value of the activity given by:* $z_c = \dfrac{\exp(2\overline{N}\underline{\varphi})}{(R^d - r^d)|B(0,1)|}$*. Any Gibbs measure* $\mu \in \mathcal{G}(z)$ *with* $0 < z < z_c$ *has its support included in* $\underline{\underline{\mathcal{A}}}$*.*

Remark 3.5 : The critical value z_c given here appears for technical reasons in corollary 5.5, where a percolation type estimate is computed.

4 Construction of approximating processes

The solution of $(\mathcal{E})$ will be constructed as a limit of approximating processes $(X^l)_{l \in \mathbb{N}^*}$. We construct here an approximation by penalization. In this whole section, $l \in \mathbb{N}^*$ is fixed.

The approximating process X^l verifies :

- if $X_i^l(0)$ does not belong to $[-l, l]^d$ then $X_i^l(\cdot)$ is constant
- if $X_i^l(0) \in [-l, l]^d$ then $X_i^l(\cdot)$ "essentially" stays in $[-l, l]^d$ (in a sense which will be clear at the end of the section).

In order to obtain such a behavior, we introduce in the equation $(\mathcal{E})$ a supplementary drift $\nabla \psi^{l, X(0)}$ which vanishes in a subset of $[-l, l]^d$ and is repulsive outside of $[-l, l]^d$. More precisely, for an allowed configuration η with support outside $[-l, l]^d$ we fix a $\mathbb{R}^+$-valued function $\psi^{l,\eta}$ on $\mathbb{R}^d$ which

- is $\mathcal{C}^2$ with bounded derivatives
- vanishes on every $y \in [-l, l]^d$ such that $y\eta = \{y\} \cup \eta$ is an allowed configuration (and only on these y), that is

$$\psi^{l,\eta}(y) = 0 \quad \Leftrightarrow \quad y \in [-l, l]^d \text{ and } y\eta \in \mathcal{A} \quad \Leftrightarrow \quad y \in [-l, l]^d \text{ and } d(y, \eta) \geqslant r \ .$$

We extend the definition of $\psi^{l,\eta}$ to configurations $\eta \in \mathcal{A}$ not necessarily belonging to $(([-l, l]^d)^c)^{\mathbb{N}}$ by taking into account only the points of η which are in $([-l, l]^d)^c$, i.e. :

$$\psi^{l,\eta} = \psi^{l, \eta \cap ([-l,l]^d)^c}.$$

We also suppose that, for every $\eta \in \mathcal{A}$,

$$\sum_{l \in \mathbb{N}^*} \int_{\mathbb{R}^d} \mathbb{1}_{\psi^{l,\eta}(y) > 0} \ \exp(-\psi^{l,\eta}(y)) \ dy \quad \leqslant \quad 1. \tag{2}$$

Such a family $(\psi^{l,\eta})_{l \in \mathbb{N}^*, \eta \in \mathcal{A}}$ exists; choose for example $\psi^{l,\eta}(y) = l^{d+1}\delta(y)$ where δ is a $\mathcal{C}^2$ function with bounded derivatives which is equivalent on $\mathbb{R}^d$ to $d(\cdot, \Lambda \smallsetminus B(\eta_{\Lambda^c}, r))$ with $\Lambda = [-l, l]^d$ (see [Ste70] p 171), that is which verifies :

$$\exists C, C' > 0 \text{ such that } \forall x \in \mathbb{R}^d \quad C \ d(x, \Lambda \smallsetminus B(\eta_{\Lambda^c}, r)) \leqslant \delta(x) \leqslant C' \ d(x, \Lambda \smallsetminus B(\eta_{\Lambda^c}, r)).$$

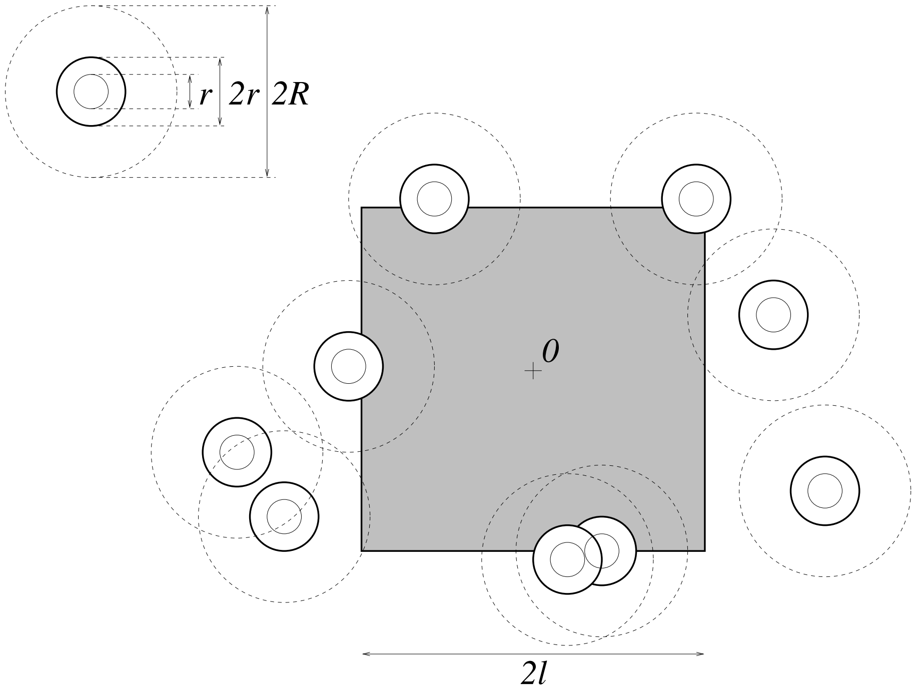

Figure 2: The grey area represents the set where $\psi^{l,\eta}$ vanishes.

For $\eta \in \mathcal{A}$ still fixed, and for $n \in \mathbb{N}^*$, let us now study the n-dimensional stochastic differential equation :

$$(\mathcal{E}_n^{l,\eta}) \left\{ \begin{array}{l} \forall i \in \{1,\dots,n\}, \quad \forall t \geqslant 0, \\ dX_i(t) = dW_i(t) - \dfrac{1}{2} \Bigg(\nabla\psi^{l,\eta}(X_i(t)) + \displaystyle\sum_{j=1,\dots,n} \nabla\varphi(X_i(t) - X_j(t)) \\ \qquad\qquad\qquad\qquad + \displaystyle\sum_{j:\eta_j \in \Lambda^c} \nabla\varphi(X_i(t) - \eta_j) \Bigg) dt \\ \qquad\qquad + \displaystyle\sum_{j=1,\dots,n} (X_i(t) - X_j(t)) dL_{ij}(t) \end{array} \right.$$

By a solution of $(\mathcal{E}_n^{l,\eta})$, we mean a family $\left((X_i)_{1\leqslant i\leqslant n}, (L_{ij})_{\substack{1\leqslant i,j\leqslant n \\ i\neq j}} \right)$ of continuous processes such that

- $\forall t \geqslant 0 \quad (X_i(t))_{1\leqslant i\leqslant n} \in \mathcal{A} \cap (\mathbb{R}^d)^n$
- $\forall i,j \quad L_{ij} = L_{ji}$
- $\forall i,j \quad \forall t \geqslant 0 \quad L_{ij}(t) = \displaystyle\int_0^t \mathbb{1}_{|X_i(s)-X_j(s)|=r} \, dL_{ij}(s)$

$(\mathcal{E}_n^{l,\eta})$ is a finite dimensional stochastic differential equation reflected in $\mathcal{A} \cap (\mathbb{R}^d)^n$ with drift $-\frac{1}{2}\nabla\beta_n^{l,\eta}$ where

$$\beta_n^{l,\eta}(x_1,\dots,x_n) = \sum_{i=1,\dots,n} \left(\psi^{l,\eta}(x_i) + \frac{1}{2} \sum_{\substack{j=1,\dots,n \\ j\neq i}} \varphi(x_i - x_j) + \sum_{j:\eta_j\in\Lambda^c} \varphi(x_i - \eta_j) \right). \tag{3}$$

Since this drift is bounded and Lipschitz continuous, the equation $(\mathcal{E}_n^{l,\eta})$ has a unique strong solution for each initial configuration $x \in \mathcal{A} \cap (\mathbb{R}^d)^n$ (see theorem 5.1 of [ST86]). We will denote this solution by $X^{l,\eta,n}(x,\cdot)$.

Proposition 4.1 *The solution of $(\mathcal{E}_n^{l,\eta})$ with initial distribution $\nu_n^{l,\eta}$ is reversible, where $\nu_n^{l,\eta}$ is the finite measure defined on $(\mathbb{R}^d)^n$ by*

$$d\nu_n^{l,\eta}(x_1,\ldots,x_n) = \exp(-\beta_n^{l,\eta}(x_1,\ldots,x_n)) \; 1\!\!1_{\mathcal{A}}(x_1,\ldots,x_n) \; dx_1 \ldots dx_n$$

The time reversible law of $X^{l,\eta,n}$ starting from $\nu_n^{l,\eta}$ will be denoted by $Q_n^{l,\eta}$:

$$Q_n^{l,\eta}(\Theta) = \int P(X^{l,\eta,n}(x,\cdot) \in \Theta) \; d\nu_n^{l,\eta}(x)$$

Like $\nu_n^{l,\eta}$, the finite measure $Q_n^{l,\eta}$ is not necessarily a Probability measure.

Proof of prop 4.1

In this proof, l, η and n are fixed, hence we drop the indices and simply write β, ν, Q for $\beta_n^{l,\eta}$, $\nu_n^{l,\eta}$, $Q_n^{l,\eta}$, etc. Again by theorem 5.1 of [ST86], the stochastic differential equation

$$\begin{aligned}
&\forall i \in \{1,\ldots,n\} \quad \forall t \geqslant 0 \\
&X_i(t) = X_i(0) + W_i(t) + \int_0^t \sum_{j=1,\ldots,n} (X_i(s) - X_j(s)) \; dL_{ij}(s) \\
&L_{ij}(t) = \textstyle\int_0^t 1\!\!1_{|X_i(s)-X_j(s)|=r} \; dL_{ij}(s)
\end{aligned}$$

has a unique strong solution. Let $\overline{P}^x$ denote the distribution on $\mathcal{C}([0,T], \mathcal{A} \cap (\mathbb{R}^d)^n)$ of the solution X starting from $x \in \mathcal{A} \cap (\mathbb{R}^d)^n$. It is known (see e.g. theorem 1 of [ST87]) that the measure $\overline{P} = \int_{\mathcal{A}\cap(\mathbb{R}^d)^n} \overline{P}^x dx$ is invariant with respect to the time reversal τ on $[0,T]$:

$$\begin{array}{rrcl}
\tau : & \mathcal{C}([0,T], \mathcal{A} \cap (\mathbb{R}^d)^n) & \longrightarrow & \mathcal{C}([0,T], \mathcal{A} \cap (\mathbb{R}^d)^n) \\
& X(\cdot) & \longrightarrow & X(T-\cdot)
\end{array}$$

Applying Girsanov theorem, we see that the process

$$t \longrightarrow \left(X_i(t) - X_i(0) - \int_0^t \sum_{j=1}^n (X_i(s) - X_j(s)) dL_{ij}(s) + \frac{1}{2} \sum_{i=1}^n \int_0^t \nabla_i \beta(X(s)) ds \right)_{1 \leqslant i \leqslant n}$$

is a Brownian motion under the measure Q^x defined on $\mathcal{C}([0,T], \mathcal{A} \cap (\mathbb{R}^d)^n)$ by

$$\frac{dQ^x}{d\overline{P}^x}(X) = \exp\left(-\frac{1}{2} \sum_{i=1,\ldots,n} \int_0^T \nabla_i \beta(X(s)) \; dW_i(s) - \frac{1}{8} \int_0^T |\nabla \beta(X(s))|^2 \; ds \right)$$

for each $x \in \mathcal{A} \cap (\mathbb{R}^d)^n$.

As consequence, Q^x is the distribution of the unique strong solution of $(\mathcal{E}_n^{l,\eta})$ starting from x, and $Q = \int_{\mathcal{A}\cap(\mathbb{R}^d)^n} Q^x \; d\nu(x)$ is the law of the solution with initial distribution ν. Using Ito's formula applied to the smooth function β, we can compute the density :

$$\begin{aligned}
\frac{dQ}{d\overline{P}}(X) = \exp\Bigg(&-\frac{1}{2} \left(\beta(X(0)) + \beta(X(T))\right) \\
&+ \frac{1}{2} \int_0^T \sum_{i,j=1,\ldots,n} \nabla_i \beta(X(s)) \; (X_i(s) - X_j(s)) \; dL_{ij}(s) \\
&+ \int_0^T \left(\frac{1}{4} \Delta \beta(X(s)) - \frac{1}{8} |\nabla \beta(X(s))|^2 \right) \; ds \Bigg)
\end{aligned}$$

Since $\overline{P}$ and $\frac{dQ}{dP}$ are invariant with respect to time reversal τ, Q is time reversal invariant too, which exactly means that the solution of $(\mathcal{E}_n^{l,\eta})$ starting from ν is reversible.

■

The finite measure $\nu_n^{l,\eta}$ on $(\mathbb{R}^d)^n$ is an approximation, up to a renormalization constant, of the distribution of n particles under $(\varphi+\psi)$-interaction in $[-l,l]^d$. We now define a Probability measure $\mu_z^{l,\eta}$ which will represent the distribution of a random number of particles in $[-l,l]^d$, this number following a Poisson distribution with intensity measure $z\,dy$.

Let us consider the direct sum $\bigcup_{n=0}^{+\infty}(\mathbb{R}^d)^n$ (by convention $(\mathbb{R}^d)^0=\{\emptyset\}$) endowed with the product $\sigma\left(\prod_{n=0}^{+\infty}\mathcal{B}or((\mathbb{R}^d)^n)\right)$ of the Borel σ-algebras on the $(\mathbb{R}^d)^n$. The Probability measure $\mu_z^{l,\eta}$ on $\bigcup_{n=0}^{+\infty}(\mathbb{R}^d)^n$ is given by :

$$\begin{aligned} &\forall A_0\times A_1\times\cdots\times A_n\times\cdots\in\prod_{n=0}^{+\infty}\mathcal{B}or((\mathbb{R}^d)^n)\\ &\mu_z^{l,\eta}(A_0\times A_1\times\cdots\times A_n\times\cdots)=\frac{e^{-z2^dl^d}}{Z_z^{l,\eta}}\sum_{n=0}^{+\infty}\frac{z^n}{n!}\,\nu_n^{l,\eta}(A_n)\end{aligned}\tag{4}$$

where $Z_z^{l,\eta}=e^{-z2^dl^d}\sum_{n=0}^{+\infty}\frac{z^n}{n!}\,\nu_n^{l,\eta}((\mathbb{R}^d)^n)$ (with the convention $\nu_0^{l,\eta}(\{\emptyset\})=1$). Similarly, consider the Probability measure on $\bigcup_{n=0}^{+\infty}\mathcal{C}(\mathbb{R}^+,(\mathbb{R}^d)^n)$ defined by

$$Q_z^{l,\eta}(\Theta)=\frac{e^{-z2^dl^d}}{Z_z^{l,\eta}}\sum_{n=0}^{+\infty}\frac{z^n}{n!}\,Q_n^{l,\eta}(\Theta)=\int P\left(X^{l,\eta,\mathrm{Card}(x)}(x,\cdot)\in\Theta\right)d\mu_z^{l,\eta}(x)\;.$$

This Probability measure is time reversal invariant, thanks to proposition 4.1, and has its support included in $\mathcal{A}$, as a mixing of $\mathcal{A}$-supported measures.

Finally, let us randomize the external configuration η and consider the following infinite dimensional stochastic equation

$$(\mathcal{E}^l)\begin{cases}\forall i\in\mathbb{N}\text{ such that }x_i\in[-l,l]^d\quad\forall t\geqslant 0\\ X_i^l(t)=x_i+W_i(t)-\frac{1}{2}\int_0^t\left(\nabla\psi^{l,x}(X_i^l(s))+\sum_j\nabla\varphi(X_i^l(s)-X_j^l(s))\right)ds\\ \qquad+\sum_{\{j:x_j\in[-l,l]^d\}}\int_0^t(X_i^l(s)-X_j^l(s))\,dL_{ij}^l(s)\\ \forall i\in\mathbb{N}\text{ such that }x_i\notin[-l,l]^d\qquad X_i^l(\cdot)\equiv x_i\\ \forall i,j\quad L_{ij}^l=L_{ji}^{l,\eta},\quad L_{ij}^l(0)=0,\quad L_{ij}^l\text{ non decreasing}\\ \text{and }\forall t\geqslant 0\quad L_{ij}^l(t)=\mathbb{1}_{x_i\in[-l,l]^d}\mathbb{1}_{x_j\in[-l,l]^d}\int_0^t\mathbb{1}_{|X_i^l(s)-X_j^l(s)|=r}dL_{ij}^l(s)\end{cases}$$

Remember that $\psi^{l,x}$ only depends on $x\cap([-l,l]^d)^c$, so that this dynamics is Markovian. For each deterministic initial configuration $x\in\mathcal{A}$, the equation $(\mathcal{E}^l)$ has a unique strong solution $(X^{l,x},L^{l,x})$ since it reduces to the dynamics of $(\mathcal{E}_n^{l,\eta})$ with $\eta=x\cap([-l,l]^d)^c$ and $n=\mathrm{Card}(x\cap[-l,l]^d)$:

$$X^{l,x}(\cdot)=X^{l,x\cap\Lambda^c,\mathrm{Card}(x\cap\Lambda)}(x\cap\Lambda,\cdot)\;x_{\Lambda^c}\qquad\text{with }\Lambda=[-l,l]^d\;.$$

5 Probability of "bad" paths

In this section, we want to prove that the probability of "bad" trajectories, i.e. trajectories of particles which interact too much, vanishes asymptotically when $l \to +\infty$. We will use this result to construct the limit of $(X^{l,x})_l$ in the next section.

Nice paths are ω's such that each particle interacts only with a finite number of particles during a finite time interval ; $X^{l,x}(\omega)$ is then the (unique) solution of a finite dimensional equation. Bad ω's are paths such that at least a particle interacts with a great number of other ones, either because it moves very fast, or because it belongs to a large chain of particles where each one interacts with its neighbors.

To simplify, we restrict the study of the paths on the time interval $[0,1]$. It is obvious that all the results in the sequel hold true on any time interval $[0,T]$, $T \geqslant 1$, up to a change of the constants.

5.1 Probability of fast motion

We obtain here an estimate of the probability that a particle moves "too fast". In order to establish this estimate, obtained in prop 5.2, we first compute the probability of fast motion between two fixed bounded domains in $\mathbb{R}^d$.

For every bounded subsets A_0 and A_1 of $\mathbb{R}^d$ and every $\varepsilon > 0$ and $\delta \in]0,1]$, let $\mathcal{F}m(A_0, A_1, \varepsilon, \delta)$ denote the event "at least a particle goes from A_0 to A_1 with an oscillation greater than ε in a time interval smaller than δ", i.e.

$$\mathcal{F}m(A_0, A_1, \varepsilon, \delta) = \{X \in \mathcal{C}([0,1], \mathcal{A}), \ \ \exists i \text{ s.t. } X_i(0) \in A_0, \ X_i(1) \in A_1 \text{ and } w(X_i, \delta) > \varepsilon\}$$

where $w(X_i, \delta) = \sup\limits_{\substack{|t-s|<\delta \\ 0 \leqslant s,t \leqslant 1}} |X_i(t) - X_i(s)|$ is the usual modulus of continuity of the path X_i on $[0,1]$.

Lemma 5.1 *For each A_0, A_1 bounded subsets of $\mathbb{R}^d$ and each $\varepsilon > 0$, $\delta \in]0,1]$, we have :*

$$\forall l \in \mathbb{N}^* \quad \forall \eta \in \mathcal{A} \quad \forall n \in \mathbb{N}^*$$

$$\begin{aligned} Q_n^{l,\eta}(\mathcal{F}m(A_0, A_1, \varepsilon, \delta))) &= \int P(X^{l,\eta,n}(x, \cdot) \in \mathcal{F}m(A_0, A_1, \varepsilon, \delta)) \, d\nu_n^{l,\eta}(x) \\ &\leqslant 41 \ n \ e^{-2\overline{N}\underline{\varphi}} \ \nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1}) \ \frac{1}{\delta} \ \exp\left(-\frac{\varepsilon^2}{5\delta}\right) \int_{\mathbb{R}^d} (\mathbb{1}_{A_0} + \mathbb{1}_{A_1}) e^{-\psi^{l,\eta}} dy \quad (5) \end{aligned}$$

and

$$\forall l \in \mathbb{N}^* \quad \forall \eta \in \mathcal{A}$$

$$Q_z^{l,\eta}(\mathcal{F}m(A_0, A_1, \varepsilon, \delta))) \leqslant 41 \ z \ e^{-2\overline{N}\underline{\varphi}} \ \frac{1}{\delta} \ \exp\left(-\frac{\varepsilon^2}{5\delta}\right) \int_{\mathbb{R}^d} (\mathbb{1}_{A_0} + \mathbb{1}_{A_1}) e^{-\psi^{l,\eta}} dy \quad (6)$$

From this lemma proved below, we easily deduce an estimate of the probability under $Q_z^{l,\eta}$ that a particle starting from $B(0,K)$ moves too fast. For every $K \in \mathbb{N}^*$, $\varepsilon > 0$ and $\delta \in]0,1]$, let $\mathcal{F}m(K, \varepsilon, \delta)$ be the following event :

$$\mathcal{F}m(K, \varepsilon, \delta) = \{X \in \mathcal{C}([0,1], \mathcal{A}) \quad \text{s.t.} \quad \exists i, \ X_i(0) \in B(0,K) \text{ and } w(X_i, \delta) > \varepsilon\}$$

Proposition 5.2 *The following upper bounds hold :*

$$\forall K \in \mathbb{N}^* \quad \forall \varepsilon > 0 \quad \forall \delta \in]0,1] \quad \forall l \in \mathbb{N}^* \quad \forall \eta \in \mathcal{A}$$

$$\begin{aligned} Q_n^{l,\eta}(\mathcal{F}m(K, \varepsilon, \delta)) &\leqslant n \ C_d \ e^{-2\overline{N}\underline{\varphi}} \ \nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1}) \ \frac{1}{\delta} \ \exp\left(-\frac{\varepsilon^2}{6\delta}\right) \ K^d \\ Q_z^{l,\eta}(\mathcal{F}m(K, \varepsilon, \delta)) &\leqslant z \ C_d \ e^{-2\overline{N}\underline{\varphi}} \ \frac{1}{\delta} \ \exp\left(-\frac{\varepsilon^2}{6\delta}\right) \ K^d \end{aligned}$$

where C_d is a constant depending only on dimension d. And similarly one has :

$$\forall K \in \mathbb{N}^* \quad \forall \varepsilon > 0 \quad \forall \delta \in]0,1] \quad \forall l \in \mathbb{N}^* \quad \forall \eta \in \mathcal{A}$$

$$Q_n^{l,\eta}(\mathcal{F}m(K,\varepsilon,\delta)) \leqslant 246\ n\ e^{-2\overline{N}\underline{\varphi}}\ \nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1})\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right)\ \int_{\mathbb{R}^d} e^{-\psi^{l,\eta}(y)}\ dy$$

$$Q_z^{l,\eta}(\mathcal{F}m(K,\varepsilon,\delta)) \leqslant 246\ z\ e^{-2\overline{N}\underline{\varphi}}\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right)\ \int_{\mathbb{R}^d} e^{-\psi^{l,\eta}(y)}\ dy$$

Proof of lemma 5.1

We first need an estimate of $Q_n^{l,\eta}(\mathcal{F}m(A_0,A_1,\varepsilon,\delta))$.

Let $(X^{l,\eta,n},L^{l,\eta,n})$ denote the unique strong solution of $(\mathcal{E}_n^{l,\eta})$ starting from $\nu_n^{l,\eta}$, and recall that the distribution $Q_n^{l,\eta}$ of $X^{l,\eta,n}$ is time reversible on $[0,1]$. By construction the processes :

$$\begin{aligned} W_i(t) = X_i^{l,\eta,n}(t) - X_i^{l,\eta,n}(0) + \frac{1}{2}\int_0^t \nabla_i \beta_n^{l,\eta}(X^{l,\eta,n}(s))ds \\ - \int_0^t \sum_{j=1,\dots,n} (X_i^{l,\eta,n}(s) - X_j^{l,\eta,n}(s)) dL_{ij}^{l,\eta,n}(s), \\ 1 \leqslant i \leqslant n, 0 \leqslant t \leqslant 1 \end{aligned}$$

and

$$\begin{aligned} \widehat{W}_i(t) = X_i^{l,\eta,n}(1-t) - X_i^{l,\eta,n}(1) + \frac{1}{2}\int_{1-t}^1 \nabla_i \beta_n^{l,\eta}(X^{l,\eta,n}(s))ds \\ - \int_{1-t}^1 \sum_{j=1,\dots,n} (X_i^{l,\eta,n}(s) - X_j^{l,\eta,n}(s)) dL_{ij}^{l,\eta,n}(s), \\ 1 \leqslant i \leqslant n, 0 \leqslant t \leqslant 1 \end{aligned}$$

are both n-dimensional Brownian motions starting from 0. Remarking that

$$\forall t \in [0,1] \quad X^{l,\eta,n}(t) - X^{l,\eta,n}(0) = \frac{1}{2}\left(W(t) + \widehat{W}(1-t) - \widehat{W}(1)\right)$$

and using the equality in law between $(X^{l,\eta,n}(1-\cdot),\widehat{W})$ and $(X^{l,\eta,n},W)$, we obtain :

$$\begin{aligned} & Q_n^{l,\eta}(\mathcal{F}m(A_0,A_1,\varepsilon,\delta)) \\ &= \int_{(\mathbb{R}^d)^n} P\left(\begin{array}{c} \exists i \leqslant n \text{ s.t. } X_i^{l,\eta,n}(x,0) \in A_0, X_i^{l,\eta,n}(x,1) \in A_1 \text{ and} \\ \sup_{\substack{|t-s|<\delta \\ 0\leqslant s,t\leqslant 1}} |W_i(t) - W_i(s) + \widehat{W}(1-t) - \widehat{W}(1-s)| > 2\varepsilon \end{array}\right) d\nu_n^{l,\eta}(x) \\ &\leqslant \int P\left(\exists i \leqslant n \text{ s.t. } X_i^{l,\eta,n}(x,0) \in A_0 \text{ and } \sup_{\substack{|t-s|<\delta \\ 0\leqslant s,t\leqslant 1}} |W_i(t) - W_i(s)| > \varepsilon\right) d\nu_n^{l,\eta}(x) \\ &\quad + \int P\left(\exists i \leqslant n \text{ s.t. } X_i^{l,\eta,n}(x,1) \in A_1 \text{ and } \sup_{\substack{|t-s|<\delta \\ 0\leqslant s,t\leqslant 1}} |\widehat{W}(t) - \widehat{W}(s)| > \varepsilon\right) d\nu_n^{l,\eta}(x) \\ &\leqslant \int P\left(\exists i \leqslant n \text{ s.t. } x_i \in A_0 \text{ and } w(W_i,\delta) > \varepsilon\right) d\nu_n^{l,\eta}(x) \\ &\quad + \int P\left(\exists i \leqslant n \text{ s.t. } x_i \in A_1 \text{ and } w(W_i,\delta) > \varepsilon\right) d\nu_n^{l,\eta}(x) \end{aligned}$$

The right hand side is smaller than

$$\begin{aligned} \sum_{i=1}^n \nu_n^{l,\eta}(x_i \in A_0)\ P\left(w(W_i,\delta) > \varepsilon\right) + \sum_{i=1}^n \nu_n^{l,\eta}(x_i \in A_1)\ P\left(w(W_i,\delta) > \varepsilon\right) \\ \leqslant n\ P\left(w(W_1,\delta) > \varepsilon\right)\ \left(\nu_n^{l,\eta}(x_1 \in A_0) + \nu_n^{l,\eta}(x_1 \in A_1)\right) \end{aligned}$$

($Q_n^{l,\eta}$ and $\nu_n^{l,\eta}$ are permutation invariant).

We know from appendix 8 that

$$P\left(w(W_1,\delta)>\varepsilon\right)\ \leqslant\ \frac{41}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right).$$

According to the definition (3) of $\beta_n^{l,\eta}$, since a particle interacts with at most $\overline{N}$ other particles (cf remark 2.1) :

$$\begin{aligned}\beta_n^{l,\eta}(x_1,\ldots,x_n) &= \psi^{l,\eta}(x_1)+\sum_{j=2}^{n}\varphi(x_1-x_j)+\sum_{j,\eta_j\in\Lambda^c}\varphi(x_1-\eta_j)+\beta_{n-1}^{l,\eta}(x_2,\ldots,x_n)\\ &\geqslant \psi^{l,\eta}(x_1)+2\overline{N}\underline{\varphi}+\beta_{n-1}^{l,\eta}(x_2,\ldots,x_n)\end{aligned}\tag{7}$$

which implies that

$$\begin{aligned}&\nu_n^{l,\eta}(x_1\in A_0)\\ &\quad=\int_{(\mathbb{R}^d)^n}\mathbb{1}_{x_1\in A_0}\ \mathbb{1}_{\mathcal{A}}(x_1,\ldots,x_n)\ e^{-\beta_n^{l,\eta}(x_1,\ldots,x_n)}\ dx_1\cdots dx_n\\ &\quad\leqslant\int_{(\mathbb{R}^d)^n}\mathbb{1}_{x_1\in A_0}\ \mathbb{1}_{\mathcal{A}}(x_2,\ldots,x_n)\ e^{-\beta_{n-1}^{l,\eta}(x_2,\ldots,x_n)}\ e^{-2\overline{N}\underline{\varphi}}\ e^{-\psi^{l,\eta}(x_1)}\ dx_1\cdots dx_n\\ &\quad\leqslant e^{-2\overline{N}\underline{\varphi}}\ \nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1})\int_{A_0}e^{-\psi^{l,\eta}(y)}\ dy\end{aligned}\tag{8}$$

and the same result holds for A_1. This leads to the estimate :

$$\begin{aligned}&Q_n^{l,\eta}(\mathcal{F}m(A_0,A_1,\varepsilon,\delta))\\ &\qquad\leqslant n\ e^{-2\overline{N}\underline{\varphi}}\ \nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1})\ 41\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right)\int_{\mathbb{R}^d}(\mathbb{1}_{A_0}+\mathbb{1}_{A_1})e^{-\psi^{l,\eta}}\ dy\ ;\end{aligned}$$

by summing this over n we obtain the desired result :

$$\begin{aligned}&Q_z^{l,\eta}(\mathcal{F}m(A_0,A_1,\varepsilon,\delta))\\ &\quad=\frac{e^{-z2^dl^d}}{Z_z^{l,\eta}}\sum_{n=1}^{+\infty}\frac{z^n}{n!}\ Q_n^{l,\eta}(\mathcal{F}m(A_0,A_1,\varepsilon,\delta))\\ &\quad\leqslant 41\ \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}}\ z\left(\sum_{n=1}^{+\infty}\frac{z^{n-1}}{(n-1)!}\ \nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1})\right)e^{-2\overline{N}\underline{\varphi}}\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right)\int(\mathbb{1}_{A_0}+\mathbb{1}_{A_1})e^{-\psi^{l,\eta}}dy\\ &\quad\leqslant 41\ z\ e^{-2\overline{N}\underline{\varphi}}\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right)\int_{\mathbb{R}^d}(\mathbb{1}_{A_0}+\mathbb{1}_{A_1})e^{-\psi^{l,\eta}}dy\ .\end{aligned}$$

■

Proof of prop 5.2

For j in $\mathbb{N}$, let $a_j=K+\sqrt{\frac{\varepsilon^2}{\delta}+5j}$. The sequence $(a_j)_j$ increases from $a_0=K+\frac{\varepsilon}{\sqrt{\delta}}$ to $+\infty$. Now for $Q=Q_n^{l,\eta}$ or $Q=Q_z^{l,\eta}$ consider

$$\begin{aligned}Q(\mathcal{F}m(K,\varepsilon,\delta)) &= Q\left(\exists i,\ |X_i(0)|\leqslant K \text{ and } w(X_i,\delta)>\varepsilon\right)\\ &\leqslant Q\left(\exists i,\ |X_i(0)|\leqslant K \text{ and } w(X_i,\delta)>\varepsilon \text{ and } |X_i(1)|\leqslant a_0\right)\\ &\qquad+\sum_{j=0}^{+\infty}Q\left(\exists i,\ |X_i(0)|\leqslant K \text{ and } a_j<|X_i(1)|\leqslant a_{j+1}\right)\end{aligned}$$

But $|X_i(0)|\leqslant K$ and $|X_i(1)| > a_j$ imply that $w(X_i,1) > a_j - K$, so this is smaller than

$$\leqslant Q\left(\exists i,\ |X_i(0)|\leqslant K,\ |X_i(1)|\leqslant a_0 \text{ and } w(X_i,\delta) > \varepsilon\right)$$
$$+\sum_{j=0}^{+\infty} Q\left(\exists i,\ |X_i(0)|\leqslant K,\ a_j < |X_i(1)| < a_{j+1} \text{ and } w(X_i,1) > a_j - K\right)$$

Using lemma 5.1 and $1\leqslant\frac{1}{\delta}$, we obtain :

$$\begin{aligned}
&Q(\mathcal{F}m(K,\varepsilon,\delta))\\
&\leqslant C(Q)\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right)\int(\mathbb{I}_{B(0,K)}+\mathbb{I}_{B(0,a_0)})e^{-\psi^{l,\eta}}\ dy\\
&\quad + C(Q)\ \sum_{j=0}^{+\infty}\exp\left(-\frac{1}{5}(\frac{\varepsilon^2}{\delta}+5j)\right)\int(\mathbb{I}_{B(0,K)}+\mathbb{I}_{B(0,a_{j+1})\smallsetminus B(0,a_j)})e^{-\psi^{l,\eta}}\ dy\\
&\leqslant C(Q)\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right)\int\left(\mathbb{I}_{B(0,K)}+\mathbb{I}_{B(0,a_0)}+\sum_{j=0}^{+\infty}e^{-j}\,\mathbb{I}_{B(0,K)}+\sum_{j=0}^{+\infty}e^{-j}\,\mathbb{I}_{B(0,a_{j+1})\smallsetminus B(0,a_j)}\right)e^{-\psi^{l,\eta}}\ dy
\end{aligned}$$

with $C(Q_n^{l,\eta}) = 41\ n\ e^{-2\overline{N}\underline{\varphi}}\ \nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1})$ and $C(Q_z^{l,\eta}) = 41\ z\ e^{-2\overline{N}\underline{\varphi}}$.
Using $e^{-\psi^{l,\eta}(y)}\leqslant 1$ and the inequality $\sqrt{\alpha+\beta}\leqslant\sqrt{\alpha}+\sqrt{\beta}$ for $\alpha,\beta>0$, one has for $j\geqslant 1$:

$$\begin{aligned}
\int\mathbb{I}_{B(0,a_{j+1})\smallsetminus B(0,a_j)}\ e^{-\psi^{l,\eta}}\ dy &\leqslant (a_{j+1})^d\ |B(0,1)|\\
&\leqslant (K+\frac{\varepsilon}{\sqrt{\delta}}+\sqrt{5(j+1)})^d\ |B(0,1)|\\
&\leqslant 3^d\ K^d\ |B(0,1)|\ \max(1,\frac{\varepsilon}{\sqrt{\delta}})^d\ \sqrt{5(j+1)}^d \qquad (9)
\end{aligned}$$

and similarly :

$$\int\mathbb{I}_{B(0,a_0)}\ e^{-\psi^{l,\eta}}\ dy \leqslant (a_0)^d\ |B(0,1)| \leqslant (K+\frac{\varepsilon}{\sqrt{\delta}})^d\ |B(0,1)| \leqslant 2^d\ K^d\ |B(0,1)|\ \max(1,\frac{\varepsilon}{\sqrt{\delta}})^d$$

Since $\sum_{j=0}^{+\infty}e^{-j}\leqslant 2$ and $\sum_{j=0}^{+\infty}(\sqrt{j+1})^d\ e^{-j}\leqslant 2$ this leads to :

$$\begin{aligned}
&Q(\mathcal{F}m(K,\varepsilon,\delta))\\
&\leqslant C(Q)\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right)\ K^d\ |B(0,1)|\left(1+2^d\max(1,\frac{\varepsilon}{\sqrt{\delta}})^d+2+2\times 3^d\max(1,\frac{\varepsilon}{\sqrt{\delta}})^d\sqrt{5}^d\right)\\
&\leqslant C(Q)\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right)\ |B(0,K)|\ 3\times 3^d\max(1,\frac{\varepsilon}{\sqrt{\delta}})^d\sqrt{5}^d
\end{aligned}$$

Finally

$$Q_n^{l,\eta}(\mathcal{F}m(K,\varepsilon,\delta)) \leqslant n\ e^{-2\overline{N}\underline{\varphi}}\ \nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1})\ C_d\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{6\delta}\right)\ K^d$$
$$Q_z^{l,\eta}(\mathcal{F}m(K,\varepsilon,\delta)) \leqslant z\ e^{-2\overline{N}\underline{\varphi}}\ C_d\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{6\delta}\right)\ K^d$$

where $C_d = 41\ |B(0,1)|\ 3\times 3^d\ \sqrt{5}^d\ \sup_{x\in\mathbb{R}^+} e^{-x^2/5}\max(1,x)^d e^{x^2/6}$.
An alternative bound for $Q(\mathcal{F}m(K,\varepsilon,\delta))$ may be obtained using the fact that each indicator function is smaller than 1 :

$$Q(\mathcal{F}m(K,\varepsilon,\delta)) \leqslant C(Q)\ \frac{1}{\delta}\ \exp\left(-\frac{\varepsilon^2}{5\delta}\right)\ \int_{\mathbb{R}^d} 6\ e^{-\psi^{l,\eta}(y)}\ dy.$$

This completes the proof.

■

5.2 Probability of large chains

Recall that two particles interact instantaneously only if the distance between their centers is smaller than R, the range of the potential φ. But more generally, a particle can have an influence on several others during any small time interval. To modelize this, we introduce the notion of $(R+\varepsilon)$-chain of particles.

Definition 5.3 *Let $x \in \mathcal{A}$ and $\varepsilon > 0$. Each subset $\{x_1, \cdots, x_n\}$ of x verifying $|x_1 - x_2| \leqslant R + \varepsilon, \cdots, |x_{n-1} - x_n| \leqslant R + \varepsilon$ is called an $(R+\varepsilon)$-chain of particles of x.*

Now, let us fix $K \in \mathbb{N}^*$, $M \in \mathbb{N}^*$ and $\varepsilon > 0$ and let $Ch(K, M, R+\varepsilon)$ denote the event that there exists an $(R+\varepsilon)$-chain of M particles with one end inside of $B(0, K)$, that is :

$$\begin{aligned}&Ch(K, M, R+\varepsilon)\\&= \{x \in \mathcal{A},\ \exists \{x_1, \cdots, x_M\} \text{ subset of } x,\ |x_1| < K \text{ and } |x_1 - x_2| \leqslant R+\varepsilon, \cdots, |x_{M-1} - x_M| \leqslant R+\varepsilon\}\end{aligned}$$

Our aim here is to estimate the $\mu_z^{l,\eta}$-probability that such a chain exists.

Proposition 5.4 *For every $M \in \mathbb{N}^*$, $K \in \mathbb{R}^+$ and $\varepsilon > 0$, and for every $l \in \mathbb{N}^*$ and $\eta \in \mathcal{A}$, we have :*

$$\begin{aligned}&\nu_n^{l,\eta}(Ch(K, M, R+\varepsilon))\\&\leqslant \frac{n!}{(n-M)!}\ e^{-2M\overline{N}\underline{\varphi}}\ \Big(((R+\varepsilon)^d - r^d)|B(0,1)| \Big)^{M-1}\ \nu_{n-M}^{l,\eta}((\mathbb{R}^d)^{n-M})\ \int_{B(0,K)} e^{-\psi^{l,\eta}(y)} dy\end{aligned}$$

and

$$\mu_z^{l,\eta}\left(\bigcup_{K=1}^{+\infty} Ch(K, M, R+\varepsilon) \right) \quad \leqslant \quad \Big(z\ |B(0,1)|\ e^{-2\overline{N}\underline{\varphi}}\ ((R+\varepsilon)^d - r^d) \Big)^{M-1}$$

From this proposition, we easily deduce the following corollary used in section 5.3.

Corollary 5.5 *There exists a critical activity z_c given by*

$$z_c = \frac{\exp(2\overline{N}\underline{\varphi})}{(R^d - r^d)\ |B(0,1)|}$$

such that for each positive z, each $\varepsilon \in]0,1[$ and each $M \in \mathbb{N}^*$

$$\sup_{l \in \mathbb{N}^*}\ \sup_{\eta \in \mathcal{A}}\ \mu_z^{l,\eta}\left(\bigcup_{K=1}^{+\infty} Ch(K, M, R+\varepsilon) \right) \leqslant \left(\frac{z}{z_c}\ \frac{(R+\varepsilon)^d - r^d}{R^d - r^d} \right)^{M-1}$$

Proof of prop 5.4

Each configuration in $(\mathbb{R}^d)^n \cap Ch(K, M, R+\varepsilon)$ has exactly $\frac{n!}{(n-M)!}$ representants in $(\mathbb{R}^d)^n$ such that $(x_{n-M+1}, \dots, x_n)$ is a fixed M-uple verifying $|x_{n-M+1}| < K$ and $|x_{n-M+1} - x_{n-M+2}| \leqslant R + \varepsilon, \cdots, |x_{n-1} - x_n| \leqslant R + \varepsilon$. In order to fix the representant of the configuration, we demand that $(x_{n-M+1}, \dots, x_n) \in \mathcal{O}$, that is for $n - M + 1 \leqslant i \leqslant n$ one has $|x_i - x_{i+1}| = \min\{|x_i - x_j|; i < j \leqslant n\}$. This fix the labelling of the points in the chain, except for the (negligible) set of configurations containing two points which are exactly at the same distance of a third one. Since $\beta_n^{l,\eta}(x_1, \dots, x_n)$ and $\mathbb{1}_{\mathcal{A}}(x_1, \dots, x_n)$ do not change by permutation of the x_i's, this leads to :

$$\begin{aligned}&\nu_n^{l,\eta}(Ch(K, M, R+\varepsilon))\\&\quad = \frac{n!}{(n-M)!} \int_{(\mathbb{R}^d)^n} \mathbb{1}_{|x_{n-M+1}| < K} \prod_{i=n-M+1}^{n-1} \mathbb{1}_{|x_i - x_{i+1}| \leqslant R+\varepsilon}\ \mathbb{1}_{\mathcal{O}}(x_{n-M+1}, \dots, x_n)\\&\qquad\qquad \mathbb{1}_{\mathcal{A}}(x_1, \dots, x_n)\ e^{-\beta_n^{l,\eta}(x_1, \dots, x_n)} dx_1 \cdots dx_n\end{aligned}$$

We use again inequality (7) established in the proof of lemma 5.1 :

$$\beta_n^{l,\eta}(x_1,\dots,x_n)\geqslant 2\overline{N}\underline{\varphi}+\beta_{n-1}^{l,\eta}(x_1,\dots,x_{n-1})$$

Reamrking that

$$\mathbb{1}_{\mathcal{A}}(x_1,\dots,x_n)\leqslant \mathbb{1}_{|x_n-x_{n-1}|\geqslant r}\ \mathbb{1}_{\mathcal{A}}(x_1,\dots,x_{n-1})\ ,$$

using again inequality (7) and integrating with respect to x_n we obtain :

$$\begin{aligned}
&\nu_n^{l,\eta}(Ch(K,M,R+\varepsilon))\\
&\leqslant n\frac{(n-1)!}{((n-1)-(M-1))!}\int_{(\mathbb{R}^d)^n}\mathbb{1}_{|x_{n-M+1}|<K}\prod_{i=n-M+1}^{n-2}\mathbb{1}_{r\leqslant|x_i-x_{i+1}|\leqslant R+\varepsilon}\ \mathbb{1}_{\mathcal{O}}(x_{n-M+1},\dots,x_{n-1})\\
&\qquad\qquad \mathbb{1}_{\mathcal{A}}(x_1,\dots,x_{n-1})\ \mathbb{1}_{r\leqslant|x_n-x_{n-1}|\leqslant R+\varepsilon}e^{-2\overline{N}\underline{\varphi}}\ e^{-\beta_{n-1}^{l,\eta}(x_1,\dots,x_{n-1})}\ dx_1\cdots dx_n\\
&\leqslant n\ e^{-2\overline{N}\underline{\varphi}}\Big(((R+\varepsilon)^d-r^d)|B(0,1)|\Big)\,\nu_{n-1}^{l,\eta}(Ch(K,M-1,R+\varepsilon))\ .
\end{aligned}$$

By iteration in n and M, we obtain for $n\geqslant M$ (which is the only interesting case, since there always are less particles in the chain than in the whole space) :

$$\begin{aligned}
&\nu_n^{l,\eta}(Ch(K,M,R+\varepsilon))\\
&\leqslant \frac{n!}{(n-M+1)!}e^{-2(M-1)\overline{N}\underline{\varphi}}\Big(((R+\varepsilon)^d-r^d)|B(0,1)|\Big)^{M-1}\ \nu_{n-M+1}^{l,\eta}(Ch(K,1,R+\varepsilon))
\end{aligned}$$

Using inequality (7) and the same idea as in the proof of inequality (8)

$$\begin{aligned}
&\nu_{n-M+1}^{l,\eta}(Ch(K,1,R+\varepsilon))\\
&=\nu_{n-M+1}^{l,\eta}(x\cap B(0,K)\neq\emptyset)\\
&\leqslant (n-M+1)\int_{B(0,K)}\int_{(\mathbb{R}^d)^{n-M}}\mathbb{1}_{\mathcal{A}}(y)\ e^{-\beta_{n-M+1}^{l,\eta}(y)}\ dy_{n-M+1}\cdots dy_1\\
&\leqslant (n-M+1)\ e^{-2\overline{N}\underline{\varphi}}\ \nu_{n-M}^{l,\eta}((\mathbb{R}^d)^{n-M})\ \int_{B(0,K)}e^{-\psi^{l,\eta}(y)}\ dy
\end{aligned}$$

Thus

$$\begin{aligned}
&\nu_n^{l,\eta}(Ch(K,M,R+\varepsilon))\\
&\leqslant \frac{n!}{(n-M)!}\ e^{-2M\overline{N}\underline{\varphi}}\ \Big(((R+\varepsilon)^d-r^d)|B(0,1)|\Big)^{M-1}\ \nu_{n-M}^{l,\eta}((\mathbb{R}^d)^{n-M})\ \int_{B(0,K)}e^{-\psi^{l,\eta}(y)}\ dy
\end{aligned}$$

By definition of $\mu_z^{l,\eta}$ (cf (4)) we have

$$\begin{aligned}
\mu_z^{l,\eta}(Ch(K,M,R+\varepsilon)) &= \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}}\sum_{n=0}^{+\infty}\frac{z^n}{n!}\ \nu_n^{l,\eta}(Ch(K,M,R+\varepsilon))\\
&= \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}}\sum_{n\geqslant M}\frac{z^n}{n!}\ \nu_n^{l,\eta}(Ch(K,M,R+\varepsilon)) \qquad (10)
\end{aligned}$$

Using this, the above inequality and iterating the result on M, we obtain :

$$\begin{aligned}
&\mu_z^{l,\eta}(Ch(K,M,R+\varepsilon))\\
&\leqslant e^{-2\overline{N}\underline{\varphi}}\ ((R+\varepsilon)^d-r^d)\ |B(0,1)|\ \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}}\ z\sum_{n\geqslant M}\frac{z^{n-1}}{(n-1)!}\ \nu_{n-1}^{l,\eta}(Ch(K,M-1,R+\varepsilon)\\
&\leqslant z\ e^{-2\overline{N}\underline{\varphi}}\ ((R+\varepsilon)^d-r^d)\ |B(0,1)|\ \mu_z^{l,\eta}(Ch(K,M-1,R+\varepsilon))\\
&\leqslant \Big(z\ e^{-2\overline{N}\underline{\varphi}}\ ((R+\varepsilon)^d-r^d)|B(0,1)|\Big)^{M-1}\ \mu_z^{l,\eta}(Ch(K,1,R+\varepsilon))\\
&\leqslant \Big(z\ e^{-2\overline{N}\underline{\varphi}}\ ((R+\varepsilon)^d-r^d)|B(0,1)|\Big)^{M-1}\ .
\end{aligned}$$

Since the event $Ch(K,M,R+\varepsilon)$ increases as a function of K

$$\mu_z^{l,\eta}\left(\bigcup_{K=1}^{+\infty} Ch(K,M,R+\varepsilon)\right) = \lim_{K\to+\infty}\mu_z^{l,\eta}(Ch(K,M,R+\varepsilon)) \leqslant \left(z\ |B(0,1)|\ \exp(-2\overline{N}\underline{\varphi})\ ((R+\varepsilon)^d - r^d)\right)^{M-1}.$$

■

5.3 Probability of too high interaction between particles

Let $\mathcal{B}(m,a,\varepsilon)$ denote the set of "Bad trajectories", that is the event that either a particle has a high oscillation in a small time interval or belongs to a large chain of interacting particles :

$$\forall m\in\mathbb{N}^* \quad \forall a\geqslant 1 \quad \forall\varepsilon>0 \qquad \mathcal{B}(m,a,\varepsilon) = \tilde{\mathcal{B}}(m,a,\varepsilon)\cup\tilde{\tilde{\mathcal{B}}}(m,\varepsilon)$$

where

$$\tilde{\mathcal{B}}(m,a,\varepsilon) = \left\{X\in\mathcal{C}([0,1],\mathcal{A}),\ \exists i,\ w(X_i,\frac{1}{m}) > \frac{\varepsilon}{4} \text{ and } \exists t\leqslant 1,\ |X_i(t)|\leqslant a+2m^2\right\}$$

and

$$\tilde{\tilde{\mathcal{B}}}(m,\varepsilon) = \left\{ X\in\mathcal{C}([0,1],\mathcal{A}),\ \begin{array}{c}\exists k\in\{0,\dots,m-1\},\ \text{ there exists an}\\ (R+\varepsilon)-\text{chain of particles of } X(\frac{k}{m})\\ \text{with diameter greater than } m-R-\varepsilon\end{array}\right\}$$

Let us remark that $a\mapsto\tilde{\mathcal{B}}(m,a,\varepsilon)$ is non-decreasing but $\tilde{\mathcal{B}}(m,a,\varepsilon)$ is not monotone as a function of ε. Our aim in this section is to estimate the probability of $\mathcal{B}(m,a,\varepsilon)$ under $Q_z^{l,\eta}$.

Proposition 5.6 *For each $m\in\mathbb{N}^*$ and each $a\geqslant 1$:*

$$\sup_{l\in\mathbb{N}^*}\sup_{\eta\in\mathcal{A}} Q_z^{l,\eta}(\tilde{\mathcal{B}}(m,a,\varepsilon)) \leqslant z\ C'_d\ e^{-2\overline{N}\underline{\varphi}}\ a^d\ m^{2d}\ \exp\left(-\frac{\varepsilon^2}{96}m\right)$$

where the constant C'_d only depends on dimension d. One also has, for each $m\in\mathbb{N}^*$:

$$\sup_{l\in\mathbb{N}^*}\sup_{\eta\in\mathcal{A}} Q_z^{l,\eta}\left(\tilde{\tilde{\mathcal{B}}}(m,\varepsilon)\right) \leqslant m\ \left(z\ |B(0,1)|\ \exp(-2\overline{N}\underline{\varphi})\ ((R+\varepsilon)^d-r^d)\right)^{[\frac{m}{R+\varepsilon}]}$$

If $z<z_c$ and ε small enough (depending on z), this implies that the left hand side decreases exponentially fast as a function of m.

Proof of proposition 5.6

We first estimate the probability

$$Q_z^{l,\eta}(\tilde{\mathcal{B}}(m,a,\varepsilon)) = Q_z^{l,\eta}\left(\exists i,\ w(X_i,\frac{1}{m}) > \frac{\varepsilon}{4} \text{ and } \exists t\leqslant 1,\ |X_i(t)|\leqslant a+2m^2\right)$$

It is clearly smaller than

$$\leqslant Q_z^{l,\eta}\left(\exists i,\ w(X_i,\frac{1}{m}) > \frac{\varepsilon}{4} \text{ and } |X_i(0)|\leqslant a+3m^2\right) + Q_z^{l,\eta}\left(\exists i,\ |X_i(0)| > a+3m^2 \text{ and } \exists t\leqslant 1,\ |X_i(t)|\leqslant a+2m^2\right)$$

But the second term of the sum is smaller than

$$\sum_{j=1}^{+\infty} Q_z^{l,\eta}\left(\exists i,\ a+(2+j)m^2 < |X_i(0)|\leqslant a+(3+j)m^2 \text{ and } w(X_i,1) > jm^2\right)$$

Thus using proposition 5.2, we obtain :

$$
\begin{aligned}
&Q_z^{l,\eta}(\tilde{\mathcal{B}}(m,a,\varepsilon))\\
&\leqslant Q_z^{l,\eta}\left(\mathcal{F}m(a+3m^2,\frac{\varepsilon}{4},\frac{1}{m})\right) + \sum_{j=1}^{+\infty} Q_z^{l,\eta}\left(\mathcal{F}m(a+(3+j)m^2,jm^2,1)\right)\\
&\leqslant z\ C_d\ e^{-2\overline{N}\underline{\varphi}}\ m\ \exp\left(-\frac{\varepsilon^2}{96}m\right)\ (a+3m^2)^d\\
&\quad + z\ C_d\ e^{-2\overline{N}\underline{\varphi}} \sum_{j=1}^{+\infty} \exp\left(-\frac{j^2m^4}{6}\right)\ (a+(3+j)m^2)^d
\end{aligned}
$$

We now use the multinomial formula $(a+3m^2)^d = (a+m^2+m^2+m^2)^d \leqslant 4^d a^d m^{2d}$. Similarly $(a+(3+j)m^2)^d \leqslant (j+4)^d a^d m^{2d} \leqslant 2^d j^d 4^d a^d m^{2d}$. This leads to :

$$
Q_z^{l,\eta}(\tilde{\mathcal{B}}(m,a,\varepsilon)) \leqslant z\ C_d\ e^{-2\overline{N}\underline{\varphi}}\ 8^d\ a^d\ m^{2d}\left(\exp\left(-\frac{\varepsilon^2}{96}m\right) + \sum_{j=1}^{+\infty}\exp\left(-\frac{j^2m^4}{6}\right)\ j^d\right)
$$

Since $j^2m^4/6 \geqslant (j^2+m^4)/12$ and $\frac{m^4}{12} \geqslant \frac{\varepsilon^2}{96}m$ for $\varepsilon \leqslant 1$ one has :

$$
\begin{aligned}
&Q_z^{l,\eta}(\tilde{\mathcal{B}}(m,a,\varepsilon))\\
&\leqslant z\ C_d\ e^{-2\overline{N}\underline{\varphi}}\ 8^d\ a^d\ m^{2d}\left(\exp\left(-\frac{\varepsilon^2}{96}m\right) + \sum_{j=1}^{+\infty} j^d \exp\left(-\frac{j^2}{12}\right)\ \exp\left(-\frac{m^4}{12}\right)\right)\\
&\leqslant z\ C_d\ e^{-2\overline{N}\underline{\varphi}}\ 8^d\ a^d\ m^{2d}\exp\left(-\frac{\varepsilon^2}{96}m\right)\left(1+\sum_{j=1}^{+\infty} j^d\exp\left(-\frac{j^2}{12}\right)\right)\ .
\end{aligned}
$$

Defining the constant C'_d by

$$
C'_d = C_d\ 8^d\left(1+\sum_{j=1}^{+\infty} j^d\exp\left(-\frac{j^2}{12}\right)\right) \quad < \quad +\infty
$$

we obtain

$$
Q_z^{l,\eta}(\tilde{\mathcal{B}}(m,a,\varepsilon)) \leqslant z\ C'_d\ e^{-2\overline{N}\underline{\varphi}}\ a^d\ m^{2d}\ \exp\left(-\frac{\varepsilon^2}{96}m\right).
$$

We now have to find a similar estimate for

$$
Q_z^{l,\eta}\left(\tilde{\tilde{\mathcal{B}}}(m,\varepsilon)\right) = Q_z^{l,\eta}\left(\exists k\in\{0,\ldots,m-1\},\ \begin{array}{l}\text{there exists an } (R+\varepsilon)-\text{chain}\\ \text{of particles of } X(\frac{k}{m}) \text{ with}\\ \text{diameter greater than } m-R-\varepsilon\end{array}\right).
$$

Thanks to the stationarity of $Q_z^{l,\eta}$, this probability is smaller than

$$
\leqslant \sum_{k=0}^{m-1}\mu_z^{l,\eta}\left(x\in\mathcal{A},\ \begin{array}{l}\text{there exists an } (R+\varepsilon)-\text{chain}\\ \text{of particles of } x \text{ with}\\ \text{diameter greater than } m-R-\varepsilon)\end{array}\right).
$$

It is necessary to have at least $[\frac{m}{R+\varepsilon}]+1$ particles ($[x]$ denotes the integer part of x) to construct a chain of length greater than $m-R-\varepsilon$ with every particle at a distance smaller than $(R+\varepsilon)$ from its neighbors. Thus the above quantity is smaller than

$$
\leqslant\ m\ \mu_z^{l,\eta}\left(\bigcup_{K=1}^{+\infty} Ch(K,[\frac{m}{R+\varepsilon}]+1,R+\varepsilon)\right).
$$

Due to proposition 5.4, this is bounded from above by

$$
\leqslant\ m\ \left(z\ |B(0,1)|\ e^{-2\overline{N}\underline{\varphi}}\ ((R+\varepsilon)^d - r^d)\right)^{[\frac{m}{R+\varepsilon}]}.
$$

■

6 Convergence of the approximations

The aim of this section is to prove the convergence of the sequence $(X^{l,x})_l$ to a limit process $X^{\infty,x}$. We shall check in the next section that X^∞ is a solution of $(\mathcal{E})$.

Through this whole section, μ denotes a fixed element of $\mathcal{G}(z)$ with $z < z_c$.

We fix $x \in \mathcal{A}$.

As usual for infinite-dimensional stochastic equations, we study $(\mathcal{E}^l)$ for each ω in a set $\Omega_0^x \subset \Omega$ and prove that the set $\Omega \setminus \Omega_0^x$ is negligible.

For each $\rho \in \mathbb{N}^*$ and $l \geqslant \rho - r + 1$, let $m(\rho,l)$ and $a(\rho,l)$ denote the following integers :

$$m(\rho,l) = \left[\sqrt{l-\rho-r}\right] - 1 \qquad \text{and} \qquad a(\rho,l) = \rho + m(\rho,l). \tag{11}$$

Remark that

$$a(\rho,l) + m(\rho,l)^2 + 1 < l - r \tag{12}$$

and

$$\forall C > 0 \qquad \sum_l a(\rho,l)^d \; m(\rho,l)^{2d} \; e^{-Cm(\rho,l)} < +\infty \; . \tag{13}$$

Let $1/\mathbb{N}$ denote the set $\{1, 1/2, 1/3, \cdots\}$ of real numbers ε such that $1/\varepsilon \in \mathbb{N}$. Let us now define the set Ω_0^x as follows :

$$\begin{aligned} \Omega_0^x &= \{\omega \in \Omega \text{ s.t. } \exists \varepsilon_0 \in 1/\mathbb{N},\ \forall \varepsilon \leqslant \varepsilon_0\ \forall \rho \in \mathbb{N}^*\ \exists l_0 \in \mathbb{N}^*,\ \forall l \geqslant l_0 \\ &\qquad X^{l,x}(\omega,\cdot) \notin \mathcal{B}\left(m(\rho,l), a(\rho,l), \varepsilon\right) \text{ and } X^{l+1,x}(\omega,\cdot) \notin \mathcal{B}\left(m(\rho,l), a(\rho,l), \varepsilon\right)\Big\} \qquad (14) \\ &= \liminf_{\varepsilon \to 0} \bigcap_{\rho \in \mathbb{N}^*} \liminf_{l \to +\infty} \Big\{X^{l,x} \notin \mathcal{B}\left(m(\rho,l), a(\rho,l), \varepsilon\right)\Big\} \cap \Big\{X^{l+1,x} \notin \mathcal{B}\left(m(\rho,l), a(\rho,l), \varepsilon\right)\Big\}. \end{aligned}$$

The set Ω_0^x satisfies the following properties :

Proposition 6.1 *For each Gibbs measure $\mu \in \mathcal{G}(z)$ with $z < z_c$ one has $\int_{\mathcal{M}} P(\Omega_0^x)\ d\mu(x) = 1$ that is, for μ-almost each x in $\mathcal{A}$, $P(\Omega_0^x) = 1$. This means that for $\underline{\mathcal{A}} = \{x \in \mathcal{A},\ P(\Omega_0^x) = 1\}$, one has*

$$\forall z < z_c \quad \forall \mu \in \mathcal{G}(z) \quad \mu(\underline{\mathcal{A}}) = 1.$$

Proposition 6.2

(i) *For every $x \in \mathcal{A}$, every ω in Ω_0^x and every $i \in \mathbb{N}$, the sequence $(X_i^{l,x}(\omega,t), L_{ij}^{l,x}(\omega,t), j \in \mathbb{N}, t \in [0,1])_{l \in \mathbb{N}^*}$ of elements of $\mathcal{C}([0,1], \mathbb{R}^d \times \mathbb{R}_+^{\mathbb{N}})$ converges (in the sense of uniform convergence of continuous paths) to a limit denoted by $(X_i^{\infty,x}(\omega,t), L_{ij}^{\infty,x}(\omega,t), j \in \mathbb{N}, t \in [0,1])$.*

Moreover, this sequence is stationary :

$$\begin{aligned} &\forall x \in \mathcal{A} \quad \forall \omega \in \Omega_0^x \quad \forall i \in \mathbb{N} \quad \exists l_0, \quad \forall l \geqslant l_0 \\ &X_i^{l,x}(\omega,\cdot) = X_i^{\infty,x}(\omega,\cdot) \text{ on } [0,1] \quad \text{and} \quad \forall j \in \mathbb{N}\ L_{ij}^{l,x}(\omega,\cdot) = L_{ij}^{\infty,x}(\omega,\cdot) \text{ on } [0,1] \end{aligned}$$

(ii) *Furthermore, the convergence takes place in $\mathcal{C}([0,1], \mathcal{M})$, i.e.*

$$\forall x \in \mathcal{A} \quad \forall \omega \in \Omega_0^x \qquad X^{\infty,x}(\omega,\cdot) = \lim_{l \to +\infty} X^{l,x}(\omega,\cdot) \text{ on } [0,1]$$

(iii) *Since $\int P(\Omega_0^x) d\mu(x) = 1$ for $\mu \in \mathcal{G}(z)$ avec $z < z_c$, the sequence of processes $(X^l)_{l \in \mathbb{N}^*} \in \mathcal{C}([0,1], \mathcal{M})$ starting from μ converges indeed in distribution to the process $X^\infty \in \mathcal{C}([0,1], \mathcal{A})$ starting from μ.*

Proof of proposition 6.1

Recall that $\mu \in \mathcal{G}(z)$ is fixed. We want to prove that $\int_{\mathcal{A}} P(\Omega \smallsetminus \Omega_0^x)\ d\mu(x) = 0$

By definition of Ω_0^x :

$$P(\Omega \smallsetminus \Omega_0^x)$$
$$= P\left(\forall \varepsilon_0 \in 1/\mathbb{N},\ \exists \varepsilon \leqslant \varepsilon_0\ \exists \rho \in \mathbb{N}^*\ \forall l_0 \in \mathbb{N}^*\ \exists l \geqslant l_0,\ \begin{array}{l} X^{l,x} \in \mathcal{B}\left(m(\rho,l), a(\rho,l), \varepsilon\right) \text{ or} \\ X^{l,x} \in \mathcal{B}\left(m(\rho,l-1), a(\rho,l-1), \varepsilon\right) \end{array} \right)$$

For each $\varepsilon_0 \in 1/\mathbb{N}$, this is smaller than

$$\leqslant \sum_{\varepsilon \leqslant \varepsilon_0} \sum_{\rho \in \mathbb{N}^*} P\Big(\limsup_{l \to +\infty} \Big\{ X^{l,x} \in \mathcal{B}\left(m(\rho,l), a(\rho,l), \varepsilon\right) \cup \mathcal{B}\left(m(\rho,l-1), a(\rho,l-1), \varepsilon\right) \Big\} \Big)$$

Thanks to Borel-Cantelli lemma, this vanishes as soon as there exists $\varepsilon_0 \in 1/\mathbb{N}$ such that

$$\forall \varepsilon \leqslant \varepsilon_0\ \forall \rho \in \mathbb{N}^* \quad \sum_{l=\rho+2}^{+\infty} \int_{\mathcal{A}} P\left(X^{l,x} \in \mathcal{B}\left(m(\rho,l), a(\rho,l), \varepsilon\right) \cup \mathcal{B}\left(m(\rho,l-1), a(\rho,l-1), \varepsilon\right) \right) d\mu(x) < +\infty\ . \tag{15}$$

We shall show (step 1) that for each $l \in \mathbb{N}^*$ and for $\Lambda = [-l,l]^d$, the following inequalities hold :

$$\begin{aligned} &\sup_{\Theta \in \mathcal{F}_1} \left| \int_{\mathcal{A}} P(X^{l,x} \in \Theta) d\mu(x) - \int_{\mathcal{A}} Q_z^{l,\eta}(\Theta) d\mu(\eta) \right| \\ &\qquad \leqslant \int_{\mathcal{A}} \sup_{\|f\| \leqslant 1} \left| \int_{\mathcal{A}} f(x)\ d\mu(x|\eta_{([-l,l]^d)^c}) - \int_{\mathcal{A}} f(x)\ d\mu_z^{l,\eta}(x) \right|\ d\mu(\eta) \\ &\text{and} \quad \forall \eta \in \mathcal{A} \quad \sup_{\|f\| \leqslant 1} \left| \int_{\mathcal{A}} f(x)\ d\mu(x|\eta_{([-l,l]^d)^c}) - \int_{\mathcal{A}} f(x)\ d\mu_z^{l,\eta}(x) \right| \leqslant 2 \left(1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{l,\eta}} \right) \end{aligned} \tag{16}$$

and (step 2) that

$$\forall \eta \in \mathcal{A} \quad 0 \leqslant 1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{l,\eta}} \leqslant z\ e^{-2\overline{N}\underline{\varphi}} \int_{\mathbb{R}^d} \mathbb{I}_{\psi^{l,\eta}(y)>0}\ \exp(-\psi^{l,\eta}(y))\ dy \tag{17}$$

Inequality (17) and assumption (2) on $\psi^{l,\eta}$ imply that

$$\sum_{l=1}^{+\infty} \int_{\mathcal{A}} \left(1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{l,\eta}} \right) d\mu(\eta) < +\infty$$

Then for each ρ and l fixed, we choose

$$\Theta = \mathcal{B}\left(m(\rho,l), a(\rho,l), \varepsilon\right)\ \cup\ \mathcal{B}\left(m(\rho,l-1), a(\rho,l-1), \varepsilon\right)$$

Thanks to (16) and (17), in order to prove (15), we only have to prove that

$$\begin{aligned} &\exists \varepsilon_0 \in 1/\mathbb{N} \quad \forall \varepsilon < \varepsilon_0 \quad \forall \rho \in \mathbb{N}^* \\ &\sum_{l=\rho+2}^{+\infty} \int Q_z^{l,\eta} \left(\tilde{\mathcal{B}}(m(\rho,l), a(\rho,l), \varepsilon) \cup \tilde{\mathcal{B}}(m(\rho,l-1), a(\rho,l-1), \varepsilon) \right) d\mu(\eta) \\ &\quad + \sum_{l=\rho+2}^{+\infty} \int Q_z^{l,\eta} \left(\tilde{\tilde{\mathcal{B}}}(m(\rho,l), \varepsilon) \cup \tilde{\tilde{\mathcal{B}}}(m(\rho,l-1), \varepsilon) \right) d\mu(\eta) < +\infty \end{aligned}$$

By proposition 5.6, this is smaller than

$$\int z\ C'_d\ e^{-2\overline{N}\underline{\varphi}}\ \sum_{l=\rho+2}^{+\infty} a(\rho,l)^d\ m(\rho,l)^{2d}\ \exp\Big(-\frac{\varepsilon^2}{96}m(\rho,l)\Big)$$
$$+a(\rho,l-1)^d\ m(\rho,l-1)^{2d}\ \exp\Big(-\frac{\varepsilon^2}{96}m(\rho,l-1)\Big)\,d\mu(\eta)$$
$$+\int \sum_{l=\rho+2}^{+\infty} m(\rho,l)\ \Big(z\ |B(0,1)|\ e^{-2\overline{N}\underline{\varphi}}\ ((R+\varepsilon)^d-r^d)\Big)^{[\frac{m(\rho,l)}{R+\varepsilon}]}$$
$$+m(\rho,l-1)\ \Big(z\ |B(0,1)|\ e^{-2\overline{N}\underline{\varphi}}\ ((R+\varepsilon)^d-r^d)\Big)^{[\frac{m(\rho,l-1)}{R+\varepsilon}]}\ d\mu(\eta).$$

Since the factors in the integrals do not depend on η and since the sums over l only differ by their first terms, we only have to prove that

$$\begin{aligned}&\exists\varepsilon_0\in 1/\mathbb{N}\quad \forall\varepsilon<\varepsilon_0\quad \forall\rho\in\mathbb{N}^*\\ &\qquad\sum_{l=\rho+2}^{+\infty} a(\rho,l)^d\ m(\rho,l)^{2d}\ \exp\Big(-\frac{\varepsilon^2}{96}m(\rho,l)\Big)<+\infty\\ &\text{and}\qquad \sum_{l=\rho+2}^{+\infty} m(\rho,l)\ \Big(z\ |B(0,1)|\ e^{-2\overline{N}\underline{\varphi}}\ ((R+\varepsilon)^d-r^d)\Big)^{[\frac{m(\rho,l)}{R+\varepsilon}]}<+\infty.\end{aligned}$$

The first series converges for each $\varepsilon\in 1/\mathbb{N}$ and each $\rho\in\mathbb{N}^*$ thanks to (13). The second one also converges thanks to (13) again, as soon as $\Big(z\ |B(0,1)|\ e^{-2\overline{N}\underline{\varphi}}\ ((R+\varepsilon_0)^d-r^d)\Big)<1$, which is true for ε_0 small enough when $z<z_c$.

It remains to prove (16) and (17).

Step 1 : Proof of (16)

Let us fix $l\in\mathbb{N}^*$ and $\Lambda=[-l,l]^d$. For each event Θ on $\mathcal{C}([0,1])$, by definition of $Q_z^{l,\eta}$:

$$\begin{aligned}&\int_{\mathcal{A}} P(X^{l,x}\in\Theta)d\mu(x)-\int_{\mathcal{A}} Q_z^{l,\eta}(\Theta)d\mu(\eta)\\ &\leqslant \int_{\mathcal{A}}\int_{\mathcal{A}} P(X^{l,x\eta_{\Lambda^c}}\in\Theta)\ d\mu(x|\eta_{\Lambda^c})\ d\mu(\eta)-\int_{\mathcal{A}}\int_{\mathcal{A}} P(X^{l,\eta,\mathrm{Card(x)}}(x,\cdot)\in\Theta)\ d\mu_z^{l,\eta}(x)\ d\mu(\eta)\end{aligned}$$

If $x\eta\in\mathcal{A}$ then $P(X^{l,\eta,\mathrm{Card(x)}}(x,\cdot)\in\Theta)=P(X^{l,x,\eta_{\Lambda^c}}\in\Theta)$ i.e. the integrated functions are equal, and since they are bounded by 1, we obtain :

$$\left|\int_{\mathcal{A}} P(X^{l,x}\in\Theta)d\mu(x)-\int_{\mathcal{A}} Q_z^{l,\eta}(\Theta)d\mu(\eta)\right|\leqslant\int_{\mathcal{A}}\sup_{\|f\|\leqslant 1}\left|\int_{\mathcal{A}} f(x)\ d\mu(x|\eta_{\Lambda^c})-\int_{\mathcal{A}} f(x)\ d\mu_z^{l,\eta}(x)\right|\ d\mu(\eta)$$

Since $\mu\in\mathcal{G}(z)$, using the conditional density of μ with respect to π^z and the definition of $\mu_z^{l,\eta}$, one has for each $f:\mathcal{A}\to\mathbb{R}$ bounded by 1 :

$$\begin{aligned}&\left|\int_{\mathcal{A}} f(x)\ d\mu(x|\eta_{\Lambda^c})-\int_{\mathcal{A}} f(x)\ d\mu_z^{l,\eta}(x)\right|\\ &=\left|\frac{e^{-z|\Lambda|}}{Z_z^{\Lambda,\eta}}\left(f(\eta_{\Lambda^c})+\sum_{n=1}^{+\infty}\frac{z^n}{n!}\int_{\Lambda^n} f(y\eta_{\Lambda^c})\ \mathbb{1}_{\mathcal{A}}(y\eta_{\Lambda^c})\ \exp\Big(-\sum_{1\leqslant i<j\leqslant n}\varphi(y_i-y_j)-\sum_{\substack{1\leqslant i\leqslant n\\ \eta_j\in\Lambda^c}}\varphi(y_i-\eta_j)\Big)dy\right)\right.\\ &\qquad\left.-\frac{e^{-z2^dl^d}}{Z_z^{l,\eta}}\left(f(\eta_{\Lambda^c})+\sum_{n=1}^{+\infty}\frac{z^n}{n!}\int_{(\mathbb{R}^d)^n} f(y\eta_{\Lambda^c})\ e^{-\beta_n^{l,\eta}(y)}\ \mathbb{1}_{\mathcal{A}}(y)\ dy\right)\right|\end{aligned}$$

Note that $\beta_n^{l,\eta}(y) = \sum_{1\leqslant i<j\leqslant n} \varphi(y_i - y_j) + \sum_{\substack{1\leqslant i\leqslant n\\ \eta_j\in\Lambda^c}} \varphi(y_i-\eta_j)$ for the $y\in\Lambda^n$ verifying $y\eta_{\Lambda^c}\in\mathcal{A}$, because $\psi^{l,\eta}(y_i)=0$ for each i in this case. Thus the above quantity is equal to

$$\left| f(\eta_{\Lambda^c}) \left(\frac{e^{-z|\Lambda|}}{Z_z^{\Lambda,\eta}} - \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}} \right) + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \left(\frac{e^{-z|\Lambda|}}{Z_z^{\Lambda,\eta}} - \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}} \right) \int_{\Lambda^n} f(y\eta_{\Lambda^c})\ \mathbb{1}_{\mathcal{A}}(y\eta_{\Lambda^c})\ e^{-\beta_n^{l,\eta}(y)}\ dy \right.$$
$$\left. - \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}} \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} f(y\eta_{\Lambda^c})\ e^{-\beta_n^{l,\eta}(y)}\ (\mathbb{1}_{\mathcal{A}}(y) - \mathbb{1}_{\mathcal{A}}(y\eta_{\Lambda^c})\ \mathbb{1}_{\Lambda^n}(y))\ dy \right|$$

Recall $e^{-z|\Lambda|} = e^{-z2^dl^d}$ and

$$Z_z^{\Lambda,\eta} = e^{-z|\Lambda|} \left(1 + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{\Lambda^n} \mathbb{1}_{\mathcal{A}}(y\eta_{\Lambda^c})\ e^{-\beta_n^{l,\eta}(y)}\ dy \right)$$
$$\leqslant e^{-z2^dl^d} \left(1 + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} \mathbb{1}_{\mathcal{A}}(y)\ e^{-\beta_n^{l,\eta}(y)}\ dy \right) = Z_z^{l,\eta}\ .$$

Since f is bounded by 1, we then obtain :

$$\left| \int_{\mathcal{A}} f(x)\ d\mu(x|\eta_{\Lambda^c}) - \int_{\mathcal{A}} f(x)\ d\mu_z^{l,\eta}(x) \right|$$
$$\leqslant \left| \frac{e^{-z|\Lambda|}}{Z_z^{\Lambda,\eta}} - \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}} \right| e^{z|\Lambda|} Z_z^{\Lambda,\eta} + \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}} \left| e^{z2^dl^d} Z_z^{l,\eta} - e^{z|\Lambda|} Z_z^{\Lambda,\eta} \right| = 2 \left(1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{l,\eta}} \right)$$

and (16) is proven.

<u>Step 2 : Proof of (17)</u>

This final step of the proof of proposition 6.1 is straightforward, simply using the definitions of $Z_z^{l,\eta}$, $Z_z^{\Lambda,\eta}$ and $\psi^{l,\eta}$:

$$1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{l,\eta}} = \frac{1}{Z_z^{l,\eta}} (Z_z^{l,\eta} - Z_z^{\Lambda,\eta})$$
$$= \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} \mathbb{1}_{\mathcal{A}}(\xi_1,\ldots,\xi_n)\ e^{-\beta_n^{l,\eta}(\xi_1,\ldots,\xi_n)} \left(1 - \prod_{i=1}^n \mathbb{1}_{\Lambda - B(\eta_{\Lambda^c},r)}(\xi_i) \right) d\xi_1\cdots d\xi_n$$
$$\leqslant \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_{(\mathbb{R}^d)^n} \mathbb{1}_{\mathcal{A}}(\xi_1,\ldots,\xi_n)\ e^{-\beta_n^{l,\eta}(\xi_1,\ldots,\xi_n)} \left(\sum_{i=1}^n \mathbb{1}_{\psi^{l,\eta}(\xi_i)>0} \right) d\xi_1\cdots d\xi_n$$

and using inequality (7), this is

$$\leqslant \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}} \sum_{n=0}^{+\infty} \frac{z^n}{n!}\ n \int_{(\mathbb{R}^d)^n} \mathbb{1}_{\mathcal{A}}(\xi_1,\ldots,\xi_n)\ e^{-\psi^{l,\eta}(\xi_1)}\ e^{-2\overline{N}\underline{\varphi}}\ e^{-\beta_{n-1}^{l,\eta}(\xi_2,\ldots,\xi_n)} \mathbb{1}_{\psi^{l,\eta}(\xi_1)>0}\ d\xi_1\cdots d\xi_n$$
$$\leqslant \frac{e^{-z2^dl^d}}{Z_z^{l,\eta}} \sum_{n=1}^{+\infty} z\ \frac{z^{n-1}}{(n-1)!}\ \nu_{n-1}^{l,\eta}((\mathbb{R}^d)^{n-1})\ e^{-2\overline{N}\underline{\varphi}} \int_{\mathbb{R}^d} \mathbb{1}_{\psi^{l,\eta}(y)>0}\ e^{-\psi^{l,\eta}(y)}\ dy$$
$$\leqslant z\ e^{-2\overline{N}\underline{\varphi}} \int_{\mathbb{R}^d} \mathbb{1}_{\psi^{l,\eta}(y)>0}\ e^{-\psi^{l,\eta}(y)}\ dy$$

■

In order to prove proposition 6.2, we need the following lemma which states that, for nice trajectories only a finite number of other particles interacts with each fixed particles. Thus dynamics $(\mathcal{E})$ reduces to an infinite number of SDE involving only a finite random number of particles up to time 1.

Lemma 6.3 *Let us assume that the path $X \in \mathcal{C}([0,1])$ does not belong to $\mathcal{B}(m, \rho+m, \varepsilon)$ for some $\varepsilon \in 1/\mathbb{N}^*$, some $m \in \mathbb{N}^*$ and some $\rho \in \mathbb{N}^*$. For k in $\{0, 1, \cdots, m-1\}$ we define $J_k(X)$ as the set of indices $i \in \mathbb{N}$ such that either $|X_i(\frac{k}{m})| \leqslant \rho + m^2 - km + R + \varepsilon$ or $X_i(\frac{k}{m})$ belongs to some $(R+\varepsilon)$-chain of particles which intersects $B(0, \rho+m^2-km+R+\varepsilon)$. Then the following inclusions hold :*

$$\{i \in \mathbb{N}, |X_i(0)| \leqslant \rho\} \subset J_{m-1}(X) \subset \cdots \subset J_{k+1}(X) \subset J_k(X) \subset \cdots \subset J_0(X).$$

Particles of $J_k(X)$ stay around the origin in the following sense :

$$\forall i \in J_k(X) \quad \forall t \in [\frac{k}{m}, \frac{k+1}{m}] \qquad |X_i(t)| \leqslant \rho + m^2 + m + 1.$$

They are also far away from the others :

$$\forall i \in J_k(X) \quad \forall j \notin J_k(X) \quad \forall t \in [\frac{k}{m}, \frac{k+1}{m}] \qquad |X_i(t) - X_j(t)| > R + \frac{\varepsilon}{2}. \tag{18}$$

Proof of lemma 6.3

The set $J_k(X)$ is defined as the set of indices $i \in \mathbb{N}$ such that $X_i(\frac{k}{m})$ belongs to $B(0, \rho + m^2 - km + R + \varepsilon)$ or is connected to $B(0, \rho + m^2 - km + R + \varepsilon)$ by some $(R+\varepsilon)$-chain of particles of $X(\frac{k}{m})$; thus

$$\forall j \notin J_k(X) \qquad |X_j(\frac{k}{m})| > \rho + m^2 - km + R + \varepsilon$$

and

$$\forall i \in J_k(X) \quad \forall j \notin J_k(X) \qquad |X_i(\frac{k}{m}) - X_j(\frac{k}{m})| > R + \varepsilon$$

Since $X \notin \mathcal{B}(m, \rho+m, \varepsilon)$ then $X(\frac{k}{m})$ does not include any $(R+\varepsilon)$-chain of particles with diameter greater than $m - R - \varepsilon$:

$$\forall i \in J_k(X) \qquad |X_i(\frac{k}{m})| \leqslant (\rho + m^2 - km + R + \varepsilon) + (m - R - \varepsilon) = \rho + m^2 - (k-1)m \tag{19}$$

Again since $X \notin \mathcal{B}(m, \rho+m, \varepsilon)$, no particle of X entering $B(0, \rho+m+2m^2)$ moves for more than $\frac{\varepsilon}{4}$ during a time period of length $\frac{1}{m}$:

$$\forall i \in J_k(X) \quad \forall j \notin J_k(X) \quad \forall t \in [\frac{k}{m}, \frac{k+1}{m}] \qquad |X_i(t) - X_j(t)| > R + \frac{\varepsilon}{2}$$

and

$$\forall i \in J_k(X) \quad \forall t \in [\frac{k}{m}, \frac{k+1}{m}] \qquad |X_i(t)| \leqslant \rho + m^2 - (k-1)m + \frac{\varepsilon}{4} \leqslant \rho + m^2 + m + 1 \ .$$

Moreover

$$\forall j \notin J_k(X) \quad |X_j(\frac{k+1}{m})| > \rho + m^2 - km + \frac{3}{4}\varepsilon > \rho + m^2 - km \ ;$$

using (19) this leads to

$$j \notin J_k(X) \quad \Longrightarrow \quad j \notin J_{k+1}(X)$$

which implies the decreasing property of the sets $J_k(X)$.

Now, using once more the "slow motion" property of X, we see that

$$|X_i(0)| \leqslant \rho \quad \Longrightarrow \quad |X_i(1)| \leqslant \rho + \frac{\varepsilon}{4}m \leqslant \rho + m^2 - (m-1)m + R + \varepsilon \quad \Longrightarrow \quad i \in J_{m-1}(X)$$

and the proof is complete. ■

Proof of proposition 6.2 (i)

In this whole proof, $x \in \mathcal{A}$, $\omega \in \Omega_0^x$ and $\rho \in \mathbb{N}^*$ are fixed; we also fix a corresponding $\varepsilon \in 1/\mathbb{N}$ as in the definition of Ω_0^x and an $l \in \mathbb{N}^*$ greater than (or equal to) l_0 associated to ω, ε, ρ in the definition of Ω_0^x. Consequently, $m(\rho, l)$ and $a(\rho, l) = \rho + m(\rho, l)$ are fixed too, and will simply be denoted by m and $a = \rho + m$.

Since $X^{l,x}(\omega, \cdot) \notin \mathcal{B}(m, a, \varepsilon)$ and $X^{l+1,x}(\omega, \cdot) \notin \mathcal{B}(m, a, \varepsilon)$, the results obtained in lemma 6.3 hold for $X^{l,x}(\omega, \cdot)$ and $X^{l+1,x}(\omega, \cdot)$. In particular, recalling (12) we have for each k in $\{0, 1, \cdots, m-1\}$:

$$\forall i \in J_k(X^{l,x}(\omega, \cdot)) \quad \forall t \in [\frac{k}{m}, \frac{k+1}{m}] \qquad |X_i^{l,x}(t)| \leqslant \rho + m^2 + m + 1 < l - r \implies \psi^{l,x}(X_i^{l,x}(t)) = 0$$

and since

$$\forall i \in J_k(X^{l,x}(\omega, \cdot)) \quad \forall j \notin J_k(X^{l,x}(\omega, \cdot)) \quad \forall t \in [\frac{k}{m}, \frac{k+1}{m}] \qquad |X_i^{l,x}(t) - X_j^{l,x}(t)| > R + \frac{\varepsilon}{2}$$

no interaction is possible during the time interval $\left[\frac{k}{m}, \frac{k+1}{m}\right]$ between the particles of $J_k(X^{l,x}(\omega, \cdot))$ and the other particles. In this case equation $(\mathcal{E}^l)$ verified by $X^{l,x}(\omega)$ during the time interval $\left[\frac{k}{m}, \frac{k+1}{m}\right]$ reduces to the following equation $(\mathcal{E}(k, J_k, X^{l,x}))$ for the indices in $J_k(X^{l,x}(\omega, \cdot))$:

$$\begin{aligned}
&\forall i \in J_k(X^{l,x}(\omega, \cdot)), \quad \forall t \in \left[\frac{k}{m}, \frac{k+1}{m}\right], \\
&X_i^{l,x}(\omega, t) = X_i^{l,x}(\omega, \frac{k}{m}) + W_i(\omega, t) - W_i(\omega, \frac{k}{m}) \\
&\qquad - \frac{1}{2} \int_{\frac{k}{m}}^t \sum_{j \in J_k(X^{l,x}(\omega, \cdot))} \nabla\varphi(X_i^{l,x}(\omega, s) - X_j^{l,x}(\omega, s))\, ds \\
&\qquad + \int_{\frac{k}{m}}^t \sum_{j \in J_k(X^{l,x}(\omega, \cdot))} (X_i^{l,x}(\omega, s) - X_j^{l,x}(\omega, s))\, dL_{ij}^{l,x}(\omega, s)
\end{aligned} \tag{20}$$

For the same reasons, the equation $(\mathcal{E}^{l+1})$ verified by $X^{l+1,x}(\omega, \cdot)$ during the time interval $\left[\frac{k}{m}, \frac{k+1}{m}\right]$ reduces to the following equation $(\mathcal{E}(k, J_k, X^{l+1,x}))$ for the indices in $J_k(X^{l+1,x}(\omega, \cdot))$:

$$\begin{aligned}
&\forall i \in J_k(X^{l+1,x}(\omega, \cdot)), \quad \forall t \in \left[\frac{k}{m}, \frac{k+1}{m}\right], \\
&X_i^{l+1,x}(\omega, t) = X_i^{l+1,x}(\omega, \frac{k}{m}) + W_i(\omega, t) - W_i(\omega, \frac{k}{m}) \\
&\qquad - \frac{1}{2} \int_{\frac{k}{m}}^t \sum_{j \in J_k(X^{l+1,x}(\omega, \cdot))} \nabla\varphi(X_i^{l+1,x}(\omega, s) - X_i^{l+1,x}(\omega, s))\, ds \\
&\qquad + \int_{\frac{k}{m}}^t \sum_{j \in J_k(X^{l+1,x}(\omega, \cdot))} (X_i^{l+1,x}(\omega, s) - X_i^{l+1,x}(\omega, s))\, dL_{ij}^{l+1,x}(\omega, s)
\end{aligned} \tag{21}$$

But since $X^{l,x}(\omega, 0) = X^{l+1,x}(\omega, 0) = x$ and $L^{l,x}(\omega, 0) = L^{l+1,x}(\omega, 0) = 0$, the sets $J_0(X^{l,x}(\omega, \cdot))$ and $J_0(X^{l+1,x}(\omega, \cdot))$ are equal and equations (20) and (21) coincide for $k = 0$. The strong uniqueness of the solution then implies that :

$$\begin{aligned}
\forall t \in [0, \frac{1}{m}] \quad &\forall i, j \in J_0(X^{l,x}(\omega, \cdot)) = J_0(X^{l+1,x}(\omega, \cdot)) \\
&X_i^{l,x}(\omega, t) = X_i^{l+1,x}(\omega, t) \text{ and } L_{ij}^{l,x}(\omega, t) = L_{ij}^{l+1,x}(\omega, t)
\end{aligned}$$

and because $J_1(X^{l,x}(\omega, \cdot)) \subset J_0(X^{l,x}(\omega, \cdot))$ (and idem for $J_1(X^{l+1,x}(\omega, \cdot))$) this in turn implies that $J_1(X^{l,x}(\omega, \cdot)) = J_1(X^{l+1,x}(\omega, \cdot))$. But again, since

$$\forall i, j \in J_1(X^{l,x}(\omega, \cdot)) = J_1(X^{l+1,x}(\omega, \cdot)) \quad X_i^{l,x}(\omega, \frac{1}{m}) = X_i^{l+1,x}(\omega, \frac{1}{m}) \text{ and } L_{ij}^{l,x}(\omega, \frac{1}{m}) = L_{ij}^{l+1,x}(\omega, \frac{1}{m})$$

equations (20) and (21) coincide for $k = 1$, and the strong uniqueness implies the equality of $X_i^{l,x}(\omega, t)$ and $X_i^{l+1,x}(\omega, t)$ (and $L_{ij}^{l,x}(\omega, t)$, $L_{ij}^{l+1,x}(\omega, t)$) for i in $J_1(X^{l,x}(\omega, \cdot)) = J_1(X^{l+1,x}(\omega, \cdot)$ and t in $[\frac{1}{m}, \frac{2}{m}]$, which in turn implies that $J_2(X^{l,x}(\omega, \cdot)) = J_2(X^{l+1,x}(\omega, \cdot)$.

By induction, we thus obtain that

$$\forall k \in \{1, \ldots, m-1\} \quad J_k(X^{l,x}(\omega, \cdot)) = J_k(X^{l+1,x}(\omega, \cdot))$$

and

$$\forall k \in \{1, \ldots, m-1\} \quad \forall i, j \in J_k(X^{l,x}(\omega, \cdot)) \quad \forall t \in \left[0, \frac{k+1}{m}\right]$$
$$X_i^{l,x}(\omega, t) = X_i^{l+1,x}(\omega, t) \text{ and } L_{ij}^{l,x}(\omega, t) = L_{ij}^{l+1,x}(\omega, t).$$

Using the inclusion chain $\{i \in \mathbb{N}, |x_i| \leqslant \rho\} \subset J_{m-1}(X) \subset \cdots \subset J_1(X) \subset J_0(X)$ which holds for $X = X^{l,x}(\omega, \cdot)$ and $X = X^{l+1,x}(\omega, \cdot)$ because $X^{l,x}(\omega, 0) = X^{l+1,x}(\omega, 0) = x$, we obtain that $X_i^{l,x}(\omega, \cdot)$ and $X_i^{l+1,x}(\omega, \cdot)$ are equal on $[0, 1]$ for i's such that $|x_i| \leqslant \rho$ and the same result holds for $(L_{ij}^{l,x}(\omega, \cdot))_{i,j}$ and $(L_{ij}^{l+1,x}(\omega, \cdot))_{i,j}$ because both local times coincide if j in $J_0(X^{l,x}(\omega, \cdot))$ and identically vanish otherwise. Since ρ may be chosen arbitrary large, proposition 6.2(i) is proven. ∎

Proof of proposition 6.2 (ii)

Recall that $\mathcal{M}$ is endowed with the vague topology, i.e.

$$(\xi^n)_n \xrightarrow[n \to +\infty]{\text{in } \mathcal{M}} \xi^\infty \iff \forall f \in \mathcal{C}_c(\mathbb{R}^d) \ \sum_i f(\xi_i^n) \xrightarrow[n \to +\infty]{} \sum_i f(\xi_i^\infty)$$

where $\mathcal{C}_c(\mathbb{R}^d)$ is the space of continuous functions with compact support.
Then the convergence of $(X^{l,x}(\omega, \cdot))_l$ takes place in $\mathcal{C}([0, 1], \mathcal{M})$ if and only if

$$\forall f \in \mathcal{C}_c(\mathbb{R}^d) \qquad \sum_i f(X_i^{l,x}(\omega, t)) \xrightarrow[l \to +\infty]{} \sum_i f(X_i^{\infty,x}(\omega, t)) \text{ uniformly in } t \in [0, 1]$$

Since f has a compact support, all the terms in the above sum vanish except at most for a finite number of indices. Thus the convergence follows directly from proposition 6.2 (i), where the stationarity was proven uniformly on compact time intervals. ∎

Proof of proposition 6.2 (iii)

The convergence in $\mathcal{C}(\mathbb{R}^+, \mathcal{M})$ is defined as the convergence in $\mathcal{C}([0, 1], \mathcal{M})$. We then have to prove that for each bounded continuous function g on $\mathcal{C}([0, 1], \mathcal{M})$

$$\int\int g(X^{l,x}(\omega, \cdot)) dP(\omega) d\mu(x) \xrightarrow[l \to +\infty]{} \int\int g(X^{\infty,x}(\omega, \cdot)) dP(\omega) d\mu(x)$$

This is obvious by proposition 6.2 (ii) and the dominated convergence theorem. ∎

7 Proofs of the main results

Theorem 3.2, theorem 3.3 and proposition 3.4 are now direct consequences of Propositions 7.1, 7.5 and 7.6 enounced and proved in this section. In order to prove these propositions, we need some more notations. We first fix $x \in \mathcal{A}$. For $\tilde{m} \in \mathbb{N}^*$, $\tilde{a} \geqslant 1$ and $\varepsilon \in 1/\mathbb{N}$ fixed, let $\Omega^x(\tilde{m}, \tilde{a}, \varepsilon)$ be the set of ω's such that $X^{l,x}(\omega, \cdot)$ does not belong to $\tilde{\mathcal{B}}(\tilde{m}, \tilde{a}, \varepsilon)$ for an infinite number of indices l :

$$\Omega^x(\tilde{m}, \tilde{a}, \varepsilon) = \left\{\omega \in \Omega \ : \forall p \in \mathbb{N} \ \exists l \geqslant p, \quad X^{l,x}(\omega, .) \notin \tilde{\mathcal{B}}(\tilde{m}, \tilde{a}, \varepsilon)\right\} = \limsup_{l \to +\infty} \{X^{l,x} \notin \tilde{\mathcal{B}}(\tilde{m}, \tilde{a}, \varepsilon)\}$$

We also define

$$\begin{aligned}\Omega_1^x &= \Big\{\omega\in\Omega \text{ s.t. } \forall\varepsilon\in 1/\mathbb{N} \text{ for } \rho \text{ large enough and for an infinite number of } l\text{'s } X^{l,x}\notin\tilde{\mathcal{B}}(\rho,R,\varepsilon)\Big\}\\ &= \bigcap_{\varepsilon\in 1/\mathbb{N}} \liminf_{\rho\to+\infty}\Omega^x(\rho,R,\varepsilon) \qquad (22)\end{aligned}$$

We have the following result :

Proposition 7.1 *For every $x\in\mathcal{A}$ and $\omega\in\Omega_0^x\cap\Omega_1^x$, the process $(X^{\infty,x}(\omega,.),L_{ij}^{\infty,x}(\omega,.))$ satisfies equation $(\mathcal{E})$ with $X^{\infty,x}(\omega,0)=x$.*
Thus, for any $x\in\underline{\underline{\mathcal{A}}}=\{\xi\in\mathcal{A}:P(\Omega_0^\xi\cap\Omega_1^\xi)=1\}$, the process $(X^{\infty,x},L_{ij}^{\infty,x})$ is a solution of $(\mathcal{E})$ with initial condition x.
Moreover for each $z<z_c$ and $\mu\in\mathcal{G}(z)$ $\mu(\underline{\underline{\mathcal{A}}})=1$.

Before proving this proposition, we first establish some useful results on $\Omega^x(\tilde{m},\tilde{a},\varepsilon)$ and Ω_1^x.

Lemma 7.2 *For each $\mu\in\mathcal{G}(z)$, for each $\varepsilon\in 1/\mathbb{N}$, $\tilde{m}\in\mathbb{N}^*$ and $\tilde{a}\geqslant 1$, one has*

$$\int P(\Omega^x(\tilde{m},\tilde{a},\varepsilon)^c)d\mu(x)\leqslant z\,C'_d\,e^{-2\overline{N}\underline{\varphi}}\,\tilde{a}^d\,\tilde{m}^{2d}\,\exp\left(-\frac{\varepsilon^2}{96}\tilde{m}\right)$$

As a corollary : $\displaystyle\int P(\Omega_1^x)d\mu(x)=1$

Lemma 7.3 *For each $\tilde{m}\in\mathbb{N}^*$, $\tilde{a}\geqslant 1$ and $\varepsilon\in 1/\mathbb{N}$ one also has*

$$\forall\omega\in\Omega_0^x\cap\Omega^x(\tilde{m},\tilde{a},\varepsilon)\qquad X^{\infty,x}(\omega,\cdot)\notin\tilde{\mathcal{B}}(\tilde{m},\tilde{a},\varepsilon)$$

and consequently

$$\forall x\in\mathcal{A}\quad\forall\omega\in\Omega_0^x\cap\Omega_1^x\quad\forall\varepsilon\in 1/\mathbb{N}\quad\exists\rho_0 \text{ s.t. } \forall\rho\geqslant\rho_0\quad X^{\infty,x}(\omega,\cdot)\notin\tilde{\mathcal{B}}(\rho,R,\varepsilon)$$

Proof of lemma 7.2 By definition of $\Omega(\tilde{m},\tilde{a},\varepsilon)$ one has

$$\Omega^x(\tilde{m},\tilde{a},\varepsilon)^c = \liminf_{l\to+\infty}\Big\{\omega\in\Omega \text{ s.t. } X^{l,x}(\omega,\cdot)\in\tilde{\mathcal{B}}(\tilde{m},\tilde{a},\varepsilon)\Big\}.$$

By Fatou lemma

$$\int P(\Omega^x(\tilde{m},\tilde{a},\varepsilon)^c)d\mu(x)\leqslant\liminf_{l\to+\infty}\int P(X^{l,x}(\omega,\cdot)\in\tilde{\mathcal{B}}(\tilde{m},\tilde{a},\varepsilon))d\mu(x).$$

Using inequality (16) (see the proof of proposition 6.1 step 1) applied to the event $\Theta=\tilde{\mathcal{B}}(\tilde{m},\tilde{a},\varepsilon)$ we obtain the following bound :

$$\int P(\Omega^x(\tilde{m},\tilde{a},\varepsilon)^c)d\mu(x)\leqslant\liminf_{l\to+\infty}\int_{\mathcal{A}}Q_z^{l,\eta}(\tilde{\mathcal{B}}(\tilde{m},\tilde{a},\varepsilon))\,d\mu(\eta)+2\int_{\mathcal{A}}\left(1-\frac{Z_z^{\Lambda,\eta}}{Z_z^{l,\eta}}\right)\,d\mu(\eta);$$

thus by proposition 5.6

$$\begin{aligned}&\int P(\Omega^x(\tilde{m},\tilde{a},\varepsilon)^c)d\mu(x)\\ &\leqslant\int_{\mathcal{A}}z\,C'_d\,e^{-2\overline{N}\underline{\varphi}}\,\tilde{a}^d\,\tilde{m}^{2d}\,\exp\left(-\frac{\varepsilon^2}{96}\tilde{m}\right)\,d\mu(\eta)+2\lim_{l\to+\infty}\int_{\mathcal{A}}\left(1-\frac{Z_z^{\Lambda,\eta}}{Z_z^{l,\eta}}\right)\,d\mu(\eta)\end{aligned}$$

Inequality (17) implies that $\lim_{l\to+\infty}\int_{\mathcal{A}}\left(1-\frac{Z_z^{\Lambda,\eta}}{Z_z^{l,\eta}}\right)\ d\mu(\eta)=0$ thus

$$\int P(\Omega^x(\tilde{m},\tilde{a},\varepsilon)^c)d\mu(x)\leqslant z\ C_d'\ e^{-2\overline{N}\underline{\varphi}}\ \tilde{a}^d\ \tilde{m}^{2d}\ \exp\left(-\frac{\varepsilon^2}{96}\tilde{m}\right).$$

Replacing $\tilde{m},\tilde{a}$ by ρ,R in the above inequality, we obtain :

$$\forall\varepsilon\in 1/\mathbb{N}\qquad \sum_{\rho=1}^{+\infty}\int P(\Omega^x(\rho,R,\varepsilon)^c)d\mu(x)<+\infty.$$

By Borel-Cantelli lemma, this leads to :

$$\forall\varepsilon\in 1/\mathbb{N}\qquad \int P(\limsup_{\rho\to+\infty}\Omega^x(\rho,R,\varepsilon)^c)d\mu(x)=0$$

and consequently

$$\int P((\Omega_1^x)^c)d\mu(x)=\int P(\bigcup_{\varepsilon\in 1/\mathbb{N}}\limsup_{\rho\to+\infty}\Omega^x(\rho,R,\varepsilon)^c)d\mu(x)=0.$$

■

Proof of lemma 7.3

According to proposition 6.2(i)

$$\forall x\in\mathcal{A}\quad\forall\omega\in\Omega_0^x\quad\forall i\in\mathbb{N}\quad\exists l_0\in\mathbb{N}\quad\forall l\geqslant l_0\qquad X_i^{\infty,x}(\omega,\cdot)=X_i^{l,x}(\omega,\cdot)\text{ on }[0,1].$$

Consequently, for $x\in\mathcal{A}$ and $\omega\in\Omega^x(\tilde{m},\tilde{a},\varepsilon)\cap\Omega_0^x$ and for $i\in\mathbb{N}$, there exists an $l\geqslant l_0$ such that $X^{l,x}(\omega,\cdot)\notin\tilde{\mathcal{B}}(\tilde{m},\tilde{a},\varepsilon)$, i.e.

$$X_i^{\infty,x}(\omega,\cdot)=X_i^{l,x}(\omega,\cdot)\text{ on }[0,1]\quad\text{ and }\quad w(X_i^{l,x}(\omega,\cdot),\frac{1}{\tilde{m}})\leqslant\frac{\varepsilon}{4}\text{ or }\forall t\leqslant 1,\ |X_i^{l,x}(\omega,t)|>\tilde{a}+2\tilde{m}^2$$

Thus

$$\forall x\in\mathcal{A}\quad\forall\omega\in\Omega^x(\tilde{m},\tilde{a},\varepsilon)\cap\Omega_0^x\quad\forall i\in\mathbb{N}\quad w(X_i^{\infty,x}(\omega,\cdot),\frac{1}{\tilde{m}})\leqslant\frac{\varepsilon}{4}\text{ or }\forall t\leqslant 1,\ |X_i^{\infty,x}(\omega,t)|>\tilde{a}+2\tilde{m}^2.$$

By definition of Ω_1^x, $\quad\forall x\in\mathcal{A}\quad\forall\omega\in\Omega_0^x\cap\Omega_1^x\quad\forall\varepsilon\in 1/\mathbb{N}\quad\exists\rho_0$ s.t. $\forall\rho\geqslant\rho_0\quad\omega\in\Omega^x(\rho,R,\varepsilon)$ thus

$$\forall x\in\mathcal{A}\quad\forall\omega\in\Omega_0^x\cap\Omega_1^x\quad\forall\varepsilon\in 1/\mathbb{N}\quad\exists\rho_0\text{ s.t. }\forall\rho\geqslant\rho_0\quad\forall i\in\mathbb{N}$$
$$w(X_i^{\infty,x}(\omega,\cdot),\frac{1}{\rho})\leqslant\frac{\varepsilon}{4}\text{ or }\forall t\leqslant 1,\ |X_i^{\infty,x}(\omega,t)|>R+2\rho^2$$

that is $\quad\forall x\in\mathcal{A}\quad\forall\omega\in\Omega_0^x\cap\Omega_1^x\quad\forall\varepsilon\in 1/\mathbb{N}\quad\exists\rho_0$ s.t. $\forall\rho\geqslant\rho_0\quad X^{\infty,x}(\omega,\cdot)\notin\tilde{\mathcal{B}}(\rho,R,\varepsilon)$. ■

Proof of proposition 7.1

Let us fix $\mu\in\mathcal{G}(z)$ for some $z<z_c$. As corollary of proposition 6.1 and lemma 7.2, $\int P(\Omega_0^x\cap\Omega_1^x)d\mu(x)=1$. This proves that for μ-almost every x in $\mathcal{A}$, $P(\Omega_0^x\cap\Omega_1^x)=1$ and then $\mu(\underline{\underline{\mathcal{A}}})=1$.

We fix now $x\in\mathcal{A}$ and $\omega\in\Omega_0^x\cap\Omega_1^x$.

We first use the fact that $\omega\in\Omega_0^x$. For $\varepsilon\in 1/\mathbb{N}$ smaller than ε_0 corresponding to ω in the definition of Ω_0^x, for each $\rho\in\mathbb{N}^*$, for each $l\geqslant\rho+1$ greater than l_0 associated to ω, ε, ρ, we have

$X^{l,x}(\omega,\cdot) \notin \mathcal{B}(m(\rho,l),\rho+m(\rho,l),\varepsilon)$. Lemma 6.3 and inequality (12) then imply, as in the proof of proposition 6.2(i), that $|X_i^{l,x}(\omega,t)| < l-r$ for $t \in [0,1]$ and for i's such that $|x_i| \leqslant \rho$. Equation $(\mathcal{E}^l)$ then reduces to the simpler equation :

$$\begin{aligned}
&\forall \omega \in \Omega_0^x \cap \Omega_1^x \quad \forall \rho \in \mathbb{N}^* \quad \exists l_0 \text{ s.t. } \forall l \geqslant l_0 \quad \forall i \text{ s.t. } |x_i| \leqslant \rho \quad \forall t \in [0,1] \\
&X_i^{l,x}(\omega,t) = x_i + W_i(\omega,t) - \frac{1}{2}\int_0^t \sum_{j\in\mathbb{N}} \nabla\varphi(X_i^{l,x}(\omega,s) - X_j^{l,x}(\omega,s))\, ds \\
&\qquad\qquad + \int_0^t \sum_{j\in\mathbb{N}} (X_i^{l,x}(\omega,s) - X_j^{l,x}(\omega,s))\, dL_{ij}^{l,x}(\omega,s).
\end{aligned} \tag{23}$$

Since ω belongs to Ω_1^x too, for each $\varepsilon \in 1/\mathbb{N}^*$ there exists ρ_0 such that $\omega \in \Omega^x(\rho,R,\varepsilon)$ for each $\rho \geqslant \rho_0$. Let us fix such a ρ. Since $\omega \in \Omega^x(\rho,R,\varepsilon)$, there exists an infinite number of indices l such that $X^{l,x}(\omega,\cdot) \notin \tilde{\mathcal{B}}(\rho,R,\varepsilon)$. Remark that $R + 2\rho^2 \geqslant \rho + \frac{\varepsilon}{4}\rho + R$ so for l's such that $X^{l,x}(\omega,\cdot) \notin \tilde{\mathcal{B}}(\rho,R,\varepsilon)$ we have :

$$\begin{aligned}
&\forall i \in \mathbb{N} \qquad |x_i| \leqslant \rho \quad \Longrightarrow \quad \forall t \in [0,1] \quad |X_i^{l,x}(\omega,t)| \leqslant \rho + \frac{\varepsilon}{4}\rho \\
&\forall j \in \mathbb{N} \qquad |x_i| > \rho + \frac{\varepsilon}{2}\rho + R \quad \Longrightarrow \quad \forall t \in [0,1] \quad |X_j^{l,x}(\omega,t)| > \rho + \frac{\varepsilon}{4}\rho + R
\end{aligned} \tag{24}$$

Equation (23) holds for these indices l provided $l \geqslant l_0(\omega,\rho,\varepsilon)$ and in this case we may replace the sums over $j \in \mathbb{N}$ by sums over $\{j, |x_j| \leqslant \rho + \frac{\varepsilon}{2}\rho + R\}$, due to (24) .

$$\begin{aligned}
&\forall \omega \in \Omega_0^x \cap \Omega_1^x \quad \forall \varepsilon \in 1/\mathbb{N} \quad \exists \rho_0 \text{ s.t. } \forall \rho \geqslant \rho_0 \\
&\text{ for an infinite number of } l\text{'s and for all } i \text{ s.t. } |x_i| \leqslant \rho \qquad \forall t \in [0,1] \\
&X_i^{l,x}(\omega,t) = x_i + W_i(\omega,t) - \frac{1}{2}\int_0^t \sum_{\{j:|x_j|\leqslant \rho+\frac{\varepsilon}{2}\rho+R\}} \nabla\varphi(X_i^{l,x}(\omega,s) - X_j^{l,x}(\omega,s))\, ds \\
&\qquad\qquad + \int_0^t \sum_{\{j:|x_j|\leqslant \rho+\frac{\varepsilon}{2}\rho+R\}} (X_i^{l,x}(\omega,s) - X_j^{l,x}(\omega,s))\, dL_{ij}^{l,x}(\omega,s)
\end{aligned} \tag{25}$$

Since the set $\{j : |x_j| \leqslant \rho + \frac{\varepsilon}{2}\rho + R\}$ is finite, using proposition 6.2(i) we can choose l large enough such that (25) holds and

$$\forall j \text{ s.t. } |x_j| \leqslant \rho + \frac{\varepsilon}{2}\rho + R \qquad X_j^{l,x}(\omega,\cdot) = X_j^{\infty,x}(\omega,\cdot) \text{ on } [0,1].$$

Consequently

$$\begin{aligned}
&\forall \omega \in \Omega_0^x \cap \Omega_1^x \quad \forall \varepsilon \in 1/\mathbb{N} \quad \exists \rho_0 \text{ s.t. } \forall \rho \geqslant \rho_0 \quad \forall i \text{ s.t. } |x_i| \leqslant \rho \quad \forall t \in [0,1] \\
&X_i^{\infty,x}(\omega,t) = x_i + W_i(\omega,t) - \frac{1}{2}\int_0^t \sum_{j,|x_j|\leqslant \rho+\frac{\varepsilon}{2}\rho+R} \nabla\varphi(X_i^{\infty,x}(\omega,s) - X_j^{\infty,x}(\omega,s))\, ds \\
&\qquad\qquad + \int_0^t \sum_{j,|x_j|\leqslant \rho+\frac{\varepsilon}{2}\rho+R} (X_i^{\infty,x}(\omega,s) - X_j^{\infty,x}(\omega,s))\, dL_{ij}^{\infty,x}(\omega,s)
\end{aligned} \tag{26}$$

On the other hand, since $\omega \in \Omega^x(\rho,R,\varepsilon) \cap \Omega_0^x$ for each $\rho \geqslant \rho_0$, lemma 7.3 leads to : $X^{\infty,x}(\omega,\cdot) \notin \tilde{\mathcal{B}}(\rho,R,\varepsilon)$. As already remarked for $X^{l,x}(\omega,\cdot)$, this implies that it is equivalent to sum over $j \in \mathbb{N}$ or over $\{j, |x_j| \leqslant \rho + \frac{\varepsilon}{2}\rho + R\}$ in the above equation :

$$\begin{aligned}
&\forall \omega \in \Omega_0^x \cap \Omega_1^x \quad \forall \varepsilon \in 1/\mathbb{N} \quad \exists \rho_0 \text{ s.t. } \forall \rho \geqslant \rho_0 \quad \forall i \text{ s.t. } |x_i| \leqslant \rho \quad \forall t \in [0,1] \\
&X_i^{\infty,x}(\omega,t) = x_i + W_i(\omega,t) - \frac{1}{2}\int_0^t \sum_{j\in\mathbb{N}} \nabla\varphi(X_i^{\infty,x}(\omega,s) - X_j^{\infty,x}(\omega,s))\, ds \\
&\qquad\qquad + \int_0^t \sum_{j\in\mathbb{N}} (X_i^{\infty,x}(\omega,s) - X_j^{\infty,x}(\omega,s))\, dL_{ij}^{\infty,x}(\omega,s)
\end{aligned}$$

Then the equation does not depend on ε and ρ any more. It is simply :

$$\begin{aligned}&\forall\omega\in\Omega_0^x\cap\Omega_1^x\quad\forall i\in\mathbb{N}\quad\forall t\in[0,1]\\&X_i^{\infty,x}(\omega,t)=x_i+W_i(\omega,t)-\frac{1}{2}\int_0^t\sum_{j\in\mathbb{N}}\nabla\varphi(X_i^{\infty,x}(\omega,s)-X_j^{\infty,x}(\omega,s))\;ds\\&\qquad+\int_0^t\sum_{j\in\mathbb{N}}(X_i^{\infty,x}(\omega,s)-X_j^{\infty,x}(\omega,s))\;dL_{ij}^{\infty,x}(\omega,s)\end{aligned}$$

This prove that for $\omega\in\Omega_0^x\cap\Omega_1^x$, $X^{\infty,x}(\omega,\cdot)$ satisfies $(\mathcal{E})$ and is such that $X^{\infty,x}(\omega,0)=x$. ■

Remark 7.4 : Ω_1^x is constructed here as $\bigcap_{\varepsilon\in 1/\mathbb{N}^*}\liminf_{\rho\to+\infty}\Omega^x(\tilde{m}(\rho),\tilde{a}(\rho),\varepsilon)$ with the choice $\tilde{m}(\rho)=\rho$ and $\tilde{a}(\rho)=R$, but any choice of $\tilde{m}(\rho),\tilde{a}(\rho)$ such that $\sum_\rho\tilde{a}(\rho)^d\tilde{m}(\rho)^{2d}\exp\left(-\frac{\varepsilon^2}{96}\tilde{m}\right)<+\infty$ is convenient to obtain $\int P(\Omega_1^x)\;d\mu(x)=1$ and any choice such that $\tilde{a}(\rho)+2\tilde{m}(\rho)^2\geqslant\rho+\frac{\varepsilon}{4}\rho+R$ suffices to construct $X^{\infty,x}$ solution of $(\mathcal{E})$ on Ω_1^x.

Proposition 7.5 *The process $(X_i^{\infty,x}(t),L_{ij}^{\infty,x}(t),i,j\in\mathbb{N},t\in\mathbb{R}^+)$ is the unique solution of equation $(\mathcal{E})$ with initial point $x\in\underline{\underline{\mathcal{A}}}$ inside the class of paths $\mathcal{C}$ defined as follows :*

$X\in\mathcal{C}(\mathbb{R}^+,\mathcal{A})$ belongs to $\mathcal{C}$ if there exists $\varepsilon>0$ and $p\in\mathbb{N}^*$ such that for all $\rho,m_0\in\mathbb{N}^*$ there exists an integer $m\geqslant m_0$, a sequence $0=t_0<t_1<\cdots<t_{m'}=1$ in $\mathbb{Q}$ verifying $t_{k+1}-t_k\leqslant\frac{1}{m}$ and bounded open sets $C_0,C_1,\cdots,C_{m'-1}$ in $\mathbb{R}^d$ which satisfy

$$B(0,\rho+m)\subset C_{m'-1}\subset B(C_{m'-1},\varepsilon)\subset C_{m'-2}\subset\cdots\subset B(C_1,\varepsilon)\subset C_0\subset B(0,\rho+m+m^p)$$

and

$$\forall k\in\{0,\cdots,m'-1\}\qquad d\left(\{X_j(u),j\in\mathbb{N}^*,u\in[t_k,t_{k+1}]\},\partial C_k\right)\geqslant\frac{R}{2}+\frac{\varepsilon}{4}$$

Proof of prop 7.5

We first check that for $\omega\in\Omega_0^x$, $\quad X^{\infty,x}(\omega,\cdot)\in\mathcal{C}$:

We choose $\varepsilon=\varepsilon_0\leqslant R$ as in the definition of Ω_0^x and $p=2$. For each ρ and m_0 in $\mathbb{N}^*$, one may find $l\geqslant l_0(\omega,\rho,\varepsilon)$ large enough to have $m(\rho,l)\geqslant m_0$. Then $m=m(\rho,l)$, $m'=m$, $t_k=\frac{k}{m}$ and

$$C_k=B\left(0,\rho+m^2-km+\frac{R+\varepsilon}{2}\right)\cup\bigcup_{i\in J_k(X^{\infty,x}(\omega,\cdot))}B\left(X_i^{\infty,x}(\omega,\frac{k}{m}),\frac{R+\varepsilon}{2}\right)$$

are convenient choices (Recall lemma 6.3 and the proof of proposition 6.2(i) which implies that

$$d\left(\{X_j^\infty(u),j\in\mathbb{N}^*,u\in[\frac{k}{m},\frac{k+1}{m}]\},\partial C_k\right)\geqslant\frac{R}{2}+\frac{\varepsilon_0}{4}$$

and that $B(C_k,\varepsilon)\subset B(0,\rho+m^2-(k-1)+m+\frac{\varepsilon}{4}+\varepsilon)\subset C_{k-1}$).

The proof of uniqueness is then a direct generalization of the proof of uniqueness for hard core potential made by Tanemura [Tan96], Lemma 5.4; so we omit it (the basic idea is to decompose the time interval in a union of intervals $[k/m,(k+1)/m]$, on which each coordinate of the process is the unique solution of a finite-dimensional stochastic differential equation like (20)). ■

Proposition 7.6 *If μ, the law of $X^\infty(0)$, belongs to $\mathcal{G}(z)$, then the process X^∞ solution of equation $(\mathcal{E})$ is a reversible process.*

Proof of proposition 7.6

We have to prove that for any $T \in [0,1]$, for $f_1, \dots, f_k$ bounded continuous functions on $\mathcal{M}$ with compact support and for $t_1, \dots, t_k \in [0,T]$:

$$\int E\left(\prod_{i=1}^{k} f_i(X^{\infty,x}(t_i))\right) \, d\mu(x) = \int E\left(\prod_{i=1}^{k} f_i(X^{\infty,x}(T-t_i))\right) \, d\mu(x) \tag{27}$$

But X^∞ is, by construction, the weak limit of X^l. Then equality (27) holds if the following equality holds :

$$\lim_{l\to+\infty} \int E\left(\prod_{i=1}^{k} f_i(X^{l,x}(t_i)) - \prod_{i=1}^{k} f_i(X^{l,x}(T-t_i))\right) \, d\mu(x) = 0$$

Like in the proof of proposition 6.1 step 1 (cf inequalities (16) and (17)), we go back to the process $X^{l,\eta}$, which is (by proposition 4.1) reversible when its initial distribution is $\mu_z^{l,\eta}$:

$$\begin{aligned}
&\left| \int E\left(\prod_{i=1}^{k} f_i(X^{l,x}(t_i)) - \prod_{i=1}^{k} f_i(X^{l,x}(T-t_i))\right) \, d\mu(x) \right| \\
&\leqslant \left| \int_{\mathcal{A}} \int_{\mathcal{A}} \prod_{i=1}^{k} f_i(X(t_i)) - \prod_{i=1}^{k} f_i(X(T-t_i)) \; dQ_z^{l,\eta}(X) \; d\mu(\eta) \right| \\
&\quad + 2 \prod_{i=1}^{k} \sup_{\xi\in\mathcal{A}} |f_i(\xi)| \int_{\mathcal{A}} \left(1 - \frac{Z_z^{\Lambda,\eta}}{Z_z^{l,\eta}}\right) \, d\mu(\eta)
\end{aligned}$$

where $\Lambda = [-l,l]^d$. The first term of the right hand side is equal to 0 and the second term tends to zero as l tends to infinity. ■

8 Appendix : Estimate of the probability of fast oscillation for Brownian motion

Proposition 8.1 *If W is a (one-dimensional) Brownian motion on $(\Omega, \mathcal{F}, P)$ then for every $\varepsilon > 0$ and every $\delta \in]0,1]$*

$$P(w(W,\delta) \geqslant \varepsilon) \quad \leqslant \quad \frac{41}{\delta} \exp\left(-\frac{\varepsilon^2}{5\delta}\right)$$

Proof of proposition 8.1

We first use Doob's inequality for the submartingale $\exp(2W(\cdot)^2/5s_0)$ and then the Gaussian property $E(\exp(aW(1)^2)) = 1/\sqrt{1-2a}$ to obtain

$$\begin{aligned}
&P\left(\exists s \leqslant s_0 \; , \; |W(s)| \geqslant \beta\right) = P\left(\sup_{0\leqslant s\leqslant s_0} \exp(\frac{2W(s)^2}{5s_0}) \geqslant \exp(\frac{2\beta^2}{5s_0})\right) \\
&\leqslant \exp(-\frac{2\beta^2}{5s_0}) E\left(\exp(\frac{2W(s_0)^2}{5s_0})\right) = \exp(-\frac{2\beta^2}{5s_0}) E\left(\exp(\frac{2W(1)^2}{5})\right) = \sqrt{5}\exp(-\frac{2\beta^2}{5s_0})
\end{aligned} \tag{28}$$

Splitting the time interval $[0,1]$ in pieces of length $\delta/8$ and using, first the translation invariance of the distribution of $W(s+u)-W(s)$, then inequality (28) and finally $1\leqslant 1/\delta$ we obtain :

$$\begin{aligned}
P(w(W,\delta)\geqslant\varepsilon) &= P\left(\sup_{\substack{|t-s|<\delta\\ 0\leqslant s,t\leqslant 1}} |W(t)-W(s)|\geqslant\varepsilon\right)\\
&\leqslant P\left(\exists i\in\{0,\frac{\delta}{8},\frac{2\delta}{8},\frac{3\delta}{8},\cdots\}\cap[0,1]\ \exists s\in[i,i+\frac{\delta}{8}[\ \exists t\in[s,s+\delta[\text{ s.t. } |W(t)-W(s)|\geqslant\varepsilon\right)\\
&\leqslant \left(\left[\frac{8}{\delta}\right]+1\right) P\left(\exists s\in[0,\frac{\delta}{8}[\ \exists t\in[0,\frac{\delta}{8}+\delta[\text{ s.t. } |W(t)-W(s)|\geqslant\varepsilon\right)\\
&\leqslant \left(\frac{8}{\delta}+1\right) P\left(\exists s\in[0,\frac{\delta}{8}[\ \exists t\in[0,\frac{9\delta}{8}[\text{ s.t. } |W(s)|\geqslant\frac{\varepsilon}{4}\text{ or } |W(t)|\geqslant\frac{3\varepsilon}{4}\right)\\
&\leqslant \left(\frac{8}{\delta}+1\right)\left(P\left(\exists s\in[0,\frac{\delta}{8}[\text{ s.t. } |W(s)|\geqslant\frac{\varepsilon}{4}\right)+P\left(\exists t\in[0,\frac{9\delta}{8}[\text{ s.t. } |W(t)|\geqslant\frac{3\varepsilon}{4}\right)\right)\\
&\leqslant \left(\frac{8}{\delta}+1\right)\left(\sqrt{5}\exp(-\frac{2}{5}\frac{\varepsilon^2}{16}\frac{8}{\delta})+\sqrt{5}\exp(-\frac{2}{5}\frac{9\varepsilon^2}{16}\frac{8}{9\delta})\right)\\
&\leqslant \frac{9}{\delta}2\sqrt{5}\exp(-\frac{\varepsilon^2}{5\delta})\\
&\leqslant \frac{41}{\delta}\exp(-\frac{\varepsilon^2}{5\delta})
\end{aligned}$$

■

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