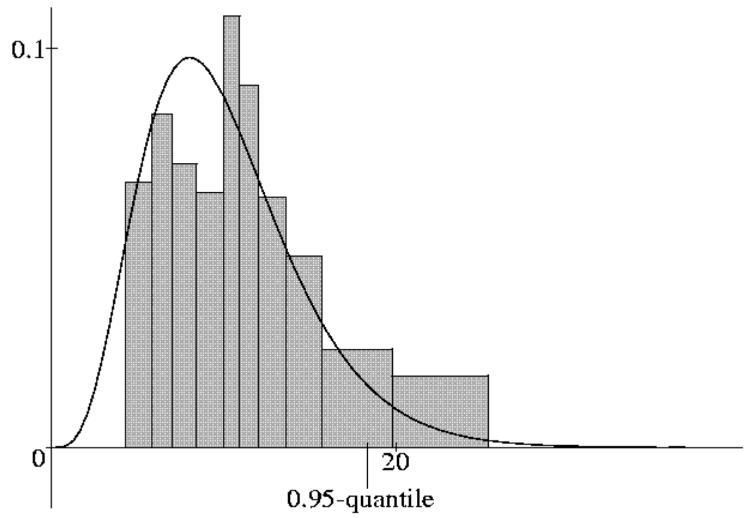
UNIVERSITÄT POTSDAM
Institut für Mathematik

On Gibbsianness of infinite-dimensional diffusions

David Dereudre

Sylvie Roelly



Mathematische Statistik und
Wahrscheinlichkeitstheorie

Universität Potsdam – Institut für Mathematik

Mathematische Statistik und Wahrscheinlichkeitstheorie

On Gibbsianness of infinite-dimensional diffusions

David Dereudre,

Université de Valenciennes et du Hainaut-Cambrésis, France
e-mail: David.Dereudre@univ-valenciennes.fr

Sylvie Roelly

Institut für Mathematik der Universität Potsdam
e-mail: roelly@math.uni-potsdam.de

Preprint 2004/01

Januar 2004

Impressum

Herausgeber:	Mathematische Statistik und Wahrscheinlichkeitstheorie am Institut für Mathematik
Adresse:	Universität Potsdam PF 60 15 53 14415 Potsdam
Telefon:	
Fax:	+49-331-977 1500
E-mail:	+49-331-977 1578
	neisse@math.uni-potsdam.de

ISSN 1613-3307

On Gibbsianness of infinite-dimensional diffusions

David Dereudre*
Université de Valenciennes et du Hainaut-Cambrésis
Le Mont Houy
59313 Valenciennes Cedex 09, France
e-mail : david.dereudre@univ-valenciennes.fr

Sylvie Rœlly**
Institut für Mathematik der Universität Potsdam
Postfach 601553
14415 Potsdam, Germany
e-mail : roelly@math.uni-potsdam.de

Abstract

We analyse different Gibbsian properties of interactive Brownian diffusions X indexed by the lattice $\mathbb{Z}^d$: $X = (X_i(t), i \in \mathbb{Z}^d, t \in [0,T], 0 < T < +\infty)$. In a first part, these processes are characterized as Gibbs states on path spaces of the form $\mathcal{C}([0,T],\mathbb{R})^{\mathbb{Z}^d}$. In a second part, we study the Gibbsian character on $\mathbb{R}^{\mathbb{Z}^d}$ of ν^t, the law at time t of the infinite-dimensional diffusion $X(t)$, when the initial law $\nu = \nu^0$ is Gibbsian.

AMS Classifications: 60G15 - 60G60 - 60H10 - 60J60.

KEY-WORDS: infinite-dimensional Brownian diffusion, Gibbs field, cluster expansion.

* D. D. thanks the University Potsdam for kind hospitality and the DAAD for financial support (Kurzstipendium A/03/02945).

** On leave of absence Centre de Mathématiques Appliquées, UMR C.N.R.S. 7641, École Polytechnique, 91128 Palaiseau Cedex, France

1 Introduction

We first recall the results of the celebrated article of Kolmogorov (1937) : *Zur Umkehrbarkeit der statistischen Naturgesetze*, rewritten by using the modern vocabulary of stochastic calculus.

Let I be a finite index set and $X = (X_i(t), i \in I, t \in [0,T])$ be an $\mathbb{R}^I$-valued diffusion solution of the following finite-dimensional stochastic differential equation (s.d.e.)

$$dX_i(t) = dB_i(t) - \frac{1}{2}U'(X_i(t))\,dt + b_i(X(t))\,dt\ ,\quad i \in I,\ t \in [0,T] \tag{1}$$

where

- U is a $\mathcal{C}^2$ *self potential* ;
- $b = (b_i)_{i\in I} : \mathbb{R}^I \to \mathbb{R}^I$ is a smooth bounded function ;
- $(B_i)_{i\in I}$ is a sequence of independent, real-valued Brownian motions.

This stochastic dynamics corresponds to a perturbation by the interactive drifts $(b_i)_{i\in I}$ of a sequence of finitely many free dynamics driven by the self potential U'.

The question posed - and solved - by Kolmogorov was : under which conditions on the drift b one can assure the existence of an equilibrium for the system, that is an initial distribution μ, probability measure on $\mathbb{R}^I$, which is invariant under the dynamics :

$$X(0) \sim \mu \Longrightarrow X(t) \sim \mu, \forall t \in [0,T].$$

The answer is the following : the drift b has to be of gradient form, i.e. it should exist a smooth function h from $\mathbb{R}^I$ into $\mathbb{R}$ such that $b_i = -\frac{1}{2}\nabla_i h, i \in I$. In this case, the unique time-invariant measures under the dynamics (1) - which are moreover also reversible - are proportional to the following measure :

$$\mu(dx) = \exp(-h(x)) \otimes_{i\in I} \exp(-U(x_i))dx_i = \exp(-h(x)) \otimes_{i\in I} m(dx_i) \tag{2}$$

where m is the equilibrium distribution for the one-dimensional free dynamics. Often can μ be renormalised to become a probability measure (f.e. when U is such that $\exp -U$ is Lebesgue integrable and h is bounded). We remark that the equilibrium distribution has a particular form : it is absolutely continuous with respect to a reference measure (here $\otimes_{i\in I} m(dx_i)$) and the density is the exponential of the function h whose gradient generates the dynamical interaction between the coordinates of the process. This form for the equilibrium distribution will remain "locally" for infinite-dimensional diffusions, if the measure μ becomes Gibbsian.

Let us then consider the so-called gradient diffusion X solution of the s.d.e. (1) with b of the gradient form :

$$dX_i(t) = dB_i(t) - \frac{1}{2}U'(X_i(t))\,dt - \frac{1}{2}\nabla_i h(X(t))\,dt\ , i \in I, \tag{3}$$

and let us have a look at the structure of the law Q^ν of X, when the initial distribution $X(0) \sim \nu$ on $\mathbb{R}^I$ is not necessarily equal to the equilibrium distribution μ. Q^ν is a probability measure on $\Omega_f = \mathcal{C}([0,T],\mathbb{R})^I$. As reference measure on the path space Ω_f we take

$P = W^{\otimes I}$ the stationary law of the free system, equal to the system (3) with $h \equiv 0$. (W is itself the stationary law on $\mathcal{C}([0,T],\mathbb{R})$ of the free one-dimensional system with m as equilibrium distribution.)

Due to Girsanov theorem, it is clear that (under some growth conditions on h and its derivatives) if ν is absolutely continuous with respect to $m^{\otimes I}$ with density on $\mathbb{R}^I$ given by $\exp -\tilde{h}$, then Q^ν is absolutely continuous with respect to the reference measure P and the density on the path level is given by

$$
\begin{aligned}
&\frac{dQ^\nu}{dP}(X) = \frac{d\nu}{dm^{\otimes I}}(X(0)) \times \\
&\exp -\Big(\frac{1}{2}\sum_{i\in I}\int_0^T \nabla_i h(X(t))dX_i(t) + \frac{1}{4}\sum_{i\in I}\int_0^T \nabla_i h(X(t))\big(\frac{1}{2}\nabla_i h(X(t)) + U'(X_i(t))\big)dt\Big) \\
&= \exp -\Bigg(\frac{1}{2}h(X(T)) + \big(\tilde{h} - \frac{1}{2}h\big)(X(0)) \\
&-\frac{1}{4}\int_0^T \Big(\big(\Delta h - \frac{1}{2}|\nabla h|^2\big)(X(t)) - \sum_{i\in I}\nabla_i h(X(t))U'(X_i(t))\Big)dt\Bigg). \qquad (4)
\end{aligned}
$$

The last expression is obtained applying Ito formula to the function $\frac{1}{2}h(X(t))$ between times 0 and T. This means that the density of Q^ν with respect to P has the following form :

$$\frac{dQ^\nu}{dP}(X) = \exp -\Big(\tilde{h}(X(0)) + H(X)\Big)$$

where H is a smooth functional on the path space Ω_f. Similar to the role of the Hamiltonian function in statistical mechanics, we can interpret here H as a Hamilton functional on Ω_f. Furthermore, still inspired by the equilibrium statistical mechanics, we remark that on the path level, even if the system in not time invariant (ν different from μ or $\tilde{h}$ different from h), its law is in some sense an equilibrium law since it has a Gibbsian form.

Our pourpose here is, starting from these remarks about well known finite dimensional diffusions, to analyse if the above structure of Q^ν is partially conserved if we replace the finite index set I of the system (3) by an infinite one, for example by $\mathbb{Z}^d$. It is clear that as soon as I is no more finite, the expression of the above density could explode. So we will be obliged to restrict ourselves on "local" properties, as usual in statistical mechanics.

The organisation of this paper is as follows.

In Section 2, we review Gibbsian properties of Q^ν as probability measure on path spaces in various situations. These results are based on former works of the second author. But they are revisited under a new angle, answering the question : *how strong is the notion of that Gibbsianness*?

In the first paragraph, the simplest case of infinite-dimensional gradient diffusions with bounded Hamiltonian h is treated. We show that, under some assumptions, Q^ν is a strong Gibbs measure on $\Omega = \mathcal{C}([0,T],\mathbb{R})^{\mathbb{Z}^d}$ in the usual sense of bounded spin systems, since it is associated to a potential on Ω which is an absolutely summable functional on this path space.

In second paragraph, we discuss the case of more general infinite-dimensional diffusions. As soon as the drift is the gradient of an *unbounded* Hamiltonian h, the Gibbsianity is present only in a weak form. In fact, we can recover in certain cases a nice Gibbsianness of Q^ν if we restrict it on a subspace of Ω. We present also a case of infinite-dimensional diffusions with non Markovian drift (and then a fortiori non-gradient). The interaction functional can not be defined on the full path space, but anyway we can identify a weak Gibbsianness property.

The Section 3 contains new results obtained by a current collaboration between both authors.

Coming back to the finite-dimensional framework presented above, it is clear that if the initial measure ν is absolutely continuous with respect to $m^{\otimes I}$ (with density given by $\exp -\tilde{h}$), this property propagates, that is : at each time $t > 0$ the law ν^t of $X(t)$ on $\mathbb{R}^I$, which is also equal to the projection at time t of Q^ν, remains absolutely continuous with respect to $m^{\otimes I}$ (with density given by $\exp -h^t$ for some function h^t). When I is replaced by $\mathbb{Z}^d$, the question whether the global absolute continuity propagates is irrelevant since the stationary measure itself is no more globally absolutely continuous with respect to $m^{\otimes \mathbb{Z}^d}$, but the question if the local absolute continuity is conserved is relevant and equivalent to the following : *does the Gibbsianness of the initial measure propagate*?

Although, on the path level, the infinite-dimensional diffusion in several cases is regular in the sense that it is strongly Gibbsian as recalled in the second Section, its projection at each fixed time t can behave badly in the sense that the sum of the interactions between the (infinitely many) components can explode. So, to obtain a positive answer to the above question, we should restrict our study to two particular regimes which can be better controlled. In the first paragraph, we present the propagation of Gibbsianness for small time t, and in the second paragraph, we analyse the case of small interactions between the coordinates - but arbitrary times.

To our knowledge, the results we present in Section 3 are the first which are related to the propagation of Gibbsianness under a continuous stochastic evolution like a diffusion with values in an infinite-dimensional vector space (here $\mathbb{R}^{\mathbb{Z}^d}$). We were inspired by the very nice paper [8] where the question of possible loss and recovery of Gibbsianness is treated in the context of particle systems with values in $\{-1,+1\}^{\mathbb{Z}^d}$ which follow a high-temperature Glauber dynamics (see also [18] for related results for Kawasaki dynamics).

Our present results are only partial, and they can certainly be developed and/or ameliorated. This is the subject of a forthcoming paper [6].

2 Infinite-dimensional diffusions as Gibbs states on the path level

2.1 Gradient diffusions with bounded interaction

Let us first introduce some definitions and notations.

An interaction potential - or **interaction** - ϕ on $\mathbb{R}^{\mathbb{Z}^d}$ is a collection of functions ϕ_Λ from $\mathbb{R}^{\mathbb{Z}^d}$ into $\mathbb{R} \cup \{+\infty\}$ where Λ varies in the set of finite subsets of $\mathbb{Z}^d$, which are measurable with respect to the canonical projections on $\mathbb{R}^\Lambda$.

The interaction ϕ is said to be of **finite range** if it satisfies :

(FR) $\quad \exists r > 0,\ \text{diameter } \Lambda \geq r \Longrightarrow \phi_\Lambda \equiv 0$

The interaction ϕ is said to be **regular bounded** if it satisfies :
(RB) $\quad \forall\Lambda$, ϕ_Λ is $\mathcal{C}^3$, bounded with bounded derivatives.

The interaction ϕ is said to be **absolutely summable** if it satisfies :
(AS) $\quad \forall i \in \mathbb{Z}^d, \sum_{\Lambda\ni i} \|\phi_\Lambda\|_\infty = \sum_{\Lambda\ni i} \sup_{x\in\mathbb{R}^{\mathbb{Z}^d}} |\phi_\Lambda(x)| < +\infty$

When an interaction ϕ is (AS) one can define the collection $h^\phi = (h^\phi_\Lambda)_{\Lambda\subset\mathbb{Z}^d}$ of associated **Hamiltonian functions** on $\mathbb{R}^{\mathbb{Z}^d}$ by

$$h^\phi_\Lambda = \sum_{\Lambda':\Lambda'\cap\Lambda\neq\varnothing} \phi_{\Lambda'}. \tag{5}$$

In fact, as soon as the serie on the right hand side converges pointwise, one can define a Hamiltonian function associated to a (possibly non absolutely) summable interaction.
To simplify we will always denote by h^ϕ_i the function $h^\phi_{\{i\}}$, $i \in \mathbb{Z}^d$.
We call ρ a **Gibbsian** measure on $\mathbb{R}^{\mathbb{Z}^d}$ associated to the reference measure m and to an interaction ϕ for which the serie (5) converges if it satisfies the following generalisation of (2); the family of its conditional expectations $(\rho(dx_i/x_j, j\neq i))_{i\in\mathbb{Z}^d}$ should satisfy the system of Dobrushin-Landford-Ruelle (DLR) equations :

$$\rho(dx_i/x_j, j\neq i) = \frac{1}{z_i}\exp -\left(h^\phi_i(x)\right) m(dx_i),\ i\in\mathbb{Z}^d.$$

The set of such measures will be denoted by $\mathcal{G}(\phi, m)$. (For general reference on Gibbs measures, see [13] and [22].)
The measure ρ will be called **strong Gibbsian** if the associated interaction is (AS).

Let φ be a so-called *dynamical interaction* on $\mathbb{R}^{\mathbb{Z}^d}$ satisfying (FR) and (RB). Then it satisfies automatically (AS). Under the above assumptions on φ, for all $i\in\mathbb{Z}^d$, the hamilton function h^φ_i, denoted by h_i to simplify, is $\mathcal{C}^3$, bounded with bounded derivatives. We can now consider the infinite-dimensional version of the system (3) given by :

$$dX_i(t) = dB_i(t) - \frac{1}{2}U'(X_i(t))\,dt - \frac{1}{2}\nabla_i h_i(X(t))\,dt\ ,\ i\in\mathbb{Z}^d,\ t\in[0,T]. \tag{6}$$

Let ν be a probability measure on $\mathbb{R}^{\mathbb{Z}^d}$ which satisfies some integrability conditions (f.e. the sequence $(\int x_i^{2p}\nu(dx))_{i\in\mathbb{Z}^d}$ belongs to the dual set of tempered sequences on $\mathbb{Z}^d$ for some $p\in\mathbb{N}$). Following [24] Theorem 4.1 (or [7] if the interaction is reduced to a pair interaction), the infinite-dimensional stochastic system (6) with initial condition $X(0)\sim\nu$ has a unique strong solution X with values in the infinite product of continuous trajectories $\Omega = \mathcal{C}([0,T],\mathbb{R})^{\mathbb{Z}^d}$.

Deuschel ([4],[5]) was the first to describe the structure of the law Q^ν of X, probability measure on Ω , as a lattice Gibbs measure, when ν itself is Gibbsian. Using a decoupling method between the ith coordinate X_i and the others and applying Girsanov formula, he proved an infinite-dimensional generalization of the equation (4). In [1], we completed and generalised his results by showing a bijection between the set of initial Gibbs measures on $\mathbb{R}^{\mathbb{Z}^d}$ and a set of Gibbs measures on the path space Ω.
Let us recall this main result ([1], Théorème 3.7):

Theorem 1 *Let Q be a probability measure on $\Omega = \mathcal{C}([0,T],\mathbb{R})^{\mathbb{Z}^d}$ with projection at time 0 a measure ν on $\mathbb{R}^{\mathbb{Z}^d}$. Q is equal to Q^ν, the diffusion solution of (6) where the initial distribution ν is in $\mathcal{G}(\tilde{\varphi}, m)$ if and only if Q is a Gibbs measure in $\mathcal{G}(\tilde{\varphi}+\Phi, W)$, that is Q^ν satisfies the following DLR equations :*

$$Q^\nu(dX_i/X_j, j \neq i) = \frac{1}{Z_i} \exp -\Big(\tilde{h}_i(X(0)) + H_i(X)\Big) W(dX_i),\ i \in \mathbb{Z}^d, \tag{7}$$

where H is the Hamilton functional on Ω associated to Φ by (5) and defined by

$$\begin{aligned} H_i(X) &= \frac{1}{2} h_i(X(T)) - \frac{1}{2} h_i(X(0)) \\ &-\frac{1}{4}\int_0^T \Big(\sum_{j:|j-i|\leq r} \big(\Delta_j h_i - \frac{1}{2}|\nabla_j h_j|^2\big)(X(t)) - \nabla_i h_i(X(t)) U'(X_i(t))\Big) dt. \end{aligned} \tag{8}$$

Moreover, we deduce from the explicit expression (8) the regularity of the underlying interaction functional Φ if we suppose the following supplementary assumption

$$\sup_{\Lambda \subset \mathbb{Z}^d} \sup_{i \in \Lambda} \sup_{x \in \mathbb{R}^\Lambda} |U'(x_i).\nabla_i \varphi_\Lambda(x)| < +\infty. \tag{9}$$

(It is a balance condition between the self-potential U and the dynamical potential φ. This is satisfied for example for any potential φ constant at infinity.)

Corollary 2 *Under condition (9), if the initial condition ν is strong Gibbsian on $\mathbb{R}^{\mathbb{Z}^d}$, Q^ν, the law of the diffusion solution of (6), is also strong Gibbsian on the path level $\Omega = \mathcal{C}([0,T],\mathbb{R})^{\mathbb{Z}^d}$ in the sense that the associated interaction functional Φ on Ω is absolutely summable (AS), bounded continuous and has a finite range bounded by $2r$ if r is the range of φ.*

In the case when φ is a pair interaction ($\varphi_\Lambda \equiv 0$ when Card $\Lambda > 2$), the reader can find the explicit form of Φ as function of φ in [1], equation (3.18). In particular, Φ is a 3-body interaction ($\Phi_\Lambda \equiv 0$ when Card $\Lambda > 3$) and, for example, $\Phi_{\{i,j,k\}}$ is given by

$$\Phi_{\{i,j,k\}}(X) = -\frac{1}{4}\int_0^T \big(\nabla_i \varphi_{\{i,j\}} \nabla_i \varphi_{\{i,k\}} + \nabla_k \varphi_{\{i,k\}} \nabla_k \varphi_{\{j,k\}} + \nabla_j \varphi_{\{i,j\}} \nabla_j \varphi_{\{j,k\}}\big)(X(t)) dt. \tag{10}$$

From the above bijection between sets of strong Gibbs measures on $\mathbb{R}^{\mathbb{Z}^d}$ and on $\mathcal{C}([0,T],\mathbb{R})^{\mathbb{Z}^d}$, we conclude that if the set $\mathcal{G}(\tilde{\varphi}, m)$ is reduced to a unique element ν, the set $\mathcal{G}(\tilde{\varphi}+\Phi, W)$ contains also a unique element, Q^ν, the law of the system (6).
In other words, **the strong Gibbsianness propagates from the initial distribution on the configuration space $\mathbb{R}^{\mathbb{Z}^d}$ to the path level, and the uniqueness property** (i.e. absence of phase transition) **also**.

2.2 Gradient diffusions with a general interaction

If φ, the dynamical interaction on $\mathbb{R}^{\mathbb{Z}^d}$, does not satisfy anymore (FR) or (RB), for example if φ is an infinite range unbounded interaction, it is non trivial at all to give a sense to the equation (6), and also to exhibit a subset of $\mathbb{R}^{\mathbb{Z}^d}$ in which a solution could live.

One classical interesting example is given by the pair potential $\varphi_{\{i,j\}}(x) = J(i-j)x_i x_j$ where J is a tempered sequence on $\mathbb{Z}^d$ and $\varphi_\Lambda \equiv 0$ if Card $\Lambda \neq 2$. If J is positive, this is the ferromagnetic Ising interaction for continuous spins.
This problem was solved in [24] and [7] (see also [23] were the above example is analysed very clearly). The authors gave a number of hypothesis (let us call them $\mathcal{H}$) on φ and its derivatives (cf. assumption $[C']$ in [24]) in such a way that existence and uniqueness of solutions for the infinite-dimensional gradient system is proved inside of the space $\mathcal{S}'(\mathbb{Z}^d)$, dual of the tempered sequences in $\mathbb{Z}^d$ (or subpolynomial sequences). This means that if the initial distribution ν carries $\mathcal{S}'(\mathbb{Z}^d)$ and satisfies some moment condition, the solution X takes its values a.s. in $\mathcal{S}'(\mathbb{Z}^d)$, and then Q^ν is a probability measure on $\mathcal{C}([0,T], \mathcal{S}'(\mathbb{Z}^d)) \subset \Omega$. In other words, the Hamilton function h associated to φ, which has no sense on the whole space $\mathbb{R}^{\mathbb{Z}^d}$, is well defined on $\mathcal{S}'(\mathbb{Z}^d)$, and at the path level the same kind of phenomena appears. Inspired by [1] Théorème 4.19, we have the following.

Proposition 3 *Under assumptions $\mathcal{H}$, if the initial condition ν is a Gibbsian measure in $\mathcal{G}(\tilde{\varphi}, m)$ with support included in $\mathcal{S}'(\mathbb{Z}^d)$, then Q^ν is a Gibbs measure in $\mathcal{G}(\tilde{\varphi} + \Phi, W)$ with support included in $\mathcal{C}([0,T], \mathcal{S}'(\mathbb{Z}^d))$. The associated interaction functional Φ is well defined on $\mathcal{C}([0,T], \mathcal{S}'(\mathbb{Z}^d))$ and it is pointwise summable.*

This furnishes an example of Gibbsian property on the path space which cannot be strong since every ingredient (spin, interaction, sum) is unbounded. Anyway the Gibbsian structure exists and contains informations on the process. In [1], one can find applications of the Gibbsianness to the identification of reversible states for the gradient dynamics, and behavior of the process under time reversal.

Let us now consider a more general equation than (6), where the drift can be non-regular and non-Markovian :

$$dX_i(t) = dB_i(t) - \frac{1}{2}U'(X_i(t))\,dt + \mathbf{b}_{i,t}(X)\,dt\ ,\quad i \in \mathbb{Z}^d,\ t \in [0,T], \tag{11}$$

where $\mathbf{b}_{i,t} : \Omega \longrightarrow \mathbb{R}$ is a bounded functional, measurable with respect to the σ-algebra generated by the canonical projections $(X(s), s \leq t)$ and local in space, that is, there exists a finite neighborhood $\mathcal{N}(i)$ of i such that $\mathbf{b}_{i,t}(X)$ depends only on the coordinates of X inside of $\mathcal{N}(i)$. Typically, $\mathbf{b}_{i,t}$ can take the form : $\mathbf{b}_{i,t}(X) = \int_0^t b_i(X(s))\pi(ds)$, where $b_i(x) = b_i(x_{\mathcal{N}(i)})$ and π is a measure on $[0,T]$.

Let us suppose that the system (11) admits a solution Q^ν, a probability measure on Ω. Using the same methods as before, it is simple to check that if the initial distribution ν is in $\mathcal{G}(\tilde{\varphi}, m)$, we can obtain for Q^ν the following representation :

$$Q^\nu(dX_i/X_j, j \neq i) = \frac{1}{Z_i}\exp -\Big(\tilde{h}_i(X(0)) + H_i(X)\Big)\,W(dX_i),\ i \in \mathbb{Z}^d, \tag{12}$$

where H can be decomposed as $H_i = \sum_{\mathcal{N}(j)\ni i} \Phi_{\mathcal{N}(i)}$ and Φ is given by :

$$\Phi_{\mathcal{N}(i)}(X) = -\int_0^T \mathbf{b}_{i,t}(X)dX_i(t) + \frac{1}{2}\int_0^T \mathbf{b}_{i,t}(X)\Big(\mathbf{b}_{i,t}(X) - U'(X_i(t))\Big)dt. \tag{13}$$

But now the functional $\mathbf{b}$ is no more of gradient type and we cannot replace the stochastic integral in (13) by a usual integral using Ito formula. Then, the interaction functional Φ is not defined a priori on the whole Ω but only for such X for which the stochastic integral $\int_0^T \mathbf{b}_{i,t}(X) dX_i(t)$ makes sense. Since W is the law of a one dimensional Brownian semi-martingale with drift $-\frac{1}{2}U'$, one has

$$\Phi_{\mathcal{N}(i)}(X) = -\left(\int_0^T \mathbf{b}_{i,t}(X)\, d\tilde{B}_i(t) - \frac{1}{2} \int_0^T \mathbf{b}_{i,t}^2(X)\, dt \right), \tag{14}$$

where $\tilde{B}_i$ are independent Brownian motions under P. The boundedness of $\mathbf{b}$ implies that $\Phi_{\mathcal{N}(i)} \in L^2(W)$ and then $\Phi_{\mathcal{N}(i)}$ makes sense W-almost surely.

We have here an example of "almost-sure Gibbsian property" on the path space which cannot be stronger since the interaction, by its nature, cannot be defined on the whole space Ω but only almost-surely. Nevertheless, the Gibbsian structure contains strong informations on the process. In [2], we studied, in a space-time stationary context, a modification of this Gibbsianness in terms of space-time Gibbsian property (cf. also [21], [20]), and proved that the DLR-approach is equivalent to the variational approach. This allowed us to prove in [3] an existence result for the system (11) for an interaction $\mathbf{b}$ sufficiently small, using the characterisation of the solution as space-time Gibbs field and constructing it by cluster expansion in the small coupling parameter. To our knowledge, no method using infinite-dimensional stochastic calculus could solve the existence problem for (11). This is an indication that the concept of Gibbsianness, even almost-sure, is really powerful.

3 Propagation of Gibbsianness during the stochastic diffusive evolution

In the last section, we have seen that under reasonable conditions, the law Q^ν of an infinite-dimensional Brownian diffusion with Gibbsian initial condition ν is a Gibbs state on the path level. Now, we would like to know if at each time t, the law of $X(t)$, which we denote by ν^t, a probability measure on $\mathbb{R}^{\mathbb{Z}^d}$, remains Gibbsian (in a strong or weak sense). Clearly, ν^t is the projection at time t of Q^ν, but projections are maps which do not conserve *a priori* the Gibbsianness (see the famous examples of [9], and also [10], [11] amoung others).

In [1], we remarked that, projecting at time 0 a general strong Gibbs measure on the path space, the image measure which is obtained on the state space preserves a Gibbsian form in the following weak sense : it is associated to a modification (cf. [13] Section 1.3, for the exact definition), roughly speaking to a family of compatible local densities with respect to a reference measure. But now the regularity of the density and the existence of an underlying nice interaction potential is completely unclear. In the Remarks after Proposition 2.5 in [1], we sent the reader to the work of Kozlov to clear this question. This will be the object of this section, not only for the projecting at time 0 but also at time $t > 0$.

The challenge is to controll the evolution of the initial (AS) interaction $\tilde{\varphi}$ under the dynamics. It is clear that, even if $\tilde{\varphi}$ is (FR) this properties immediately disappears for time $t > 0$ since the Brownian motions carry instantaneously the information between the coordinates. So, to assure that at time t, the process is still Gibbsian and associated to a "good" interaction, i.e. an (AS) one, we are obliged to restrict our study to two cases; first for small times t, which implies that the process stays close to the initial Gibbsian condition. Secondly, for small dynamical interaction φ between the coordinates, which assures that the sum of the initial interaction and the interaction induced by the dynamics does not explode.

To explore this problematic, we were inspired by the work of van Enter, Fernandez, den Hollander and Redig, who consider in [8] the question of Gibbsianness/non Gibbsianness in the context of particle systems with values in $\{-1,+1\}^{\mathbb{Z}^d}$ which follow a high-temperature Glauber dynamics. They treat several cases and can exhibit situations where the process at time t is strong Gibbsian, and other situations where it is not. Unfortunately, since our state space $\mathbb{R}^{\mathbb{Z}^d}$ is unbounded, we cannot use all the criteria they have at their disposition (in particular, the criterion of non Gibbsianness contained in [11]) to test the Gibbsianness/non Gibbsianness of ν^t. So our present results only concern situations for which the Gibbsianness is conserved. We hope to extend them soon to some non-Gibbsian example.

3.1 Small times

Let us consider the infinite-dimensional gradient system introduced in the section 2.1, where the self-potential U is included in the hamiltonian h and which is induced by an interaction φ through (5).

$$\begin{cases} dX_i(t) = dB_i(t) - \frac{1}{2}\nabla_i h_i(X(t))\,dt\ ,\ i \in \mathbb{Z}^d,\ \ t\in[0,T], \\ X(0) \sim \nu \in \mathcal{G}(\tilde{\varphi}, dx) \end{cases} \tag{15}$$

We have the following result.

Theorem 4 *Let us suppose that*

- *the initial interaction $\tilde{\varphi}$ is of finite range (FR), $\mathcal{C}^1$ with bounded derivatives*
- *the dynamical interaction φ is of finite range (FR), $\mathcal{C}^2$ with bounded derivatives.*

Then, there exists a time $t_0 > 0$ depending only on $\tilde{\varphi}$ and φ such that, for any $t \le t_0$,

$$\{\nu^t = \mathcal{L}(X(t)) : \nu \in \mathcal{G}(\tilde{\varphi}, dx)\} \subset \mathcal{G}(\varphi^t, dx)$$

where φ^t is an absolutely summable (AS) interaction depending only on $\tilde{\varphi}$ and φ.

We do not enter into the details of the proof since the reader will find them in [6]. We only give a sketch of the **steps of the proof :**
Our aim is to represent the family of conditional expectations of ν^t as follows :

$$\nu^t(dx_i/x_j, j\neq i) = \frac{1}{z_i}\exp-\left(h_i^t(x)\right)dx_i, i\in\mathbb{Z}^d,$$

where h^t derives from an (AS) interaction φ^t.

- The existence of the nice interaction φ^t will be a consequence of the regularity of h^t by using a result of Kozlov ([17] Theorem 1, cf. also [13] Theorem 2.30). He proved the existence of an underlying (AS) interaction under two assumptions : for each $\Lambda \subset \mathbb{Z}^d$,
  (i) *uniform boundedness* : $\exists c_\Lambda, C_\Lambda, \forall x \in \mathbb{R}^{\mathbb{Z}^d}, 0 < c_\Lambda \le h_\Lambda^t(x) \le C_\Lambda < +\infty$
  (ii) *quasilocality* : $\lim_{\Delta \nearrow \mathbb{Z}^d} \sup_{x,\bar{x}:x_\Delta=\bar{x}_\Delta} |h_\Lambda^t(x) - h_\Lambda^t(\bar{x})| = 0$

- One approximates uniformly h^t by a sequence $(h^t_\Delta)_{\Delta\subset\mathbb{Z}^d}$ of finite volume Hamiltonians, for which the verification of (i) and (ii) is easier. For each Δ, the Hamiltonian $h^t_\Delta = (h^t_{i,\Delta})_{i\in\mathbb{Z}^d}$ is obtained as follows : $exp(-h^t_{i,\Delta}(x))$ is proportional to the density of the Δ-finite volume approximation ν^t_Δ of ν^t with respect to the Δ-finite volume measure $\nu^{i,t}_\Delta$ obtained by decoupling the ith coordinate.

- One can express ν^t_Δ in the following way:

$$\frac{d\nu^t_\Delta}{dx_\Delta} = E\big(\frac{d\nu_\Delta}{dx_\Delta}(X_\Delta(t))/X(0)=x\big) = E\Big(\exp -\big(H^t_\Delta(X)+\tilde{h}_\Delta(X(t))\big)/X(0)=x\Big)$$

where the first expression is obtained using the reversibility of Lebesgue measure under the Brownian dynamics and the last expression comes from Girsanov formula. So, the density of ν^t_Δ can be interpreted as a partition function, say $Z^t_\Delta(x)$, and as such, one expands it in clusters with respect to the small time parameter t .

- Since $h^t_{i,\Delta}(x)$ is the sum of a regular function and the function $\log\frac{Z^t_\Delta(x)}{Z^t_{\Delta\setminus\{i\}}(x)}$, using the criterium of Kotecký-Preiss ([16]) one can write a cluster expansion in t for $h^t_{i,\Delta}(x)$, with nice cluster estimates. This implies that $h^t_\Delta(x)$ satisfies - uniformly in Δ - the above conditions (i) and (ii) of Step 1, and then, due to Step 2, h^t too. $\square$

Let us remark that unfortunately we are not able to prove that, in general, each element of $\mathcal{G}(\varphi^t, dx)$ corresponds to the law ν^t of the diffusion at time t for some adequate initial condition, which would correspond to the following set equality :

$$\{\nu^t = \mathcal{L}(X(t)) : \nu \in \mathcal{G}(\tilde{\varphi}, dx)\} \equiv \mathcal{G}(\varphi^t, dx).$$

This is obviously true when the cardinal of $\mathcal{G}(\varphi^t, dx)$ is reduced to 1.

3.2 Small dynamical interactions

Let us now consider the infinite-dimensional gradient dynamics (6) where the dynamical interaction has a small uniform norm. To this aim, we introduce a small parameter $\beta > 0$ as follows :

$$\begin{cases} dX_i(t) = dB_i(t) - \frac{1}{2}U'(X_i(t))\,dt - \frac{\beta}{2}\nabla_i h_i(X(t))\,dt\ ,\ i\in\mathbb{Z}^d,\ t\in[0,T] \\ X(0)\sim\nu\in\mathcal{G}(\tilde{\varphi}, m). \end{cases} \tag{16}$$

This dynamics is then a small perturbation of a free system, which is furthermore supposed to have nice ergodic properties, in such a way that its behavior for large times is close to the stationary one.

In the previous section, no particular assumption was given on the set of Gibbs measures $\mathcal{G}(\tilde{\varphi}, dx)$, which contains the initial distribution ν. Thus $\mathcal{G}(\tilde{\varphi}, dx)$ could be a singleton or it could have more than one element (phase transition). On the contrary, in this section, to control the evolution of the interaction at each time t we use techniques involving Dobrushin's uniqueness condition, and therefore, we should suppose that $\mathcal{G}(\tilde{\varphi}, m) = \{\nu\}$. Let us then introduce two definitions.

The self potential U is said to be **ultracontractive** if the semi-group associated to the one-dimensional diffusion process $dx(t) = dB(t) - \frac{1}{2}U'(x(t))\,dt$, where B is a real-valued Brownian motion, is ultracontractive.

A set of sufficient conditions for U to be ultracontractive is the following (cf. Theorem 1.4 in [15]) :

$$U'' - \frac{1}{2}(U')^2 \text{ is bounded from above, } 0 < \underline{\lim}_{|x|\to\infty} U''(x) \text{ and } \int^{\infty} \frac{1}{U'(x)}dx < +\infty,$$

and a typical example is given by $U(x) = |x|^{s+2}$ for some $s > 0$.

We say that an interaction ϕ on $\mathbb{R}^{\mathbb{Z}^d}$ satisfies the **strong Dobrushin's condition** if :

(SDC) $\quad \sup_{i\in\mathbb{Z}^d} \sum_{\Lambda\ni i}(\text{Card}\,\Lambda - 1)\sup_{x,y\in\mathbb{R}^\Lambda} |\phi_\Lambda(x) - \phi_\Lambda(y)| < 2.$

In [8] such an interaction is called a "high temperature interaction". It is well known that if an interaction ϕ satisfies (SDC), then it satisfies the Dobrushin's uniqueness condition which implies that $\mathcal{G}(\phi, m)$ contains at most one element (cf. [13], Proposition (8.8)).

We can now present our result.

Theorem 5 *Let us suppose that*

- *the self-potential U is ultracontractive*
- *the initial interaction $\tilde{\varphi}$ satisfies (SDC)*
- *the dynamical interaction φ is of finite range (FR), regular bounded (RB) and satisfies condition (9).*

Then, there exists $\beta_0 > 0$ depending only on $\tilde{\varphi}$ and φ such that, for any $\beta \leq \beta_0$ and for all $t > 0$,

$$\nu^t = \mathcal{L}\big(X(t)/\{\nu\} = \mathcal{G}(\tilde{\varphi}, m)\big) \in \mathcal{G}(\varphi^t, m)$$

where φ^t is an absolutely summable (AS) interaction.

Moreover, for large times t,

$$\text{Card}\,\mathcal{G}(\varphi^t, m) = 1.$$

As for the last theorem, we refer the reader to [6] for details and give only a sketch of the **steps of the proof :**
Let us first remark, that for $\tilde{\varphi}$ small enough, we could use similar techniques as in the proof of Theorem 4, writing the cluster expansion no more with respect to the time but with respect to both small parameters $\tilde{\varphi}$ and β. Anyway, we want to obtain more than a perturbative result around the free dynamics case. Therefore we have to develop other techniques than before.

As already remarked, it is not directly useful to consider ν^t as projection of Q^ν in the study of its Gibbsianness. Nevertheless, it is effective if we introduce one more step : ν^t is the projection on the second coordinate of the joint distribution of $(X(0), (X(t))$ denoted by $Q^{0,t}$, which is itself the projection of Q^ν on the *bi-space* $\mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}$. (In the framework of Probabilistic Cellular Automata, the idea to analyse the properties of the process on a bi-space was already powerful, cf. [26]).

- Similar to the previous section, our aim is to decompose ν^t as follows :

$$\nu^t(dy_i/y_j, j \neq i) = \frac{1}{z_i} \exp - \big(h_i^t(y)\big)\, m(dy_i),\ i \in \mathbb{Z}^d,$$

  where h^t derives from an (AS) interaction φ^t. The existence of φ^t will come from the regularity of h^t, i.e. boundedness and quasilocality in the sense of Kozlov ([17] Theorem 1).

- Let us compute the conditional probability of $Q^{0,t}$ freezing an exterior condition outside a *bi-volume* $\Delta \times \Lambda \subset \mathbb{Z}^d \times \mathbb{Z}^d$. Using Girsanov formula which is available since the volumes Λ and Δ are finite, we obtain :

$$\begin{aligned} &Q^{0,t}\big((dx_\Delta, dy_\Lambda)/x_{\Delta^c}, y_{\Lambda^c}\big) \\ &\quad = Q^\nu\big((X(0),X(t)) = (dx,dy)/X_{\Delta^c}(0) = x_{\Delta^c}, X_{\Lambda^c}(t) = y_{\Lambda^c}\big) \\ &\quad = C \exp - \Big(\tilde{h}_\Delta(x) + H^{0,t}_{\Delta,\Lambda}(x,y)\Big)\, m^{\otimes\Delta}(dx_\Delta) m^{\otimes\Lambda}(dy_\Lambda) \qquad (17) \end{aligned}$$

  where $H^{0,t}$ is directly related to the Hamiltonian H in (8). It is then clear that

$$\nu^t(dy_i/y_j, j \neq i) = C \lim_{\Delta \nearrow \mathbb{Z}^d} \int_{\mathbb{R}^\Delta} \exp - \Big(\tilde{h}_\Delta(x) + H^{0,t}_{\Delta,\{i\}}(x,y)\Big)\, m^{\otimes\Delta}(dx_\Delta)\, m(dy_i). \qquad (18)$$

  So, boundedness and quasilocality of $\tilde{h}_\Delta(x)+H^{0,t}_{\Delta,\{i\}}(x,.)$ uniformly in Δ and x would imply the regularity of h^t.

- Suppose first that β vanishes, that is the dynamics is free. Then, one proves the uniform regularity of the function $\tilde{h}+H^{0,t}$ using the smoothness of the free semigroup and the uniqueness, for each fixed $\Lambda \subset \mathbb{Z}^d$ and $y_{\Lambda^c} \in \mathbb{R}^{\Lambda^c}$, of a Gibbs measure on $\mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^\Lambda$ associated to the Hamiltonian $\tilde{h} + H^{0,t}_{.,\Lambda}(.,.y^c_\Lambda)$.

- When the dynamics is not independent but β remains small, one has a small perturbation of the above Hamilton functional. Making a space-time cluster expansion of H with respect to β in the same spirit as in [3] (cf. also [14]), we obtain the regularity of $\tilde{h} + H^{0,t}$ even uniformly in the time t.

- The uniqueness criterion for t large is obtained using the fact that the free dynamics is ultracontractive; thus the Hamiltonian of $Q^{0,t}$ on the bi-space satisfies Dobrushin's uniqueness criterium, which implies in particular that the specifications of $Q^{0,t}$ are not only local but also global ([11] and [12]). As consequence, one obtains that the projection at time t of $Q^{0,t}$ is Gibbsian and Card $\mathcal{G}(\varphi^t, m) = 1$. □

Remark 6 : *i)* If the self potential U is not ultracontractive, it is still true that Gibbsianness propagates at time t but the critical value β_0 depends on t.

ii) The above proof implies in fact more general results in the case of independent dynamics ($\beta = 0$). In this situation, for any U (not necessarily ultracontractive) and any time t, initial uniqueness propagates in time, that is Card $\mathcal{G}(\varphi^t, m) = 1$.

Acknowledgements. We thank A.C.D. van Enter, A. Le Ny and F. Redig for the very interesting Workshop they organized in Eurandom, where a talk was given on which this paper is based. We thank also F. den Hollander and R. Fernandez for fruitful discussions.

References

[1] P. Cattiaux, S. Rœlly and H. Zessin, *Une approche Gibbsienne des diffusions Browniennes infini-dimensionnelles*, Probab. Th. Rel. Fields 104 (1996), 147-179.

[2] P. Dai Pra, S. Rœlly and H. Zessin, *A Gibbs variational principle in space-time for infinite-dimensional diffusions*, Probab. Th. Rel. Fields 122 (2002), 289-315.

[3] P. Dai Pra and S. Rœlly, *An existence result for infinite-dimensional Brownian diffusions with non-regular and non-Markovian drift*, To appear in Markov Proc. Rel. Fields (2004).

[4] J.D. Deuschel, *Non-linear smoothing of infinite-dimensional diffusion processes*, Stochastics Vol. 19 (1986), 237-261.

[5] J.D. Deuschel, *Infinite-dimensional diffusion processes as Gibbs measures on* $C[0,1]^{\mathbb{Z}^d}$, Probab. Th. Rel. Fields 76 (1987), 325-340.

[6] D. Dereudre and S. Rœlly, *Propagation of Gibbsianness for infinite-dimensional Brownian diffusions*, in preparation.

[7] H. Doss and G. Royer, *Processus de diffusion associé aux mesures de Gibbs*, Z. Wahrsch. Verw. Geb. 46 (1978), 125-158.

[8] A.C.D. van Enter, R. Fernandez, F. den Hollander and F. Redig, *Possible loss and recovery of Gibbsianness during the stochastic evolution of Gibbs measures*, Commun. Math. Phys. 226 (2002), 101-130.

[9] A.C.D. van Enter, R. Fernandez and A.D. Sokal, *Regularity properties and pathologies of position-space renormalisation-group transformations : scope and limitations of Gibbsian theory*, J. Stat. Phys. 72 (1993), 879-1167.

[10] A.C.D. van Enter and J. Lörinczi, *Robustness of non-Gibbsian property : some examples*, J. Phys. A 29 (1996), 2465-2473.

[11] R. Fernandez and C.-E. Pfister, *Global specifications and nonquasilocality of projections of Gibbs measures*, Annals of Prob. 25-3 (1997), 1284-1315.

[12] H. Föllmer, *On the global Markov property*, In *Quantum fields: Algebras, Processes*, Ed. Streit, Springer (1980), 293-302.

[13] H.-O. Georgii, *Gibbs measures and phase transitions*, Berlin, W. de Gruyter (1988).

[14] I.A. Ignatyuk, V.A. Malyshev and V. Sidoravicius, *Convergence of the stochastic quantization method I**, Theory Prob. Appl. 37-2 (1990), 209-221.

[15] O. Kavian, G. Kerkyacharian and B. Roynette, *Quelques remarques sur l'ultracontractivité*, J. Func. Anal. 111 (1993), 155-196.

[16] R. Kotecký and D. Preiss, *Cluster expansions for abstract polymer models*, Commun. Math. Phys. 103 (1986), 491-498.

[17] O.K. Kozlov, *Gibbs description of a system of random variables*, Probl. Info. Trans. 10 (1974), 258-265.

[18] A. Le Ny and F. Redig, *Short time conservation of Gibbsianness under local stochastic evolutions*, J. Stat. Phys. 109, 5-6 (2002), 1073-1090.

[19] V.A. Malyshev and R.A. Minlos, *Gibbs Random Fields, Cluster expansions*, Mathematics and Its Applications Vol. 44, Kluwer Ac. Pub. (1991).

[20] R.A. Minlos, S. Rœlly and H. Zessin, *Gibbs states on space-time*, Potential Analysis 13 (2000), 367-408.

[21] R.A. Minlos, A. Verbeure and V. Zagrebnov, *A quantum cristal model in the light mass limit: Gibbs states*, Rev. Math. Phys. Vol. 12-7 (2000), 981-1032.

[22] C. Preston, *Random fields*, L.N. in Math. 534, Springer (1976).

[23] G. Royer, *Une initiation aux inegalités de Sobolev logarithmiques*, Cours Spécialisés, Soc. Math. France, Paris (1999).

[24] T. Shiga and A. Shimizu, *Infinite dimensional stochastic differential equations and their applications*, J. Math. Kyoto Univ. 20, 3 (1980), 395-416.

[25] D.W. Stroock, *Probability Theory, An analytic view*, Cambridge University Press (1993).

[26] A.L. Toom, N.B. Vasilyev, O.N. Stavskaya, L.G. Mityushin, G.L. Kurdyumov and S.A. Pirogov, *Locally interactive systems and their application in biology*, L.N. in Math. 653, Springer (1978).