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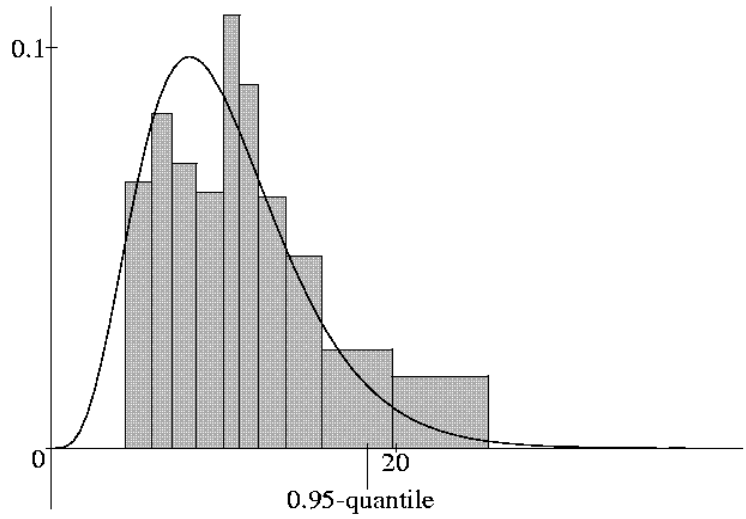

Mathematische Statistik und Wahrscheinlichkeitstheorie

Universität Potsdam – Institut für Mathematik

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Increasing coupling of Probabilistic Cellular Automata

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Abstract

We give a necessary and sufficient condition for the existence of an increasing coupling of N ($N \geqslant 2$) synchronous dynamics on $S^{\mathbb{Z}^d}$(PCA). Increasing means the coupling preserves stochastic ordering. We present our main construction theorem in the case when S is totally ordered; applications to attractive PCA are given. When S is only partially ordered we compare our results with previous ones. We also prove an extension of our main result to some class of partially ordered spaces.

Key words: Probabilistic Cellular Automata, Stochastic ordering, Monotone Coupling
2000 MSC: 60K35, 60E15, 60J10, 82C20, 37B15, 68W10

1 Introduction

Probabilistic Cellular Automata (usually abbreviated in PCA) are discrete-time Markov chains on a product space S^Λ (*configuration space*) whose transition probability is a *product measure*. Usually S is assumed to be a finite set (so called *spin space*). We denote by Λ (set of *sites*) a subset, finite or infinite, of $\mathbb{Z}^d$. Since the transition probability kernel $P(d\sigma|\sigma')$ ($\sigma, \sigma' \in S^\Lambda$) is a product measure, all interacting elementary components called spins $\{\sigma_k : k \in \Lambda\}$ are simultaneously and independently updated (*parallel updating*). This synchronous transition is the main feature of PCA and differs from the one in the most common Gibbs samplers, where only one site is updated at each time step (*sequential updating*). In opposition to these dynamics with sequential updating, it is simple to define PCA's on the infinite set $S^{\mathbb{Z}^d}$ without passing to continuous time.

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Probabilistic Cellular Automata were first studied as Markov chains in the 70's, these results are collected in Toom et al. (1978). We refer to Louis (2002) for recent detailed historical informations and list of possible applications of Cellular Automata dynamics, which, like for continuous time interacting particle systems are to be found in physics, biology... Let us here just mention their important use in image restoration (see Younes (1998)) The theoretical works cited above emphasise the variety of behaviours PCA dynamics may present. Let us only mention the following fact: in opposition to usual discrete time sequential updating dynamics, for a given measure μ, there is no canonical way to construct a PCA for which μ is stationary. Moreover, there exists Gibbs measures on $S^{\mathbb{Z}^2}$ such that no PCA admits them as stationary reversible measures (cf. Theorem 4.2 in Dawson (1974)).

Coupling means the construction of a product probability space on which several dynamics may evolve at the same time and with the property the marginals coincide with each one of these dynamics. The original idea comes from the pioneer work of Doeblin (Doeblin (1938)) who used such construction to investigate the ergodic behaviour of Markov Chains. Coupling techniques for stochastic processes are now established powerful tools of investigation (cf. Liggett (1993)). We refer to Lindvall (1992) and Thorisson (2000) to a more extensive review and applications to a large scope of probabilistic objects. For instance, coupling techniques are used in fiability theory (cf. Cocozza-Thivent and Roussignol (1995)) and in statistical mechanics (see Chapter 4 in Georgii et al. (2001)). For the analysis of the time asymptotics of markovian dynamics (cf Griffeath (1978)) or more precisely Interacting Particle Systems, coupling arguments were developed and used (see for instance Liggett (1985); Chen (1992)). The idea goes back to Harris (1955) and Spitzer (1970).

The first use of a coupling of Probabilistic Cellular Automata is in Vasershtein (1969) (see also Chapter 3 in Toom et al. (1978)). It was also used in Maes (1993) to state some ergodicity criterion for PCA when there is a weak dependence between the sites. Recently, the coupling constructed in this paper was used to state some necessary and sufficient condition for exponential ergodicity for attractive PCA (see Louis (2004)). This last result relies on the fact that our coupling preserves the stochastic order between configurations (so called *increasing coupling*). In López and Sanz (2000) the authors give some necessary and sufficient condition for the existence of a preserving stochastic ordering coupling of two (possible different) PCA on $S^{\mathbb{Z}^d}$ where S admits a partial order. As emphasised in this work, the synchronous evolution of all the components leads to a more complicated situation as in the sequential updating case. In this paper, we give necessary and sufficient condition for the existence of an increasing coupling of any finite number of possibly different PCA dynamics. As some counter examples state it in section 5, there is a gap between increasing coupling of two PCA and a number N of PCA with $N \geqslant 3$. Moreover, we give here an explicit algorithmic construction of this coupling. It is a kind of graphical construction, as the usual one introduced in Harris (1972, 1978) to construct Interacting Particle Systems. We also give several examples and general applications of the constructed coupling. The motivation of coupling three or more PCA comes indeed from the paper Louis (2004) were comparison between four different PCA dynamics was useful.

In section 2 we state our main result, namely the existence, under some necessary and sufficient condition of monotonicity (Definition 2.2), of an increasing coupling of several PCA dynamics (Theorem 2.3). Corollary 2.4 states the existence of some universal coupling of an attractive PCA. Important examples are also presented. In section 3 we prove these results, and state some important property (Lemma 3.2) of coherence between the different coupling. We then present in section 4 some useful consequences of the coupling. In section 5 we consider the case where S is a partially ordered set. Despite the fact that a coupling of two PCA may be considered, two counter-examples state that it can happen that an increasing coupling of N PCA dynamics do not exist when $N \geqslant 3$. A generalisation of Theorem 2.3 and Corollary 2.4 to the case where S is partially linearly ordered is presented.

Finally, let us notice that the motivation for considering partially ordered spin spaces comes from considering 'block dynamics' where not all sites are updated at the same time, but group of them. It means considering PCA on $(S^r)^{\mathbb{Z}^d}$ where r is the number of sites in these blocks. And even if S is totally ordered, (S^r) is not naturally totally ordered.

Pay attention all the measures considered in this paper are probability measures.

2 Definitions and main results

Let S be a finite set, with a partial order denoted by $\preccurlyeq$. Let $s \preccurlyeq s'$ such that $s \neq s'$ be denoted with $s \precneqq s'$. Let P denotes a PCA dynamics on the product space $S^{\mathbb{Z}^d}$, which means a time-homogeneous Markov Chain on $S^{\mathbb{Z}^d}$ whose transition probability kernel P verifies, for all configuration $\eta \in S^{\mathbb{Z}^d}$, $\sigma = (\sigma_k)_{k \in \mathbb{Z}^d} \in S^{\mathbb{Z}^d}$,

$$P(\ d\sigma \mid \eta\ ) = \bigotimes_{k \in \mathbb{Z}^d} p_k(\ d\sigma_k \mid \eta\ ),$$

where for all site $k \in \mathbb{Z}^d$, $p_k(\ .\ |\eta)$ is a probability measure on S, called *updating rule*. In other words, *given the previous time step* $(n-1)$, all the spin values $(\omega_k(n))_{k \in \mathbb{Z}^d}$ at time n are *simultaneously and independently updated*, each one according to the probabilistic rule $p_k(\ .\ |\ (\omega_{k'}(n-1))_{k' \in \mathbb{Z}^d})$. We will denote $P = \bigotimes_{k \in \mathbb{Z}^d} p_k$. Let us assume that the PCA dynamics considered here are local which means $\forall k \in \mathbb{Z}^d, \exists\, V_k \Subset \mathbb{Z}^d, p_k(\ .\ |\eta) = p_k(\ .\ |\eta_{V_k})$, where the notation $\Lambda \Subset \mathbb{Z}^d$ means Λ is a finite subset of $\mathbb{Z}^d$. For any subset Δ of $\mathbb{Z}^d$, and for all configurations σ and η of $S^{\mathbb{Z}^d}$, the configuration $\sigma_\Delta \eta_{\Delta^c}$ is defined by σ_k for $k \in \Delta$, η_k elsewhere. Let $\sigma_\Delta \equiv (\sigma_k)_{k \in \Delta}$ too.

For ν probability measure on $S^{\mathbb{Z}^d}$ (equipped with the Borel σ-field associated to the product topology), νP refers to the law at time 1 of the PCA dynamics with law ν at time 0: $\nu P(d\sigma) = \int P(d\sigma|\eta)\nu(d\eta)$. Recursively $\nu P^{(n)} = (\nu P^{(n-1)})P$ is the law at time

n of the system evolving according to the PCA dynamics P and initial law ν. For each function f on $S^{\mathbb{Z}^d}$, $P(f)$ denotes the function defined by $P(f)(\eta) = \int f(\sigma)P(d\sigma|\eta)$.

Let us now define some notions of stochastic ordering $\preccurlyeq$. Two configurations σ and η of S^Λ (with $\Lambda \subset \mathbb{Z}^d$) satisfy $\sigma \preccurlyeq \eta$ if $\forall k \in \Lambda, \sigma_k \preccurlyeq \eta_k$. A real function f on S^Λ will to be increasing if $\sigma \preccurlyeq \eta \Rightarrow f(\sigma) \leqslant f(\eta)$. Thus two probability measures ν_1 and ν_2 satisfy the stochastic ordering $\nu_1 \preccurlyeq \nu_2$ if, for all increasing functions f on S^Λ, $\nu_1(f) \leqslant \nu_2(f)$, with the notation $\nu_i(f) = \int f(\sigma)\nu_i(d\sigma)$. As Markov chain, a PCA dynamics P on S^Λ ($\Lambda \subset \mathbb{Z}^d$) is said to be *attractive* if for all increasing function f, $P(f)$ is still increasing. It is equivalent to the property $\mu_1 \preccurlyeq \mu_2 \Rightarrow \mu_1 P \preccurlyeq \mu_2 P$ where μ_1, μ_2 are probability measures on $S^{\mathbb{Z}^d}$.

Definition 2.1 (Synchronous coupling of PCA dynamics)
Let $P^1, P^2, \ldots, P^N$ be N probabilistic cellular automata dynamics, with $P^i = \bigotimes_{k \in \mathbb{Z}^d} p_k^i$. We call synchronous coupling *of the $(P^i)_{1 \leqslant i \leqslant N}$ a Markovian dynamics Q on $(S^{\mathbb{Z}^d})^N \sim (S^N)^{\mathbb{Z}^d}$, which is a PCA dynamics too, and with marginals the P^i. It means that $Q = \bigotimes_{k \in \mathbb{Z}^d} q_k$ with*

$$\forall i \in \{1, \ldots, N\}, \quad \forall s^i \in S, \quad \forall \zeta^i \in S^{\mathbb{Z}^d},$$
$$p_k^i(s^i \mid \zeta^i) = \sum_{s^j \in S, j \neq i} q_k\Big((s^1, \ldots, s^N) \Bigm| (\zeta^1, \ldots, \zeta^N)\Big). \tag{1}$$

Let us now introduce a notion of order between PCA dynamics on $S^{\mathbb{Z}^d}$.

Definition 2.2 (Increasing N-uple of PCA dynamics) *Let $(P^1, P^2, \ldots, P^N)$ be a N-uple of PCA dynamics where $N \geqslant 2$ and $P^i = \bigotimes_{k \in \mathbb{Z}^d} p_k^i$ $(1 \leqslant i \leqslant N)$. It is said (monotone)* increasing *if: $\forall \zeta^1 \preccurlyeq \zeta^2 \preccurlyeq \ldots \preccurlyeq \zeta^N$,*

$$P^1(\ .\mid \zeta^1) \preccurlyeq P^2(\ .\mid \zeta^2) \preccurlyeq \ldots \preccurlyeq P^N(\ .\mid \zeta^N) \tag{2}$$

Since $P(\ .\ |\sigma)$ is a product measure, according to Proposition 2.9 in Toom et al. (1978) condition (2) is equivalent to:
$\forall \zeta^1 \preccurlyeq \zeta^2 \preccurlyeq \ldots \preccurlyeq \zeta^N$, $\forall k \in \mathbb{Z}^d$,

$$p_k^1(\ .\mid \zeta^1) \preccurlyeq p_k^2(\ .\mid \zeta^2) \preccurlyeq \ldots \preccurlyeq p_k^N(\ .\mid \zeta^N) \tag{3}$$

Here is now our main statement:

Theorem 2.3 *Let S be a totally ordered space. Let $(P^i)_{1 \leqslant i \leqslant N}$ be a N-uple of PCA dynamics on $S^{\mathbb{Z}^d}$. It exists a synchronous coupling $\mathbf{Q}$ called* increasing coupling *of $(P^1, P^2, \ldots, P^N)$ with the following property: for any initial configuration $\sigma^1 \preccurlyeq \sigma^2 \preccurlyeq \ldots \preccurlyeq \sigma^N$, $\forall n \geqslant 1$,*

$$\mathbf{Q}\ \Big(\ \omega^1(n) \preccurlyeq \ldots \preccurlyeq \omega^N(n) \Bigm| (\omega^1, \ldots, \omega^N)(0) = (\sigma^1, \ldots, \sigma^N)\ \Big) = 1 \tag{4}$$

if and only if $(P^1, \dots, P^N)$ *is increasing.*
We will denote the one we construct by $P^1 \circledast P^2 \circledast \dots \circledast P^N$.

Note that the property of preserving the order implies the coupling has the *coalescence property*. It means, if two components take the same value for some time, then they (and all the components inbetween) will have the same value from this time on.

Lemma 3.3 will state, for a PCA dynamics P, that if P is an attractive dynamics then for all $N \geqslant 2$, the N-uple $(P, P, \dots, P)$ is increasing. An immediate consequence of Theorem 2.3 is the

Corollary 2.4 *Let* S *be a totally ordered space,* P *be a PCA dynamics on* $S^{\mathbb{Z}^d}$ *and* $N \geqslant 2$. *It exists an increasing coupling*

$$\mathbf{P} = P^{\circledast N} \tag{5}$$

if and only if P *is attractive.*

Lemma 3.1 to be shown in section 3 gives a practical constructive criteria for testing if an N-uple of PCA dynamics is increasing or the attractiveness of some PCA P. We use it in the following examples.

Example of a family of different PCA dynamics
Let $S = \{-1, +1\}$ and the $(P^{\beta_i, h_i})_{1 \leqslant i \leqslant N}$ a family of N PCA dynamics on $\{-1, +1\}^{\mathbb{Z}^d}$ defined by $\forall k \in \mathbb{Z}^d$, $\forall \eta \in \{-1, +1, \}^{\mathbb{Z}^d}$, $\forall s \in \{-1, +1\}$

$$p_k^i(s \mid \eta) = \frac{1}{2}\Big(1 + s \tanh(\beta_i \sum_{k' \in \mathbb{Z}^d} \mathcal{K}(k' - k)\eta_{k'} + \beta_i h_i)\Big), \tag{6}$$

where $(\beta_i)_{1 \leqslant i \leqslant N}$ are positive real numbers, $(h_i)_{1 \leqslant i \leqslant N}$ real numbers, and $\mathcal{K} : \mathbb{Z}^d \to \mathbb{R}$ is an interaction function between sites which is symmetric and has finite range $R > 0$ (*i.e.* for all k of $\mathbb{Z}^d$ such that $\|k\|_1 > R$ then $\mathcal{K}(k) = 0$, with $\|k\|_1 = \sum_{i=1}^d |k_i|$).

This example is an important family of PCA dynamics. PCA of this form are the most general ones among the reversible PCA dynamics on $\{-1, +1\}^{\mathbb{Z}^d}$ (which means it admits at least one reversible probability measure) (see subsection 4.1.1 in Louis (2002)). When $\beta_i = \beta$, for $h_1 \leqslant \dots \leqslant h_N$, the N-uple $(P^{\beta, h_i})_{1 \leqslant i \leqslant N}$ is increasing. On the other hand, note that in the case $h_i = 0$, the assumption $\beta_1 \leqslant \dots \leqslant \beta_N$ does not imply the N-uple $(P^{\beta_i, 0})_{1 \leqslant i \leqslant N}$ is increasing. Consider for instance $\beta_1 = \frac{1}{2}$, $\beta_2 = 3$, $d = 2$, $\mathcal{K}$ such that $V_k = \{k - e_1, k + e_1, k - e_2, k + e_2\}$ where (e_1, e_2) is a basis of $\mathbb{R}^2$. Condition (3) is false considering $k = 0$, $\zeta^1_{V_0}$ consisting of four -1, and $\zeta^2_{V_0}$ of three -1 and one $+1$.

Example of an attractive PCA dynamics
Let $P^{\beta, h}$ some PCA dynamics defined thanks to the updating rule (6) ($\beta \geqslant 0$, $h \in \mathbb{R}$). This dynamics is attractive if and only if $\mathcal{K}(.) \geqslant 0$ holds (cf. Proposition 4.1.2 in Louis (2002)). For a more systematic study of this class, let us refer to Dai Pra et al. (2002) and Louis (2004).

Example of an attractive PCA dynamics with $\#S = q$, $q \geqslant 2$
Let $S = \{1, \ldots, q\}$ ($q \geqslant 2$), and consider the updating rule

$$\forall k \in \mathbb{Z}^d, \quad \forall s \in S, \quad \forall \sigma \in S^{\mathbb{Z}^d}, \quad p_k(s|\sigma) = \frac{e^{\beta N_k(s,\sigma)}}{\sum_{s' \in S} e^{\beta N_k(s',\sigma)}} \tag{7}$$

where $\beta \geqslant 0$ and $N_k(s, \sigma)$ is the number of $\sigma_{k'}$ ($k' \in V_k$) which are larger than s. It is attractive for any β non-negative.

3 Proof of the main results

Assume in this section that S is a totally ordered set. Let us then enumerate the spin set elements with $S = \{-, \ldots, s, s+1, \ldots, +\}$ where we denote with $+$ (resp. $-$) the (necessarily unique) maximum (resp. minimum) value of S and for $s \in S$, $(s+1)$ denotes the only element in S such that there is no $s'' \in S$, $s \precneqq s'' \precneqq s+1$.
A real valued function f on $S^{\mathbb{Z}^d}$is said *local* if $\exists \Lambda_f \Subset \mathbb{Z}^d$, $\forall \sigma \in S^{\mathbb{Z}^d}$, $f(\sigma) = f(\sigma_{\Lambda_f})$.

Lemma 3.1 *When S is a totally ordered space, the monotonicity condition (3) is equivalent to*

$$\begin{gathered} \forall k \in \mathbb{Z}^d, \forall \zeta^1 \preccurlyeq \zeta^2 \preccurlyeq \ldots \preccurlyeq \zeta^N \in (S^{\mathbb{Z}^d})^N, \forall s \in S \\ F_k^1(s, \zeta^1) \geqslant F_k^2(s, \zeta^2) \geqslant \ldots \geqslant F_k^N(s, \zeta^N), \end{gathered} \tag{8}$$

where $F_k^i(s, \sigma)$ is the repartition function of $p_k^i(.|\sigma)$:

$$F_k^i(s, \sigma) = \sum_{s' \leqslant s} p_k^i(s'|\sigma) \qquad (s \in S, \sigma \in S^{\mathbb{Z}^d}, i \in \{1, \ldots, N\}). \tag{9}$$

Proof. The implication (3) $\Rightarrow$ (8) is straightforward using the increasing function $f(s') = \mathbb{1}_{s'>s}$. To prove (8) $\Rightarrow$ (3) it is enough to remark that, for any function $f : S \to \mathbb{R}$,

$$p_k^i(f|\sigma) = f(+) + \sum_{s \precneqq +} (f(s) - f(s+1)) F_k^i(s, \sigma). \tag{10}$$

□

Proof of Theorem 2.3 We explain the way to construct explicitly the coupling $P^1 \circledast P^2 \circledast \ldots \circledast P^N$. Let n be a fixed time, and let us describe how to construct $(\omega^1, \ldots, \omega^N)(n+1)$ $\in S^N$, knowing the configuration $(\omega^1, \ldots, \omega^N)(n)$. Let $(U_k)_{k \in \Lambda}$ be a family of independent identically distributed uniform laws on $]0, 1[$. Since we are constructing a synchronous coupling, it is enough to define the rule for a fixed site $k \in \mathbb{Z}^d$. Let call r a realization of the random variable U_k. Use the following *algorithmic rule* to choose the value $\omega_k^i(n+1)$

for any i $(1 \leqslant i \leqslant N)$:

$$\begin{cases} \text{if } F_k^i(s-1,\omega^i(n)) < r \leqslant F_k^i(s,\omega^i(n)), \quad s \gneqq -, \text{ assign } \omega_k^i(n+1) = s \\ \text{if } 0 \leqslant r \leqslant F_k^i(-,\omega^i(n)) \text{ assign } \omega_k^i(n+1) = - \ . \end{cases} \tag{11}$$

This rule corresponds to the definition of the coupling between times n and $n+1$ according to

$$\forall k \in \mathbb{Z}^d, \quad \Big(\omega_k^i(n+1)\Big)_{1\leqslant i\leqslant N} = \Big(\big(F_k^i(\ .\ ,\omega^i(n))\big)^{-1}(U_k)\Big)_{1\leqslant i\leqslant N}$$

where $(F_k^i)^{-1}$ denotes the Lévy probability transform (generalised inverse probability transform) of the F_k^i repartition function

$$(F_k^i)^{-1}(t) = \inf_{\preccurlyeq}\{s \in S : F_k^i(s) \geqslant t\}, \quad t \in]0,1[.$$

Finally, remark that the stochastic dependence between the components i comes from the fact that we use the *same* realisation r of U_k for *all* the components. The fact that this coupling preserves stochastic ordering is then easy to check according to this construction, when the monotonicity of $(P^1,\ldots,P^N)$ is assumed, since it is equivalent to check (8) (Lemma 3.1).

Reciprocally, the condition (8) is necessary. Assume it exists a synchronous coupling $(q_k)_{k\in\mathbb{Z}^d}$ of N PCA dynamics on $S^{\mathbb{Z}^d}$ which preserves the stochastic ordering. It means that for $\zeta^1 \preccurlyeq \ldots \preccurlyeq \zeta^N$, $q_k(\ .\ |(\zeta^1,\ldots,\zeta^N)) > 0$ only on $(S^N)^+$ where $(S^N)^+$ is the subset $\{(s^1,\ldots,s^N) : s^1 \preccurlyeq \ldots \preccurlyeq s^N\}$ of S^N. Let $s \in S$, $1 \leqslant i < N$, and $\zeta^1 \preccurlyeq \ldots \preccurlyeq \zeta^N$ be fixed. Using the condition (1) on the i-th marginal of a coupling, we have

$$F_k^i(s,\zeta^i) = \sum_{(s^1,\ldots,s^N)\in A_s^i} q_k((s^1,\ldots,s^N)|(\zeta^1,\ldots,\zeta^N)),$$

where $A_s^i = \{(s^1,\ldots,s^N) \in (S^N)^+ : s^i \preccurlyeq s\}$. Decompose $A_s^i = A_s^{i+1} \sqcup \Delta_s^i$ with $\Delta_s^i = \{s^1,\ldots,s^N) \in (S^N)^+ : s^i \precneqq s \preccurlyeq s^{i+1}\}$ ($\sqcup$ denotes the disjoint union). Finally note that

$$F_k^i(s,\zeta^i) = F_k^{i+1}(s,\zeta^{i+1}) + \sum_{(s^1,\ldots,s^N)\in\Delta_s^i} q_k((s^1,\ldots,s^N)|(\zeta^1,\ldots,\zeta^N))$$

where the last term is non negative. $\square$

Pay attention to the *compatibility property* that the introduced coupling presents:

Lemma 3.2 *Let N and N' be two integers such that $1 \leqslant N < N'$. Let $(P^1,\ldots,P^{N'})$ be N' PCA dynamics. The projection of the coupling $P^1 \circledast P^2 \ldots \circledast P^{N'}$ on any N components $(i_1,\ldots,i_N)$ coincides with the coupling $(P^{i_1},\ldots,P^{i_N})$.*

Proof. According to the construction of the increasing coupling, this result can be checked straightforward. □

Lemma 3.3 *Let P be a PCA dynamics on $S^{\mathbb{Z}^d}$. It is an attractive dynamics if and only if, for all $N \geqslant 2$, the N-uple $(P, P, \dots, P)$ is increasing.*

Proof. Assume P is attractive. Let $k \in \mathbb{Z}^d$ be fixed, and let f be a local increasing function on S. It may be considered as a function on $S^{\mathbb{Z}^d}$such that $\forall \sigma \in S^{\mathbb{Z}^d}$, $f(\sigma) = f(\sigma_k)$. Since $P(f) = p_k(f)$ is an increasing function, relation (3) holds with $p_k^i = p_k, \forall i$. The equivalence (3) $\iff$ (2) gives $(P, \dots, P)$ increasing for any $N \geqslant 2$.
Reciprocally, (P, P) is assumed to be increasing. Then relation (8) holds with the same dynamics on the two components. Let f be an increasing function on $S^{\mathbb{Z}^d}$such that $\exists k \in \mathbb{Z}^d$, $\forall \sigma \in S^{\mathbb{Z}^d}$, $f(\sigma) = f(\sigma_k)$. According to the formula (10), we conclude that $P(f)$ is increasing. Recursively, we can state the same result for all local functions, because of the product form of the kernels. Since S is finite, $S^{\mathbb{Z}^d}$is compact, and a density argument gives the conclusion. □

Proof of Corollary 2.4 This result is a direct consequence of Theorem 2.3 and Lemma 3.3. One only needs to justify the notation $\mathbf{P}$ to denote the coupling $P \circledast P \circledast \dots \circledast P$ of N times the same attractive PCA dynamics P. Using the compatibility property of the constructed coupling (Lemma 3.2), when the dynamics on each components are identical, the marginal of $P^{\circledast N'}$ on N components chosen in $\{1, .., N'\}$ is the same as the coupling $P^{\circledast N}$. So the notation $\mathbf{P}$ can be used to denote the coupling $P^{\circledast N}$ for N large enough. □

4 Applications

In this section, let us first, for dynamics on S^Λ ($\Lambda \Subset \mathbb{Z}^d$) associated to a PCA on $S^{\mathbb{Z}^d}$(cf. formul (12)), give a structural property for some of their stationary measures (Proposition 4.1). The relation between these measures and the stationary measures for the PCA dynamics on $S^{\mathbb{Z}^d}$is then established (Proposition 4.2). It is analogous to Theorem 2.3 in Liggett (1985) with the advantage here of the finite volume approach. In particular, note we state the coincidence between spatial limits and temporal limits (cf. equations (15) and (16)). Proposition 4.4 state inequalities comparing the behaviour of the PCA with the one of these associated finite volume PCA. See also Louis (2004) for applications.

Let $P = \bigotimes_{k \in \mathbb{Z}^d} p_k$ be an attractive PCA dynamics on $S^{\mathbb{Z}^d}$. where S is, as in the previous section, a totally ordered space.

Let $\Lambda \Subset \mathbb{Z}^d$ be a finite subset of $\mathbb{Z}^d$, called finite volume. We call *finite volume PCA dynamics* with *boundary condition* τ ($\tau \in S^{\mathbb{Z}^d}$ or $\tau \in S^{\Lambda^c}$), the Markov Chain on S^Λ whose transition probability P_Λ^τ is defined by:

$$P_\Lambda^\tau(d\sigma_\Lambda \mid \eta_\Lambda) = \bigotimes_{k\in\Lambda} p_k(\, d\sigma_k \mid \eta_\Lambda \tau_{\Lambda^c}\,). \tag{12}$$

It may be identified with the following infinite volume PCA dynamics on $S^{\mathbb{Z}^d}$:

$$P_\Lambda^\tau(d\sigma \mid \eta_\Lambda) = \bigotimes_{k\in\Lambda} p_k(\, d\sigma_k \mid \eta_\Lambda \tau_{\Lambda^c}\,) \otimes \delta_{\tau_{\Lambda^c}}(d\sigma_{\Lambda^c}) \tag{13}$$

where the spins of Λ evolve according to P_Λ^τ, and those of Λ^c are almost surely 'frozen' at the value τ. We assume that the finite volume PCA dynamics P_Λ^τ are irreducible and aperiodic Markov Chains. They then admit one and only one stationary probability measure, called ν_Λ^τ (*i.e.* $\nu_\Lambda^\tau P_\Lambda^\tau = \nu_\Lambda^t$); furthermore P_Λ^τ is ergodic, which means $\lim_{n\to\infty} \rho_\Lambda (P_\Lambda^\tau)^{(n)} = \nu_\Lambda^\tau$ in the weak sense, for any initial condition ρ_Λ.
A sufficient condition for the irreducibility and aperiodicity of P_Λ^τ is for instance to assume that PCA dynamics studied are *non degenerate* ones. It means: $\forall k \in \mathbb{Z}^d$, $\forall \eta \in S^{\mathbb{Z}^d}$, $\forall s \in S$, $p_k(\, s \mid \eta\,) > 0$. The following Proposition states that the finite volume stationary measures associated with extremal boundary conditions satisfy some sub/super-DLR relation, which means are sub/super-Gibbs measures. In the very special case $S = \{-1, +1\}$ and for P reversible this result was shown in Dai Pra et al. (2002).

Proposition 4.1 *Let ν_Λ^+ (resp. ν_Λ^-) be the unique stationary probability measure associated with the finite volume PCA dynamics P_Λ^+ (resp. P_Λ^-) with $+$ (resp. $-$) extremal boundary condition . Let $\Lambda \subset \Lambda' \Subset \mathbb{Z}^d$. Following inequalities hold for any σ:*

$$\nu_{\Lambda'}^-(.|\sigma_{\Lambda'\setminus\Lambda}) \succcurlyeq \nu_\Lambda^-(.) \quad \textit{and} \quad \nu_{\Lambda'}^+(.|\sigma_{\Lambda'\setminus\Lambda}) \preccurlyeq \nu_\Lambda^+(.). \tag{14}$$

Proof. First remark, using (3) that the pair of PCA $(P_{\Lambda'}^+, P_\Lambda^+ \otimes \delta_{+_{\Lambda'\setminus\Lambda}})$ (resp. $(P_\Lambda^- \otimes \delta_{-_{\Lambda'\setminus\Lambda}}, P_{\Lambda'}^-)$ on $S^{\Lambda'}$ is increasing. Using the increasing coupling defined in Theorem 2.3, we state, for any initial condition σ, and for $n \geqslant 1$,

$$P_{\Lambda'}^+ \circledast \Big(P_\Lambda^+ \otimes \delta_{+_{\Lambda'\setminus\Lambda}}\Big)\Big(f(\omega^2(n)) - f(\omega^1(n))|\ (\omega^1,\omega^2)(0) = (\sigma,\sigma)\ \Big) \geqslant 0\ ,$$

where f is any increasing function on $S^{\mathbb{Z}^d}$. Thus

$$P_{\Lambda'}^+(f(\omega(n)) \mid \omega(0) = \sigma) \leqslant P_\Lambda^+ \otimes \delta_{+_{\Lambda'\setminus\Lambda}}(f(\omega(n)) \mid \omega(0) = \sigma).$$

Letting n going to infinity, and using finite volume ergodicity, it holds $\nu_{\Lambda'}^+ \preccurlyeq \nu_\Lambda^+ \otimes \delta_{+_{\Lambda'\setminus\Lambda}}$. Analogously, $\nu_\Lambda^- \otimes \delta_{-_{\Lambda'\setminus\Lambda}} \preccurlyeq \nu_{\Lambda'}^-$.

Let $\sigma_{\Lambda'\setminus\Lambda} \in S^{\Lambda'\setminus\Lambda}$. Let B be the event $B = \{\omega \in S^{\Lambda'} : \omega_{\Lambda'\setminus\Lambda} = \sigma_{\Lambda'\setminus\Lambda}\}$. Consider a sequence of independent, identically distributed random variables $(Z_n)_{(n\geqslant 1)}$, with distribution $\nu^+_{\Lambda'}$. Let Y be a random variable with distribution $\nu^+_\Lambda \otimes \delta_{+_{\Lambda'\setminus\Lambda}}$. Let N be the stopping time $\inf\{n \geqslant 1 : Z_n \in B\}$. We prove that almost surely, $\forall n \geqslant 1$, $Z_n \preccurlyeq Y$. So it holds $Z_N \preccurlyeq Y$, which in distribution means $\nu^+_{\Lambda'}(.|\sigma_{\Lambda'\setminus\Lambda}) \preccurlyeq \nu^+_\Lambda(.)$. The other inequality is proved in the same way. □

Proposition 4.2 *Let $\Lambda \Subset \mathbb{Z}^d$. The measure ν^+_Λ (resp. ν^-_Λ) is the maximal (resp. minimal) measure of the set $\{\nu^\tau_\Lambda : \tau \in S^{\Lambda^c}\}$. Let ν^+ and ν^- denote the maximal and the minimal elements of the set $\mathcal{S}$ of stationary measures on $S^{\mathbb{Z}^d}$ associated to the PCA dynamics P. Following relations hold:*

$$\nu^+ = \lim_{L\to\infty} \nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}} = \lim_{n\to\infty} \delta_+ P^{(n)} \tag{15}$$

$$\nu^- = \lim_{L\to\infty} \nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} = \lim_{n\to\infty} \delta_- P^{(n)}, \tag{16}$$

where for L integer, $\mathcal{B}(L)$ is the ball $\mathcal{B}(0, L)$ with respect to the norm $\|k\|_1 = \sum_{i=1}^d |k_i|$, $k = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d$.
In particular, P admits a unique stationary measure ν if and only if $\nu^- = \nu^+$.

Proof. First prove that $\tau \preccurlyeq \tau' \Rightarrow \nu^\tau_\Lambda \preccurlyeq \nu^{\tau'}_\Lambda$. Let τ et τ' be two boundary conditions such that $\tau \preccurlyeq \tau'$ and let f be an increasing function on $S^{\mathbb{Z}^d}$. It is easy to check that $(P^\tau_\Lambda, P^{\tau'}_\Lambda)$ is a increasing pair, thus $P^\tau_\Lambda \circledast P^{\tau'}_\Lambda$ preserves stochastic order. Let $\sigma \in S^{\mathbb{Z}^d}$ be an initial condition. Because, $\sigma_\Lambda \tau_{\Lambda^c} \preccurlyeq \sigma_\Lambda \tau'_{\Lambda^c}$, at time n this inequality is preserved, and using the monotonicity of f, we have:

$$P^\tau_\Lambda \circledast P^{\tau'}_\Lambda \Big(f(\omega^2(n)) - f(\omega^1(n)) |\ (\omega^1, \omega^2)(0) = (\sigma, \sigma)\ \Big) \geqslant 0\ .$$

Thus

$$P^\tau_\Lambda(f(\omega(n)) \mid \omega(0) = \sigma) \leqslant P^{\tau'}_\Lambda(f(\omega(n)) \mid \omega(0) = \sigma).$$

The first result follows letting n going to infinity, and using finite volume ergodicity. The extremality of ν^+_Λ and ν^-_Λ follows.
Then, note that $\lim_{L\to\infty}(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})$ and $\lim_{L\to\infty}(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}})$ exist due to monotonicity of the following sequences: $(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})_L$ and $(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}})_L$. This comes from the fact that $\wp_\Lambda\ \nu^+_{\Lambda'} \preccurlyeq \nu^+_\Lambda$ where $\Lambda \Subset \Lambda' \Subset \mathbb{Z}^d$, and $\wp_\Lambda$ denotes the projection on Λ, which is easily checked using the increasing coupling $(P^+_{\Lambda'}, P^+_\Lambda)$. Since $\nu^+_{\mathcal{B}(L)}$ is P^+_Λ-stationary, (resp. $\nu^-_{\mathcal{B}(L)}$ is P^-_Λ-stationary the limits $\lim_{L\to\infty}(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})$ and $\lim_{L\to\infty}(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}})$ are P-stationary.
Let ν be a P-stationary measure, and L any positive integer. Since the coupling $P^-_{\mathcal{B}(L)} \circledast P \circledast P^+_{\mathcal{B}(L)}$ preserves stochastic order, using finite volume ergodicity, one can state:

$$\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} \preccurlyeq \nu \preccurlyeq \nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}}.$$

We then have:

$$\lim_{L\to\infty} \nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} \preccurlyeq \nu \preccurlyeq \lim_{L\to\infty} \nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}}. \tag{17}$$

On the other hand, it is easy to check $\delta_+ P \preccurlyeq \delta_+$, so using P's attractivity, $(\delta_+ P^{(n)})_{n\in\mathbb{N}}$ is decreasing. Analogously, $(\delta_- P^{(n)})_{n\in\mathbb{N}}$ is increasing. Thus, the limits $\lim_{n\to\infty} \delta_- P^{(n)}$ and $\lim_{n\to\infty} \delta_+ P^{(n)}$ exist, and then are obviously P-stationary measures.
Let ν be a P-stationary measure. Because P is attractive and $\delta_- \preccurlyeq \nu \preccurlyeq \delta_+$, we have:

$$\lim_{n\to\infty} \delta_- P^{(n)} \preccurlyeq \nu \preccurlyeq \lim_{n\to\infty} \delta_+ P^{(n)}. \tag{18}$$

Using the fact that all measures $\lim_{L\to\infty}(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})$, $\lim_{L\to\infty}(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\Lambda^c}})$, $\lim_{n\to\infty} \delta_- P^{(n)}$ and $\lim_{n\to\infty} \delta_+ P^{(n)}$ are P-stationary, we apply to them inequalities (17) and (18). Conclusions follow. □

4.2 Comparison of finite & infinite volume PCA

The PCA dynamics P on the infinite volume space $S^{\mathbb{Z}^d}$ considered in this subsection is assumed to be translation invariant (or *space homogeneous*). It means: $\forall k \in \mathbb{Z}^d$, $\forall s \in S$, $\forall \eta \in S^{\mathbb{Z}^d}$, $p_k(\ s \mid \eta\ ) = p_0(\ s \mid \theta_{-k}\eta\ )$, where $\theta_{k_0}(\sigma)$ defines the translation of a configuration σ of $S^{\mathbb{Z}^d}$ with $\theta_{k_0}(\sigma) = (\sigma_{k-k_0})_{k\in\mathbb{Z}^d}$. Remark that according to the construction of the coupling in Theorem 2.3, if the PCA dynamics to be coupled are translation invariant, so is the coupled dynamics.

As Proposition 4.2 shows, in order to study the behaviour of a PCA dynamics P on $S^{\mathbb{Z}^d}$ there is advantage to use finite volume associated dynamics P^τ_Λ on S^Λ with $\Lambda \Subset \mathbb{Z}^d$. In particular, their time asymptotics is known. So, let us state the following Proposition 4.4 where important relations are stated. They are extensively used in the paper Louis (2004) to show that exponential ergodicity of the dynamics P is equivalent to the decrease, with exponential speed in L, of the quantity $\left(\int \sigma_0\ d\nu^+_{\mathcal{B}(L)} - \int \sigma_0\ d\nu^-_{\mathcal{B}(L)}\right)$. To this aim, the behaviour of this sequence is related to the one of $(\rho(n))_{n\geqslant 0}$ where

$$\rho(n) = \mathbf{P}\left(\omega^1_0(n) \neq \omega^2_0(n) \middle| (\omega^1, \omega^2)(0) = (-,+)\right), \tag{19}$$

with $\mathbf{P}$ the coupling introduced in Corollary 2.4.
Let $\Lambda \subset \mathbb{Z}^d$. Let P^+_Λ (resp. P^-_Λ) be the dynamics on S^Λ defined in (13) with the maximal (resp. minimal) boundary condition $+$ (resp. $-$). First note the easily checked fact:

Lemma 4.3 *If the PCA dynamics P is attractive then $(P^-_\Lambda, P, \ldots, P, P^+_\Lambda)$ is increasing, and thus the increasing coupling $P^-_\Lambda \circledast P \circledast \ldots \circledast P \circledast P^+_\Lambda$ can be defined.*

Proposition 4.4 *Let $\sigma, \eta \in S^{\mathbb{Z}^d}$ be such that $\sigma \preccurlyeq \eta$. The following inequality holds:*

$$\mathbf{P}\Big(\omega_0^1(n) \neq \omega_0^2(n)\Big|(\omega^1,\omega^2)(0) = (\sigma,\eta)\Big)$$
$$\leqslant \rho(n) \leqslant P_\Lambda^- \circledast P_\Lambda^+(\omega_0^1(n) \neq \omega_0^2(n) \;|(\omega^1,\omega^2)(0) = (-,+)) \tag{20}$$

where $(\rho(n))_{n\in\mathbb{N}^*}$ *is defined by (19).*
For each initial condition ξ *on* $S^{\mathbb{Z}^d}$ *and for any time* n*, it holds:*

$$P_\Lambda^-\Big(\omega(n) \in .\Big|\omega(0) = \xi_\Lambda(-)_{\Lambda^c}\Big) \tag{21}$$
$$\preccurlyeq P\Big(\omega(n) \in .\Big|\omega(0) = \xi\Big) \preccurlyeq P_\Lambda^+\Big(\omega(n) \in .\Big|\omega(0) = \xi_\Lambda(+)_{\Lambda^c}\Big).$$

The sequence $(\rho(n))_{n\in\mathbb{N}^*}$ *is decreasing, and* P *is ergodic if and only if* $\lim_{n\to\infty}\rho(n) = 0$*. Moreover,*

$$\sup_\sigma \Big|P(f(\omega(n))|\omega(0) = \sigma) - \nu(f)\Big| \leqslant 2 \;|||\; f \;|||\;\; \rho(n) \tag{22}$$

where ν *denotes the unique stationary measure and where we define, for each* f *continuous function on the compact* $S^{\mathbb{Z}^d}$ *and for all* k *in* $\mathbb{Z}^d$*,*

$$\Delta_f(k) = \sup\Big\{\Big|f(\sigma) - f(\eta)\Big| \;:\; (\sigma,\eta) \in (S^{\mathbb{Z}^d})^2, \sigma_{\{k\}^c} \equiv \eta_{\{k\}^c}\Big\}, \tag{23}$$

and the semi-norm $|||\; f \;||| = \sum_{k\in\mathbb{Z}^d}\Delta_f(k)$*.*

Proof. The proof of the left inequality in (20) is straightforward using the compatibility property 3.2. The right inequality comes from the preserving stochastic order property and compatibility property of the coupling $P_\Lambda^- \circledast P \circledast P \circledast P_\Lambda^+$.
Since the coupling $P_\Lambda^- \circledast P \circledast P_\Lambda^+$ is increasing, (21) is a consequence of the fact that any initial condition ξ in $S^{\mathbb{Z}^d}$ is such that $\xi_\Lambda(-)_{\Lambda^c} \preccurlyeq \xi \preccurlyeq \xi_\Lambda(+)_{\Lambda^c}$.
The monotonicity of the sequence $(\rho(n))_{n\in\mathbb{N}^*}$ comes from the coalescence property of the increasing coupling $\mathbf{P}$.
If P is ergodic, there can be only one stationary measure on $S^{\mathbb{Z}^d}$and so $\lim_{n\to\infty}\rho(n) = 0$.
Reciprocally, let f be a local function. For any σ,η configurations in $S^{\mathbb{Z}^d}$, let us write:

$$\Big|P(f(\omega(n))|\omega(0) = \sigma) - P(f(\omega(n))|\omega(0) = \eta)\Big|$$
$$\leqslant \Big|\mathbf{P}\Big(f(\omega^1(n)) - f(\omega^2(n))\;\Big|(\omega^1,\omega^2)(0) = (-,\sigma)\Big)\Big| \tag{24}$$
$$+\Big|\mathbf{P}\Big(f(\omega^1(n)) - f(\omega^2(n))\;\Big|(\omega^1,\omega^2)(0) = (-,\eta)\Big)\Big|.$$

Since f is local, for all ξ^1,ξ^2, $\Big|f(\xi^1) - f(\xi^2)\Big|$ depends only on $\xi^1_{\Lambda_f}$ and $\xi^2_{\Lambda_f}$ which differ only in a finite number of sites. Using interpolating configurations between $\xi^1_{\Lambda_f}$ and $\xi^2_{\Lambda_f}$ we write:

$|f(\xi^1) - f(\xi^2)| \leqslant \sum_{k \in \Lambda_f} \Delta_f(k) \mathbb{1}_{\sigma_k \neq \eta_k}$, and so using the translation invariance assumption and the left part of (20), we obtain

$$\left| P(f(\omega(n))|\omega(0) = \sigma) - P(f(\omega(n))|\omega(0) = \eta) \right| \leqslant \ 2 \ ||| \ f \ ||| \ \rho(n), \tag{25}$$

which inequality is enough to conclude the reciprocal and to prove (22). □

5 Partially ordered spin space case

In all this section, S is a partially ordered space. When S is totally ordered, a necessary and sufficient condition for existence of an increasing coupling of PCA dynamics is given in section 3 by the inequality (8). It is done in term of the repartition function $F_k(., \sigma)$ (σ given) of the probability $p_k(\ . \ |\sigma)$ (of the subset $[x, +]$ of S). In what follows, we first recall (Proposition 5.2) previous result of 26), who gave a necessary and sufficient condition for the existence of an increasing coupling of two PCA dynamics. The quantity which now makes sense is the generalised function $F_k(\Gamma, \sigma)$ defined by $\sum_{s' \in \Gamma} p_k(s'|\sigma)$ where Γ is an upset of S (see Definition 5.1).

Nevertheless, there is a gap between coupling two PCA or at least three PCA. The examples D and E presented here show that, even when the PCA are the same the coupling of three such dynamics may not exist. So we deduce that condition (26) of López and Sanz (2000) is not sufficient for the existence of an increasing 3-coupling when S is any partially ordered space.
These counter-examples rely on examples 1.1 and 5.7 in Fill and Machida (2001) of stochastically monotone family of distributions, indexed by a partially ordered set, which are not realisable monotone. Let $(Q_\alpha)_{\alpha \in \mathcal{A}}$ be a family of probability distributions on a finite set $\mathcal{S}$ indexed by a partially ordered set $\mathcal{A}$. In Fill and Machida (2001), the authors define the system $(Q_\alpha)_{\alpha \in \mathcal{A}}$ as stochastically monotone if $\alpha_1 \preccurlyeq_{\mathcal{A}} \alpha_2$ implies $Q_{\alpha_1} \preccurlyeq_{\mathcal{S}} Q_{\alpha_2}$. It is said realisable monotone if it exist a system of $\mathcal{S}$-valued random variables $(X_\alpha)_{\alpha \in \mathcal{A}}$, defined on some probability space, such that the distribution of X_α is Q_α and $\alpha_1 \preccurlyeq_{\mathcal{A}} \alpha_2$ implies $X_{\alpha_1} \preccurlyeq_{\mathcal{S}} X_{\alpha_2}$ a.s.
In our case the existence of a coupling of the N PCA dynamics $(P^1, \ldots, P^N)$ implies, for any $k \in \mathbb{Z}^d$ fixed, the system of probability distributions on S $(p_k(.|\sigma_{V_k}))_{\sigma_{V_k} \in S^{V_k}}$, which is indexed by the partially ordered set S^{V_k}, is realisable monotone. In the counter-examples presented here, the distributions are stochastically monotone but not realisable monotone.

Let us define:

Definition 5.1 *A subset Γ of S is said to be an* up-set *(or increasing set) (resp. down-set or decreasing set) if $x \in \Gamma, y \in S, x \preccurlyeq y \Rightarrow y \in \Gamma$ (resp. $x \in \Gamma, y \in S, x \succcurlyeq y \Rightarrow y \in \Gamma$).*

Note that the indicator function of an up-set (resp. down-set) is an increasing (resp. decreasing) function. Moreover, Theorem 1 in Kamae et al. (1977) state that two measures on S μ_1, μ_2 are such that $\mu_1 \preccurlyeq \mu_2$ if and only if $\mu_1(\Gamma) \leqslant \mu_2(\Gamma)$ for all up-sets Γ of S, which is equivalent to $\mu_1(\Gamma) \geqslant \mu_2(\Gamma)$ for all down-sets Γ of S.

Proposition 5.2 (Lopez-Sanz)
Let P^1 and P^2 be PCA dynamics on $S^{\mathbb{Z}^d}$, where S is a partially ordered finite set. It exists an increasing synchronous coupling of these two PCA dynamics if and only if

$$\forall k \in \mathbb{Z}^d, \forall \sigma \preccurlyeq \eta, \forall \Gamma \text{ up-set in} S \Rightarrow \sum_{s\in\Gamma} p_k^1(s|\sigma) \leqslant \sum_{s\in\Gamma} p_k^2(s|\eta). \tag{26}$$

In particular, if $P^1 = P^2 = P$ we obtain the following: P is attractive if and only if (26) holds.

Example A
Let $S = \{0,1\}^2$ with the natural partial order represented in Figure 1(a) (where $(0,1)$ and $(1,0)$ are not comparable). Let $P = \bigotimes_{k\in\mathbb{Z}^d} p_k$ (for any dimension d) with $p_k(\,.\,|\sigma) = p_k(\,.\,|\sigma_k)$ as follows

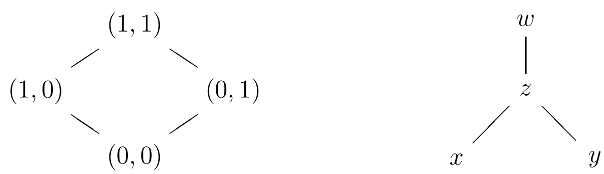
$$\begin{cases} p_k(\,.\,|(0,0)) = \frac{1}{2}(\delta_{(0,0)} + \delta_{(1,0)}) \\ p_k(\,.\,|(1,0)) = \frac{1}{2}(\delta_{(0,0)} + \delta_{(1,1)}) \\ p_k(\,.\,|(0,1)) = \frac{1}{2}(\delta_{(0,1)} + \delta_{(1,0)}) \\ p_k(\,.\,|(1,1)) = \frac{1}{2}(\delta_{(1,0)} + \delta_{(1,1)}). \end{cases} \tag{27}$$

It is simple to check this PCA dynamics is attractive. Nevertheless, $P^{\circledast 4}$ can not exist since $(p_k(\,.\,|(0,0)), p_k(\,.\,|(1,0)), p_k(\,.\,|(0,1)), p_k(\,.\,|(1,1)))$ is a stochastically monotone family which is not realisable monotone (cf. example 1.1 in Fill and Machida (2001). Remark this PCA is in fact a collection of independent S-valued Markov Chains, whose transition probability is $p_0(.|.)$. This example state the non-existence of a coupling of four times this Markov Chain.

Example B
Let $S = \{x, y, z, w\}$, considered with the following partial order $x \preccurlyeq z, y \preccurlyeq z, z \preccurlyeq w$ and x and y are not comparable (cf. Figure 1(B)). Let take the dimension $d = 1$. Consider the PCA $P = \bigotimes_{k\in\mathbb{Z}} p_k$ with $p_k(\,.\,|\sigma) = p_k(\,.\,|\sigma_{\{k,k+1\}})$ as follows:

$$\begin{cases} p_k(\,.\,|(x,y)) = \frac{1}{2}(\delta_x + \delta_y) & p_k(\,.\,|(y,z)) = \delta_z \\ p_k(\,.\,|(x,z)) = \frac{1}{2}(\delta_x + \delta_w) & p_k(\,.\,|(z,x)) = \delta_z \\ p_k(\,.\,|(z,y)) = \frac{1}{2}(\delta_y + \delta_w) & p_k(\,.\,|(x,x)) = \delta_x \\ p_k(\,.\,|(z,z)) = \frac{1}{2}(\delta_z + \delta_w) & p_k(\,.\,|(y,x)) = \delta_z \\ p_k(\,.\,|(y,y)) = \delta_y & p_k(\,.\,|\text{otherwise}) = \delta_w \end{cases} \tag{28}$$

It is an attractive PCA, nevertheless an increasing markovian coupling $P^{\circledast 4}$ can not exist since $(p_k(\ .\ |(x,y)), p_k(\ .\ |(x,z)), p_k(\ .\ |(z,y)), p_k(\ .\ |(z,z)))$ is a stochastically monotone family which is also non realisable monotone (cf. example 5.7 in Fill and Machida (2001)).



(A) Example A (B) Example B

Figure 1.

Let us now present some generalisation of our main results, Theorem 2.3 and Corollary 2.4, when the spin space S belongs to a special class Z of partially ordered sets introduced in Fill and Machida (2001) and called *linearly ordered spaces*. The figure 2(a) gives an example of such a space, and the spin space represented in Figure 3 do not belong to this class.
We call *predecessor* (resp. *successor*) of s ($s \in S$) any element s' such that $s \preccurlyeq s'$ (resp. $s \succcurlyeq s'$) and $s \preccurlyeq s'' \preccurlyeq s' \Rightarrow s'' \in \{s, s'\}$ (resp. $s \succcurlyeq s'' \succcurlyeq s' \Rightarrow s'' \in \{s, s'\}$). S is said to belong to class Z when, for any $s \in S$ only one of the following situations occurs:

- s admits exactly one successor and one predecessor ;
- s admits no predecessor and at most two successors ;
- s admits no sucessor and at most two predecessors.

It means one can define on S a natural linear order $\leqslant_n$ by numbering the elements of S: $\{s_1, \dots, s_n\}$ (where $n = \#S$) and saying $s_i \leqslant_n s_j$ if $i \leqslant j$. The figure 2(b) gives the natural linear order corresponding to the example of figure 2(a). Note that such linear order is in general not consistent with the partial order $\preccurlyeq$ originally defined on S.
Define, for $s_i \in S$ ($1 \leqslant i \leqslant n$), the subset $(\leftarrow, s_i]$ of S with

$$(\leftarrow, s_i] = \{s_j \in S\ : s_j \leqslant_n s_i\}. \tag{29}$$

Remark the sets $(\leftarrow, s]$ (with $s \in S$) are either upsets or downsets of S. For instance, in Figure 2(a), $(\leftarrow, s_5]$ is an up-set and $(\leftarrow, s_6]$ is a down-set. The generalised function $F_k(\Gamma, \sigma)$ introduced before becomes in that case the probability of $(\leftarrow, s]$:

$$F_k(s, \sigma) = p_k(\ (\leftarrow, s]\ |\ \sigma) = \sum_{s' \in (\leftarrow, s]} p_k(s'|\sigma) \qquad s \in S, \sigma \in S^{\mathbb{Z}^d}. \tag{30}$$

When S is a linearly ordered set the monotonicity condition (3) is equivalent to the following conditions for the generalised associated repartition functions (Lemma 5.5 in Fill and Machida (2001)):

$\forall k \in \mathbb{Z}^d, \forall (\zeta^1, \zeta^2, \dots, \zeta^N) \in (S^{\mathbb{Z}^d})^N$ such that $\zeta^1 \preccurlyeq \zeta^2 \preccurlyeq \dots \preccurlyeq \zeta^N, \forall s \in S,$

$$\forall s \text{ such that } (\leftarrow, s] \text{ downset } F_k^1(s \mid \zeta^1) \geqslant F_k^2(s \mid \zeta^2) \geqslant \dots \geqslant F_k^N(s \mid \zeta^N) \tag{31}$$
$$\forall s \text{ such that } (\leftarrow, s] \text{ upset } F_k^1(s \mid \zeta^1) \leqslant F_k^2(s \mid \zeta^2) \leqslant \dots \leqslant F_k^N(s \mid \zeta^N) \tag{32}$$

We can now state:

Proposition 5.3
When S is a linearly ordered spin space, Theorem 2.3 and Corollary 2.4 hold.

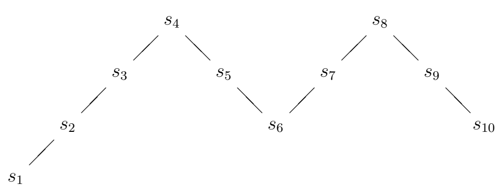
Proof. The proof of such results relies on the following construction. Let us define the generalised probability transform, for $\sigma \in S^{\mathbb{Z}^d}$ and $k \in \mathbb{Z}^d$ fixed:

$$(F_k(\,.\,,\sigma))^{-1}(t) = \inf_{\leqslant_n} \{s_k : t < F_k(s_k, \sigma)\} \qquad t \in]0,1[,$$

where the infimum is given in term of the linear order $\leqslant_n$. Construction of the increasing coupling hold as before thanks to the following evolution rule between time n and $n+1$:

$$\forall k \in \mathbb{Z}^d, \quad \Big(\omega_k^i(n+1)\Big)_{1 \leqslant i \leqslant N} = \Big(\big(F_k^i(\,.\,,\omega^i(n))\big)^{-1}(U_k)\Big)_{1 \leqslant i \leqslant N}.$$

The coherence of this coupling with the partial orderd $\preccurlyeq$ is insured by Lemma 6.2 in Fill and Machida (2001). $\square$



(B) Example C with total order $\leqslant_n$

(A) Example C with partial order $\preccurlyeq$

Figure 2.

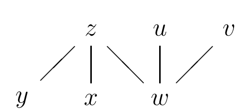


Figure 3.

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