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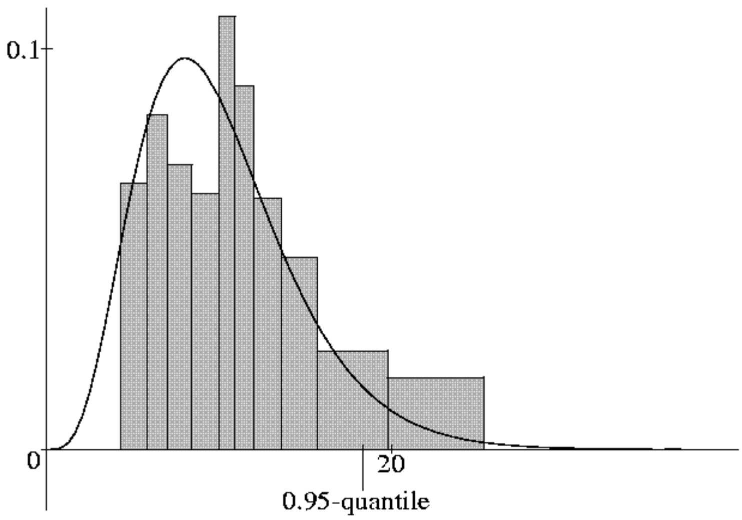

Mathematische Statistik und Wahrscheinlichkeitstheorie

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Pierre-Yves Louis

Institut für Mathematik der Universität Potsdam
e-mail: louis@math.uni-potsdam.de

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Adresse: Universität Potsdam
PF 60 15 53
14415 Potsdam
Telefon: +49-331-977 1500
Fax: +49-331-977 1578
E-mail: neisse@math.uni-potsdam.de

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Coupling, space and time Mixing for parallel stochastic dynamics

P.-Y. Louis[*‡]
E-mail: louis@math.uni-potsdam.de

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Abstract

We first introduce some *coupling* of a finite number of Probabilistic Cellular Automata dynamics (PCA), *preserving the stochastic ordering.* Using this tool, for a general attractive probabilistic cellular automata on $S^{\mathbb{Z}^d}$, where S is finite, we prove that a condition $(\mathcal{A})$ is equivalent to the (time-) convergence towards equilibrium of this Markovian parallel dynamics, in the uniform norm, exponentially fast. This condition $(\mathcal{A})$ means the exponential decay of the influence from the boundary for the invariant measures of the system restricted to finite 'box'-volume.

For a class of reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$, with a naturally associated Gibbsian potential φ, we prove that a Weak Mixing condition for φ implies the validity of the assumption $(\mathcal{A})$; thus the '*exponential ergodicity*' of the dynamics towards the unique Gibbs measure associated to φ holds. On some particular examples of this PCA class, we verify that our assumption $(\mathcal{A})$ is weaker than the Dobrushin-Vasershtein ergodicity condition. For some special PCA, the 'exponential ergodicity' holds as soon as there is no phase transition.

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[†]address: Institut für Mathematik, Potsdam Universität, Am neuen Palais, Sanssouci, Postfach 60 15 53, D-144 15 Potsdam

[‡]on leave from Graduiertenkolleg Stochastik, Technische Universität Berlin

Contents

1 Introduction

The main feature of Probabilistic Cellular Automata dynamics (usually abbreviated in PCA) is the parallel, or synchronous, evolution of all interacting elementary components. They are precisely discrete-time Markov chains on a product space S^Λ (*configuration space*) whose transition probability is a *product measure*. In this paper, S is assumed to be a finite set (so called *spin space*), and Λ (set of *sites*) a subset, finite or infinite, of $\mathbb{Z}^d$. The fact that the transition probability kernel $P(d\sigma|\sigma')$ $(\sigma, \sigma' \in S^\Lambda)$, is a product measure means that all spins $\{\sigma_k : k \in \Lambda\}$ are simultaneously and independently updated (*parallel updating*). This transition mechanism differs from the one in the most common Gibbs samplers, where only one site is updated at each time step (*sequential updating*). In opposition to these dynamics with sequential updating, it is simple to define PCA's on the infinite set $S^{\mathbb{Z}^d}$ without passing to continuous time.

Probabilistic Cellular Automata were first studied as Markov chains in the 70's under the name *locally interacting Markov systems* or *discrete local Markov systems*. Most of these results may be found in [32]. They were also called *synchronous dynamics* by D. Dawson (see [4]). The terminology used here arose with [11]. We refer to [22] for detailed historical informations and list of possible applications of Cellular Automata dynamics.

In this article we will focus on *local* PCA *i.e.* each site interacts at each time only with a finite number of neighbouring sites and *non degenerate* PCA, whose local behaviour is never deterministic. Let us however first mention some recent works on other probabilistic cellular automata classes. In [8], the non-Gibbsian nature of equilibrium state of a degenerate PCA is established. In [24] numerical simulations' investigation for this model is done. In [10], some non-local PCA are studied and applied to mathematical finance, following the idea introduced by Föllmer in [9], to use PCA as random media for financial stochastic models. For PCA dynamics considered in this paper, an application to credit risk modelling is in preparation.

The main purpose of this article is to study the convergence towards an equilibrium state of PCA dynamics on $S^{\mathbb{Z}^d}$ where S is a finite totally ordered set. The expression '*equilibrium state*' designs a stationary probability measure ν on $S^{\mathbb{Z}^d}$ characterised by the relation $\nu P = \nu$ with the notations defined below. As usual, the Markov process P is said *ergodic* if it exists a unique stationary measure ν such that for all initial measure π on $S^{\mathbb{Z}^d}$: $\lim_{n\to\infty} \pi P^{(n)} = \nu$, for the weak convergence topology. A slightly stronger definition of ergodicity, which will be satisfied here, is: it exists a unique stationary measure ν such that for all local function f,

$$\lim_{n\to\infty} \sup_{\sigma} \Big| \int f(\omega(n))\ P(\ d\omega(n) \mid \omega(0) = \sigma) - \int f\ d\nu \Big| = 0.$$

Let us emphasise that the non-degeneracy hypothesis implies that the asymptotical behaviour of PCA dynamics on S^{Λ} where $\Lambda \Subset \mathbb{Z}^d$ (called *finite volume PCA dynamics*) is perfectly known. It is a classical result for finite state space aperiodic irreducible Markov Chains. Such discrete time processes admits a unique stationary probability measure, and are ergodic. However, if the PCA dynamics is considered on $S^{\mathbb{Z}^d}$ (*infinite volume dynamics*), some non-ergodic behaviour may arise (see for instance example 2 section *III in* [15]). The most famous condition which insures ergodicity of the PCA dynamics on $S^{\mathbb{Z}^d}$ is due to Dobrushin and Vasershtein's work (see [6, 33]), and applies in the high-temperature regime. Others conditions of ergodicity for general PCA can be found in the following works: [29, 17, 14, 25, 23]. See for instance Sections 6.1.2 and 6.1.3 in [22] for details. They all are effective when some high-temperature condition holds or in some perturbative cases.

We will here adopt another approach, partially inspired by Martinelli and Olivieri's work for a class of continuous time Interacting Particle Systems called Glauber dynamics (see [26]), and based on a famous statement of Holley about rate of convergence ([13]). We introduce a general condition ($\mathcal{A}$) which means the *exponential decay of the influence from the boundary for the invariant measures of the system restricted to finite 'box'-volume* which will be here proved to be equivalent to the exponentially fast ergodicity (Theorem 3.4). The condition ($\mathcal{A}$) we use is not a constructive criterium as the beautiful Dobrushin-Vasershtein condition, or its generalised versions developped in [23] and numerically studied in [5]. But, theoretically, comparaison of spatial and time mixing are always interesting (cf. [26, 27, 30]). Furthermore, at the end of this paper, different examples are treated to show that the condition ($\mathcal{A}$) is sometimes satisfied in some larger domain than Dobrushin-Vasershtein condition, and is moreover optimal for some models.

In section 2, we develop some *coupling* of a finite number of Probabilistic Cellular Automata dynamics, *preserving the stochastic ordering* (Theorem 2.2). In section 3, we then establish four equivalent conditions, sufficient to insure ergodicity for *attractive* probabilistic cellular automata (Proposition 3.3). Moreover, we establish our main result (Theorem 3.4): convergence towards equilibrium in the uniform norm, with an *exponential rate* is *equivalent* to the condition $(\mathcal{A})$. In other words *exponential mixing in space is equivalent to exponential mixing in time*. It will then be illustrated in section 4, on a class of reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$, associated in a natural way to a Gibbsian potential φ. We prove that Weak Mixing condition for φ implies the validity of this assumption $(\mathcal{A})$, thus the '*exponential ergodicity*' of the dynamics towards the unique Gibbs measure associated to φ holds (Theorem 4.3). For some special PCA of this class, we verify that our assumption $(\mathcal{A})$ is weaker than the Dobrushin-Vasershtein ergodicity condition and note that the exponential ergodicity holds as soon as there is no phase transition.

2 Stochastic order preserving coupling of PCA

2.1 Definitions and general assumptions

Let the *spin space* S be a finite set, with total order denoted by $\leqslant$. Let P denotes a PCA dynamics on the product space $S^{\mathbb{Z}^d}$, which means a time-homogeneous Markov Chain on $S^{\mathbb{Z}^d}$ whose transition probability kernel P verifies, for all *configuration* $\eta \in S^{\mathbb{Z}^d}$, $\sigma = (\sigma_k)_{k\in\mathbb{Z}^d} \in S^{\mathbb{Z}^d}$,

$$P(\ d\sigma \mid \eta\ ) = \underset{k\in\mathbb{Z}^d}{\otimes}\ p_k(\ d\sigma_k \mid \eta\ ),$$

where for all *site* $k \in \mathbb{Z}^d$, for all η, $p_k(\ .\ |\eta)$ is a probability measure on S, called *updating rule*. In other words, *given the previous time step* $(n-1)$, all the spin values $(\omega_k(n))_{k\in\mathbb{Z}^d}$ at time n are *simultaneously and independently updated*, each one according to the probabilistic rule $p_k(\ .\mid(\omega_k(n-1))_{k\in\mathbb{Z}^d})$. For any subset Δ of $\mathbb{Z}^d$, and for all configurations σ and η of $S^{\mathbb{Z}^d}$, the configuration $\sigma_\Delta\eta_{\Delta^c}$ is defined by σ_k if $k\in\Delta$, else η_k. Let the notation σ_Δ design $(\sigma_k)_{k\in\Delta}$ too. Let Λ be a finite subset of $\mathbb{Z}^d$, which is denoted by $\Lambda \Subset \mathbb{Z}^d$. We call *finite volume PCA dynamics* with *boundary condition* τ ($\tau \in S^{\mathbb{Z}^d}$ or $\tau \in S^{\Lambda^c}$), the Markov Chain on S^Λ whose transition probability P^τ_Λ is defined by:

$$P^\tau_\Lambda(d\sigma_\Lambda \mid \eta_\Lambda\ ) = \underset{k\in\Lambda}{\otimes}\ p_k(\ d\sigma_k \mid \eta_\Lambda\tau_{\Lambda^c}\ ).$$

It may be identified with the following infinite volume PCA dynamics on $S^{\mathbb{Z}^d}$:

$$P^\tau_\Lambda(d\sigma \mid \eta_\Lambda\ ) = \underset{k\in\Lambda}{\otimes}\ p_k(\ d\sigma_k \mid \eta_\Lambda\tau_{\Lambda^c}\ ) \otimes \delta_{\tau_{\Lambda^c}}(d\sigma_{\Lambda^c}) \tag{2.1}$$

where the spins of Λ evolve according to P^τ_Λ, and those of Λ^c are almost surely 'freezed' on the value τ.

Let us then recall some usual notations. For ν probability measure on $S^{\mathbb{Z}^d}$ (equipped with the Borel σ-field associated to the product topology), νP refers to the law at time 1 of the PCA dynamics with law ν at time 0, in other words $\nu P(d\sigma) = \int P(d\sigma|\eta)\nu(d\eta)$. Recursively $\nu P^{(n)} = (\nu P^{(n-1)})P$ is the law at time n of the system evolving according to the PCA dynamics P and initial law ν at time 0. For each function f on $S^{\mathbb{Z}^d}$, $P(f)$ denotes the function defined by $P(f)(\eta) = \int f(\sigma)P(d\sigma|\eta)$.

In the sections 3 and 4, PCA dynamics studied are *non degenerate* ones. It means the following condition holds:

$$\forall k \in \mathbb{Z}^d,\ \forall \eta \in S^{\mathbb{Z}^d}, \forall s \in S, \quad p_k(\ s \mid \eta\ ) > 0\ . \tag{2.2}$$

PCA dynamics are said to be *local* if

$$\forall k \in \mathbb{Z}^d, \exists\ V_k \Subset \mathbb{Z}^d, p_k(\ .\ |\eta) = p_k(\ .\ |\eta_{V_k}),$$

that is the probabilistic evolution rule p_k depends only of the spin values of the finite number of the 'neighbouring sites' in V_k. PCA considered in sections 3 and 4 will be assumed to be local.

A PCA dynamics P on the infinite volume space $S^{\mathbb{Z}^d}$ is said to be translation invariant (or *space homogeneous*) if the following condition holds:

$$\forall k \in \mathbb{Z}^d,\ \forall s \in S,\ \forall \eta \in S^{\mathbb{Z}^d}, \quad p_k(\ s \mid \eta\ ) = p_0(\ s \mid \theta_{-k}\eta\ )\ ,$$

where $\theta_{k_0}(\sigma)$ defines the translation of a configuration σ of $S^{\mathbb{Z}^d}$ with $\theta_{k_0}(\sigma) = (\sigma_{k-k_0})_{k\in\mathbb{Z}^d}$. PCA dynamics will in sections 3 and 4 be assumed to be translation invariant too.

Let us now defined some notions of stochastic ordering $\preccurlyeq$. Two configurations σ and η of S^Λ (with $\Lambda \subset \mathbb{Z}^d$) are said to satisfy $\sigma \preccurlyeq \eta$ if $\forall k \in \Lambda, \sigma_k \leqslant \eta_k$. A real function f on S^Λ will then be said to be increasing if $\sigma \preccurlyeq \eta$ implies $f(\sigma) \leqslant f(\eta)$. Thus two probability measures ν_1 and ν_2 satisfy the stochastic ordering $\nu_1 \preccurlyeq \nu_2$ if, for all increasing functions f on S^Λ, $\nu_1(f) \leqslant \nu_2(f)$, with the notation $\nu_i(f) = \int f(\sigma)\nu_i(d\sigma)$. As Markov chain, a PCA dynamics P on S^Λ ($\Lambda \subset \mathbb{Z}^d$) is said to be *attractive* if for all increasing function f, $P(f)$ is still increasing. Let us define too, for $s \in S, \sigma \in S^\Lambda$, the function $G_k(s,\sigma)$ by:

$$G_k(s,\sigma) = \sum_{s' \geqslant s} p_k(s'|\sigma), \tag{2.3}$$

and note that $G_k(s,\sigma)$ is always a decreasing function in s since $G_k(s,\sigma) = 1 - F_k(s,\sigma)$, where $F_k(s,\sigma)$ is the repartition function of $p_k(.|\sigma)$. It is then easy to prove that a PCA dynamics is attractive if, and only if, for all k in Λ, and all value $s \in S$, the quantity $G_k(s,\sigma)$ defined by (2.3) is increasing in σ.

2.2 Increasing synchronous coupling of PCA

Coupling techniques for stochastic processes are now established powerful tools for the analyse of the time asymptotic behaviour of Interacting Particle Systems (see for instance [18]). It means the construction of a probability space on which several dynamics may evolve at the same time. The original idea for general coupling techniques and their applications comes from the pioneer work of Doeblin ([7]). See the references [20, 19] for more detailed informations. Here we construct in a new way a coupling of a finite number of (possible different) PCA dynamics which will be a PCA dynamics too and which has the property to preserve stochastic ordering. As far as we know, this kind of coupling was only mentioned in the following works: Steif (see [31]) defines such a coupling but just for two PCA and S restricted to $\{-1,+1\}$; and Lopez and Sanz (see [21]) proposed a general-but not easy to use-approach. In both of those works, none of the properties we need were studied. Moreover we give in this section a simple way to construct such a coupling which is efficient for numerical simulations' algorithm.

By coupling of two time homogeneous Markovian dynamics P and P' defined on a state space E we mean a Markov Chain Q on $E \times E$, such that marginal dynamics coincide respectively with P and P'. Generalisation to coupling of a finite number of Markovian dynamics follows easily. A particular important case for coupling PCA dynamics is when Q has a PCA form too. Let $P^1, P^2, \ldots, P^N$ be N probabilistic cellular automata dynamics, each P^i being defined on $S^{\mathbb{Z}^d}$ thanks to its updating rule $(p^i_k)_{k\in\mathbb{Z}^d}$. We call *synchronous coupling* of the PCA dynamics $P^1, P^2, \ldots, P^N$ a Markovian dynamics Q on $(S^{\mathbb{Z}^d})^N$, coupling of the $(P^i)_{1\leqslant i\leqslant N}$, which is a PCA dynamics too. It means that Q's updating rules $(q_k)_{k\in\mathbb{Z}^d}$ are such that:

$$\begin{aligned}\forall i \in \{1,\ldots,N\}, \quad \forall s^i \in S, \qquad & \forall \zeta^i \in S^{\mathbb{Z}^d}, \\ p^i_k(s^i \mid \zeta^i) \quad = \quad & \sum_{s^j\in S, j\neq i} q_k\big(\,(s^1,\ldots,s^N) \bigm| (\zeta^1,\ldots,\zeta^N)\big).\end{aligned}$$

To study ergodicity, a coupling which preserves stochastic ordering is convenient. Before establishing the main result of this section, we introduce a notion of order between N PCA dynamics on $S^{\mathbb{Z}^d}$.

Definition 2.1 *Let $(P^1, P^2, \ldots, P^N)$ be a N-uple of PCA dynamics where $N \geqslant 2$ and $P^i = (p^i_k)_{k\in\mathbb{Z}^d}$ $(1 \leqslant i \leqslant N)$. It is said* increasing *if:*

$$\forall k \in \mathbb{Z}^d, \forall (\zeta^1, \zeta^2, \ldots, \zeta^N) \in (S^{\mathbb{Z}^d})^N \text{ such that } \zeta^1 \preccurlyeq \zeta^2 \preccurlyeq \ldots \preccurlyeq \zeta^N, \forall s \in S$$

$$G^1_k(s \mid \zeta^1) \leqslant G^2_k(s \mid \zeta^2) \leqslant \ldots \leqslant G^N_k(s \mid \zeta^N),$$

where, according to (2.3), $G^i_k(s,\sigma) = \sum_{s'\geqslant s} p^i_k(s'|\sigma)$.

A fundamental example of an increasing N-uple of PCA dynamics is: if P is an attractive PCA dynamics then for all $N \geqslant 2$, the N-uple $(P, P, \ldots, P)$ is increasing.

Here is now the statement:

Theorem 2.2 *Let $(P^i)_{1\leqslant i\leqslant N}$ be N probabilistic cellular automata dynamics on S^Λ. It exists a synchronous coupling written $P^1 \circledast P^2 \circledast \ldots \circledast P^N$ with the following property: for all initial configuration $(\sigma^1, \ldots, \sigma^N)$ such that $\sigma^1 \preccurlyeq \sigma^2 \preccurlyeq \ldots \preccurlyeq \sigma^N$ and for all time $n \geqslant 1$,*

$$P^1 \circledast \ldots \circledast P^N \ \big(\ \omega^1(n) \preccurlyeq \ldots \preccurlyeq \omega^N(n) \bigm| (\omega^1,\ldots,\omega^N)(0) = (\sigma^1,\ldots,\sigma^N)\ \big) = 1. \qquad (2.4)$$

Such a coupling $P^1 \circledast P^2 \circledast \ldots \circledast P^N$ will be called increasing synchronous coupling *of $(P^1, P^2, \ldots, P^N)$.*

Proof: We explain here the way to construct explicitly the coupling $P^1 \circledast P^2 \circledast \ldots \circledast P^N$, the fact that it preserves stochastic ordering is then easy to check. Because S is a totally ordered set, let us enumerate the spin set elements with:

$$S = \{-, \ldots, s, s+1, \ldots, +\},$$

where $+$ (resp. $-$) denotes-symbolically-the maximal (resp. minimal) of S, and 's+1' denotes the successive element of s according to the increasing ($\leqslant$) enumeration.

Let n be a fixed step time. We now explain how to construct the configuration $(\omega^1, \ldots, \omega^N)(n+1)$, knowing the configuration $(\omega^1, \ldots, \omega^N)(n)$. Let $(U_k)_{k\in\Lambda}$ be a family of independent identically distributed uniform laws on $[0,1]$. Since we are constructing a synchronous coupling, it is enough to define the rule for a fixed site $k \in \Lambda$. Let call r a realization of the random variable U_k. Use the following algorithmic rule to choose the value $\omega^i_k(n+1)$ for any i ($1 \leqslant i \leqslant N$):

$$\text{if } G^i_k(s+1, \omega^i(n)) \leqslant r < G^i_k(s, \omega^i(n)) \text{ then assign } \omega^i_k(n+1) = s \ . \tag{2.5}$$

Note that $G^i_k(+, \omega^i(n)) = p^i_k(+|\omega^i(n))$ and $G^i_k(-, \omega^i(n)) = 1$.

Remark that the stochastic dependence between the components i comes from the fact that we use the *same* realization r of U_k for *all* the components. ■

Pay attention to the following *compatibility property*, easy to check (see Proposition 5.3.1 *in* [22]), that the introduced coupling presents. Let N and N' be two integers such that $1 \leqslant N < N'$. Let $(P^1, \ldots, P^{N'})$ be N' PCA dynamics. The projection of the coupling $P^1 \circledast P^2 \ldots \circledast P^{N'}$ on any N components coincides with the direct coupling of these N dynamics. In particular, when these dynamics are identical (let us say, to P), the marginal of $P^{\circledast N'}$ on N components chosen in $\{1, .., N'\}$ is the same as the coupling $P^{\circledast N}$. Using this property, from now on, the notation $\mathbb{P}$ will denote the coupling $P \circledast P \circledast \ldots \circledast P$ of N times the same PCA dynamics P, where N will be a finite large enough number. It means:

$$\mathbb{P} = P^{\circledast N}. \tag{2.6}$$

Moreover, if P is attractive, then, using Theorem 2.2, we known that the coupling $\mathbb{P}$ will preserve stochastic ordering.

2.3 Comparison of finite & infinite volume PCA

In order to study, in section 3, the behaviour of a PCA dynamics P on $S^{\mathbb{Z}^d}$ using finite volume associated dynamics P^τ_Λ on S^Λ with $\Lambda \Subset \mathbb{Z}^d$, we need some preliminary remarks and establish two lemmas.

Remark first the following property, which is characteristics of discrete time Interacting Particle Systems. Let define $\overline{\Lambda} = \cup_{k\in\Lambda} V_k = \overline{\Lambda}^{(1)}$, and:

$$\overline{\Lambda}^{(2)} = \cup_{k\in\overline{\Lambda}} V_k = \overline{\overline{\Lambda}^{(1)}}^{(1)}, \ \ldots \ \overline{\Lambda}^{(n)} = \cup_{k\in\overline{\Lambda}^{(n-1)}} V_k \ .$$

For n fixed, for all finite subset Λ of $\mathbb{Z}^d$, for all configurations $(\sigma, \eta) \in (S^{\mathbb{Z}^d})^2$ such that $\sigma_{\overline{\Lambda}^{(n)}} \equiv \eta_{\overline{\Lambda}^{(n)}}$ we then have:

$$\mathbb{P}\left(\omega^1_\Lambda(n) \equiv \omega^2_\Lambda(n) \middle| (\omega^1, \omega^2)(0) = (\sigma, \eta)\right) = 1. \tag{2.7}$$

We now establish the following useful lemma. For any time $n \in \mathbb{N}$, let us define the quantity, which will be used in section 3 in order to control the ergodicity:

$$\rho(n) = \mathbb{P}\left(\omega_0^1(n) \neq \omega_0^2(n) \Big| (\omega^1, \omega^2)(0) = (\boldsymbol{-}, \boldsymbol{+})\right), \tag{2.8}$$

where $\boldsymbol{+}$ (resp. $\boldsymbol{-}$) denotes the configuration of $S^{\mathbb{Z}^d}$ equal, in all sites, to $+$ (resp. $-$).

Lemma 2.3 *Let P be an attractive PCA dynamics, and $\mathbb{P}$ denotes its coupling introduced in (2.6). Let $\sigma, \eta \in S^{\mathbb{Z}^d}$ be such that $\sigma \preccurlyeq \eta$. The following inequality holds:*

$$\mathbb{P}\left(\omega_0^1(n) \neq \omega_0^2(n) \Big| (\omega^1, \omega^2)(0) = (\sigma, \eta)\right) \leqslant \rho(n) \ .$$

Proof: The proof is straightforward using the compatibility property (stated at the end of Sub-section 2.2) and $\mathbb{P}$'s property (2.4). ■

From now on, let Λ be a finite subset of $\mathbb{Z}^d$. Let $P_\Lambda^{\boldsymbol{+}}$ (resp. $P_\Lambda^{\boldsymbol{-}}$) be the dynamics on S^Λ defined in (2.1) with the maximal (resp. minimal) boundary condition $\boldsymbol{+}$ (resp. $\boldsymbol{-}$). If the PCA dynamics P is attractive, it is easy to check that, $(P_\Lambda^{\boldsymbol{-}}, P, \ldots, P, P_\Lambda^{\boldsymbol{+}})$ is increasing, and thus the coupling $P_\Lambda^{\boldsymbol{-}} \circledast P \circledast \ldots \circledast P \circledast P_\Lambda^{\boldsymbol{+}}$ has the property of preserving stochastic order.

Lemma 2.4 *Let P be an attractive PCA dynamics and $\Lambda \Subset \mathbb{Z}^d$. Then,*

- *for each initial condition ξ on $S^{\mathbb{Z}^d}$ and for any time n, we have:*

$$\begin{aligned} &P_\Lambda^{\boldsymbol{-}}\big(\omega(n) \in . \big| \omega(0) = \xi_\Lambda(\boldsymbol{-})_{\Lambda^c}\big) \\ &\qquad \preccurlyeq P\big(\omega(n) \in . \big| \omega(0) = \xi\big) \preccurlyeq P_\Lambda^{\boldsymbol{+}}\big(\omega(n) \in . \big| \omega(0) = \xi_\Lambda(\boldsymbol{+})_{\Lambda^c}\big) \end{aligned} \tag{2.9}$$

- *the following inequality holds:*

$$\rho(n) \leqslant P_\Lambda^{\boldsymbol{-}} \circledast P_\Lambda^{\boldsymbol{+}}(\omega_0^1(n) \neq \omega_0^2(n) \ |(\omega^1, \omega^2)(0) = (\boldsymbol{-}, \boldsymbol{+})), \tag{2.10}$$

where $(\rho(n))_{n \in \mathbb{N}^*}$ is defined by (2.8).

Proof: Since the coupling $P_\Lambda^{-} \circledast P \circledast P_\Lambda^{+}$ preserves stochastic ordering, (2.9) is a consequence of the fact that any initial condition ξ in $S^{\mathbb{Z}^d}$ is such that $\xi_\Lambda(\boldsymbol{-})_{\Lambda^c} \preccurlyeq \xi \preccurlyeq \xi_\Lambda(\boldsymbol{+})_{\Lambda^c}$.

(2.10) comes from the preserving stochastic order property and compatibility property of the coupling $P_\Lambda^{\boldsymbol{-}} \circledast P \circledast P \circledast P_\Lambda^{\boldsymbol{+}}$. ■

3 Ergodicity for attractive PCA dynamics

Let us first emphasise the fact that all the measures considered here are probability measures. From now on, PCA dynamics considered will always be local, translation invariant, non degenerate and attractive.

3.1 Stationary measures

Before stating the main result in the next section, we prove two results, using dynamics' attractivity. The first one (Proposition 3.1) establishes that the unique finite volume stationary measure ν_Λ^τ associated to finite volume dynamics P_Λ^τ increases (in the sense of stochastic order) when the boundary condition τ increases. It is a usual result for Glauber dynamics, but note that in our context, neither the explicit form of these measures is known, nor any (ferromagnetic) Gibbsian nature. This property will be fundamental for the development of our argumentation, and is essentially a consequence of the existence of the preserving order coupling.

The second result (Proposition 3.2) identifies extremal measures–with respect to the stochastic order–of the set of infinite volume stationary measures. They coincide with spatial limit of finite volume stationary measures with extremal boundary conditions, and with infinite volume temporal asymptotics of deterministic initial conditions $+$ and $-$.

Proposition 3.1 *Let Λ be a finite subset of $\mathbb{Z}^d$. For all attractive PCA dynamics, stationary measures of finite volume associated dynamics P_Λ^τ have the following monotonicity property: $\tau \preccurlyeq \tau' \Rightarrow \nu_\Lambda^\tau \preccurlyeq \nu_\Lambda^{\tau'}$. In particular, the measures ν_Λ^+ (resp. ν_Λ^-) is the maximal (resp. minimal) measure of the set $\{\nu_\Lambda^\tau \;:\; \tau \in S^{\Lambda^c}\}$.*

Proof: Let τ et τ' be two boundary conditions such that $\tau \preccurlyeq \tau'$ and let f be an increasing function on $S^{\mathbb{Z}^d}$. It is easy to check that $(P_\Lambda^\tau, P_\Lambda^{\tau'})$ is an increasing couple, thus $P_\Lambda^\tau \circledast P_\Lambda^{\tau'}$ preserves stochastic order. Let $\sigma \in S^{\mathbb{Z}^d}$ be an initial condition. Because, $\sigma_\Lambda \tau_{\Lambda^c} \preccurlyeq \sigma_\Lambda \tau'_{\Lambda^c}$, at time n inequality is preserved, and using monotonicity of f, we have:

$$P_\Lambda^\tau \circledast P_\Lambda^{\tau'}\big(f(\omega^2(n)) - f(\omega^1(n))|\;(\omega^1,\omega^2)(0) = (\sigma,\sigma)\;\big) \geqslant 0\;.$$

Thus

$$P_\Lambda^\tau(f(\omega(n)) \mid \omega(0) = \sigma) \leqslant P_\Lambda^{\tau'}(f(\omega(n)) \mid \omega(0) = \sigma).$$

Conclusion follows letting n going to infinity, and using finite volume ergodicity. ■

For L integer, let us now denote by $\mathcal{B}(L)$ the ball $\mathcal{B}(0, L)$:

$$\mathcal{B}(L) = \{k \in \mathbb{Z}^d \;:\; \|k\|_1 \leqslant L\}\;, \tag{3.1}$$

where $\|k\|_1 = \sum_{i=1}^d |k_i|$ with $k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$.

Proposition 3.2 *Let P be an attractive PCA dynamics and $\nu_{\mathcal{B}(L)}^\tau$ be the stationary measure of the finite volume associated dynamics $P_{\mathcal{B}(L)}^\tau$. Then, the volume limits $\lim_{L\to\infty} \nu_{\mathcal{B}(L)}^- \otimes \delta_{(-)_{\mathcal{B}(L)^c}}$ and $\lim_{L\to\infty} \nu_{\mathcal{B}(L)}^+ \otimes \delta_{(+)_{\mathcal{B}(L)^c}}$ exist, respectively coincide with the temporal limits: $\lim_{n\to\infty} \delta_- P^{(n)}$ and $\lim_{n\to\infty} \delta_+ P^{(n)}$. Furthermore they are the maximal and the minimal elements (eventually equal) of the set $\mathcal{S}$ of stationary measures for P. This means: all P-stationary measure ν verifies:*

$$\nu^- \preccurlyeq \nu \preccurlyeq \nu^+ \tag{3.2}$$

where:

$$\nu^+ = \lim_{n\to\infty} \delta_+ P^{(n)} = \lim_{L\to\infty} \;\nu_{\mathcal{B}(L)}^+ \otimes \delta_{(+)_{\mathcal{B}(L)^c}}$$

and

$$\nu^- = \lim_{n\to\infty} \delta_- P^{(n)} = \lim_{L\to\infty} \nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} .$$

In particular, P admits a unique stationary measure ν if and only if $\nu^- = \nu^+$.

Proof: Note that the limits $\lim_{L\to\infty}(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})$ and $\lim_{L\to\infty}(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}})$ exist due to monotonicity of the following sequences: $(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-\mathbf{1})_{\mathcal{B}(L)^c}})_L$ and $(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+\mathbf{1})_{\Lambda^c}})_L$. This comes from the fact that $\wp_\Lambda\ \nu^+_{\Lambda'} \preccurlyeq \nu^+_\Lambda$ where Λ and Λ' are two finite subsets of $\mathbb{Z}^d$ such that $\Lambda \Subset \Lambda'$, and $\wp_\Lambda$ denotes the projection on Λ. This last relation is easily checked using the increasing coupling $(P^+_{\Lambda'}, P^+_\Lambda)$. Since $\nu^{\mathbf{s}}_{\mathcal{B}(L)}$ is $P^{\mathbf{s}}_\Lambda$-stationary, the limits $\lim_{L\to\infty}(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})$ and $\lim_{L\to\infty}(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}})$ are P-stationary.

Let ν be a P-stationary measure, and L any positive integer. Since the coupling $P^-_{\mathcal{B}(L)} \circledast P \circledast P^+_{\mathcal{B}(L)}$ preserves stochastic order, and using finite volume ergodicity, one can state:

$$\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} \preccurlyeq \nu \preccurlyeq \nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}}. \tag{3.3}$$

We then have:

$$\lim_{L\to\infty} \nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} \preccurlyeq \nu \preccurlyeq \lim_{L\to\infty} \nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}}. \tag{3.4}$$

On the other hand, it is easy to check $\delta_+ P \preccurlyeq \delta_+$, so using P's attractivity, $(\delta_+ P^{(n)})_{n\in\mathbb{N}}$ is decreasing. Analogously, $(\delta_- P^{(n)})_{n\in\mathbb{N}}$ is increasing. Thus, the limits $\lim_{n\to\infty} \delta_- P^{(n)}$ and $\lim_{n\to\infty} \delta_+ P^{(n)}$ exist, and then are obviously P-stationary measures.

Let ν be a P-stationary measure. Because P is attractive, and $\delta_- \preccurlyeq \nu \preccurlyeq \delta_+$, we have:

$$\lim_{n\to\infty} \delta_- P^{(n)} \preccurlyeq \nu \preccurlyeq \lim_{n\to\infty} \delta_+ P^{(n)}. \tag{3.5}$$

Using the fact that all measures $\lim_{L\to\infty}(\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}})$, $\lim_{L\to\infty}(\nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\Lambda^c}})$, $\lim_{n\to\infty} \delta_- P^{(n)}$ and $\lim_{n\to\infty} \delta_+ P^{(n)}$ are P-stationary, we apply to them inequalities (3.4) and (3.5). Conclusions follow. ■

Pay attention to the immediate corollary of the Proposition 3.2: Because δ_- and δ_+ are translation invariant, so are $\lim_{n\to\infty} \delta_- P^{(n)} = \nu^-$ and $\lim_{n\to\infty} \delta_+ P^{(n)} = \nu^+$. And then, $\mathcal{S} = \{\nu\} \iff \mathcal{S}_s = \{\nu\}$, where $\mathcal{S}_s$ denotes the subset of $\mathcal{S}$ which are translation invariant.

Thanks to Proposition 3.1, note that $\nu^-_{\mathcal{B}(L)} \preccurlyeq \nu^+_{\mathcal{B}(L)}$ so: $\int \sigma_0\, d\nu^+_{\mathcal{B}(L)} - \int \sigma_0\, d\nu^-_{\mathcal{B}(L)} \geqslant 0$.

Proposition 3.3 *Let S be a totally ordered finite set with maximal (resp. minimal) element denoted by $+$(resp. $-$). Let P be an attractive, translation invariant, non degenerate, local PCA dynamics on $S^{\mathbb{Z}^d}$. The following statements are then equivalent:*

(i) the PCA dynamics P is ergodic;

(ii) it exists only one stationary measure ν;

(iii) it exists only one translation invariant stationary measure ν;

(iv) $\lim_{L\to\infty}\left(\int \sigma_0\, d\nu^+_{\mathcal{B}(L)} - \int \sigma_0\, d\nu^-_{\mathcal{B}(L)}\right) = 0$,

where $\nu^+_{\mathcal{B}(L)}$ (resp. $\nu^-_{\mathcal{B}(L)}$) is the stationary measure of $P^+_{\mathcal{B}(L)}$ (resp. $P^-_{\mathcal{B}(L)}$).

Proof: Implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are trivial. Proof of the implication $(iv) \Rightarrow (i)$ is a consequence of forthcoming Lemma 3.7 and Lemma 3.8. ■

3.2 Main result

In Theorem 3.4 we present our main result. Let f be a real valued function on $S^{\mathbb{Z}^d}$. It is said *local* if it depends only on a finite number of sites, that is:

$$\exists \Lambda_f \Subset \mathbb{Z}^d,\ \forall \sigma \in S^{\mathbb{Z}^d},\ f(\sigma) = f(\sigma_{\Lambda_f}).$$

We define, for each f continuous function on the compact $S^{\mathbb{Z}^d}$and for all k in $\mathbb{Z}^d$,

$$\Delta_f(k) = \sup\left\{\left|f(\sigma) - f(\eta)\right| \ :\ (\sigma,\eta) \in (S^{\mathbb{Z}^d})^2, \sigma_{\{k\}^c} \equiv \eta_{\{k\}^c}\right\},$$

and the semi-norm $|||\ f\ ||| = \sum_{k\in\mathbb{Z}^d} \Delta_f(k)$.

Theorem 3.4 *Let S be a totally ordered finite set with maximal (resp. minimal) element denoted by $+$(resp. $-$). Let P be an attractive, translation invariant, non degenerate, local PCA dynamics on $S^{\mathbb{Z}^d}$. Let $\nu^+_{\mathcal{B}(L)}$ (resp. $\nu^-_{\mathcal{B}(L)}$) be the stationary measure of $P^+_{\mathcal{B}(L)}$ (resp. $P^-_{\mathcal{B}(L)}$). The following spatial mixing condition $(\mathcal{A})$:*
$\exists C > 0,\ \exists M > 0,\ \exists L_1 \in \mathbb{N}^*, \forall L \in \mathbb{N}^*, L \geqslant L_1,$

$$\left(\int \sigma_0\, d\nu^+_{\mathcal{B}(L)} - \int \sigma_0\, d\nu^-_{\mathcal{B}(L)}\right) \leqslant Ce^{-ML}, \qquad (\mathcal{A})$$

is equivalent to the convergence of the dynamics P towards the unique equilibrium state ν with exponential rate: $\exists \lambda > 0$, $\exists n_1$, $\forall n \geqslant n_1$, $\forall f$ local function on $S^{\mathbb{Z}^d}$:

$$\sup_{\sigma}\left|\delta_\sigma P^{(n)}(f) - \nu(f)\right| \leqslant 2|||\ f\ |||\ e^{-\lambda n}. \tag{3.6}$$

Sketch of the different steps of the proof : The fact that (3.6) implies $(\mathcal{A})$ is quite straightforward using a usual strategy and the advantage of the coupling $P \circledast P^+_{\mathcal{B}(L)}$ for PCA: see Lemma 3.6.

The more delicate part is then to establish the exponential rate of convergence towards equilibrium. The main framework is partly analogous to Martinelli and Olivieri proof of exponential ergodicity for continuous time Glauber dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$ (see [26]). If we assume the exponential bound $(\mathcal{A})$, then thanks to Lemma 3.8, we know $\lim_{n\to\infty} \rho(n) = 0$. Reporting then assumption $(\mathcal{A})$ in the inequality (3.13), we can use Lemma 3.10 to deduce that $(\rho(n))_{n\in\mathbb{N}^*}$ converge to 0 faster than $\frac{1}{n^d}$. Finally, using inequality (3.12) and Lemma 3.11, we conclude that $\rho(n)$ converges to 0 exponentially fast; thus, thanks to inequality (3.8), conclusion holds. ■.

3.3 Proof of previously mentioned lemmas

In this section, we state several results used to prove that *(iv)* implies *(i)* in Proposition 3.3 and the previous main result. In all this subsection P denotes a PCA dynamics as stated in Theorem 3.4. First let state the following lemma, whose proof is straightforward:

Lemma 3.5 *Let* $(\Omega, \mathfrak{A}, \mathcal{P})$ *be a probability space, and* Z *a random variable with values in a finite set* $\{z_1 < \ldots < z_m\}$ *of* $\mathbb{R}$*, such that* $\mathcal{P}(Z \geqslant 0) = 1$*. Then, defining* $\kappa = \max\{\frac{1}{z_i}, z_i > 0, 1 \leqslant i \leqslant m\}$ *and* $\kappa' = max\{z_i, 1 \leqslant i \leqslant m\}$ *(which do not depend on the law of* Z *under* $\mathcal{P}$*) we have:* $\mathcal{P}(Z \neq 0) \leqslant \kappa \int Z d\mathcal{P}$ *and* $\int Z d\mathcal{P} \leqslant \kappa' \mathcal{P}(Z \neq 0)$*.*

We now state the previously announced

Lemma 3.6 *The convergence towards equilibrium* ν *of* P *with exponential rate (3.6) implies* $(\mathcal{A})$ *for finite volume stationary measures.*

Proof: Let L be a fixed integer, larger than $L_1 = n_1$ where n_1 is defined in (3.6). Using (3.3) for the positivity of each term, we write:

$$0 \leqslant \int \sigma_0 \, d\nu^{+}_{\mathcal{B}(L)} - \int \sigma_0 \, d\nu^{-}_{\mathcal{B}(L)} = \Big(\int \sigma_0 \, d\nu^{+}_{\mathcal{B}(L)} - \int \sigma_0 \, d\nu \Big) + \Big(\int \sigma_0 \, d\nu - \int \sigma_0 \, d\nu^{-}_{\mathcal{B}(L)} \Big),$$

and we will state that each part is lower than $2||| \, f_0 \, ||| \, e^{-\lambda L}$ (where $f_0(\sigma) = \sigma_0$). We only give the proof for $\int \sigma_0 \, d\nu - \int \sigma_0 \, d\nu^{+}_{\mathcal{B}(L)}$ since the proof with respect to the $-$ boundary condition is analogous.

Note that, for any $n \in \mathbb{N}^*$,

$$\begin{aligned}
\int \sigma_0 \, d\nu^{+}_{\mathcal{B}(L)} - \int \sigma_0 \, d\nu &= \int \sigma_0 d\nu^{+}_{\mathcal{B}(L)} - \delta_{+} {P^{+}_{\mathcal{B}(L)}}^{(n)}(f_0) \\
&+ \delta_{+} {P^{+}_{\mathcal{B}(L)}}^{(n)}(f_0) - \delta_{+} P^{(n)}(f_0) \\
&+ \delta_{+} P^{(n)}(f_0) - \int \sigma_0 \, d\nu.
\end{aligned}$$

Using the monotonicity of $P^{+}_{\mathcal{B}(L)} \circledast P^{+}_{\mathcal{B}(L)}$ it is easy to check that the first term is non positive. Using the assumption (3.6) the third term is bounded from above by $2||| \, f_0 \, ||| \, e^{-\lambda n}$ ($\forall n \geqslant n_1$).

Choose now $n = L$. Rewrite the second term as $\mathbb{Q}^{+,+}(\omega_0^2(n) - \omega_0^1(n))$ where

$$\mathbb{Q}^{+,+}(\, . \,) = P \circledast P^{+}_{\mathcal{B}(L)}\big(\, . \, |(\omega^1, \omega^2)(0) = (+,+)\big).$$

Using Lemma 3.5, we bound the second term from above with $\kappa' \, \mathbb{Q}^{+,+}(\omega_0^2(n) \neq \omega_0^1(n))$. According to the construction of the coupling, remark that, with respect to $\mathbb{Q}^{+,+}(.)$, $\omega_0^2(n) \neq \omega_0^1(n)$ is possible only if it exists a previous time n' $(0 < n' < n)$ and a site k in $\mathcal{B}(L)^c \cap \overline{\{0\}}^{(n')}$ such that $\omega_k^2(n') = \omega_k^1(n') < +$. By taking $n = L$, we have $\overline{\{0\}}^{(n')} \subset \mathcal{B}(L)$; so is this event not possible. It ensures $\mathbb{Q}^{+,+}(\omega_0^2(n) \neq \omega_0^1(n)) = 0$. Thus is $(\mathcal{A})$ proved. ■

Lemma 3.7 *Let $\rho(n)$ be the quantity defined in (2.8). The sequence $(\rho(n))_{n\in\mathbb{N}^*}$ is decreasing, and for all local functions f, and all configurations σ and η:*

$$\Big|P(f(\omega(n))|\omega(0)=\sigma)-P(f(\omega(n))|\omega(0)=\eta)\Big| \leqslant 2 \;|||\; f \;|||\; \rho(n)\ . \tag{3.7}$$

Thus, if $\lim_{n\to\infty}\rho(n)=0$, the dynamics P is ergodic, and:

$$\sup_{\sigma}\Big|P(f(\omega(n))|\omega(0)=\sigma)-\nu(f)\Big| \leqslant 2 \;|||\; f \;|||\;\; \rho(n)\ , \tag{3.8}$$

where ν denotes the unique stationary measure.

Proof: The monotonicity of the sequence $(\rho(n))_{n\in\mathbb{N}^*}$ comes from the coalescence property of the increasing coupling $\mathbb{P}$. For any σ,η configurations in $S^{\mathbb{Z}^d}$, let us write:

$$\begin{aligned}
&\Big|P(f(\omega(n))|\omega(0)=\sigma)-P(f(\omega(n))|\omega(0)=\eta)\Big| \\
&\leqslant\ \Big|P(f(\omega(n))|\omega(0)=-)-P(f(\omega(n))|\omega(0)=\sigma)\Big| \\
&\qquad +\Big|P(f(\omega(n))|\omega(0)=-)-P(f(\omega(n))|\omega(0)=\eta)\Big| \\
&=\ \Big|\mathbb{P}\Big(f(\omega^1(n))-f(\omega^2(n))\ \Big|(\omega^1,\omega^2)(0)=(-,\sigma)\Big)\Big| \\
&\qquad +\Big|\mathbb{P}\Big(f(\omega^1(n))-f(\omega^2(n))\ \Big|(\omega^1,\omega^2)(0)=(-,\eta)\Big)\Big|.
\end{aligned} \tag{3.9}$$

On the other hand, because f is local, for all ξ^1,ξ^2, $\Big|f(\xi^1)-f(\xi^2)\Big|$ depends only on $\xi^1_{\Lambda_f}$ and $\xi^2_{\Lambda_f}$. Using interpolating configurations between $\xi^1_{\Lambda_f}$ and $\xi^2_{\Lambda_f}$ we write:
$|f(\xi^1)-f(\xi^2)|\leqslant\sum_{k\in\Lambda_f}\Delta_f(k)\mathbb{1}_{\sigma_k\neq\eta_k}$, and so:

$$\begin{aligned}
&\Big|\mathbb{P}\big(f(\omega^1(n))-f(\omega^2(n))\ \big|\ (\omega^1,\omega^2)(0)=(-,\sigma)\big)\Big| \\
&\leqslant\ \sum_{k\in\Lambda_f}\|\nabla_k(f)\|_\infty\mathbb{P}\Big(\omega^1_k(n)\neq\omega^2_k(n)\Big|(\omega^1,\omega^2)(0)=(-,\sigma)\Big)\ .
\end{aligned}$$

Because P is translation invariant, so is $\mathbb{P}$, and then:

$$\begin{aligned}
&\Big|\mathbb{P}\big(f(\omega^1(n))-f(\omega^2(n))\ \big|\ (\omega^1,\omega^2)(0)=(-,\sigma)\big)\Big| \\
&\leqslant\ \sum_{k\in\Lambda_f}\|\nabla_k(f)\|_\infty\mathbb{P}\Big(\omega^1_0(n)\neq\omega^2_0(n)\Big|(\omega^1,\omega^2)(0)=(-,\theta_{-k}\sigma)\Big) \\
&\leqslant\ |||\ f\ |||\ \rho(n),
\end{aligned}$$

where the last inequality comes from Lemma 2.3. Equation (3.9) then gives:

$$\Big|P(f(\omega(n))|\omega(0)=\sigma)-P(f(\omega(n))|\omega(0)=\eta)\Big| \leqslant 2 \;|||\; f \;|||\; \rho(n)\ .$$

If we then assume $\lim_{n\to\infty}\rho(n)=0$, this implies the ergodicity of the dynamics, and then integrating with respect to the unique stationary measure ν, and taking the supremum in the other configuration, inequality (3.8) holds. ■

Note that due to the monotonicity of $\rho(.)$, we can restrict ourselves to the case $\rho(.)>0$.

Lemma 3.8 *It exists κ such that, for each Λ subset of $\mathbb{Z}^d$, the following inequality holds:*

$$\lim_{n\to\infty} \rho(n) \leqslant \kappa\Big(\int \sigma_0 \; d\nu_\Lambda^+ - \int \sigma_0 \; d\nu_\Lambda^-\Big). \tag{3.10}$$

Proof: Let Λ be a subset of $\mathbb{Z}^d$. Since the coupling preserves the order:

$$P_\Lambda^- \circledast P_\Lambda^+\Big(\omega_0^1(n) \leqslant \omega_0^2(n)) \;\Big|\; (\omega^1(0), \omega^2(0)) = (-,+)\Big) = 1.$$

So, thanks to Lemma 3.5, applied with
$\mathcal{P} = P_\Lambda^- \circledast P_\Lambda^+(\; . \; |(\omega^1(0), \omega^2(0)) = (-,+))$ and $Z = \omega_0^2(n) - \omega_0^1(n)$ we have:

$$\begin{aligned} & P_\Lambda^- \circledast P_\Lambda^+\Big(\omega_0^1(n) \neq \omega_0^2(n)\Big|(\omega^1(0), \omega^2(0)) = (-,+)\Big) \\ \leqslant \; & \kappa\Big(P_\Lambda^+(\omega_0(n)|\omega(0) = +) - P_\Lambda^-(\omega_0(n)|\omega(0) = -)\Big), \end{aligned} \tag{3.11}$$

where $\kappa = (\min\{s - s' \; : \; s > s', s \in S, s' \in S\})^{-1}$. By (2.10), $\rho(n)$ is bounded from above by the l.h.s of equation (3.11). Taking the limit in n, and using the finite volume ergodicity, the r.h.s of equation (3.11) converges to $(\int \sigma_0 d\nu_\Lambda^+ - \int \sigma_0 d\nu_\Lambda^-)$, which concludes the proof. ■

Let us denote by $R = \sup_{k\in\mathbb{Z}^d} \max_{k'\in V_k} \|k' - k\|_1$ the finite range of the local PCA dynamics P.

Lemma 3.9 *The following two inequalities hold:*

$$\forall n \in \mathbb{N}^*, \;\; \rho(2n) \leqslant (2nR + 1)^d \rho^2(n) \; ; \tag{3.12}$$

$$\forall n, \forall L \in \mathbb{N}^*, \;\; \rho(2n) \leqslant 2(2L+1)^d \rho^2(n) + 2\kappa\Big(\int \sigma_0 \; d\nu_{\mathcal{B}(L)}^+ - \int \sigma_0 \; d\nu_{\mathcal{B}(L)}^-\Big) \; , \tag{3.13}$$

where κ is defined in Lemma 3.5.

Proof: Let n be a fixed integer.
Proof of inequality (3.12) Let $\nu_n^{-,+}$ denote the distribution on $S^{\mathbb{Z}^d} \times S^{\mathbb{Z}^d}$:

$$\nu_n^{-,+}(\; . \;) = \mathbb{P}\Big((\omega^1, \omega^2)(n) \in . \; \Big|(\omega^1, \omega^2)(0) = (-,+)\Big) \; .$$

Using Markov property of $\mathbb{P}$:

$$\rho(2n) = \int \mathbb{P}\Big(\omega_0^1(2n) \neq \omega_0^2(2n)\Big|(\omega^1, \omega^2)(n) = (\xi^-, \xi^+)\Big)\nu_n^{-,+}(d\xi^-, d\xi^+) \; .$$

Note that $\nu_n^{-,+}$ almost surely, $\xi^- \preccurlyeq \xi^+$, thus $\mathbb{P}(\; . \; | \; (\omega^1, \omega^2)(n) = (\xi^-, \xi^+))$ preserves stochastic order. Let A be the subset of $S^{\mathbb{Z}^d} \times S^{\mathbb{Z}^d}$ defined by

$$A = \{(\xi^-, \xi^+) \;\; : \; \exists k \in \mathbb{Z}^d, \; \|k\|_1 \leqslant nR, \; \xi_k^- \neq \xi_k^+\} \; .$$

So: $A^c = \{(\xi^-, \xi^+) \ : \ \forall k \in \mathcal{B}(nR), \ \xi^-_k = \xi^+_k\}$. We decompose the integral representation of $\rho(2n)$ into two parts, respectively on A and A^c. Thanks to (2.7), observe that the exact control of information's propagation for PCA implies that the integral on A^c vanishes because $\mathcal{B}(nR) \supset \overline{\{0\}}^{(n)}$ so $\xi^-_{\mathcal{B}(nR)} \equiv \xi^+_{\mathcal{B}(nR)}$. Then:

$$\rho(2n) = \int_A \mathbb{P}\Big(\omega^1_0(n) \neq \omega^2_0(n) \Big| (\omega^1, \omega^2)(0) = (\xi^-, \xi^+)\Big) \ \nu^{-,+}_n(d\xi^-, d\xi^+) \ .$$

Using Lemma 2.3, we find $\rho(2n) \leqslant \rho(n) \ \nu^{-,+}_n(A)$. Writing $A = \cup_{\{k \in \mathbb{Z}^d \ : \ \|k\|_1 \leqslant nR\}} \{(\xi^-, \xi^+) \ : \ \xi^-_k \neq \xi^+_k\}$ we deduce:

$$\nu^{-,+}_n(A) \leqslant \sum_{k \in \mathbb{Z}^d, \|k\|_1 \leqslant nR} \mathbb{P}\Big(\omega^1_k(n) \neq \omega^2_k(n) \Big| (\omega^1, \omega^2)(0) = (-, +)\Big).$$

Since P is translation invariant, the conclusion follows from:

$$\begin{aligned} \nu^{-,+}_n(A) &\leqslant \rho(n) \ \#\mathcal{B}(nR) \\ &\leqslant \rho(n) \#\mathcal{B}(nR) = \rho(n)(2nR+1)^d \ , \end{aligned}$$

where $\|k\|_{\max} = \max_{1 \leqslant i \leqslant d} |k_i|$, with $k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$, and $\#\mathcal{B}(nR)$ denotes the cardinality of $\mathcal{B}(nR)$.

Proof of inequality (3.13) Let us first write:

$$\rho(2n) = \int \mathbb{P}\Big(\omega^1_0(2n) \neq \omega^3_0(2n) \Big| (\omega^1, \omega^2, \omega^3)(0) = (-, \eta, +)\Big) \nu(d\eta)$$

where ν denotes a P-stationary measures. Note that $\omega^1_0(n) \leqslant \omega^2_0(n) \leqslant \omega^3_0(n)$, $\mathbb{P}\Big((\omega^1, \omega^2, \omega^3) \in \ . \ \Big| \ (\omega^1, \omega^2, \omega^3)(0) = (-, \eta, +)\Big)$ almost surely, so that

$$\{\omega^1_0(n) \neq \omega^3_0(n)\} = \{\omega^1_0(n) \neq \omega^2_0(n)\} \ \cup \ \{\omega^2_0(n) \neq \omega^3_0(n)\},$$

where the union is non necessarily disjoint (unless cardinality of S is 2). Thus, following decomposition holds:

$$\begin{aligned} \rho(2n) \leqslant &\int \mathbb{P}\Big(\omega^1_0(2n) \neq \omega^2_0(2n) \Big| (\omega^1, \omega^2)(0) = (-, \eta)\Big) \ \nu(d\eta) \\ + &\int \mathbb{P}\Big(\omega^1_0(2n) \neq \omega^2_0(2n) \Big| (\omega^1, \omega^2)(0) = (\eta, +)\Big) \ \nu(d\eta) \ . \end{aligned} \qquad (3.14)$$

It is then enough to prove that each of these quantities are bounded from above by half the quantity wanted.

Consider first the second term in the r.h.s. Let $\nu^{\eta,+}_n$ be the law on $S^{\mathbb{Z}^d} \times S^{\mathbb{Z}^d}$:

$$\nu^{\eta,+}_n = \mathbb{P}\Big((\omega^1, \omega^2)(n) = \ . \ \Big| \ (\omega^1, \omega^2)(0) = (\eta, +)\Big) \ .$$

$$\int \mathbb{P}\Big(\omega_0^1(2n) \neq \omega_0^2(2n) \Big| (\omega^1,\omega^2)(0) = (\eta,+)\Big)\ \nu(d\eta)$$
$$= \iint \mathbb{P}\Big(\omega_0^1(n) \neq \omega_0^2(n) \Big| (\omega^1,\omega^2)(0) = (\xi^1,\xi^2)\Big)\ \nu_n^{\eta,+}(d\xi^1,d\xi^2)\ \nu(d\eta)\ .$$

Let $L \in \mathbb{N}^*$ and $A_L = \{(\xi^1,\xi^2) \in (S^{\mathbb{Z}^d})^2\ :\ (\xi^1)_{\mathcal{B}(L)} \equiv (\xi^2)_{\mathcal{B}(L)}\}$. Let decompose the integration with respect to (ξ^1,ξ^2) into an integration on A_L^c (part (I)) and an integration on A_L (part (II)). We will prove that:

$$(I) = \iint_{A_L^c} \mathbb{P}\Big(\omega_0^1(n) \neq \omega_0^2(n) \Big| (\omega^1,\omega^2)(0) = (\xi^1,\xi^2)\Big)\ \nu_n^{\eta,+}(d\xi^1,d\xi^2)\ \nu(d\eta)$$
$$\leqslant (2L+1)^d \rho^2(n) \tag{3.15}$$

and

$$(II) = \iint_{A_L} \mathbb{P}\Big(\omega_0^1(n) \neq \omega_0^2(n) \Big| (\omega^1,\omega^2)(0) = (\xi^1,\xi^2)\Big)\ \nu_n^{\eta,+}(d\xi^1,d\xi^2)\ \nu(d\eta)$$
$$\leqslant \kappa\Big(\int \sigma_0\ d\nu^+_{\mathcal{B}(L)} - \int \sigma_0\ d\nu^-_{\mathcal{B}(L)}\Big). \tag{3.16}$$

Let us consider part (I). Thanks to $\nu_n^{\eta,+}(\xi^1 \preccurlyeq \xi^2) = 1$, and using Lemma 2.3, we have $(I) \leqslant \rho(n) \int \nu_n^{\eta,+}(A_L^c)\ \nu(d\eta)$. Note that A_L^c may also be written $\cup_{k\in\mathcal{B}(L)}\{(\xi^1,\xi^2)\ :\ (\xi^1)_k \neq (\xi^1)_k\}$. Thus we have:

$$\nu_n^{\eta,+}(A_L^c) \leqslant \sum_{k\in\mathcal{B}(L)} \nu_n^{\eta,+}\{(\xi^1,\xi^2)\ :\ (\xi^1)_k \neq (\xi^2)_k\}\ .$$

At this point, using translation invariance of the coupling, and Lemma 2.3, it comes:

$$\begin{aligned}\nu_n^{\eta,+}\{(\xi^1,\xi^2)\ :\ (\xi^1)_k \neq (\xi^2)_k\} &= \mathbb{P}\Big(\omega_k^1(n) \neq \omega_k^2(n) \Big| (\omega^1,\omega^2)(0) = (\eta,+)\Big)\\ &\leqslant \mathbb{P}\Big(\omega_k^1(n) \neq \omega_k^2(n) \Big| (\omega^1,\omega^2)(0) = (-,+)\Big)\\ &\leqslant \rho(n)\ .\end{aligned}$$

So $\nu_n^{\eta,+}(A_L^c) \leqslant \rho(n)\ (\#\mathcal{B}(L))$, and then (3.15) follows.

Part (II): let $\tau \in S^{\mathcal{B}(L)}$ be fixed, and define sets $A_{L,\tau}$ by:

$$A_{L,\tau} = \{(\xi^1,\xi^2)\ :\ (\xi^1)_{\mathcal{B}(L)} \equiv (\xi^2)_{\mathcal{B}(L)} \equiv \tau\}\ .$$

So $A_L = \bigsqcup_{\tau\in S^{\mathcal{B}(L)}} A_{L,\tau}$, and following decomposition holds:

$$(II) = \int \sum_{\tau\in S^{\mathcal{B}(L)}} \int \mathbb{P}\Big(\omega_0^1(n) \neq \omega_0^2(n) \Big| (\omega^1,\omega^2)(0) = (\xi_1,\xi_2)\Big)\ \mathbb{1}_{A_{L,\tau}}(\xi^1,\xi^2)$$
$$\nu_n^{\eta,+}(d\xi^1,d\xi^2)\ \nu(d\eta)\ .$$

Let us now use the finite volume dynamics. $\nu_n^{\eta,+}$ almost surely, we have $\xi^1 \preccurlyeq \xi^2$, $(\xi^1)_{\mathcal{B}(L)} = (\xi^2)_{\mathcal{B}(L)} = \tau$ and also:

$$\xi^2 = \tau(\xi^2)_{\mathcal{B}(L)^c} \preccurlyeq \tau(+)_{\mathcal{B}(L)^c} \text{ and } \tau(-)_{\mathcal{B}(L)^c} \preccurlyeq \xi^1 = \tau(\xi^1)_{\mathcal{B}(L)^c} \ .$$

Then:

$$P^-_{\mathcal{B}(L)} \circledast P \circledast P \circledast P^+_{\mathcal{B}(L)} \Big(\omega^1 \preccurlyeq \omega^2 \preccurlyeq \omega^3 \preccurlyeq \omega^4 \ \Big| \ \Big| (\omega^1_{\mathcal{B}(L)}, \omega^2, \omega^3, \omega^4_{\mathcal{B}(L)})(0) = (\tau, \tau(\xi^1)_{\mathcal{B}(L)^c}, \tau(\xi^2)_{\mathcal{B}(L)^c}, \tau) \Big) = 1,$$

which implies:

$$\mathbb{P}\Big(\omega^1_0(n) \neq \omega^2_0(n) \Big| (\omega^1, \omega^2)(0) = (\tau\xi^1_{\mathcal{B}(L)^c}, \tau\xi^2_{\mathcal{B}(L)^c})\Big) \leqslant \ P^-_{\mathcal{B}(L)} \circledast P^+_{\mathcal{B}(L)}\big(\omega^1_0(n) \neq \omega^2_0(n) \mid (\omega^1, \omega^2)(0) = (\tau, \tau)\big) \ .$$

We can now write:

$$(II) \leqslant \int \sum_{\tau \in S^{\mathcal{B}(L)}} P^-_{\mathcal{B}(L)} \circledast P^+_{\mathcal{B}(L)}\Big(\omega^1_0(n) \neq \omega^2_0(n) \ \Big| \ (\omega^1, \omega^2)(0) = (\tau, \tau)\Big) \ \nu_n^{\eta,+}(A_{L,\tau}) \ \nu(d\eta) \ . \tag{3.17}$$

Use now the following inequality:

$$\begin{aligned} \nu_n^{\eta,+}(A_{L,\tau}) &= \mathbb{P}\Big(\omega^1(n)_{\mathcal{B}(L)} \equiv \omega^2_{\mathcal{B}(L)}(n) \equiv \tau \Big| (\omega^1, \omega^2)(0) = (\eta, +)\Big) \\ &\leqslant \nu_n^{\eta,+}\Big(\{(\xi^1, \xi^2) \ : \ (\xi^1)_{\mathcal{B}(L)} \equiv \tau\}\Big) = P(\omega_{\mathcal{B}(L)}(n) = \tau | \ \omega_{\mathcal{B}(L)}(0) = \eta) \ . \end{aligned}$$

On the other hand, $P^-_{\mathcal{B}(L)} \circledast P^+_{\mathcal{B}(L)}\Big(\ . \ \Big| \ (\omega^1, \omega^2)(0) = (\tau, \tau)\Big)$ almost surely, we have $\omega^1_0(n) \leqslant \omega^2_0(n)$; so, using Lemma 3.5, we can write

$$P^-_{\mathcal{B}(L)} \circledast P^+_{\mathcal{B}(L)}\Big(\omega^1_0(n) \neq \omega^2_0(n) \ \Big| \ (\omega^1, \omega^2)(0) = (\tau, \tau)\Big) \leqslant \kappa\Big(P^+_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau) - P^-_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau)\Big) \ .$$

Reporting the two last estimates in the equation (3.17) we find

$$(II) \leqslant \kappa \int \sum_{\tau \in S^{\mathcal{B}(L)}} \Big(P^+_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau) - P^-_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau)\Big) P(\omega_{\mathcal{B}(L)}(n) = \tau | \ \omega_{\mathcal{B}(L)}(0) = \eta) \ \nu(d\eta) \ .$$

Let us now denote with (a) and (b) the quantities:

$$(a) = \int \sum_{\tau \in S^{\mathcal{B}(L)}} P^+_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau) \ P(\omega_{\mathcal{B}(L)}(n) = \tau | \ \omega_{\mathcal{B}(L)}(0) = \eta) \ \nu(d\eta)$$

and

$$(b) = \int \sum_{\tau \in S^{\mathcal{B}(L)}} P^-_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau)\ P(\omega_{\mathcal{B}(L)}(n) = \tau \mid \omega_{\mathcal{B}(L)}(0) = \eta)\ \nu(d\eta)\ ,$$

so $(II) \leqslant \kappa\big((a) - (b)\big)$. Let us write $(a) = \int P\Big(f_{n,+}(\omega_{\mathcal{B}(L)}(n)) \Bigm| \omega_{\mathcal{B}(L)}(0) = \eta\Big)\ \nu(d\eta)$ with $f_{n,+}(\tau) = P^+_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau)$. Using the fact that the function $f_{n,+}(.)$ is increasing, and (2.9), we state:

$$(a) \leqslant \int \sum_{\tau \in S^{\mathcal{B}(L)}} P^+_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau)\ P^+_{\mathcal{B}(L)}(\omega_{\mathcal{B}(L)}(n) = \tau \mid \omega_{\mathcal{B}(L)}(0) = \eta_{\mathcal{B}(L)})\ \nu(d\eta)\ .$$

Using Markov property for the $P^+_{\mathcal{B}(L)}$ finite volume dynamics, we find: $(a) \leqslant \nu(f_{2n,+})$. The function $f_{2n,+}$ is increasing; thanks to inequality (3.3), we thus have $(a) \leqslant \nu^+_{\mathcal{B}(L)}(f_{2n,+})$. We can now write:

$$\begin{aligned}(a) &\leqslant \int P^+_{\mathcal{B}(L)}(\omega_0(2n) | \omega_{\mathcal{B}(L)}(0) = \eta_{\mathcal{B}(L)})\ \nu^+_{\mathcal{B}(L)}(d\eta_{\mathcal{B}(L)}) \\ &= \int \sigma_0\ \left(\nu^+_{\mathcal{B}(L)} {P^+_{\mathcal{B}(L)}}^{(2n)}\right)(d\sigma) = \int \sigma_0\ d\nu^+_{\mathcal{B}(L)},\end{aligned}$$

where the last equality comes from the fact that $\nu^+_{\mathcal{B}(L)}$ is stationary for the $P^+_{\mathcal{B}(L)}$ dynamics.

Analogously we prove $(b) \geqslant \int \sigma_0\ d\nu^-_{\mathcal{B}(L)}$ using inequality (2.9), the fact that $f_{n,-}(\xi) = P^-_{\mathcal{B}(L)}\Big(\omega_0(n) \Bigm| \omega(0) = \xi\Big)$ is increasing, and inequality (3.3).

Thus, the following inequality holds:

$$(II) \leqslant \kappa\big((a) - (b)\big) \leqslant \kappa\left(\int \sigma_0\ d\nu^+_{\mathcal{B}(L)} - \int \sigma_0\ d\nu^-_{\mathcal{B}(L)}\right),$$

which gives the estimate of the second term in inequality (3.14). The first term is treated in the same way. So the recursive inequality (3.13) is established. ■

As in [26], we state now some general analytic lemmas:

Lemma 3.10 *If* $\lim_{n\to\infty} \rho(n) = 0$ *and if* $\exists\ (\tilde{C}, M) \in (\mathbb{R}^+_*)^2,\ \exists L_1 \in \mathbb{N}^*, \forall L \in \mathbb{N}^*, L \geqslant L_1, \forall n \in \mathbb{N}^*$

$$\rho(2n) \leqslant 2(2L+1)^d \rho(n)^2 + 2\tilde{C}e^{-ML}$$

then $\lim_{n\to\infty} n^d \rho(n) = 0$.

Proof: Let L be defined according to n: $L(n) = [\![-\frac{2}{M}\log\rho(n)]\!] \in \mathbb{N}$ (where $[\![x]\!]$ denotes the integer part of the real x). Let n be fixed, large enough so that $L(n) \geqslant L_1$. Thanks to the recursive inequality, one easily checks:

$$\rho(2n) \leqslant 2\left[\left(-\frac{4}{M}\log\rho(n) + 1\right)^d + \tilde{C}e^M\right]\rho(n)^2\ .$$

$(\rho(n))_{n\in\mathbb{N}^*}$ is then a decreasing sequence of real positive numbers, with limit 0 and verifying

$$\exists n_0,\ \forall n \geqslant n_0,\ \rho(2n) \leqslant \rho(n)^{\frac{3}{2}}\ . \tag{3.18}$$

It is then quite easy to deduce that $n^d\rho(n)$ tends to 0. ■

Note that inequality (3.12) may also by written:

$$\forall n \in \mathbb{N}^*,\ \rho(2n) \leqslant \hat{C} n^d \rho^2(n)\ ,$$

where we use $(2nR+1)^d \leqslant (3R)^d n^d$ and state $\hat{C} = (3R)^d$.

Lemma 3.11 *If* $\lim_{n\to\infty} n^d\rho(n) = 0$*, and if inequality (3.12) holds then, for all* n_1 *such that* $(2^d\hat{C})\ n_1^d\rho(n_1) < 1$*, we have:*

$$\forall n \geqslant n_1,\ \rho(n) \leqslant e^{-\lambda n}$$

where $\lambda = -\frac{1}{2n_1}\log(2^d\hat{C} n_1^d \rho(n_1)) > 0$.

Proof: Let $(u(n)_{n\in\mathbb{N}})$ be a sequence or real positive numbers defined by $u(n) = n^d\rho(n)$. Thanks to inequality (3.12), we have $u(2n) \leqslant (2^d\hat{C})\ u^2(n)$. Because $\lim_{n\to\infty} n^d\rho(n) = 0$, it exists $n_1 \in \mathbb{N}^*$ such that $\forall n \geqslant n_1,\ (2^d\hat{C})\ n_1^d\rho(n_1) < 1$. Let the sequence $(a_m)_{m\in\mathbb{N}}$ be defined by $a_m = u(2^m n_1)$. Then, one easily checks that $a_{m+1} \leqslant (2^d\hat{C})\ a_m^2$, thus recursively,

$$\forall m \in \mathbb{N},\ a_m \leqslant \frac{\left((2^d\hat{C})\ u(n_1)\right)^{2^m}}{2^d\hat{C}}\ .$$

So:

$$\forall m \geqslant 1,\ \rho(2^m n_1) \leqslant \frac{e^{2^m \ln((2^d\hat{C})\ n_1^d\rho(n_1))}}{(2^{(m+1)d}\hat{C})\ n_1^d}\ .$$

Using $\hat{C} \geqslant 1$, we conclude:

$$\forall m \geqslant 1,\ \rho(2^m n_1) \leqslant e^{-2^{m+1} n_1 \lambda}\ ,$$

which is immediately extended to the whole sequence $(\rho(n))_n$ since $\rho(.)$ is decreasing. ■

4 Reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$

In order to better interpret the meaning of condition $(\mathcal{A})$, and the relevance of Theorem 3.4, we now apply it to a wide class of reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$. This class is defined in subsection 4.1, the main result is stated in subsection 4.2 and comments are to be found in subsection 4.3.

First, let us recall some known facts about *reversible PCA dynamics* (that is to say PCA dynamics whose set of reversible measures is not empty). The study of the qualitative nature of their equilibrium states, as Gibbs measures, was initiated by Kozlov and Vasilyev (see [15, 34]). Gibbs measures, with respect to some dynamics' naturally associated potential, are indeed natural candidates as stationary states. See also [16] for more general ‘Gibbsian’ dynamics. In [3, 22], precise relations were established between the sets of stationary measures, reversible measures and some Gibbs measures (see Proposition 3.3 in [3]). Moreover, unlike what is done (or expected to hold) for continuous time Interacting Particle Systems like Glauber dynamics, or gradient diffusions, it is shown that Gibbs measures may be non stationary for PCA’s dynamics, which is a characteristic manifestation of the discrete time case. Finally, let us recall that a characterisation of the laws of stationary PCA as space-time Gibbs measure on $S^{\mathbb{Z}^d\times\mathbb{Z}}$ was also previously established in [11, 17] for non degenerate PCA.

4.1 Class $\mathcal{C}_0$ of PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$

From now on, assume $S=\{-1,+1\}$. We call class $\mathcal{C}_0$ the family of PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$ whose updating rule $(p_k)_{k\in\mathbb{Z}^d}$ is given for all site k of $\mathbb{Z}^d$, for all configuration $\eta\in S^{\mathbb{Z}^d}$, and for all $s\in S$, by:

$$p_k(s\mid\eta)=\frac{1}{2}\Big(1+s\tanh(\beta\sum_{k'\in\mathbb{Z}^d}\mathcal{K}(k'-k)\eta_{k'})\Big), \tag{4.1}$$

where $\mathcal{K}(.)$ is an interaction function between sites $\mathcal{K}\,:\,\mathbb{Z}^d\to\mathbb{R}$ which is symmetric and has finite range $R>0$ (*i.e.* for all k of $\mathbb{Z}^d$ such that $\|k\|_1>R$ then $\mathcal{K}(k)=0$), and where β is a positive real parameter. Remark that $\beta=0$ is the independent case (sites don’t interact), and that when β increases, the dynamics becomes less and less random. So β may be thought as a kind of inverse temperature parameter. See subsection 4.1.1 in [22] for the generality of the class $\mathcal{C}_0$ among reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$. Due to their definition, PCA dynamics in $\mathcal{C}_0$ are local, translation invariant, non degenerate. It is known (see [15, 1] and [3]) that any PCA dynamics P in $\mathcal{C}_0$ admits at least one reversible measure which is a Gibbs measure associated to the following translation invariant potential φ:

$$\begin{array}{rcl}\varphi_{U_k}(\sigma_{U_k}) &=& -\log\cosh\Big(\beta\sum_j\mathcal{K}(k-j)\sigma_j\Big)\\ \varphi_\Lambda(\sigma_\Lambda) &=& 0\ \ \text{otherwise,}\end{array} \tag{4.2}$$

where $U_k=\{j:\mathcal{K}(k-j)\neq 0\}$ is finite by assumption, and coincide in fact with the set V_k previously associated to PCA dynamics. Moreover Proposition 3.3 in [3] stated the precise relations (see also [2]):

$$\mathcal{R}=\mathcal{S}\cap\mathcal{G}(\varphi)\text{ and }\mathcal{R}_s=\mathcal{S}_s, \tag{4.3}$$

where $\mathcal{S}$ (resp. $\mathcal{R}$) denotes the set of P-stationary (resp. P-reversible) measures, $\mathcal{S}_s$ and $\mathcal{R}_s$ their respective space-translation invariant measures’ parts, and $\mathcal{G}(\varphi)$ the set of Gibbs measures on $S^{\mathbb{Z}^d}$ associated to the potential φ.

One also checks that such a PCA dynamics P is attractive, if and only if function $\mathcal{K}(.)$ is non-negative (see Property 4.1.2 in [22]). From now on, let us assume that $\mathcal{K}$ is non negative.

4.2 Ergodicity under Weak Mixing condition

Mixing conditions define different regions in the domain of absence of phase transition. Strong Mixing Conditions are usually related to the Dobrushin's uniqueness domain, and Weak Mixing conditions are expected to be valid in the main part of the uniqueness domain. See [26, 28] for detailed information. Here, we call Weak Mixing condition for the potential φ, the condition:
$\exists C > 0,\ \exists M > 0,\ \forall L \geqslant 2,$

$$\Big(\int \sigma_0 \ \mu(d\sigma_{\mathcal{B}(L)} | \sigma_{\mathcal{B}(L)^c} = +1) - \int \sigma_0 \ \mu(d\sigma_{\mathcal{B}(L)} | \sigma_{\mathcal{B}(L)^c} = -1) \Big) \leqslant C e^{-ML}, \tag{4.4}$$

where μ is the unique Gibbs measure associated to φ. For ferromagnetic potentials, it is equivalent to usual Weak Mixing condition.

For general PCA in finite volume, reversible measures are not explicitely known; for the class $\mathcal{C}_0$ here considered, explicit form was computed: the unique reversible measure for the PCA dynamics P_Λ^τ is defined by

$$\nu_\Lambda^\tau(\sigma) = \frac{1}{\mathcal{W}_\Lambda^\tau} \prod_{k\in\Lambda} \cosh \left(\beta \sum_{j\in\mathbb{Z}^d} \mathcal{K}(k-j) \tilde{\sigma}_j \right) e^{\beta \sigma_k \sum_{j\in\Lambda^c} \mathcal{K}(k-j)\tau_j}, \tag{4.5}$$

where $\tilde{\sigma} = \sigma_\Lambda \tau_{\Lambda^c}$, and $\mathcal{W}_\Lambda^\tau$ is the normalisation factor (see Proposition 3.1 *in* [3]). Such measure does not coincide with the finite volume Gibbs measures contrary to what happens for Glauber dynamics when detailed balance holds. Nevertheless, they are related as relation (4.6) attempts.

We will not write down all technical computations which prove relations (4.6), (4.8), (4.9), and (4.10). Interested reader may refer respectively to Proposition 4.1.8, Proposition 4.1.9, and Property 4.1.12 *in* [22].

Let Λ, Λ' two finite subsets of $\mathbb{Z}^d$ such that $\Lambda \subset \Lambda'$ and $\partial_i \Lambda \cap \partial_i \Lambda' = \varnothing$, where $\partial_i \Lambda \triangleq \{k \in \Lambda \ : \ V_k \cap \Lambda^c \neq \varnothing\}$. Let τ be a boundary condition of Λ'. The notation $\mu_{\Lambda'}^\tau$ denotes the finite volume Gibbs distribution associated to the potential φ on the volume Λ' with boundary condition τ. We then state:

$$\nu_{\Lambda'}^\tau(d\sigma_\Lambda | \sigma_{\Lambda'\setminus\Lambda}) = \mu_\Lambda^{\sigma_{\Lambda'\setminus\Lambda}\tau_{\Lambda'^c}}(d\sigma_\Lambda) \ . \tag{4.6}$$

In particular, for $\Lambda = \{k\} \subset \Lambda'$ such that $k \notin \partial_i \Lambda'$,we get, for all $\sigma_k \in S$:

$$\nu_{\Lambda'}^\tau(d\sigma_k | \sigma_{\Lambda'\setminus k}) = \mu_{\{k\}}^{\sigma_{\Lambda'\setminus k}\tau_{\Lambda'^c}}(d\sigma_k) \ . \tag{4.7}$$

Pay attention that the potential φ is not really a ferromagnetic potential in the usual sense. However we can check that associated finite volume Gibbs measures verify a kind of monotone behaviour:

$$\tau_1 \preccurlyeq \tau_2 \Rightarrow \mu_\Lambda^{\tau_1} \preccurlyeq \mu_\Lambda^{\tau_2}. \tag{4.8}$$

In particular, Gibbs measures on $S^{\mathbb{Z}^d}$ obtained as the infinite volume limit with $+1$ boundary condition (resp. -1), and, denoted with μ^+ (resp. μ^-), are extremal states in the sense of stochastic ordering, of the set $\mathcal{G}(\varphi)$. Finally, let us state the following lemma:

Lemma 4.1 *If the Weak Mixing Condition (4.4) holds for the potential φ associated to the PCA dynamics P, then assumption $(\mathcal{A})$ holds for P.*

Proof: It is enough to show the following inequality:

$$\left(\int \sigma_0 \; d\nu^+_{\mathcal{B}(L)} - \int \sigma_0 \; d\nu^-_{\mathcal{B}(L)}\right) \leqslant \left(\int \sigma_0 \; d\mu^+_{\mathcal{B}(L)} - \int \sigma_0 \; d\mu^-_{\mathcal{B}(L)}\right) .$$

Let us first check $\int \sigma_0 \; d\nu^+_{\mathcal{B}(L)} \leqslant \int \sigma_0 \; d\mu^+_{\mathcal{B}(L)}$. Let f_0 be the increasing function defined on $S^{\mathbb{Z}^d}$ by $f_0(\sigma) = \sigma_0$. According to the finite range R, let L be large enough such that $0 \notin \partial_i \mathcal{B}(L)$. Note $\int \sigma_0 \; d\nu^+_{\mathcal{B}(L)} = \nu^+_{\mathcal{B}(L)}(\; \nu^+_{\mathcal{B}(L)}(\; f_0 \; |\sigma_{\mathcal{B}(L)\setminus 0}))$. Using relation (4.7), we then have

$$\nu^+_{\mathcal{B}(L)}(f_0) = \nu^+_{\mathcal{B}(L)}(\mu^{\sigma_{\mathcal{B}(L)\setminus 0}(+\mathbf{1})_{\mathcal{B}(L)^c}}_{\{0\}}(f_0)).$$

On the other hand, using the monotonicity in the boundary condition of the finite volume Gibbs measures, we find:

$$\mu^{\sigma_{\mathcal{B}(L)\setminus 0}(+\mathbf{1})_{\mathcal{B}(L)^c}}_{\{0\}}(f_0) \leqslant \mu^{(+\mathbf{1})_{0^c}}_{\{0\}}(f_0) = \mu^+_{\mathcal{B}(L)}(f_0) \; .$$

So desired inequality holds, and $\nu^-_{\mathcal{B}(L)}(f_0) \geqslant \mu^-_{\mathcal{B}(L)}(f_0)$ is analogously checked. ■

Lemma 4.2 *For a PCA dynamics P of class $\mathcal{C}_0$ with $\mathcal{K}(.)$ non negative, the extremal stationary measures ν^-, ν^+ coincide respectively with extremal Gibbs measure μ^-, μ^+ of $\mathcal{G}(\varphi)$, that is $\mu^+ = \nu^+$ and $\mu^- = \nu^-$ (eventually these two relations coincide)*

Proof: Let Λ, Λ' be two finite subsets of $\mathbb{Z}^d$ such that $\Lambda \subset \Lambda'$. Then, for all configurations $\sigma_{\Lambda'\setminus\Lambda} \in S^{\Lambda'\setminus\Lambda}$, finite volume reversible measures with extremal boundary condition are such that:

$$\nu^+_{\Lambda'}\left((.)_\Lambda|\sigma_{\Lambda'\setminus\Lambda}\right) \preccurlyeq \nu^+_\Lambda(.) \; ; \qquad (4.9)$$

$$\nu^-_{\Lambda'}\left((.)_\Lambda|\sigma_{\Lambda'\setminus\Lambda}\right) \succcurlyeq \nu^-_\Lambda(.) \; . \qquad (4.10)$$

Using relation (4.6), we can deduce from the previous result the following inequalities between finite volume Gibbs measure and reversible measure, with extremal boundary condition: $\mu^+_\Lambda \preccurlyeq \nu^+_\Lambda$ and $\mu^-_\Lambda \preccurlyeq \nu^-_\Lambda$. Taking now the limit in volume, we find: $\mu^+ \preccurlyeq \nu^+$ and $\mu^- \preccurlyeq \nu^-$.

On the other hand, ν^+_Λ is P^+_Λ reversible, so taking the limit, ν^+ is P-reversible. Analogously, ν^- is P-reversible. From (4.3) we conclude ν^- and ν^+ are Gibbs measures, so thanks to the fact that μ^- and μ^+ are stochastic ordering extremal states for Gibbs measures, we deduce: $\nu^+ \preccurlyeq \mu^+$ and $\mu^- \preccurlyeq \nu^-$. Thus the conclusion follows. ■

Theorem 4.3 *Let P be an attractive PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$ of the class $\mathcal{C}_0$ defined by (4.1), let φ denote the potential canonically associated defined in (4.2), and $\mathcal{G}(\varphi)$ the set of Gibbs measures w.r.t φ:*

- *if there is phase transition (i.e. $\#\mathcal{G}(\varphi) > 1$) then the extremal Gibbs states ν^- and ν^+ are different, and the dynamics P is non ergodic;*

- *otherwise, when there is no phase transition (i.e. $\mathcal{G}(\varphi) = \{\mu\}$ and $\mu = \mu^- = \mu^+ = \nu^- = \nu^+$), the dynamics P is ergodic towards the unique Gibbs measure μ.*
  Moreover if we assume the Weak Mixing condition (4.4), then the convergence towards μ happens with exponential rate.

Proof: When there is phase transition, thanks to the fact that μ^- and μ^+ are stochastic order extremal states for $\mathcal{G}(\varphi)$, we have that $\mu^- \neq \mu^+$. So, using Lemma 4.2, the two reversible (so stationary) measures ν^- and ν^+ are different. Then, dynamics P can not be ergodic.

When there is no phase transition, then $\mathcal{G}(\varphi) = \{\mu\}$ where $\mu = \mu^- = \mu^+$ is the unique Gibbs state. Thanks to Lemma 4.2, $\nu^- = \mu^- = \mu^+ = \nu^+$, and using (3.2) uniqueness of P-stationary measure holds. Thanks to Theorem 3.4, it implies ergodicity of the PCA dynamics P.

Finally, if Weak Mixing condition (4.4) is assumed, then Lemma 4.1 implies that the exponential bound ($\mathcal{A}$) holds (assumption ($\mathcal{A}$)). We conclude using Theorem 3.4. ■

Note φ is a multi-body potential. In [3], we established that, for nearest neighbour interaction function $\mathcal{K}$, phase transition holds for β large. For instance, when $d = 2$, let P_0 be the PCA dynamics of the class $\mathcal{C}_0$ obtained taking:

$$\mathcal{K}(\pm e_1) = \mathcal{K}(\pm e_2) = K > 0, \ \mathcal{K}(k) = 0 \text{ otherwise,} \tag{4.11}$$

where (e_1, e_2) is a basis of $\mathbb{R}^2$ and K a positive constant. The canonically associated (4.2) potential φ_0 is the following four body potential:

$$\varphi_{0,V_k}(\sigma_{V_k}) = -\log\cosh(\beta K \sum_{j \in V_k} \sigma_j), \qquad \varphi_\Lambda(\sigma_\Lambda) = 0 \text{ otherwise,}$$

where $V_k = \{k - e_1, k + e_1, k - e_2, k + e_2\}$. We conclude that for β large, the PCA P_0 is non ergodic since it has at least two different stationary states ν^- and ν^+. Thanks to Proposition 3.2, we also know that $\delta_+ P^{(n)}$ (resp. $\delta_- P^{(n)}$) converges weakly, as n goes to $+\infty$, towards the stationary measure ν^+ (resp. ν^-).

4.3 Comments

One conjectures Weak Mixing condition for Gibbs measure is valid up to the critical temperature, that is, as soon as there is no phase transition. In that sense, our main result would give ergodicity with exponential rate on a much larger region as the region where the Dobrushin-Vasershtein criterion holds. In fact, let us mention the reference [12], where, using percolation techniques, it is proved that in dimension $d = 2$, for a ferromagnetic nearest neighbour Ising model without extremal magnetic field, the associated Gibbs measure is weak mixing as soon as it is unique (*i.e.* $\forall \beta, \beta < \beta_c$). In order to precise the previous assertion, let us consider P_0 given by (4.11).

In that case, a tricky argumentation relates the potential φ_0, associated to the P_0 dynamics, with the usual Ising ferromagnetic potential (see [34]). So, Higuchi's result applies, and we know that the Gibbs state associated to this potential φ_0 is weak mixing as soon as there is no phase transition, which happens for β lower than a critical β_c, which coincide with the Ising critical temperature $\beta_c = \frac{\log(1+\sqrt{2})}{2K}$. In other words, we know that the PCA dynamics P_0 is ergodic with exponential rate for $\beta < \beta_c$ and non ergodic for $\beta > \beta_c$. Taking $K = 1$, $\beta_c \simeq 0.441$; Dobrushin-Vasershtein criteria applies only for

$$\gamma = \frac{1}{2} \sum_{j \in V_0} \sup_{\eta \in S^{V_0}} \Big| \tanh(\beta \sum_{k' \in V_0} \mathcal{K}(k')\ \eta^j_{k'}) \ - \ \tanh(\beta \sum_{k' \in V_0} \mathcal{K}(k')\ \eta_{k'}) \Big| < 1,$$

where $\eta^j_k = \eta_k$ if $k \neq j$ and $\eta^j_j = -\eta_j$, which means $\beta < \frac{1}{2}\mathrm{Argth}(\frac{1}{2}) \simeq 0.275$ (*cf.* part 6.1.2 *in* [22]).

For another PCA dynamics P_1 defined by

$$\mathcal{K}(\pm e_1) = \mathcal{K}(\pm e_2) = \mathcal{K}(0) = +1,\ \mathcal{K}(k) = 0 \text{ otherwise},$$

Dobrushin-Vasershtein criteria applies for $\gamma = 5 \tanh \beta < 1$, *i.e.* $\beta <\simeq\ 0.203$. Numerical simulations (see Matlab© code *in* chapter 7 *in* [22]) for this P_1 PCA dynamics give an approximation of a critical parameter $\beta_c \simeq\ 0.3$.

We conclude that for PCA dynamics P_0, our result states ergodicity on a region which is strictly larger than the one of Dobrushin-Vasershtein condition, and which is moreover optimal. Numerical simulations confirm this fact for the P_1 dynamics too.

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