GLOBAL SOLUTIONS TO BUBBLE GROWTH IN POROUS MEDIA

LAVI KARP

Abstract. We study a moving boundary problem modeling an injected fluid into another viscous fluid. The viscous fluid is withdrawn at infinity and governed by Darcy's law. We present solutions to the free boundary problem in terms of time-derivative of a generalized Newtonian potentials of the characteristic function of the bubble. This enables us to show that the bubble occupies the entire space as the time tends to infinity if and only if the internal generalized Newtonian potential of the initial bubble is a quadratic polynomial. Howison [7], and DiBenedetto and Friedman [2], studied such behavior, but for bounded bubbles. We extend their results to unbounded bubbles.

1. Introduction

The propose of this note is to apply the technique of generalized Newtonian potential to the study of bubble growth in porous media and Hele-Shaw flows. This technique enables the computation a potential of measures with unbounded support in a similar manner to the ordinary Newtonian potential . It is a multi-valued right inverse to the Laplacain and unique up to a harmonic polynomial of degree not exceeding two (see [8, 10, 13]). We can therefore compute the potential of the characteristic function of arbitrary large set $\mathbb{R}^n$. The Newtonian potential theory is a basic tool in the studies of Hele-Shaw flows [5, 6, 14, 18, 21], and since in a bubble growth in porous media the fluid motion is in unbounded domains, the application of the generalized Newtonian potential is rather natural in this type of moving boundary problems.

Consider $\mathbb{R}^n$ as homogeneous porous medium filled with a viscous fluid. Another fluid is injected and forms a bubble which occupies a

2010 *Mathematics Subject Classification.* Primary 35R37; Secondary 31B20, 35Q86.

Key words and phrases. Bubble growth, Hele-Shaw flows, generalized Newton potential, quadratic polynomial.

*Supported by DFG Ref: SCHU 808/21-1 and by ORT Braude College's Research Authority.

domain $D(t)$ at each time t. The fluid withdrawn at infinity at a certain rate and the bubble $D(t)$ increases with the time t. Let $\Omega(t) = \mathbb{R}^n \setminus D(t)$ and $\Psi(x,t)$ denotes the pressure of the incompressible fluid in $\Omega(t)$. We will prove that $\Psi(x,t)$, the solution to this moving boundary problem (1.1) below, is a time-derivative of a generalized Newtonian potential of $\chi_{D(t)}$, the characteristic function of bubble $D(t)$. In this formulation it does not matter whether the bubble is bounded or unbounded.

The representation of solutions by potentials has several advantages specially in higher dimensions where the tool of conformal mappings is not available. For example, this enables us to construct in a simple manner solutions for which the bubble exists for all for all $t > 0$ and occupies the entire space as as $t \to \infty$. This problem was settled by Howison [7], and DiBenedetto and Friedman [2], but under the condition that the initial bubble is a bonded domain. Here we show that this type of fluids motions exists if and only if the internal generalized Newtonian potential of the initial bubble $\chi_{D(0)}$ is a quadratic polynomial. It thus extends their result to unbounded bubbles.

The known examples of domains in $\mathbb{R}^n$ for which internal generalized Newtonian potential of their characteristic set equals to a quadratic polynomial are (a) ellipsoids, (b) convex domains bounded by elliptic paraboloids, (c) domains bounded by two parallel hyperplanes, (d) cylinders over (a) and (b), and (e) half-spaces [10]. The complements of these domains are null quadrature domains, that is, an open set Ω for which

$$\int_\Omega h dx = 0$$

for all harmonic and integrable functions h in Ω. In the two dimensional plane Sakai classified those domains [17], but the classification in higher dimensional spaces is an open problem. Dive [4], and Nikliborc [15], proved that if the internal Newtonian potential of a bounded domain is a quadratic polynomial, then it must be an ellipsoid (their proof in given in $\mathbb{R}^3$, for proofs in arbitrary dimension see [2, 9]). The classification problem of this type of domains is also settled under the a priori assumption that the domain is contained in a cylinder of codimension two [10]. A recent progress in the classification problem was obtained in [11] and for its current state see Remark 4.3 below.

1.1. **The formulation of the moving boundary problem.** The bubble is formed by injected a fluid of negligible viscosity into another incompressible viscous fluid. We denote the set occupied by the bubble at time t by $D(t)$ and $\Omega(t) = \mathbb{R}^n \setminus D(t)$. Suppose $\mathbb{R}^n$ to consists of homogeneous porous medium, then the motion of the flow in $\Omega(t)$ is subject to the Darcy's law which asserts that the velocity $\vec{v}$ is proportional to the gradient of the pressure Ψ:

$$\vec{v} = -\kappa \nabla_x \Psi.$$

Since the flow is incompressible, div $\vec{v} = 0$, hence

$$\Delta\Psi = 0 \quad \text{in} \quad \Omega(t) \quad \text{for all} \quad t > 0.$$

Assuming there is no surface tension, then the pressure is constant on the boundary $\partial\Omega(t)$, so we may set

$$\Psi(x,t) = 0 \quad x \in D(t) \quad \text{for all} \quad t > 0.$$

The free boundary moves with velocity $-\nabla_x \Psi$, that is

$$\frac{\partial \Psi}{\partial \eta} = -\mathcal{V}_\eta \quad \text{on} \quad \partial D(t),$$

where η is the outward normal pointing into $\Omega(t)$ and $\mathcal{V}_\eta$ is is the outward velocity of the free boundary $\partial D(t)$. If $D(t) = \{g(x,t) < 0\}$, then $\frac{\partial \Psi}{\partial \eta} = \nabla_x \Psi \cdot \left(\frac{\nabla_x g}{|\nabla_x g|}\right)$ and $\mathcal{V}_\eta = \frac{-\partial_t g}{|\nabla_x g|}$. Generalized Newtonian potential of densities in $L^\infty(\mathbb{R}^n)$ have $|x|^2 \log|x|$ growth as $x \to \infty$ [8, 20], and since in our setting the bubble $D(t)$ may has infinite volume we allow $\Psi(x,t) = o(|x|^3)$ at infinity. We summarize these conditions:

$$\Delta\Psi = 0 \quad \text{in} \quad \Omega(t), \tag{1.1a}$$

$$\Psi = 0 \quad \text{in} \quad D(t), \tag{1.1b}$$

$$\frac{\partial \Psi}{\partial \eta} = -\mathcal{V}_\eta \quad \text{on} \quad \partial\Omega(t), \tag{1.1c}$$

$$\Psi(x,t) = o(|x|^3) \quad \text{as} \quad |x| \to \infty, \tag{1.1d}$$

$$\nabla\Psi(x,t) = o(|x|^2) \quad \text{as} \quad |x| \to \infty. \tag{1.1e}$$

2. Generalized Newtonian Potential

The definition and establishment of basic properties of this potential were carried out in [10]. Here we recall the definition and present an estimate which is needed for the current application.

The Newtonian potential of a measure μ with compact support is defined by means of the convolution

$$V(\mu)(x) = (J * \mu)(x) = \int J(x-y)\,d\mu(y) \tag{2.1}$$

where

$$J(x) = \begin{cases} -\dfrac{1}{2\pi}\log|x| \ , & n = 2 \\ \dfrac{1}{(n-2)\omega_n |x|^{n-2}} \ , & n \geq 3 \end{cases}$$

and ω_n is the area of the unit sphere in $\mathbb{R}^n$. The potential $V(\mu)$ satisfies the Poisson equation $\Delta V(\mu) = -\mu$ in the distributional sense. Similarly, the *generalized Newtonian potential* is a right inverse of the Laplacian, but it is multi-valued and acting on the space $\mathcal{L}$, the space of all Radon measures μ in $\mathbb{R}^n$ satisfying condition

$$\|\mu\|_{\mathcal{L}} := \int \frac{d|\mu|(x)}{1 + |x|^{n+1}} < \infty. \tag{2.2}$$

Note that $\mathcal{L}$ contains all measures with densities in $L^\infty(\mathbb{R}^n)$. The linear space $\mathcal{L}$ is the Banach space with the norm defined by (2.2). For $\mu \in \mathcal{L}$ we first define $V^\alpha(\mu)$, the operator of the third order generalized derivatives of the potential:

$$\langle V^\alpha(\mu), \varphi\rangle := -\int \partial^\alpha V(\varphi)(x) d\mu(x), \quad \varphi \in \mathcal{S}, \ |\alpha| = 3. \tag{2.3}$$

Here $\mathcal{S}$ is the Schwartz class of rapidly decreasing functions and $V^\alpha(\mu)$ is a tempered distribution.

Definition 2.1. The generalized Newtonian potential $V[\mu]$ of a measure $\mu \in \mathcal{L}$ is the set of all solutions to the system

$$\begin{cases} \Delta u = -\mu \\ \partial^\alpha u = V^\alpha(\mu) \ , \quad |\alpha| = 3 \ . \end{cases} \tag{2.4}$$

The existence of solutions to system (2.4) was proved in [10] and it is unique modulo $\mathcal{H}_2$, the space of all harmonic polynomials of degree at most two. The operator $V \colon \mathcal{L} \to \mathcal{S}'/\mathcal{H}_2$ is continuous [10].

We will not distinguish here between ordinary and generalized potentials. By a potential of a set A we mean the potential of its characteristic function χ_A.

Let k be a positive integer, then

$$\|\varphi\|_k = \sup_{\{|\alpha| \leq k\}} \sup_{\mathbb{R}^n} \Big((1 + |x|)^k \left|\partial^k \varphi(x)\right| \Big)$$

is a semi-norm on $\mathcal{S}$. We shall need the following estimate.

Proposition 2.2. Let α and β be multi-indexes such that $|\alpha| = 3$. Then for any $\varphi \in \mathcal{S}$

$$|\partial^{\alpha+\beta} V(\varphi)(x) \leq C \left(1 + |x|\right)^{-(n+1+|\beta|)} \|\varphi\|_{2(n+1+|\beta|)}, \tag{2.5}$$

where the constant C does not depend on φ.

For $\beta = 0$ this was proved in [10, §1]. Since only a slightly modification is needed in order to extend it for (2.5) we leave it to the reader.

3. The Generalized Newtonian Potential of the Bubble

In this section we show that solutions to the moving boundary problem (1.1) can be expressed in terms of the generalized potential.

Proposition 3.1. Let $V^\alpha(\chi_{D(t)})$ be the third order derivatives operator defined in (2.3) and assume Ψ satisfies (1.1), then

$$\frac{d}{dt}V^\alpha(\chi_{D(t)}) = \partial^\alpha \Psi, \quad |\alpha| = 3, \tag{3.1}$$

where the identity (3.1) is in the distributional sense.

Proof. Let $\varphi \in \mathcal{S}$, then Proposition 2.2 implies that $\partial V^\alpha(\varphi) \in L^1(\mathbb{R}^n)$ and therefore $V^\alpha(\chi_{\mathbb{R}^n})(\varphi) = -\int_{\mathbb{R}^n} \partial^\alpha V(\varphi)dx = 0$. Since $\Omega(t) = \mathbb{R}^n \setminus D(t)$,

$$\langle V^\alpha(\chi_{D(t)}), \varphi\rangle = \int_{D(t)} \partial^\alpha V(\varphi)(x)dx = -\int_{\Omega(t)} \partial^\alpha V(\varphi)(x)dx \tag{3.2}$$

and

$$\frac{d}{dt}\left(\langle V^\alpha(\chi_{D(t)}), \varphi\rangle\right) = -\int_{\partial\Omega(t)} \partial^\alpha V(\varphi)(x)\mathcal{V}_\eta dS. \tag{3.3}$$

Let B_R be a ball of radius R and apply the Green's formula, then

$$\begin{aligned}
&\int_{\Omega(t)\cap B_R} \Delta\left(\partial^\alpha V(\varphi)\right)\Psi dx \\
&= -\int_{\partial\Omega(t)} \left(\frac{\partial\left(\partial^\alpha V(\varphi)\right)}{\partial\eta}\Psi - \partial^\alpha V(\varphi)\frac{\partial\Psi}{\partial\eta}\right)dS \\
&\quad + \int_{\Omega(t)\cap\partial B_R} \left(\frac{\partial\left(\partial^\alpha V(\varphi)\right)}{\partial\eta}\Psi - \partial^\alpha V(\varphi)\frac{\partial\Psi}{\partial\eta}\right)dS.
\end{aligned} \tag{3.4}$$

By the estimate (2.5) and the growth assumptions (1.1d) and (1.1e) of Ψ, the second term of the right hand site of (3.4) tends to zero as R goes to infinity. Thus, letting $R \to \infty$ and using the boundary conditions of (1.1) we get

$$\int_{\Omega(t)} \Delta\left(\partial^\alpha V(\varphi)\right)\Psi dx = \int_{\partial\Omega(t)} \partial^\alpha V(\varphi)\mathcal{V}_\eta dS. \tag{3.5}$$

Since $\Delta V(\varphi) = -\varphi$ for any test function, we see from (3.3) and (3.5) that

$$\frac{d}{dt}\left(\langle V^{\alpha}(\chi_{D(t)}), \varphi\rangle\right) = -\int_{\Omega(t)} \Delta\left(\partial^{\alpha} V(\varphi)\right)\Psi dx = -\int \partial^{\alpha}\varphi \Psi dx \tag{3.6}$$

which is the distributional meaning of (3.3). □

Corollary 3.2. *If the bubble $D(t)$ occupies the entire space as t tends to infinity, that is,*

$$\lim_{t\to\infty} D(t) = \mathbb{R}^n, \tag{3.7}$$

then the potential of the initial bubble $D(0)$ coincides with a quadratic polynomial on $D(0)$.

Proof. Taking the integral of (3.1) we get

$$\langle V^{\alpha}(\chi_{D(t)}), \varphi\rangle - \langle V^{\alpha}(\chi_{D(0)}), \varphi\rangle = -\int_0^t \int \partial^{\alpha}\varphi(x)\Psi(x,s)dxds. \tag{3.8}$$

Since $\Psi(x,s) = 0$ for all $x \in D(0)$ and all $s \geq 0$, we get from (3.8) that

$$\langle V^{\alpha}(\chi_{D(0)}), \varphi\rangle = \langle V^{\alpha}(\chi_{D(t)}), \varphi\rangle \quad \text{for all } \varphi \in C_0^{\infty}(D(0)). \tag{3.9}$$

Now if (3.7) holds, then by the continuity of the generalized potentials we have that

$$\langle V^{\alpha}(\chi_{D(0)}), \varphi\rangle = \lim_{t\to\infty}\langle V^{\alpha}(\chi_{D(t)}), \varphi\rangle = \langle V^{\alpha}(\chi_{\mathbb{R}^n}), \varphi\rangle = 0$$

for all $\varphi \in C_0^{\infty}(D(0))$. Hence the third order derivatives for any $u \in V[\chi_{D(0)}]$ vanish in $D(0)$. □

The following formula is the analogous of Richardson's theorem [16], but for suction at infinity. This was previously proved by Entov and Etingof [5] (see also [21, §4.6]), in the plane and for a bounded bubble. We thus generalized their result to the space $\mathbb{R}^n$ and for unbounded bubbles.

Theorem 3.3. *Suppose the solution of (1.1) exists for $t \in [0,T])$, then there exists a generalized Newtonian potential $u(\cdot,t) \in V[\chi_{D(t)}]$ such that*

$$\frac{d}{dt}u(x,t) = a(t) + \Psi(x,t). \tag{3.10}$$

Proof. Let $w \in V[\chi_{D(t)}]$ and $|\alpha| = 3$, then by Proposition 3.1

$$\frac{d}{dt}\partial^{\alpha} w(x,t) = \partial^{\alpha}\Psi(x,t), \tag{3.11}$$

which implies

$$\partial^\alpha w(x,t) = F_\alpha(x) + \int_0^t \partial^\alpha \Psi(x,s)ds. \tag{3.12}$$

Since $\partial_i F_\alpha = \partial_j F_\beta$ for any α, β with $\partial_i \partial^\alpha = \partial_j \partial^\beta$, there is a function F such that $\partial^\alpha F = F_\alpha$. Hence

$$w(x,t) = F(x) + \int_0^t \Psi(x,s)ds + q(x,t), \tag{3.13}$$

where $q(x,t)$ is a quadratic polynomial. We can write it in the form

$$q(x,t) = |x|^2 M(t) + h(x,t) + A(t),$$

where $h(x,t)$ is harmonic polynomial of degree not exceeding two (see e.g. [1]). Now $\Psi(x,t) = 0$ for $x \in D(0)$ and $t \in [0,T]$, therefore in $D(0)$ we have

$$-1 = \Delta w(x,t) = \Delta F(x) + 2nM(t). \tag{3.14}$$

Differentiating with respect to t gives

$$0 = \frac{d}{dt}\Delta w(x,t) = 2n\frac{d}{dt}M(t). \tag{3.15}$$

So $M(t)$ does not depend on t. Letting $u(x,t) = w(x,t) - h(x,t)$, then u belongs to $V[\chi_{D(t)}]$ and satisfies (3.10) with $a(t) = \frac{d}{dt}A(t)$. □

We shall now see that the converse statement to Theorem 3.3 is also true.

Theorem 3.4. *Let $D(t)$ be a continuous family of domains such that $D(t_1) \subset D(t_2)$ for $t_1 < t_2$ and $\partial D(t)$ is sufficiently smooth so that the Green's identity holds. If for $t \in [0,T]$ there are potentials $u(\cdot,t) \in V[\chi_{D(t)}]$ such that*

$$u(x,t_2) - u(x,t_1) \quad \textit{does not depend on } x \textit{ for } x \in D(t_1), \tag{3.16}$$

then the solution to (1.1) is given by

$$\Psi(x,t) := \frac{d}{dt}u(x,t) - a(t). \tag{3.17}$$

Proof. Here we argue similar to [2]. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $t_1 < t_2$. Then

$$\lim_{t_2 \to t_1} \frac{1}{t_2 - t_1} \int_{D(t_2)\setminus D(t_1)} \varphi dx = \int_{\partial D(t_1)} \mathcal{V}_\eta dS,$$

where $\mathcal{V}_\eta$ is is the outward velocity of the free boundary $\partial D(t_1)$. Since

$$\frac{1}{t_2 - t_1} \int_{D(t_2)\setminus D(t_1)} \varphi dx = \frac{-1}{t_2 - t_1} \int \left(u(x,t_2) - u(x,t_1)\right) \Delta\varphi dx,$$

$\frac{d}{dt}u(x,t)$ exists and

$$\langle \frac{d}{dt}u(\cdot,t), \Delta\varphi\rangle = -\langle \mathcal{V}_n, \varphi\rangle. \tag{3.18}$$

By condition (3.16), $u(x,t)-u(x,0)$ does not depend on x for $x \in D(0)$. So we denote $u(x,t)-u(x,0)$ by $A(t)$ and define

$$\Psi(x,t) := \frac{d}{dt}u(x,t) - A'(t).$$

We shall now verify that this $\Psi(x,t)$ satisfies (1.1). Since $u(x,t) \in V[\chi_{D(t)}]$, it is harmonic in $\Omega(t)$ and therefore (1.1a) holds. Condition (3.16) implies that $\Psi(x,t) = 0$ for $x \in D(t)$, so (1.1b) holds too and the growth properties (1.1d) and (1.1e) are consequence of the known estimates of the generalized potential of $L^\infty(\mathbb{R}^n)$-densities [8, 20]. In order to verify (1.1c) we use (1.1b), (3.18) and the Green's formula, these yield

$$\begin{aligned} -\langle \mathcal{V}_\eta, \varphi\rangle = \langle \frac{d}{dt}u(\cdot,t), \Delta\varphi\rangle &= \int_{\Omega(t)} \frac{d}{dt}u(x,t)\Delta\varphi(x)dx \\ &= \int_{\partial D(t)} \left(\frac{\partial}{\partial\eta}\left(\frac{d}{dt}u(x,t)\right)\varphi(x)\right) dS. \end{aligned} \tag{3.19}$$

This completes the proof. □

4. Construction of global solutions

In this section we use Theorem 3.4 in order to construct solutions to the moving boundary problem (1.1) such that the bubble $D(t)$ will occupy the entire space as t goes to infinity. By Corollary 3.2 a necessary condition for that is

$$u(x) = q(x) \quad \text{for } x \in D(0), \qquad u \in V[\chi_{D(0)}],$$

where q is a quadratic polynomial. Here we shall prove that this is also a sufficient condition.

We first recall few known facts. A classical theorem of Newton says that the gravitational force of a homogeneous shell between two similar ellipsoids vanishes in the cavity of the shell (see e.g. [3, 12]). This is equivalent to the property that the ordinary internal Newtonian potential of ellipsoids is a quadratic polynomial (see e.g. [2]). We also know that the following unbounded domains have their internal generalized potential coincides with a quadratic polynomial: (*i*) strips, (*ii*) half-spaces, (*iii*) convex domains bounded by elliptical paraboloid, and (*iv*) cylinders over ellipsoids and over these domains [10, 19].

The first example shows that there are many motions of bubbles which fill the entire space as t runs to infinity.

Example 4.1. Let $E(t)$ be any continuous family of ellipsoids such that $E(t_1) \subset E(t_2)$ whenever $t_1 < t_2$ and $\lim_{t\to\infty} E(t) = \mathbb{R}^n$. Then the Newtonian potential $V(\chi_{E(t)})(x) = q(x,t)$ for $x \in E(t)$, where q is a quadratic polynomial satisfying $\Delta q = -1$. So we may write

$$q(x,t) = \sum_{i=1}^{n} a_i(t)x_i^2 + p_2(x,t), \quad a_1(t) + \cdots + a_n(t) = -\frac{1}{2}, \tag{4.1}$$

where p_2 is a harmonic polynomial of degree not exceeding two. Then

$$h(x,t) := \sum_{i=1}^{n-1} a_i(t)x_i^2 + \left(a_n(t) + \frac{1}{2}\right) x_n^2$$

is harmonic polynomial of degree two and therefore

$$u(x,t) = V(\chi_{E(t)})(x) - h(x,t) - p_2(x,t)$$

is a generalized Newton potential of $\chi_{E(t)}$ satisfying $u(x,t) = -\frac{1}{2}x_n^2$ on $E(t)$. Therefore, condition (3.16) holds so we conclude that $\Psi(x,t) = \frac{d}{dt}u(x,t)$ is a global solution to the moving boundary problem (1.1).

Of course these solutions may not have a meaningful physical interpretation, since they have a quadratic growth at infinity. It is therefore reasonable to look for solutions which have a minimal growth at infinity. We see from (4.1) that a necessary condition for that is that the coefficients $a_i(t)$ are independent of t. Such behavior of the coefficients $a_i(t)$ occurs when $E(t)$ are concentric ellipsoids having there center at the origin (see e.g. [15]).

Thus the ellipsoids of the form

$$E(t) = \left\{ x \in \mathbb{R}^n : \left(\frac{x_1}{\alpha_1}\right)^2 + \cdots + \left(\frac{x_n}{\alpha_n}\right)^2 < (1+t)^2 \right\}$$

will form the bubble for which $\Psi(x,t)$ will have a minimal growth at infinity. This example where also obtained by Friedman and DiBenedetto [2]. In addition, any cylinder

$$E_k(t) = \left\{ x \in \mathbb{R}^k : \left(\frac{x_1}{\alpha_1}\right)^2 + \cdots + \left(\frac{x_k}{\alpha_k}\right)^2 < (1+t)^2 \right\} \times \mathbb{R}^{n-k}$$

will also gives global solutions which has minimal a growth in directions off the cylinder.

We now turn showing that elliptical paraboloid also form a global solution.

Example 4.2. Let

$$F = \left\{ x : \ x_n > \left(\frac{x_1}{\alpha_1}\right)^2 + \cdots + \left(\frac{x_{n-1}}{\alpha_{n-1}}\right)^2 \right\}, \tag{4.2}$$

then any generalized Newton potential of χ_F coincides with a quadratic polynomial on F. We could therefore argue like as in Example 4.1, but that will not have a minimal growth. We claim that there is $u \in V[\chi_F]$ such that $u(x)$ does not depend on x_n for $x \in F$. Assuming it for the moment, and define the bubble $F(t)$ by a translation in the x_n axis, that is,

$$F(t) = \{x \in \mathbb{R}^n : (x_1, ..., x_{n-1}, x_n + t) \in D\}.$$

Then obviously $u(x,t) := u(x_1, ..., x_{n-1}, x_n + t)$ is a generalized potential of $\chi_{F(t)}$ and since $u(x)$ does not depend on x_n, $u(x,t) = u(x)$ for $x \in F(t)$. Thus $u(x,t)$ satisfies condition (3.16) and by Theorem 3.4 $\frac{d}{dt}u(x,t)$ is a global solution to (1.1). In addition, $\lim_{t\to\infty} F(t) = \mathbb{R}^n$.

In order to prove the claim, we adopt the following argument from [11, §4]. There we showed that if $V[\chi_D]$ coincides with a quadratic polynomial in D and

$$\lim_{\rho\to\infty} \frac{\mathrm{Vol}(D \cap B_\rho)}{\mathrm{Vol}(B_\rho)} = 0, \tag{4.3}$$

then there is a potential $w \in V[\chi_D]$, such that

$$w(x) = q(x) := \sum_{ij=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + c, \quad \text{for } x \in D$$

and

$$\lim_{\rho\to\infty} \frac{w(\rho x) - q(\rho x)}{\rho^2} = \sum_{ij=1}^{n} a_{ij} x_i x_j. \tag{4.4}$$

Here $\mathrm{Vol}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^n$ and B_ρ a ball with radius ρ.

Now the set F given by (4.2) certainly satisfies condition (4.3). So let $w \in V[\chi_F]$ be the potential which satisfies (4.4). Since $w(x) - q(x)$ vanishes on F, $w(\rho x) - q(\rho x)$ vanishes on the positive x_n axis for every ρ and therefore the limit in (4.4) also vanishes there. Hence $a_n = 0$. Letting $u(x) = w(x) - \sum_{i=1}^{n}(a_{in} + a_n i) x_i x_n - b_n x_n$ gives the required potential which also provides a solution with minimal growth at the direction of the negative x_n axis.

Remark 4.3. The complete characterization of domains D with internal potential equal to a quadratic polynomial is not known. We recently proved in [11] that if D is such domain, then it must be one of the following type:

1. If $\limsup_{\rho\to\infty} \dfrac{\mathrm{Vol}\,(D \cap B_\rho)}{\mathrm{Vol}(B_\rho)} > 0$, then D is half-space;
2. If $\lim_{\rho\to\infty} \dfrac{\mathrm{Vol}\,(D \cap B_\rho)}{\mathrm{Vol}(B_\rho)} = 0$, then there are two possibilities:
   - 2a. If D is contained between two parallel hyperplanes, then D is an ellipsoid, or a cylinder over an ellipsoid;
   - 2b. If D is not contained between two parallel hyperplanes, then $D = \{x_n > f(x_1, ..., x_k),\ 1 \le k < n\}$ and f is real analytic convex function.

It is obvious that we can extend Example 4.2 to domains of type 2b. above. We thus proved:

Theorem 4.4. *Let $\Psi(x,t)$ be a solution to the moving boundary problem (1.1). Then the bubble $D(t)$ occupies the entire space as t goes to infinity if and only if the generalized Newtonian potential of the initial bubble $V[\chi_{D(0)}]$ coincides with a quadratic polynomial in $D(0)$.*

Finally, these techniques can be used also for contractions of flows. For example, we can construct solution to (1.1) so that the paraboloid D in (4.2) will contract to a half-line. Here we use again the fact that there is $u \in V[\chi_D]$ that does not depend on x_n for $x \in D$.

Acknowledgment. I would like to thanks A. Margulis for many valuable conversations. This project was completed during the author's visit at the Department of Mathematics at Potsdam Universität and I am grateful to Professor B.-W. Schulze for his support and kind hospitality.

References

1. S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, Second Edition, Springer-Velag,New York, 2001.
2. E. DiBenedetto and A. Friedman, Bubble growth in porous media, *Indiana Univ. Math. J.* **35** (1986), 5730-606.
3. S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium*, Dover, 1987.
4. P. Dive, Attrcation des ellipsoides homogénes et rèciproque d'um théorème de Newton, *Bull Soc. Math. France* **59** (1931),128-140.
5. V.M. Entov and P.I. Etingof, Bubble contraction in Hele-Shaw cells, *Quart. J. Mech. Appl. Math.* **44**, no. 4 (1991) 507-535.
6. B. Gustafsson, Existence of weak backward solutions to a generalized Hele-Shaw flow moving boundary problem, *Nonlinear Anal.* **9**, no. 2 (1985), 203-215.
7. S. D. Howison, Bubble growth in porous media and Hele Shaw flow, *Proc. Roy. Soc. Edinburgh* **A102** (1985), 141-148.
8. L. Karp, Generalized Newtonian potential and its applications, *J. Math. Anal. Appl.* **174** (1993), 480-497.

9. L. Karp, On the Newtonian potential of ellipsoids, *Complex Variables* **25** (1995), 367-372.
10. L. Karp and A. Margulis, Newtonian potential theory for unbounded sources and applications to free boundary problems, *J. Analyse Math.* **70** (1996), 1-63.
11. L. Karp and A. Margulis, Null quadrature domains and a free boundary problem for the Laplacian, http://arxiv.org/abs/1003.1054v1, 2010.
12. O.D. Kellog, *Foundation of Potential Theory*, Reprint from the First Edition 1929, Springer-Verlag, Berlin-Heidelberg-New York, 1967.
13. A.S. Margulis, *Potential theory for L^p-densities and its applications to inverse problems of gravimetry*, in Theory and Practice of Gravitational and Magnetic Fields Interpretation in USSR, Naukova Dumka Press, Kiev, 1983, pages 188-197 (Russian).
14. A.S. Margulis, *The moving boundary problem of potential theory*, Adv. Math. Sci. Appl. **5(2)** (1995), 603-629.
15. W. Nikliborc, Eine Bemerkung über die Volumpotentiale, *Math Z*, **35** (1932), 625-631.
16. S. Richardson, Some Hele-Show flows with a free boundary produced by the injection of fluid into a narrow channel, *J. Fluid Mech.* **56** (1972), 609-618.
17. M. Sakai, Null quadrature domains, *J. Analyse Math.* **40** (1981), 144-154.
18. M. Sakai, *Quadrature Domains*, Lecture Notes in Math. **934**, Springer-Verlag, Berlin-Heidelberg-New York, 1982.
19. H. Shahgholian, *On quadrature domains and the Schwarz Potential*, J. Math. Anal. Appl. **177** (1992), 61-78.
20. H. S. Shapiro, A weighted L^1 estimate for the Laplace Operator, in *Recent Advance in Fourier Analysis and its Applications* (Il Ciocco 1989), Kluwer Acad Publ. Dordrecht 1990, pp. 563-577.
21. A. N. Varchenko, P. I. Etingof, *Why the Boundary of a Round Drop Becomes a Curve of Order Four*, American Mathematical Society University Lecture Series, Volume 3, Providence, Rhode Island 1992.

E-mail address: karp@braude.ac.il

Department of Mathematics, ORT Braude College, P.O. Box 78, 21982 Karmiel, Israel