On the Well-Posedness of the Vacuum Einstein's Equations

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Abstract

The Cauchy problem of the vacuum Einstein's equations aims to find a semi-metric $\mathbf{g}_{\alpha\beta}$ of a spacetime with vanishing Ricci curvature $\mathbf{R}_{\alpha,\beta}$ and prescribed initial data. Under the harmonic gauge condition, the equations $\mathbf{R}_{\alpha,\beta} = 0$ are transferred into a system of quasi-linear wave equations which are called *the reduced Einstein equations*. The initial data for Einstein's equations are a proper Riemannian metric $\mathbf{h}_{ab}$ and a second fundamental form $\mathbf{K}_{ab}$. A necessary condition for the reduced Einstein equation to satisfy the vacuum equations is that the initial data satisfy Einstein constraint equations. Hence the data $(\mathbf{h}_{ab}, \mathbf{K}_{ab})$ cannot serve as initial data for the reduced Einstein equations.

Previous results in the case of asymptotically flat spacetimes provide a solution to the constraint equations in one type of Sobolev spaces, while initial data for the evolution equations belong to a different type of Sobolev spaces. The goal of our work is to resolve this incompatibility and to show that under the harmonic gauge the vacuum Einstein equations are well-posed in one type of Sobolev spaces.

1 Introduction

This paper deals with well-posedness of the Cauchy problem of Einstein vacuum equations

$$\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0, \qquad \alpha, \beta = 0, 1, 2, 3. \tag{1.1}$$

Here $\mathbf{R}_{\alpha\beta}(\mathbf{g})$ denotes the Ricci curvature tensors of a Lorentzian metric $\mathbf{g}$. The unknowns are the coefficients $\mathbf{g}_{\alpha\beta}$ of the semi-metric $\mathbf{g}$.

The initial data consist of the triple $(M, \mathbf{h}_{ab}, \mathbf{K}_{ab})$, where M is a space-like manifold, $\mathbf{h}_{ab}$ is a proper Riemannian metric on M and $\mathbf{K}_{ab}$ is its second fundamental form (extrinsic curvature). The semi-metric $\mathbf{g}_{\alpha\beta}$ takes the following data on M:

$$\begin{cases} \mathbf{g}_{00}|_M = -1, \qquad \mathbf{g}_{0a}|_M = 0, \qquad \mathbf{g}_{ab}|_M = \mathbf{h}_{ab} \\ \qquad -\frac{1}{2}\partial_0 \mathbf{g}_{ab}|_M = \mathbf{K}_{ab}, \end{cases} \qquad a, b = 1, 2, 3. \tag{1.2}$$

*Research partially supported by DFG: SCHU 808/19-1 and ORT Braude College's Research Authority

The rest of the data $\partial_0 \mathbf{g}_{\alpha 0}$ are determined through the conditions

$$F^{\mu} = \mathbf{g}^{\beta\gamma}\Gamma^{\mu}_{\beta\gamma} = 0. \tag{1.3}$$

Here $\Gamma^{\mu}_{\beta\gamma}$ denotes the Christoffel symbols and $\mathbf{g}^{\beta\gamma}$ the inverse matrix of $\mathbf{g}_{\beta\gamma}$. Condition (1.3) is known as the *harmonic gauge*. Since (1.1) is a characteristic (see e.g. [14]), it is impossible to solve it in the present form. However, under the harmonic gauge (1.3), the vacuum Einstein equations (1.3) is equivalent to the *reduced Einstein equations:*

$$\mathbf{g}^{\alpha\beta}\partial_{\alpha}\partial_{\beta}\mathbf{g}_{\gamma\delta} = Q_{\gamma\delta}(\mathbf{g}, \partial\mathbf{g}). \tag{1.4}$$

Since $\mathbf{g}_{\alpha\beta}$ has a Lorentzian signature, this is a system of semi-linear wave equations. The expressions $Q_{\gamma\delta}(\mathbf{g}, \partial\mathbf{g})$ are quadratic functions of the semi-metric $\mathbf{g}_{\alpha\beta}$ and its first order partial derivatives $\partial\mathbf{g}_{\alpha\beta}$.

It is well known that the initial data $(\mathbf{h}_{ab}, \mathbf{K}_{ab})$ cannot be prescribed arbitrarily and, in fact, the Codazzi equations, which relate the curvature of the manifold $(V, \mathbf{g}_{\alpha\beta})$ to the one of $(M, \mathbf{h}_{ab})$, lead to the Einstein constraint equations:

$$\begin{cases} R(\mathbf{h}) - \|\mathbf{K}_{ab}\|^2 + (\mathrm{Tr}_{\mathbf{h}}\mathbf{K}_{ab})^2 = 0 & \text{(Hamilton constraint)} \\ D_a\mathbf{K}^a_b - D_b(\mathrm{Tr}_{\mathbf{h}}\mathbf{K}_{ab}) = 0 & \text{(momentum constraint)} \end{cases} \tag{1.5}$$

Here $R(\mathbf{h})$ and D_a are the scalar and the covariant derivative with respect to the metric $\mathbf{h}_{ab}$.

The fulfillment of the constraint equations (1.5) is a necessary condition for the solution of the wave equations (1.4) with the initial data (1.2) to satisfy the vacuum Einstein vacuum equations (1.1). We refer to [1], [24], [16] or [27] for a discussion of this fact.

Conclusion: We conclude that any solution of the Cauchy problem for the Einstein equations (1.1) includes the treatment of the following two problems:

(i) Solutions to constraints (1.5), which can be reduced to elliptic equations;

(ii) Solutions to the reduced Einstein equations (1.4) with the initial data (1.2).

We will deal with these two problems in asymptotically flat manifolds (AF). A Riemanian 3-manifold $(M, \mathbf{h}_{ab})$ is (AF) if there is a compact subset K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$ and the metric $\mathbf{h}_{ab}$ tends to the identity $\mathbf{e}_{ab}$ at infinity.

1.1 Existence of the evolution equations

The unknowns $\mathbf{g}_{\alpha\beta}$ are functions of $t = x^0$ and x^a, $a = 1, 2, 3$. We assume that for $t = 0$, the initial data (1.2) are given on a space-like (AF) hypersurface $M \simeq \mathbb{R}^3$ and denote the Bessel potential space on $\mathbb{R}^3$ by H^s.

The following classical result was first proved by Y. Choquet-Bruhat [7] for $s \geq 3$ and improved by T. Hughes, T. Kato and J. Marsden [17].

Theorem A. *Assume the following hold:*

$$(\mathbf{g}_{\alpha\beta}(0) - \mathbf{m}_{\alpha\beta}) \in H^{s+1}(\mathbb{R}^3), \quad \partial_t \mathbf{g}_{\alpha\beta}(0) \in H^s(\mathbb{R}^3); \tag{1.6}$$

$$\sup_{|x|=r} |g_{\alpha\beta}(0, x^1, x^2, x^3) - \mathbf{m}_{\alpha\beta}| \to 0 \quad \text{as } r \to \infty, \tag{1.7}$$

here $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ *and* $\mathbf{m}_{\alpha\beta}$ *is the Minkowski metric.*

Then for $s > \frac{3}{2}$, *there is a positive* $T > 0$ *and a unique semi-metric* $\mathbf{g}_{\alpha\beta}(t)$ *which satisfies (1.4) and such that*

$$(\mathbf{g}_{\alpha\beta}(t) - \mathbf{m}_{\alpha\beta}) \in C([0, T], H^{s+1}) \quad \text{and} \quad \partial_t \mathbf{g}(t) \in C([0, T], H^s). \tag{1.8}$$

In addition, if the pair $(\mathbf{g}_{\alpha\beta}(0), \partial_t \mathbf{g}_{\alpha\beta}(0))$ *satisfyies the constraint the equations (1.5), then the metric* $\mathbf{g}_{\alpha\beta}(t)$ *is a solution to the vacuum Einstein equation (1.1).*

S. Klainerman and I. Rodnianski [20] succeeded in improving regularity below the critical index $\frac{3}{2}$.

Theorem B. *Assume the conditions (1.6) and (1.7) of Theorem A hold and* $\mathbf{g}_{\alpha\beta}(t)$ *is a classical solution to (1.4) such that* $(\mathbf{g}_{\alpha\beta}(0), \partial_t \mathbf{g}_{\alpha\beta}(0))$ *satisfy the constraint the equations (1.5). Then for* $s > 1$ *there is a positive* $T > 0$ *depending on* $\|\partial \mathbf{g}_{\alpha\beta}(0)\|_{H^s}$ *such that*

$$\|\partial \mathbf{g}_{\alpha\beta}(t)\|_{L_t^2 L_x^\infty} \leq C \|\partial \mathbf{g}_{\alpha\beta}(0)\|_{H^s}. \tag{1.9}$$

1.2 Solutions of the constraint equations

The H^s spaces are inappropriate for solutions of the constraint equations, roughly speaking, because in these spaces the Laplacian is not invertible. It turns out that the Nirenberg-Walker-Cantor weighted Sobolev spaces $H_{m,\delta}$ ([23],[5]) are suitable for asymptotically flat manifolds and indeed these spaces have been widely used in General Relativity. We denote the norm of the weighted Sobolev spaces by

$$\|u\|_{H_{m,\delta}}^2 = \sum_{|\alpha| \leq m} \|(1 + |x|)^{(\delta + |\alpha|)} \partial^\alpha u\|_{L^2(\mathbb{R}^3)}^2, \quad -\infty < \delta < \infty, \tag{1.10}$$

and the space $H_{m,\delta}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm (1.10).

We may express now the asymptotically flat condition for a Riemannian metric $\mathbf{h}_{ab}$ by means of $(\mathbf{h}_{ab} - \mathbf{e}_{ab}) \in H_{m,\delta}$. In this paper the identity is denoted by $\mathbf{e}_{ab}$.

There is an extensive literature for the solutions of the constraint equations in asymptotically flat manifolds. The following theorem was proved under various assumptions in [1], [6],[10][8], [9], [12], [22].

Theorem C. *Let m be an integer greater or equal to one, $-\frac{3}{2} < \delta < -\frac{1}{2}$. Given a set of free data $(\bar{\mathbf{h}}_{ab}, \bar{\mathbf{K}}_{ab})$ such that $((\bar{\mathbf{h}}_{ab} - \mathbf{e}_{ab}), \bar{\mathbf{K}}_{ab}) \in H_{m+1,\delta} \times H_{m,\delta+1}$. Then there are exists a conformally equivalent data $(\mathbf{h}_{ab}, \mathbf{K}_{ab})$ which satisfies the constraint equations (1.5). Moreover, there is a constant C such that*

$$\|(\mathbf{h}_{ab} - \mathbf{e}_{ab}, \mathbf{K}_{ab})\|_{H_{m+1,\delta}\times H_{m,\delta+1}} \leq C \left\|\left(\bar{\mathbf{h}}_{ab} - \mathbf{e}_{ab}, \bar{\mathbf{K}}_{ab}\right)\right\|_{H_{m+1,\delta}\times H_{m,\delta+1}}.$$

1.3 The main result

The disadvantage of the present situation is the inconsistency of the Sobolev spaces of Theorems A and B with those of the constraint equations. The initial data for the semi-linear wave equations (1.4) are given in H^s-spaces, while Theorem C provides the initial data (solutions to the constraint equations (1.5)) in $H_{m,\delta}$. Therefore it is impossible to obtain a solution to the Cauchy problem for the vacuum Einstein equations with initial data which are given in one type of Sobolev spaces. *Our goal is to unify the Sobolev spaces of the constraint and the evolution equations.*

Before stating the main theorem we need to introduce the extension of the spaces $H_{m,\delta}$ into fractional order. We denote a scaling with ϵ by $f_\epsilon(x) = f(\epsilon x)$.

Definition 1.1 ***($H_{s,\delta}$ Sobolev spaces)*** *For $s \geq 0$ and $-\infty < \delta < \infty$, we define the $H_{s,\delta}$ norm by*

$$\left(\|u\|_{H_{s,\delta}}\right)^2 = \sum_j 2^{(\frac{3}{2}+\delta)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^s}. \tag{1.11}$$

The sequence $\{\psi_j\} \subset C_0^\infty(\mathbb{R}^3)$ satisfies the following: $\psi_j(x) = 1$ on $K_j = \{x : 2^{j-3} \leq |x| \leq 2^{j+2}\}$, $j = 1, 2, ...,$ $K_0 = \{x : |x| \leq 4\}$; $\sup(\psi_j) \subset \{x : 2^{j-4} \leq |x| \leq 2^{j+3}\}$, for $j \geq 1$, $\sup(\psi_0) \subset \{x : |x| \leq 2^3\}$; $|\partial^\alpha \psi_j(x)| \leq C_\alpha 2^{-|\alpha|j}$, where the constant C_α does not depend on j.

The space $H_{s,\delta}$ is the set of all temperate distributions having a finite norm given by (1.11).

Triebel [26] proved that whenever s is equal to an integer m, then

$$\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2^j}\|^2_{H^m} \sim \sum_{|\alpha|\leq m} \|(1+|x|)^{(\delta+|\alpha|)} \partial^\alpha u\|^2_{L^2(\mathbb{R}^3)}. \tag{1.12}$$

Thus whenever the parameter s is an integer, the norms (1.10) and (1.11) are equivalent.

Theorem 1.2 ***(Main results)*** *Let $s > \frac{3}{2}$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Given a set of free data $(\bar{\mathbf{h}}_{ab}, \bar{\mathbf{K}}_{ab})$ such that $((\bar{\mathbf{h}}_{ab} - \mathbf{e}_{ab}), \bar{\mathbf{K}}_{ab}) \in H_{s+1,\delta} \times H_{s,\delta+1}$.*

(i) *Then there exists a conformally equivalent data $(\mathbf{h}_{ab}, \mathbf{K}_{ab})$ which satisfies the constraint equations (1.5). Moreover $((\mathbf{h}_{ab} - \mathbf{e}_{ab}), \mathbf{K}_{ab}) \in H_{s+1,\delta} \times H_{s,\delta+1}$ and depend continuously on the norms of $((\bar{\mathbf{h}}_{ab} - \mathbf{e}_{ab}), \bar{\mathbf{K}}_{ab})$.*

(ii) *Then there exists a $T > 0$ and a semi-metric $\mathbf{g}_{\alpha\beta}(t)$ solution to the vacuum Einstein equations (1.1) such that*

$$(\mathbf{g}_{\alpha\beta}(t) - \mathbf{m}_{\alpha\beta}) \in C([0,T], H_{s+1,\delta}) \cap C^1([0,T], H_{s,\delta+1}) \tag{1.13}$$

and

$$\left.\begin{array}{l} \|(\mathbf{g}_{\alpha\beta}(t) - \mathbf{m}_{\alpha\beta})\|_{H_{s+1,\delta}} \\ \|\partial_t \mathbf{g}_{\alpha\beta}(t)\|_{H_{s,\delta+1}} \end{array}\right\} \leq C \left\| \left(\bar{\mathbf{h}}_{ab} - \mathbf{e}_{ab}, \bar{\mathbf{K}}_{ab}\right) \right\|_{H_{s+1,\delta} \times H_{s,\delta+1}}. \tag{1.14}$$

for $t \in [0,T]$. The metric $\mathbf{g}_{\alpha\beta}(t)$ is the unique solution to the reduce Einstein (1.4) with initial data $(\mathbf{h}_{ab}, \mathbf{K}_{ab})$.

Remark 1.3 ***(Uniqueness)*** *Since the Ricci tensor is invariant under diffeomorphisms, it is impossible to get a unique solution to the vacuum Einstein equation. Because if $\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0$ and ϕ is a diffeomorphism, then the pull-back $\phi^*\mathbf{g}_{\alpha\beta}$ also satisfies (1.1). However, it can be shown that if two metrics $\mathbf{g}_{\alpha\beta}$ and $\widetilde{\mathbf{g}}_{\alpha\beta}$ satisfy (1.1) and $(\mathbf{g}_{\alpha\beta} - \mathbf{m}_{\alpha\beta}), (\widetilde{\mathbf{g}}_{\alpha\beta} - \mathbf{m}_{\alpha\beta}) \in H_{s+1,\delta}$, then there is a coordinates transformation $x^\alpha \to y^\alpha = f^\alpha(x^\mu)$, which preserve the harmonic condition (1.3) and such that $\widetilde{\mathbf{g}}_{\mu\nu}(y) = \mathbf{g}_{\alpha\beta}(x(y)) \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu}$. This transformation can be established by means of solutions to linear wave equations (see [15], [10]) with coefficients in the $H_{s,\delta}$-spaces. Thus we can apply the energy estimate Lemma 4.3 and the tools of Section 2 to establish the existence of this transformation in $H_{s+1,\delta}$. Previously this procedure has been applied with one more degree of differentiability, but recently, Planchon and Rodnianski found a trick which allows the obtaining of the diffeomorphisms without losing regularity, see Section 4 in the monograph [13] for details. Thus we conclude that for asymptotically flats metrics which preserve the harmonic condition (1.3) the uniqueness holds up to a diffeomorphism.*

Remark 1.4 *We would like to mention that the results of Christodoulou [11] and Christodoulou and O'Murchadha [12] differ from ours. They assume $\left(\bar{\mathbf{h}}_{ab} - \mathbf{e}_{ab}, \bar{\mathbf{K}}_{ab}\right) \in H_{s+1,\delta+\frac{1}{2}} \times H_{s+1,\delta+\frac{3}{2}}$, while the solutions $(\mathbf{g}_{\alpha\beta}(t) - \mathbf{m}_{\alpha\beta})$ belong to $H_{s+1,\delta}(\Omega_\theta)$, where Ω_θ is a certain unbounded region of $\mathbb{R}^4$. Thus in their setting, the rates of fall-off of the initial data and the semi-metric are different. In addition, they require the regularity condition $s \geq 3$.*

The idea to solve both the evolution and the constraint equations in the weighted Sobolev spaces of fractional order $H_{s,\delta}$ has previously appeared in [2] and [3], but for the Einstein-Euler systems. The regularity condition for these systems is higher since they are coupled with a fluid.

The plan of the paper is as follows. In Section 2 we present several properties of the fractional weighted Sobolev spaces. Section 3 deals with the reduction of the wave equations into a first order symmetric hyperbolic systems. The specific form of these hyperbolic system has an essential role in our approach. The energy estimates are established in Section 4. In Section 5 we treat the existence, uniqueness and continuity of semi-linear first order symmetric hyperbolic systems in the $H_{s,\delta}$-spaces and the main result is proved in Section 6. In this paper Greek indices will take the values $0, 1, 2, 3$ while Latin indices $1, 2, 3$.

2 Weighted Sobelev spaces of fractional order

Here we present the basic properties of these spaces and the equivalence between various norms. All these results were established in the appendices of [4] and [3]. At the end of the section we define a norm on product spaces.

Definition 2.1 ***(Definitions of norms)***

- *Let $\{\psi_j\}$ be the sequence of functions in Definition 1.1. For any positive γ we set*

$$\|u\|^2_{H_{s,\delta,\gamma}} = \sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^\gamma u)_{2^j}\|^2_{H^s} \tag{2.1}$$

  and we will use the convention $\|u\|_{H_{s,\delta,1}} = \|u\|_{H_{s,\delta}}$. The subscripts 2^j mean a scaling with 2^j, that is, $(\psi_j^\gamma u)_{2^j}(x) = (\psi_j^\gamma u)(2^j x)$.

- *For a non-negative integer m and $\beta \in \mathbb{R}$, the space C^m_β is the set of all functions having continuous partial derivatives up to order m and such that the norm (2.2) is finite:*

$$\|u\|_{C^m_\beta} = \sum_{|\alpha|\leq m} \sup_{\mathbb{R}^3} \left((1+|x|)^{\beta+|\alpha|}|\partial^\alpha u(x)|\right). \tag{2.2}$$

2.1 Some Properties of $H_{s,\delta}$

Proposition 2.2

1. **Equivalence of norms $H_{s,\delta}$ and $H_{s,\delta,\gamma}$:** *For any positive γ,*

$$\|u\|^2_{H_{s,\delta}} = \sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j u)_{2^j}\|^2_{H^s} \simeq \sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j^\gamma u)_{2^j}\|^2_{H^s} = \|u\|^2_{H_{s,\delta,\gamma}}. \tag{2.3}$$

2. **Equivalence of norms (2.1) and (1.10):** *For any nonnegative integer m, positive γ and δ there holds*

$$\|u\|^2_{H_{m,\delta,\gamma}} = \sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j^\gamma u)_{2^j}\|^2_{H^m} \sim \sum_{|\alpha|\le m} \|(1+|x|)^{(\delta+|\alpha|)}\partial^\alpha u\|^2_{L^2(\mathbb{R}^3)}. \tag{2.4}$$

3. **$H_{s,\delta}$-norm of a derivative:**

$$\|\partial_i u\|_{H_{s-1,\delta+1}} \le \|u\|_{H_{s,\delta}}. \tag{2.5}$$

4. **Algebra:** *If $s_1, s_2 \ge s$, $s_1 + s_2 > s + \frac{3}{2}$ and $\delta_1 + \delta_2 \ge \delta - \frac{3}{2}$, then*

$$\|uv\|_{H_{s,\delta}} \le C\|u\|_{H_{s_1,\delta_1}}\ \|v\|_{H_{s_2,\delta_2}}. \tag{2.6}$$

5. **Compact embedding:** *Let $0 \le s' < s$ and $\delta' < \delta$, then the embedding*

$$H_{s,\delta} \hookrightarrow H_{s',\delta'}. \tag{2.7}$$

is compact.

6. **Embedding into the continuous:** *If $s > \frac{3}{2} + m$ and $\delta + \frac{3}{2} \ge \beta$, then*

$$\|u\|_{C^m_\beta} \le C\|u\|_{H_{s,\delta}}. \tag{2.8}$$

7. **Third Moser's inequality:** *Let $F : \mathbb{R}^m \to \mathbb{R}^l$ be C^{N+1}-function such that $F(0) \in H_{s,\delta}$ and where $N \ge [s] + 1$. Then there is a constant C such that for any $u \in H_{s,\delta}$*

$$\|F(u)\|_{H_{s,\delta}} \le C\|F\|_{C^{N+1}}\left(1 + \|u\|^N_{L^\infty}\right)\|u\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}}. \tag{2.9}$$

In particular, if $s > \frac{3}{2}$, then $u \in L^\infty$ and $\|F(u)\|_{H_{s,\delta}} \le C\|u\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}}$.

8. **Difference estimate:** *Suppose F is a C^{N+2}-function and $u, v \in H_{s,\delta} \cap L^\infty$. Then*

$$\|F(u) - F(v)\|_{H_{s,\delta}} \le C(\|u\|_{L^\infty}, \|v\|_{L^\infty})\left(\|u\|_{H_{s,\delta}} + \|v\|_{H_{s,\delta}}\right)\|u - v\|_{H_{s,\delta}}. \tag{2.10}$$

9. **Density:**

   (a) The class $C_0^\infty(\mathbb{R}^3)$ is dense in $H_{s,\delta}$.

(b) Given $u \in H_{s,\delta}$, $s' > s \geq 0$ and $\delta' \geq \delta$. Then for $\rho > 0$ there is $u_\rho \in C_0^\infty(\mathbb{R}^3)$ and a positive constant $C(\rho)$ such that

$$\|u_\rho - u\|_{H_{s,\delta}} \leq \rho \quad \textit{and} \quad \|u_\rho\|_{H_{s',\delta'}} \leq C(\rho)\|u\|_{H_{s,\delta}}. \tag{2.11}$$

10. **Mixed norm estimate:** *If $u \in H_{s,\delta}$ and $\partial_x u \in H_{s,\delta+1}$, then*

$$\|u\|_{H_{s+1,\delta}} \lesssim \left(\|u\|_{H_{s,\delta}} + \|\partial_x u\|_{H_{s,\delta+1}}\right). \tag{2.12}$$

The proof of (2.12) follows from the integral representation of the norm (1.11) (see [4], [3] in the Appendix).

The density property (b) where proved in [3], [4] for $\delta' = \delta$. Only a slight modification is needed to include it also for $\delta' \geq \delta$ and therefore we leave it to the reader.

2.2 Product spaces

Definition 2.3 ***(Product spaces)*** *We set $X_{s,\delta} = H_{s,\delta} \times H_{s,\delta+1} \times H_{s,\delta+1}$, and the norm of a vector valued function $V = (v_1, v_2, v_3) \in X_{s,\delta}$ is defined by*

$$\|V\|^2_{X_{s,\delta}} = \|v_1\|^2_{H_{s,\delta,2}} + \|v_2\|^2_{H_{s,\delta+1,2}} + \|v_3\|^2_{H_{s,\delta+1,2}}. \tag{2.13}$$

We will use the following convention: for a vector valued function $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^N$, we set $U = (u, \partial_t u, \partial_x u)$, where $\partial_x u$ denotes the set of all partial derivatives in the space variable $x \in \mathbb{R}^3$. Thus $\|U\|^2_{X_{s,\delta}} = \|u\|^2_{H_{s,\delta,2}} + \|\partial_t u\|^2_{H_{s,\delta+1,2}} + \|\partial_x u\|^2_{H_{s,\delta+1,2}}$.

Essential of our approach is the following observation.

Remark 2.4 *It follows from the Mixed norm estimate (2.12) above that if $U(t,\cdot) \in X_{s,\delta}$, then*

$$\|u(t,\cdot)\|_{H_{s+1,\delta}} \lesssim \|U(t,\cdot)\|_{X_{s,\delta}}.$$

3 First order hyperbolic symmetric systems

The system of wave equations (1.4) can be transferred into a first order symmetric hyperbolic system. The specific form of the hyperbolic system play en important role in our approach.

Letting $\mathbf{h}_{\alpha\beta\gamma} = \partial_\gamma \mathbf{g}_{\alpha\beta}$, reduces the wave equations (1.4) into

$$\begin{aligned}
&\partial_t \mathbf{g}_{\alpha\beta} = \mathbf{h}_{\alpha\beta 0} \\
&\partial_t \mathbf{h}_{\gamma\delta 0} = \tfrac{1}{-\mathbf{g}^{00}} \left\{ 2\mathbf{g}^{0a}\partial_a \mathbf{h}_{\gamma\delta 0} + \mathbf{g}^{ab}\partial_a \mathbf{h}_{\gamma\delta b} + C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\gamma\delta\alpha\beta\rho\sigma} \mathbf{h}_{\epsilon\zeta\eta}\mathbf{h}_{\kappa\lambda\mu}\mathbf{g}^{\alpha\beta}\mathbf{g}^{\rho\sigma} \right\} \\
&(-\mathbf{g}^{00})^{-1}\mathbf{g}^{ab}\partial_t \mathbf{h}_{\gamma\delta a} = (-\mathbf{g}^{00})^{-1}\mathbf{g}^{ab}\partial_a \mathbf{h}_{\gamma\delta 0},
\end{aligned} \tag{3.1}$$

where the objects $C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\gamma\delta\alpha\beta\rho\sigma}$ are a combination of Kronecker deltas with integer coefficients. We would like now to write system (3.1) in a matrix form. We set

$$\tilde{g}^{\alpha\beta} = (-\mathbf{g}^{00})^{-1}\mathbf{g}^{\alpha\beta},$$

where $\mathbf{g}^{\alpha\beta}$ denotes the inverse matrix of $\mathbf{g}_{\alpha\beta}$. By introducing the auxiliary vector valued functions

$$U = \begin{pmatrix} u \\ \partial_t u \\ \partial_x u \end{pmatrix} = \begin{pmatrix} \mathbf{g}_{\alpha\beta} - \mathbf{m}_{\alpha\beta} \\ \partial_t \mathbf{g}_{\alpha\beta} \\ \partial_x \mathbf{g}_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_{\alpha\beta} - \mathbf{m}_{\alpha\beta} \\ \mathbf{h}_{\alpha\beta 0} \\ \mathbf{h}_{\alpha\beta a} \end{pmatrix}, \quad a = 1,2,3,$$

we can write the system (3.1) in the form

$$\mathcal{A}^0(u)\partial_t U = \sum_{a=1}^{3} \left(\mathcal{A}^\alpha(u) + \mathcal{C}^a\right)\partial_a U + \mathcal{B}(U)U, \tag{3.2}$$

where

$$\mathcal{A}^0(u) = \begin{pmatrix} \mathbf{e}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & \mathbf{e}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & \mathbf{0}_{10} & \tilde{g}^{11}\mathbf{e}_{10} & \tilde{g}^{12}\mathbf{e}_{10} & \tilde{g}^{13}\mathbf{e}_{10} \\ \mathbf{0}_{10} & \mathbf{0}_{10} & \tilde{g}^{21}\mathbf{e}_{10} & \tilde{g}^{22}\mathbf{e}_{10} & \tilde{g}^{23}\mathbf{e}_{10} \\ \mathbf{0}_{10} & \mathbf{0}_{10} & \tilde{g}^{31}\mathbf{e}_{10} & \tilde{g}^{32}\mathbf{e}_{10} & \tilde{g}^{33}\mathbf{e}_{10} \end{pmatrix}, \tag{3.3}$$

$$\mathcal{A}^a(u) = \begin{pmatrix} \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & 2\tilde{g}^{a0}\mathbf{e}_{10} & (\tilde{g}^{a1} - \delta^{a1})\mathbf{e}_{10} & (\tilde{g}^{a2} - \delta^{a2})\mathbf{e}_{10} & (\tilde{g}^{a3} - \delta^{a3})\mathbf{e}_{10} \\ \mathbf{0}_{10} & (\tilde{g}^{a1} - \delta^{a1})\mathbf{e}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & (\tilde{g}^{a2} - \delta^{a2})\mathbf{e}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & (\tilde{g}^{a3} - \delta^{a3})\mathbf{e}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \end{pmatrix}, \tag{3.4}$$

$$\mathcal{C}^a = \begin{pmatrix} \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & \mathbf{0}_{10} & \delta^{a1}\mathbf{e}_{10} & \delta^{a2}\mathbf{e}_{10} & \delta^{a3}\mathbf{e}_{10} \\ \mathbf{0}_{10} & \delta^{a1}\mathbf{e}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & \delta^{a2}\mathbf{e}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & \delta^{a3}\mathbf{e}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \end{pmatrix}, \tag{3.5}$$

and

$$\mathcal{B}(U) = \begin{pmatrix} \mathbf{0}_{10} & \mathbf{e}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & C^{\kappa\lambda 0} & C^{\kappa\lambda 1} & C^{\kappa\lambda 2} & C^{\kappa\lambda 3} \\ \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \\ \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} & \mathbf{0}_{10} \end{pmatrix}. \tag{3.6}$$

Here $C^{\kappa\lambda\mu} = (-\mathbf{g}^{00})^{-1}C^{\varepsilon\zeta\eta\kappa\lambda\mu}_{\gamma\delta\alpha\beta\rho\sigma}\partial_\varepsilon \mathbf{g}_{\zeta\eta}\mathbf{g}^{\alpha\beta}\mathbf{g}^{\rho\sigma}$.

Apart from the facts that $\mathcal{A}^\alpha(u)$ and $\mathcal{C}^\alpha$ are symmetric matrices and $\mathcal{A}^0(u)$ is positive definite, they hold three additional properties: (i) the matrices $\mathcal{A}^\alpha(u)$ doest not depend on the derivatives of u; (ii) the coefficients of $\partial_t u$ in $\mathcal{A}^0(u)$ do not depend on t; (iii) it follows from Moser type estimate 7 and Algebra 4 of Proposition 2.2 that if $\mathbf{g}_{\alpha\beta} - \mathbf{m}_{\alpha\beta} \in H_{s+1,\delta}$, then $(\tilde{g}^{aa} - 1) \in H_{s+1,\delta}$ $(a = 1, 2, 3)$ and $\tilde{g}^{\alpha\beta} \in H_{s+1,\delta}$ whenever $\alpha \neq \beta$. Thus the matrices $(\mathcal{A}^0(u) - \mathbf{e}), \mathcal{A}^a(u) \in H_{s+1,\delta}$ whenever $\mathbf{g}_{\alpha\beta} - \mathbf{m}_{\alpha\beta} \in H_{s+1,\delta}$, while $\mathcal{C}^a$ are a constant matrices. These facts are crucial for the energy estimates.

4 Energy Estimates

We consider here energy estimates for a first order linear hyperbolic system of the form

$$\mathcal{A}^0 \partial_t \begin{pmatrix} u \\ \partial_t u \\ \partial_x u \end{pmatrix} = \sum_{a=1}^{3} \left(\mathcal{A}^a + \mathcal{C}^a\right) \partial_a \begin{pmatrix} u \\ \partial_t u \\ \partial_x u \end{pmatrix} + \mathcal{B} \begin{pmatrix} u \\ \partial_t u \\ \partial_x u \end{pmatrix} + \mathcal{F}. \tag{4.1}$$

Here $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^N$, $\mathcal{A}^\alpha = \left(\mathbf{a}^\alpha_{i,j}\right)_{i,j=1,2,3}$, $\mathcal{C}^a = \left(\mathbf{c}^a_{i,j}\right)_{i,j=1,2,3}$ and $\mathcal{B} = (\mathbf{b}_{ij})_{i,j=1,2,3}$ are $5N \times 5N$ block matrices having the sizes of their blocks according to the following structure

$$\left(\begin{array}{c|c|c} N \times N & N \times N & N \times 3N \\ \hline N \times N & N \times N & N \times 3N \\ \hline 3N \times N & 3N \times N & 3N \times 3N \end{array}\right). \tag{4.2}$$

We assume the following conditions:

$$\mathbf{a}^0_{ij} = 0 \quad \text{for } i \neq j; \;\; \mathbf{a}^0_{11} = \mathbf{a}^0_{22} = \mathbf{e}; \tag{4.3a}$$

$$\mathbf{a}^0_{33} \quad \text{is symmetric and } \frac{1}{c_0} v^T v \leq v^T \mathbf{a}^0_{33} v \leq c_0 v^T v \quad \forall v \in \mathbb{R}^{3N}; \tag{4.3b}$$

$$\mathcal{A}^0(t, \cdot) - \mathbf{e} \in H_{s+1,\delta}; \tag{4.3c}$$

$$\partial_t \mathcal{A}^0(t, \cdot) \in L^\infty; \tag{4.3d}$$

$$\mathcal{A}^a \quad \text{are symmetric} \quad \text{with} \quad \mathbf{a}^a_{i1} = \mathbf{a}^a_{1j} = \mathbf{0}, \;\; a = 1, 2, 3; \tag{4.3e}$$

$$\mathcal{A}^a(t, \cdot) \in H_{s+1,\delta}, \quad a = 1, 2, 3; \tag{4.3f}$$

$$\mathcal{C}^a \quad \text{are constant and symmetric} \quad \text{with} \quad \mathbf{c}^a_{i1} = \mathbf{c}^a_{1j} = \mathbf{0}, \;\; a = 1, 2, 3; \tag{4.3g}$$

$$\mathbf{b}_{i1} = \mathbf{0} \text{ and } \;\mathbf{b}_{1,j} \quad \text{are constant}, \quad i, j = 1, 2, 3; \tag{4.3h}$$

$$\widetilde{\mathcal{B}}(t, \cdot) := (\mathbf{b}_{ij})_{i,j=2,3} \in H_{s,\delta+1}; \tag{4.3i}$$

$$\mathcal{F}(t, \cdot) \in H_{s,\delta+1}. \tag{4.3j}$$

Note that any system which is originated from a linearization of (3.1) meets the above requirements.

4.1 $H_{s,\delta}$ - energy estimates

We define an inner-product on $X_{s,\delta}$ in accordance with the equations (4.1). Let

$$\Lambda^s(u) = (1-\Delta)^{\frac{s}{2}}(u) = \mathcal{F}^{-1}\left((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}\right)(u),$$

where $\mathcal{F}$ denote the Fourier transform.

Definition 4.1 ***(Inner-product)***

- **Inner-product on L^2:** *For vector valued functions $f, g \in L^2$, we set*

$$\langle f, g\rangle_{L^2} = \int f^T g dx, \tag{4.4}$$

where f^T denotes the transpose matrix.

- **Inner-product on $H_{s,\delta}$:** *For $v_1, \phi_1 \in H_{s,\delta}$, we set*

$$\langle v_1, \phi_1\rangle_{s,\delta} = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\langle \Lambda^s\left(\psi_j^2 v_1\right)_{2^j}, \Lambda^s\left(\psi_j^2\phi_1\right)_{2^j}\right\rangle_{L^2}. \tag{4.5}$$

Recall that the subscripts 2^j mean a scaling (see Definition 2.1).

- **A weighted inner-product on $H_{s,\delta+1}\times H_{s,\delta+1}$:** *For a matrix $\mathbf{a}_{33}^0$ which satisfies (4.3b) and $(v_2, v_3), (\phi_2, \phi_3) \in H_{s,\delta+1}\times H_{s,\delta+1}$, we set*

$$\begin{aligned}&\left\langle \begin{pmatrix} v_2 \\ v_3\end{pmatrix}, \begin{pmatrix} \phi_2 \\ \phi_3\end{pmatrix}\right\rangle_{s,\delta+1,\mathbf{a}_{33}^0} \\ &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+1)2j}\left\langle \Lambda^s\left(\psi_j^2\begin{pmatrix} v_2 \\ v_3\end{pmatrix}\right)_{2^j}, \begin{pmatrix} \mathbf{e} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_{33}^0\end{pmatrix}_{2^j}\Lambda^s\left(\psi_j^2\begin{pmatrix} \phi_2 \\ \phi_3\end{pmatrix}\right)_{2^j}\right\rangle_{L^2}.\end{aligned} \tag{4.6}$$

- **Inner-product on $X_{s,\delta}$:** *For a matrix $\mathcal{A}^0$ which satisfies (4.3a)-(4.3b) and $V = (v_1, v_2, v_3), \Phi = (\phi_1, \phi_2, \phi_3) \in X_{s,\delta}$, we set*

$$\langle V, \Phi\rangle_{X_{s,\delta,\mathcal{A}^0}} = \langle v_1, \phi_1\rangle_{s,\delta} + \left\langle \begin{pmatrix} v_2 \\ v_3\end{pmatrix}, \begin{pmatrix} \phi_2 \\ \phi_3\end{pmatrix}\right\rangle_{s,\delta+1,\mathbf{a}_{33}^0} \tag{4.7}$$

We denote by $\|V\|_{X_{s,\delta,\mathcal{A}^0}}$ the norm which is associated with the inner-product (4.7).

From condition (4.3b) we see that

$$
\begin{aligned}
\|V\|^2_{X_{s,\delta,\mathcal{A}^0}} &\leq c_0 \left\{ \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \| \left(\psi_j^2 v_1\right)_{2^j} \|^2_{H^s} \right. \\
&\left. + \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+1)2j} \left[\| \left(\psi_j^2 v_2\right)_{2^j} \|^2_{H^s} + \| \left(\psi_j^2 v_3\right)_{2^j} \|^2_{H^s} \right] \right\} \\
&= c_0 \left\{ \|v_1\|^2_{H_{s,\delta,2}} + \|v_2\|^2_{H_{s,\delta+1,2}} + \|v_3\|^2_{H_{s,\delta+1,2}} \right\} \\
&= c_0 \|V\|^2_{X_{s,\delta}}.
\end{aligned} \tag{4.8}
$$

Thus we have shown:

Corollary 4.2 ***(Equivalence of norms)*** *The norms which are defined by the inner-product (4.7) and (2.13), satisfy*

$$
\frac{1}{\sqrt{c_0}} \|V\|_{X_{s,\delta}} \leq \|V\|_{X_{s,\delta,\mathcal{A}^0}} \leq \sqrt{c_0} \|V\|_{X_{s,\delta}}. \tag{4.9}
$$

For a vector valued function $u(t,x)$, we set $u(t) = u(t,x)$, $U(t) = (u(t), \partial_t u(t), \partial_x u(t))$ and the energy of $U(t)$ is denoted by

$$
E(t) = \frac{1}{2} \langle U(t), U(t) \rangle_{X_{s,\delta,\mathcal{A}^0}}. \tag{4.10}
$$

The energy estimate in the product space $X_{s,\delta}$ is indispensable tool of our method. The next Lemma establishes it and its proof relies on tedious computations. The essential point is that fact that $\mathcal{A}^\alpha \in H_{s+1,\delta}$. This enables to use the Kato-Ponce commutator estimate (Theorem 4.4) with the pseudodifferential operator $\Lambda^s \partial_x$ rather then Λ^s as in [4].

Lemma 4.3 ***(Energy estimates)*** *Let $s > \frac{3}{2}$, $\delta \geq -\frac{3}{2}$ and assume the coefficients of (4.1) satisfy conditions (4.3). If $U(t,\cdot) \in C_0^\infty(\mathbb{R}^3)$ is a solution to the linear system (4.1), then*

$$
\frac{d}{dt} E(t) \leq C c_0 \left(E(t) + 1 \right), \tag{4.11}
$$

where the constant C depends on $\|(\mathcal{A}^0 - \mathbf{e})\|_{H_{s+1,\delta}}$, $\|\mathcal{A}^a\|_{H_{s+1,\delta}}$, $\|\widetilde{\mathcal{B}}\|_{H_{s,\delta+1}}$, $\|\mathcal{F}\|_{H_{s,\delta+1}}$, $\|\partial_t \mathcal{A}^0\|_{L^\infty}$, s and δ.

An essential tool for deriving these estimates is the Kato & Ponce commutator estimate [25].

Theorem 4.4 ***(Kato and Ponce)*** *Let P be a pseudodifferential operator in the class $S^s_{1,0}$, $f \in H^s \cap C^1$, $g \in H^s \cap L^\infty$ and $s > 0$. Then*

$$
\|P(fg) - fP(g)\|_{L^2} \leq C \left\{ \|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}} + \|f\|_{H^s} \|g\|_{L^\infty} \right\}. \tag{4.12}
$$

Proof (of Lemma 4.3)**.** Taking into account the structure of the inner-product (4.7) we see that

$$\begin{aligned}\frac{d}{dt}E(t) &= \langle u, \partial_t u\rangle_{s,\delta} + \left\langle \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix}, \partial_t \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right\rangle_{s,\delta+1,\mathbf{a}_{33}^0} \\ &+\frac{1}{2}\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+1)2j} \left\langle \Lambda^s \left(\psi_j^2 \left(\partial_x u\right)\right)_{2^j}, \partial_t \left(\mathbf{a}_{33}^0\right)_{2^j} \Lambda^s \left(\psi_j^2 \left(\partial_x u\right)\right)_{2^j} \right\rangle_{L^2}\end{aligned} . \tag{4.13}$$

The infinite sum of the right hand side of (4.13) is less than

$$\sqrt{3N}\|\partial_t \left(\mathbf{a}_{33}^0\right)\|_{L^\infty} \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+1)2j} \| \left(\psi_j^2 \partial_x u\right)_{2^j} \|_{H^s}^2 = \sqrt{3N}\|\partial_t \mathbf{a}_{33}^0\|_{L^\infty} \|\partial_x u\|^2_{H_{s,\delta+1,2}} \tag{4.14}$$

and

$$\langle u, \partial_t u\rangle_{s,\delta} \leq \|u\|_{H_{s,\delta,2}} \|\partial_t u\|_{H_{s,\delta,2}} \leq \frac{1}{2}\left(\|u\|^2_{H_{s,\delta,2}} + \|\partial_t u\|^2_{H_{s,\delta+1,2}}\right). \tag{4.15}$$

We turn now to difficult task, this is the estimation of $\left\langle \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix}, \partial_t \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right\rangle_{s,\delta+1,\mathbf{a}_{33}^0}$. Setting

$$E_{\partial_t}(j) = \left\langle \Lambda^s \left(\psi_j^2 \partial_t u\right)_{2^j}, \Lambda^s \left(\psi_j^2 \partial_t(\partial_t u)\right)_{2^j} \right\rangle_{L^2}, \tag{4.16}$$

$$E_{\partial_x}(j) = \left\langle \Lambda^s \left(\psi_j^2 \partial_x u\right)_{2^j}, \left(\mathbf{a}_{33}^0\right)_{2^j} \Lambda^s \left(\psi_j^2 \partial_t(\partial_x u)\right)_{2^j} \right\rangle_{L^2}, \tag{4.17}$$

and using the specific form of (4.6) we see that

$$\left\langle \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix}, \partial_t \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right\rangle_{s,\delta+1,\mathbf{a}_{33}^0} = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+1)2j} \left[E_{\partial_t}(j) + E_{\partial_x}(j)\right]. \tag{4.18}$$

We first estimate $E_{\partial_x}(j)$. For that purpose we define a sequence of functions

$$\Psi_k(x) = \left(\sum_{j=0}^{\infty} \psi_j(x)\right)^{-1} \psi_k(x), \tag{4.19}$$

where $\{\psi_j\}$ is the sequence defined in Definition 1.1. This sequence has the following properties: $\Psi_k \in C_0^\infty(\mathbb{R}^3)$, $|\partial_\alpha \Psi_k(x)| \leq C_\alpha 2^{-k}$, $\sum_{k=0}^{\infty} \Psi_k(x) = 1$ and

$$\Psi_k(x)\psi_j(x) \neq 0 \quad \text{only for} \quad k = j-3, ..., j+4. \tag{4.20}$$

We will use the convention that $\Psi_{j-m} \equiv 0$ whenever $j - m < 0$. Then

$$
\begin{aligned}
E_{\partial_x}(j) &= \left\langle \Lambda^s \left[\left(\psi_j^2 \partial_x u \right)_{2^j} \right], \left(\mathbf{a}_{33}^0 \right)_{2^j} \Lambda^s \left[\left(\sum_{k=0}^{\infty} \Psi_k \right)_{2^j} \left(\psi_j^2 \partial_t (\partial_x u) \right)_{2^j} \right] \right\rangle_{L^2} \\
&= \sum_{k=j-3}^{j+4} \left\langle \Lambda^s \left[\left(\psi_j^2 \partial_x u \right)_{2^j} \right], \left(\mathbf{a}_{33}^0 \right)_{2^j} \Lambda^s \left[\left(\Psi_k \psi_j^2 \partial_t (\partial_x u) \right)_{2^j} \right] \right\rangle_{L^2} \\
&=: \sum_{k=j-3}^{j+4} E_{\partial_x}(j,k).
\end{aligned} \tag{4.21}
$$

Our aim now is to take $\mathbf{a}_{33}^0$ across Λ^s in (4.17), and then we can use the fact that U satisfies equation (4.1). In order to do it we will use the commutator estimate, Theorem 4.4. However, if use the commutator (4.12) directly with $f = (\Psi_k)_{2^j}$ and $g = \left(\Psi_k \psi_j^2 \partial_t \partial_x u \right)_{2^j}$, then we get $\| \left(\Psi_k \psi_j^2 \partial_t \partial_x u \right)_{2^j} \|_{L^\infty} \lesssim \| \left(\Psi_k \psi_j^2 \partial_t \partial_x u \right)_{2^j} \|_{H^s}$ by Sobolev inequality. That would leads to the condition $s - 1 > \frac{3}{2}$, and therefore we would not obtain the desired regularity. In order to avoid this, we write

$$
\left(\Psi_k \psi_j^2 \partial_t \partial_x u \right)_{2^j} = \frac{1}{2^j} \partial_x \left(\Psi_k \psi_j^2 \partial_t u \right)_{2^j} - \left(\partial_x \left(\Psi_k \psi_j^2 \right) \right)_{2^j} \left(\partial_t u \right)_{2^j}, \tag{4.22}
$$

then

$$
\begin{aligned}
& \Lambda^s \left[\left(\Psi_k \psi_j^2 \partial_t \partial_x u \right)_{2^j} \right] \\
&= \frac{1}{2^j} \Lambda^s \left[\partial_x \left(\Psi_k \psi_j^2 \partial_t u \right)_{2^j} \right] - \Lambda^s \left[\left(\partial_x \left(\Psi_k \psi_j^2 \right) \right)_{2^j} \left(\partial_t u \right)_{2^j} \right] \\
&= \frac{1}{2^j} \left\{ \left(\Lambda^s \partial_x \right) \left[\left(\Psi_k \psi_j^2 \partial_t u \right)_{2^j} \right] - (\Psi_k)_{2^j} \left(\Lambda^s \partial_x \right) \left[\left(\psi_j^2 \partial_t u \right)_{2^j} \right] \right\} \\
&+ \frac{1}{2^j} \left[(\Psi_k)_{2^j} \left(\Lambda^s \partial_x \right) \left(\psi_j^2 \partial_t u \right)_{2^j} \right] \\
&- \Lambda^s \left[\left(\partial_x \left(\Psi_k \psi_j^2 \right) \right)_{2^j} \left(\partial_t u \right)_{2^j} \right].
\end{aligned} \tag{4.23}
$$

Inserting the last three expressions in each term of $E_{\partial_x}(j,k)$ in the right hand side of (4.21) results in

$$
\begin{aligned}
& E_{\partial_x}(j,k) = \\
& \frac{1}{2^j} \left\langle \Lambda^s \left[\left(\psi_j^2 \partial_x u \right)_{2^j} \right], \left(\mathbf{a}_{33}^0 \right)_{2^j} \left\{ \left(\Lambda^s \partial_x \right) \left[\left(\Psi_k \psi_j^2 \partial_t u \right)_{2^j} \right] \right. \right. \\
& \left. \left. - \ (\Psi_k)_{2^j} \left(\Lambda^s \partial_x \right) \left[\left(\psi_j^2 \partial_t u \right)_{2^j} \right] \right\} \right\rangle_{L^2} \\
& + \frac{1}{2^j} \left\langle \Lambda^s \left[\left(\psi_j^2 \partial_x u \right)_{2^j} \right], \left(\mathbf{a}_{33}^0 \right)_{2^j} (\Psi_k)_{2^j} \left(\Lambda^s \partial_x \right) \left[\left(\psi_j^2 \partial_t u \right)_{2^j} \right] \right\rangle_{L^2} \\
& - \left\langle \Lambda^s \left[\left(\psi_j^2 \partial_x u \right)_{2^j} \right], \left(\mathbf{a}_{33}^0 \right)_{2^j} \Lambda^s \left[\left(\partial_x \left(\Psi_k \psi_j^2 \right) \right)_{2^j} \left(\partial_t u \right)_{2^j} \right] \right\rangle_{L^2} \\
& =: E_{\partial_x}(a,j,k) + E_{\partial_x}(b,j,k) + E_{\partial_x}(c,j,k).
\end{aligned} \tag{4.24}
$$

Estimation of $E_{\partial_x}(a,j,k)$: Applying the Cauchy Schwarz inequality we get

$$\begin{aligned} |E_{\partial_x}(a,j,k)| &\leq \frac{\sqrt{3N}}{2^j} \left\| \Lambda^s \left(\psi_j^2 \partial_x u\right)_{2^j} \right\|_{L^2} \left\| \left(\mathbf{a}_{33}^0\right)_{2^j} \right\|_{L^\infty} \\ &\times \left\| \left(\Lambda^s \partial_x\right) \left(\Psi_k \psi_j^2 \partial_t u\right)_{2^j} - \left(\Psi_k\right)_{2^j} \left(\Lambda^s \partial_x\right) \left(\psi_j^2 \partial_t u\right)_{2^j} \right\|_{L^2}. \end{aligned} \tag{4.25}$$

The advantage of (4.22) is that $\Lambda^s \partial_x \in OPS_{1,0}^{s+1}$, hence the Kato-Ponce (4.12) is applied with $s+1$ rather than s. Therefore

$$\begin{aligned} &\left\| \left(\Lambda^s \partial_x\right) \left(\Psi_k \psi_j^2 \partial_t u\right)_{2^j} - \left(\Psi_k\right)_{2^j} \left(\Lambda^s \partial_x\right) \left(\psi_j^2 \partial_t u\right)_{2^j} \right\|_{L^2} \\ &\lesssim \left\| \nabla (\Psi_k)_{2^j} \right\|_{L^\infty} \left\| (\psi_j^2 \partial_t u)_{2^j} \right\|_{H^s} + \left\| (\Psi_k)_{2^j} \right\|_{H^{s+1}} \left\| (\psi_j^2 \partial_t u)_{2^j} \right\|_{L^\infty} \end{aligned}. \tag{4.26}$$

From the properties of ψ_j (see Definition 1.1) and (4.19), we see that $\|\partial^\alpha (\Psi_k)_{2^j}\|_{L^\infty} \leq C$, where the constant C is independent of j and k. Hence both $\|\nabla (\Psi_k)_{2^j}\|_{L^\infty}$ and $\|(\Psi_k)_{2^j}\|_{H^s}$ are bounded by a certain constant independent of j and k. For $s > \frac{3}{2}$ the Sobolev inequality yields $\left\|(\psi_j^2 \partial_t u)_{2^j}\right\|_{L^\infty} \lesssim \left\|(\psi_j^2 \partial_t u)_{2^j}\right\|_{H^s}$, and combining these with inequality (4.25) we get

$$|E_{\partial_x}(a,j,k)| \lesssim \left\| \left(\mathbf{a}_{33}^0\right)_{2^j} \right\|_{L^\infty} \left\| \left(\psi_j^2 \partial_x u\right)_{2^j} \right\|_{H^s} \left\| \left(\psi_j^2 \partial_t u\right)_{2^j} \right\|_{H^s}. \tag{4.27}$$

Estimation of $E_{\partial_x}(c,j,k)$: Since $\left(\partial_x(\Psi_k \psi_j^2)\right)_{2^j} (\partial_t u)_{2^j} = (F_{j,k})_{2^j} (\psi_j \partial_t u)_{2^j}$ and the partial derivatives of $(F_{j,k})_{2^j}$ up to order $[s]$ are bounded by a constant C independent of j and k, there holds

$$|E_{\partial_x}(c,j,k)| \lesssim \left\| \left(\mathbf{a}_{33}^0\right)_{2^j} \right\|_{L^\infty} \left\| \left(\psi_j^2 \partial_x u\right)_{2^j} \right\|_{H^s} \left\| (\psi_j \partial_t u)_{2^j} \right\|_{H^s}. \tag{4.28}$$

Estimation of $E_{\partial_x}(b,j,k)$: We see from (4.24) that in order to use equation (4.1) we need to commute $(\Psi_k \mathbf{a}_{33}^0)_{2^j}$ with $(\Lambda^s \partial_x)$. Therefore we write

$$\begin{aligned} &\left(\Psi_k \mathbf{a}_{33}^0\right)_{2^j} \left(\Lambda^s \partial_x\right) \left[(\psi_j^2 \partial_t u)_{2^j}\right] \\ &= \left(\Psi_k \mathbf{a}_{33}^0\right)_{2^j} \left(\Lambda^s \partial_x\right) \left[(\psi_j^2 \partial_t u)_{2^j}\right] - \left(\Lambda^s \partial_x\right) \left[\left(\Psi_k \mathbf{a}_{33}^0\right)_{2^j} (\psi_j^2 \partial_t u)_{2^j}\right] \\ &+ \left(\Lambda^s \partial_x\right) \left[\left(\Psi_k \mathbf{a}_{33}^0\right)_{2^j} (\psi_j^2 \partial_t u)_{2^j}\right] \\ &= \left(\Psi_k \mathbf{a}_{33}^0\right)_{2^j} \left(\Lambda^s \partial_x\right) \left[(\psi_j^2 \partial_t u)_{2^j}\right] - \left(\Lambda^s \partial_x\right) \left[\left(\Psi_k \mathbf{a}_{33}^0\right)_{2^j} (\psi_j^2 \partial_t u)_{2^j}\right] \\ &+ \Lambda^s \left[\partial_x \left(\Psi_k \mathbf{a}_{33}^0 \psi_j^2\right)_{2^j} (\partial_t u)_{2^j}\right] \\ &+ 2^j \Lambda^s \left[\left(\Psi_k \psi_j^2 \mathbf{a}_{33}^0\right)_{2^j} (\partial_t \partial_x u)_{2^j}\right]. \end{aligned} \tag{4.29}$$

Thus $E_{\partial_x}(b,j,k)$ is sum of three terms: $E_{\partial_x}(b,j,k) = E_{\partial_x}(d,j,k) + E_{\partial_x}(e,j,k) + E_{\partial_x}(f,j,k)$. The first one will be estimated by means Theorem 4.4, in the second one we use algebra property of H^s and in the last one brings us to equation (4.1).

We recall that $(\Lambda^s \partial_x) \in OPS^{s+1}_{1,0}$, therefore by Kato-Ponce commutator estimate (4.12),

$$\begin{aligned} &\left\| \left(\Psi_k \mathbf{a}^0_{33}\right)_{2^j} (\Lambda^s \partial_x) \left[(\psi_j^2 \partial_t u)_{2^j}\right] - (\Lambda^s \partial_x) \left[\left(\Psi_k \mathbf{a}^0_{33}\right)_{2^j} (\psi_j^2 \partial_t u)_{2^j}\right] \right\|_{L^2} \\ \lesssim &\left\{ \left\| \nabla (\Psi_k \mathbf{a}^0_{33})_{2^j} \right\|_{L^\infty} \left\| (\psi_j^2 \partial_t u)_{2^j} \right\|_{H^s} + \left\| (\Psi_k \mathbf{a}^0_{33})_{2^j} \right\|_{H^{s+1}} \left\| (\psi_j^2 \partial_t u)_{2^j} \right\|_{L^\infty} \right\}. \end{aligned} \tag{4.30}$$

From (4.19) and (4.20) we see that

$$\left\| \nabla (\Psi_k \mathbf{a}^0_{33})_{2^j} \right\|_{L^\infty} \leq C_1 \left\| (\mathbf{a}^0_{33})_{2^j} \right\|_{L^\infty} + C_2 2^j \left\| \nabla (\mathbf{a}^0_{33})_{2^j} \right\|_{L^\infty} \tag{4.31}$$

and

$$\begin{aligned} &\left\| (\Psi_k \mathbf{a}^0_{33})_{2^j} \right\|_{H^{s+1}} \\ \leq\, & C_3 \left\| (\psi_k \mathbf{a}^0_{33})_{2^j} \right\|_{H^{s+1}} = C_3 \left\| \left(\psi_k \left(\mathbf{a}^0_{33} - \mathbf{e}\right)\right)_{2^j} + (\psi_k \mathbf{e})_{2^j} \right\|_{H^{s+1}} \\ =\, & C_3 \left\| \left(\left(\psi_k \left(\mathbf{a}^0_{33} - \mathbf{e}\right)\right)_{2^k}\right)_{2^{j-k}} + \left((\psi_k \mathbf{e})_k\right)_{2^{j-k}} \right\|_{H^{s+1}} \\ \leq\, & C_3 \left\| \left(\left(\psi_k \left(\mathbf{a}^0_{33} - \mathbf{e}\right)\right)_{2^k}\right)_{2^{j-k}} \right\|_{H^{s+1}} + C_3 \left\| \left((\psi_k \mathbf{e})_k\right)_{2^{j-k}} \right\|_{H^{s+1}} \\ \simeq\, & C_3 \left\{ \left\| \left(\psi_k \mathbf{a}^0_{33} - \mathbf{e}\right)_{2^k} \right\|_{H^{s+1}} + 1 \right\}. \end{aligned} \tag{4.32}$$

Thus, the combination of (4.31) and (4.32) with the inequality $\left\| (\psi_j^2 \partial_t u)_{2^j} \right\|_{L^\infty} \lesssim \left\| (\psi_j^2 \partial_t u)_{2^j} \right\|_{H^s}$ (keeping in mind the factor 2^{-j} in (4.24)), leads to

$$\begin{aligned} &|E_{\partial_x}(d, j, k)| \\ =& \left\langle \Lambda^s \left[(\psi_j^2 \partial_x u)_{2^j}\right], \left(\Psi_k \mathbf{a}^0_{33}\right)_{2^j} (\Lambda^s \partial_x) \left[(\psi_j^2 \partial_t u)_{2^j}\right] \right. \\ & \left. - (\Lambda^s \partial_x) \left[\left(\Psi_k \mathbf{a}^0_{33}\right)_{2^j} (\psi_j^2 \partial_t u)_{2^j}\right] \right\rangle_{L^2} \\ \lesssim & \left\{ \left\| \left(\mathbf{a}^0_{33}\right)_{2^j} \right\|_{L^\infty} + \left\| \left(\nabla \mathbf{a}^0_{33}\right)_{2^j} \right\|_{L^\infty} + \left\| \left(\psi_k (\mathbf{a}^0_{33} - \mathbf{e})\right)_{2^k} \right\|_{H^{s+1}} \right\} \\ & \times \left\| (\psi_j^2 \partial_x u)_{2^j} \right\|_{H^s} \left\| (\psi_j^2 \partial_t u)_{2^j} \right\|_{H^s}. \end{aligned} \tag{4.33}$$

We turn now to $E_{\partial_x}(e, j, k) = \left\langle \Lambda^s \left[(\psi_j^2 \partial_x u)_{2^j}\right], (\Lambda^s \partial_x) \left[(\Psi_k \mathbf{a}^0_{33})_{2^j} (\psi_j^2 \partial_t u)_{2^j}\right] \right\rangle_{L^2}$. Noting that

$$\begin{aligned} &\partial_x \left(\Psi_k \mathbf{a}^0_{33} \psi_j^2\right)_{2^j} (\partial_t u)_{2^j} \\ =&\partial_x \left(\Psi_k \mathbf{a}^0_{33} \psi_j\right)_{2^j} (\psi_j \partial_t u)_{2^j} + 2^j \left(\Psi_k \mathbf{a}^0_{33} \partial_x \psi_j\right)_{2^j} (\psi_j \partial_t u)_{2^j}, \end{aligned} \tag{4.34}$$

and applying the algebra property of H^s, we get

$$\begin{aligned} &\left\| \partial_x \left(\Psi_k \mathbf{a}^0_{33} \psi_j^2\right)_{2^j} (\partial_t u)_{2^j} \right\|_{H^s} \lesssim \left\| \partial_x \left(\Psi_k \mathbf{a}^0_{33} \psi_j\right)_{2^j} \right\|_{H^s} \left\| (\psi_j \partial_t u)_{2^j} \right\|_{H^s} \\ &+ 2^j \left\| \left(\Psi_k \mathbf{a}^0_{33} \partial_x \psi_j\right)_{2^j} \right\|_{H^s} \left\| (\psi_j \partial_t u)_{2^j} \right\|_{H^s}. \end{aligned} \tag{4.35}$$

Now $\left\| \partial_x (\Psi_k \mathbf{a}^0_{33} \psi_j)_{2^j} \right\|_{H^s} \lesssim \left\| (\Psi_k \mathbf{a}^0_{33})_{2^j} \right\|_{H^{s+1}}$ and $\left\| (\Psi_k \mathbf{a}^0_{33} \partial_x \psi_j)_{2^j} \right\|_{H^s} \lesssim \left\| (\Psi_k \mathbf{a}^0_{33})_{2^j} \right\|_{H^{s+1}}$, hence by (4.35) and inequality (4.32) we get

$$\begin{aligned} &|E_{\partial_x}(e, j, k)| \\ \lesssim & \left\{ \left\| \left(\psi_k (\mathbf{a}^0_{33} - \mathbf{e})\right)_{2^k} \right\|_{H^{s+1}} + 1 \right\} \left\| (\psi_j^2 \partial_x u)_{2^j} \right\|_{H^s} \left\| (\psi_j \partial_t u)_{2^j} \right\|_{H^s}. \end{aligned} \tag{4.36}$$

In order to use equation (4.1) we write

$$\mathcal{A}^a = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \widetilde{\mathcal{A}}^a \end{array}\right), \qquad \mathcal{C}^a = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \widetilde{\mathcal{C}}^a \end{array}\right), \quad a = 1,2,3, \tag{4.37}$$

where $\widetilde{\mathcal{A}}^a = (\mathbf{a}^a_{ij})_{i,j=2,3}$, $\widetilde{\mathcal{C}}^a = (\mathbf{c}^a_{ij})_{i,j=2,3}$ are symmetric block matrix and $\mathbf{c}^a_{ij}$ are constant. Further, let $\{\Psi_k\}$ be the sequence which is defined by (4.19), then

$$\begin{aligned} E_{\partial_t}(j) &= \langle \Lambda^s(\psi_j^2\partial_t u)_{2^j}, \Lambda^s(\psi_j^2\partial_t(\partial_t u))_{2^j} \rangle_{L^2} \\ &= \left\langle \Lambda^s(\psi_j^2\partial_t u)_{2^j}, \Lambda^s \left[\left(\sum_{k=0}^{\infty} \Psi_k \right)_{2^j} (\psi_j^2 \partial_t(\partial_t u))_{2^j} \right] \right\rangle_{L^2} \\ &= \sum_{k=j-3}^{j+4} \left\langle \Lambda^s(\psi_j^2\partial_t u)_{2^j}, \Lambda^s \left[\left(\Psi_k \psi_j^2 \partial_t(\partial_t u) \right)_{2^j} \right] \right\rangle_{L^2} \\ &=: \sum_{k=j-3}^{j+4} E_{\partial_t}(j,k). \end{aligned} \tag{4.38}$$

From (4.21), (4.24) and (4.29) we see that

$$E_{\partial_x}(f,j,k) = \left\langle \Lambda^s \left[(\psi_j^2 \partial_x u)_{2^j} \right], \Lambda^s \left[\left(\Psi_k \mathbf{a}^0_{33} \right)_{2^j} (\psi_j^2 \partial_t \partial_x u)_{2^j} \right] \right\rangle_{L^2}$$

and since $U(t)$ satisfies (4.1), we have obtained

$$\begin{aligned} &\{E_{\partial_t}(j,k) + E_{\partial_x}(f,j,k)\} = \\ &\left\langle \Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right], \Lambda^s \left[\left(\Psi_k \psi_j^2 \begin{pmatrix} \mathbf{e} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}^0_{33} \end{pmatrix} \partial_t \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right\rangle_{L^2} = \\ &\sum_{a=1}^{3} \left\langle \Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right], \Lambda^s \left[\left(\Psi_k \psi_j^2 \left(\left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a \right) \partial_a \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right) \right)_{2^j} \right] \right\rangle_{L^2} \\ &+ \left\langle \Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right], \Lambda^s \left[\left(\Psi_k \psi_j^2 \begin{pmatrix} \mathbf{b}_{22} & \mathbf{b}_{23} \\ \mathbf{b}_{32} & \mathbf{b}_{33} \end{pmatrix} \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right\rangle_{L^2} \\ &+ \left\langle \Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right], \Lambda^s \left[\left(\Psi_k \psi_j^2 \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \right)_{2^j} \right] \right\rangle_{L^2}. \end{aligned} \tag{4.39}$$

The main difficulty is the estimation of the first term of the right hand side of (4.39). We recall that $\widetilde{\mathcal{C}}^a$ are constant and $\widetilde{\mathcal{A}}^a \in H_{s+1,\delta}$, therefore we may write

$$\begin{aligned} &\left(\Psi_k \psi_j^2 \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a \right) \partial_a \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \\ =& \frac{1}{2^j} \partial_a \left(\Psi_k \psi_j^2 \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a \right) \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} - \left(\partial_a \left(\Psi_k \psi_j^2 \widetilde{\mathcal{A}}^a \right) \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \end{aligned} \tag{4.40}$$

and hence

$$
\begin{aligned}
&\Lambda^s\left[\left(\Psi_k\psi_j^2\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right)\partial_a\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right]\\
=&\frac{1}{2^j}\left\{(\Lambda^s\partial_a)\left[\left(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right)\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right]\right.\\
-&\left.\left(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right)\right)_{2^j}(\Lambda^s\partial_a)\left[\left(\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right]\right\}\\
+&\frac{1}{2^j}\left(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right)\right)_{2^j}\Lambda^s\left[\partial_a\left(\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right]\\
-&\frac{1}{2^j}\Lambda^s\left[\partial_a\left(\left(\Psi_k\psi_j^2\widetilde{\mathcal{A}}^a\right)\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right].
\end{aligned}
\tag{4.41}
$$

The first term of the right hand side of (4.41) will be estimate by Theorem 4.4 with $P=\Lambda^s\partial_a$, in the second one the symmetry of $\widetilde{\mathcal{A}}^a$ will be exploited and in the third one we will use algebra property of H^s. In both the first and third we take the advantage that $\widetilde{\mathcal{A}}^a\in H_{s+1,\delta}$.

$$
\begin{aligned}
&\left\|(\Lambda^s\partial_a)\left[(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right))\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right]-\left(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right)\right)_{2^j}(\Lambda^s\partial_a)\left[\left(\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right]\right\|_{L^2}\\
\lesssim&\left\|\nabla(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right))_{2^j}\right\|_{L^\infty}\left\{\left\|(\psi_j^2\partial_t u)_{2^j}\right\|_{H^s}+\left\|(\psi_j^2\partial_x u)_{2^j}\right\|_{H^s}\right\}\\
+&\left\|(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right))_{2^j}\right\|_{H^{s+1}}\left\{\left\|(\psi_j^2\partial_t u)_{2^j}\right\|_{L^\infty}+\left\|(\psi_j^2\partial_x u)_{2^j}\right\|_{L^\infty}\right\}.
\end{aligned}
$$

Now, for $k=j-3,...,j+4$,

$$
\left\|\nabla\left(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right)\right)_{2^j}\right\|_{L^\infty}\lesssim\left(\left\|\widetilde{\mathcal{A}}^a\right\|_{L^\infty}+1+2^j\left\|\nabla\widetilde{\mathcal{A}}^a\right\|_{L^\infty}\right),
\tag{4.42}
$$

and

$$
\begin{aligned}
&\left\|\left(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right)\right)_{2^j}\right\|_{H^{s+1}}\lesssim\left\{\left\|(\psi_k\widetilde{\mathcal{A}}^a)_{2^j}\right\|_{H^{s+1}}+1\right\}\\
=&\left\{\left\|\left((\psi_k\widetilde{\mathcal{A}}^a)_{2^k}\right)_{2^{j-k}}\right\|_{H^{s+1}}+1\right\}\lesssim\left\{\left\|(\psi_k\widetilde{\mathcal{A}}^a)_{2^k}\right\|_{H^{s+1}}+1\right\}.
\end{aligned}
\tag{4.43}
$$

In addition, since $s>\frac{3}{2}$,

$$
\left\|(\psi_j^2\partial_t u)_{2^j}\right\|_{L^\infty}\lesssim\left\|(\psi_j^2\partial_t u)_{2^j}\right\|_{H^s}\quad\text{and}\quad\left\|(\psi_j^2\partial_x u)_{2^j}\right\|_{L^\infty}\lesssim\left\|(\psi_j^2\partial_x u)_{2^j}\right\|_{H^s}.
$$

Thus,

$$
\begin{aligned}
&\frac{1}{2^j} \mid \left\langle \Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right], (\Lambda^s \partial_a) \left[\left(\Psi_k \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a \right) \psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right. \\
&\left. - \left(\Psi_k \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a \right) \right)_{2^j} (\Lambda^s \partial_a) \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right\rangle_{L^2} \mid \\
&\lesssim \left\{ \left\| \nabla \widetilde{\mathcal{A}}^a \right\|_{L^\infty} + \left\| \widetilde{\mathcal{A}}^a \right\|_{L^\infty} + \left\| (\psi_k \widetilde{\mathcal{A}}^a)_{2^k} \right\|_{H^{s+1}} + 1 \right\} \\
&\times \left\{ \left\| (\psi_j^2 \partial_t u)_{2^j} \right\|_{H^s}^2 + \left\| (\psi_j^2 \partial_x u)_{2^j} \right\|_{H^s}^2 \right\}.
\end{aligned}
\tag{4.44}
$$

As to the third term of (4.41), writing

$$
\partial_a \left(\Psi_k \psi_j^2 \widetilde{\mathcal{A}}^a \right) \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} = \left(\partial_a (\Psi_k \psi_j \ \widetilde{\mathcal{A}}^a) + 2 (\Psi_k \widetilde{\mathcal{A}}^a \partial_a \psi_j \right) \psi_j \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix},
$$

and noting that

$$
\left\| \partial_a \left(\left(\Psi_k \psi_j \widetilde{\mathcal{A}}^a \right) \right)_{2^j} \right\|_{H^s} \leq \left\| \left(\Psi_k \psi_j \widetilde{\mathcal{A}}^a \right)_{2^j} \right\|_{H^{s+1}} \lesssim \left\| \left(\psi_k \widetilde{\mathcal{A}}^a \right)_{2^k} \right\|_{H^{s+1}},
$$

then by the embedding $H^{s+1} \hookrightarrow H^s$ we get that

$$
\begin{aligned}
&\mid \left\langle \Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right], \Lambda^s \left[\left(\partial_a \left(\Psi_k \psi_j^2 \widetilde{\mathcal{A}}^a \right) \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right\rangle_{L^2} \mid \\
&\lesssim \left\{ \left\| (\psi_k \widetilde{\mathcal{A}}^a)_{2^k} \right\|_{H^{s+1}} + \left\| (\psi_j \widetilde{\mathcal{A}}^a)_{2^j} \right\|_{H^{s+1}} \right\} \\
&\times \left(\left\| (\psi_j^2 \partial_t u)_{2^j} \right\|_{H^s} + \left\| (\psi_j^2 \partial_x u)_{2^j} \right\|_{H^s} \right) \left(\| (\psi_j \partial_t u)_{2^j} \|_{H^s} + \| (\psi_j \partial_x u)_{2^j} \|_{H^s} \right).
\end{aligned}
\tag{4.45}
$$

We turn now the second term of (4.41). Recall U is a $C_0^\infty(\mathbb{R}^3)$, therefore $\Lambda^s(\partial_t u), \Lambda^s(\partial_x u)$ are rapidly decreasing functions. This allows us to make the following operations:

$$
\begin{aligned}
&\int \partial_a \left\{ \left(\Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right)^T \left(\Psi_k \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a \right) \right)_{2^j} \Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right\} dx \\
&= \int \left\{ \left(\Lambda^s \left[\partial_a \left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right)^T \left(\Psi_k \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a \right) \right)_{2^j} \Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right\} dx \\
&+ \int \left\{ \left(\Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right)^T \partial_a \left(\Psi_k \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a \right) \right)_{2^j} \Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right\} dx \\
&+ \int \left\{ \left(\Lambda^s \left[\left(\psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right)^T \left(\Psi_k \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a \right) \right)_{2^j} \Lambda^s \left[\left(\partial_a \psi_j^2 \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right)_{2^j} \right] \right\} dx \\
&= 0.
\end{aligned}
\tag{4.46}
$$

Since $\widetilde{\mathcal{A}}^a$ and $\widetilde{\mathcal{C}}^a$ are symmetric the first and the third terms of the right hand side of (4.46) are equal and hence

$$\begin{aligned}
&\frac{2}{2^j}\,|\left\langle\Lambda^s\left[\left(\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right],\left(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right)\right)_{2^j}\Lambda^s\left[\left(\partial_a\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right]\right\rangle_{L^2}|\\
=&\frac{1}{2^j}\,|\left\langle\Lambda^s\left[\left(\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right],\partial_a\left(\Psi_k\left(\widetilde{\mathcal{A}}^a+\widetilde{\mathcal{C}}^a\right)\right)_{2^j}\Lambda^s\left[\left(\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right]\right\rangle_{L^2}|\\
\lesssim&\left\{\left\|\left(\widetilde{\mathcal{A}}^a\right)_{2^j}\right\|_{L^\infty}+\left\|\left(\partial_a\widetilde{\mathcal{A}}^a\right)_{2^j}\right\|_{L^\infty}+1\right\}\left\{\|(\psi_j^2\partial_t u)_{2^j}\|_{H^s}^2+\|(\psi_j^2\partial_x u)_{2^j}\|_{H^s}^2\right\}.
\end{aligned}\tag{4.47}$$

That completes the estimation of the first term of the right hand side of (4.39). The second and the third terms of are easier to handle since they do not contain derivatives of high order. Recalling conditions (4.3i) and (4.3j) and using algebra in H^s for $s>\frac{3}{2}$, we have

$$\begin{aligned}
&|\left\langle\Lambda^s\left[\left(\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right],\Lambda^s\left[\left(\Psi_k\psi_j^2\begin{pmatrix}\mathbf{b}_{22} & \mathbf{b}_{23}\\ \mathbf{b}_{32} & \mathbf{b}_{33}\end{pmatrix}\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right]\right\rangle_{L^2}|\\
\lesssim&\left\|\left(\psi_j\widetilde{\mathcal{B}}\right)_{2^j}\right\|_{H^s}\left(\left\|(\psi_j^2\partial_t u)_{2^j}\right\|_{H^s}+\left\|(\psi_j^2\partial_x u)_{2^j}\right\|_{H^s}\right)\\
&\times\left(\|(\psi_j\partial_t u)_{2^j}\|_{H^s}+\|(\psi_j\partial_x u)_{2^j}\|_{H^s}\right)
\end{aligned}\tag{4.48}$$

and

$$\begin{aligned}
&|\left\langle\Lambda^s\left[\left(\psi_j^2\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right)_{2^j}\right],\Lambda^s\left[\left(\Psi_k\psi_j^2\begin{pmatrix}f_2\\ f_3\end{pmatrix}\right)_{2^j}\right]\right\rangle_{L^2}|\\
\lesssim&\left\|(\psi_j^2\partial_t u)_{2^j}\right\|_{H^s}\left\|(\psi_j^2 f_2)_{2^j}\right\|_{H^s}+\left\|(\psi_j^2\partial_x u)_{2^j}\right\|_{H^s}\left\|(\psi_j^2 f_3)_{2^j}\right\|_{H^s}.
\end{aligned}\tag{4.49}$$

To complete the proof we need to summarize

$$\sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+4}2^{(\frac{3}{2}+\delta+1)2j}\left(E_{\partial_t}(j,k)+E_{\partial_x}(j,k)\right)\right).\tag{4.50}$$

We see from the inequalities (4.27), (4.28), (4.33), (4.36), (4.44), (4.45), (4.47), (4.48) and (4.49) that the estimation of (4.50) consists of the following types of series:

Type 1:

$$\sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+4}2^{(\frac{3}{2}+\delta+1)2j}\left(\|(h)_{2^j}\|_{L^\infty}\|(\psi_j^2 f)_{2^j}\|_{H^s}\|(\psi_j^2 g)_{2^j}\|_{H^s}\right)\right),\tag{4.51}$$

where f and g belong to $H_{s,\delta+1}$ and h is in L^∞;

Type 2:

$$\sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+4}2^{(\frac{3}{2}+\delta+1)2j}\left(\|(h)_{2^j}\|_{L^\infty}\|(\psi_j^2 f)_{2^j}\|_{H^s}\|(\psi_j g)_{2^j}\|_{H^s}\right)\right),\tag{4.52}$$

where f and g belong to $H_{s,\delta+1}$ and h is in L^∞;

Type 3:

$$\sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta+1)2j}\left(\|(\psi_k h)_{2^k}\|_{H^{s+1}}\left\|(\psi_j^2 f)_{2^j}\right\|_{H^s}\left\|(\psi_j^2 g)_{2^j}\right\|_{H^s}\right)\right), \tag{4.53}$$

where f and g belong to $H_{s,\delta+1}$ and h is in $H_{s+1,\delta}$;

Type 4:

$$\sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta+1)2j}\left(\|(\psi_k h)_{2^k}\|_{H^{s+1}}\left\|(\psi_j^2 f)_{2^j}\right\|_{H^s}\|(\psi_j g)_{2^j}\|_{H^s}\right)\right), \tag{4.54}$$

where f and g belong to $H_{s,\delta+1}$ and h is in $H_{s+1,\delta}$;

Type 5:

$$\sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta+1)2j}\left(\|(\psi_j h)_{2^j}\|_{H^{s+1}}\left\|(\psi_j^2 f)_{2^j}\right\|_{H^s}\left\|(\psi_j^2 g)_{2^j}\right\|_{H^s}\right)\right), \tag{4.55}$$

where f and g belong to $H_{s,\delta+1}$ and h is in $H_{s+1,\delta}$;

Type 6:

$$\sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta+1)2j}\left(\|(\psi_j h)_{2^j}\|_{H^{s+1}}\left\|(\psi_j^2 f)_{2^j}\right\|_{H^s}\|(\psi_j g)_{2^j}\|_{H^s}\right)\right), \tag{4.56}$$

where $f \in H_{s,\delta}$, $g \in H_{s,\delta+1}$ and $h \in H_{s,\delta+1}$.

The estimation (4.51)- (4.56) will be done by means of the Cauchy-Schwarz and Hölder's inequalities and the equivalence property (2.3), of the $H_{s,\delta}$-norm. Starting with type 1, we see from (4.51) that

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta+1)2j}\left(\left\|(\psi_j^2 f)_{2^j}\right\|_{H^s}\left\|(\psi_j^2 g)_{2^j}\right\|_{H^s}\right)\\
&\leq 7\|h\|_{L^\infty}\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+1)2j}\left(\left\|(\psi_j^2 f)_{2^j}\right\|_{H^s}^2+\left\|(\psi_j^2 g)_{2^j}\right\|_{H^s}^2\right)\\
&\simeq\|h\|_{L^\infty}\left(\|f\|_{H_{s,\delta+1,2}}^2+\|g\|_{H_{s,\delta+1,2}}^2\right).
\end{aligned} \tag{4.57}$$

Similarly we estimate type 2, the only difference is the use of the equivalence (2.3) in the final step. Number 3 is more sophisticated, we first note that $(\frac{3}{2}+\delta+1)2j \leq (\frac{3}{2}+\delta)j+(\frac{3}{2}+\delta+1)j+(\frac{3}{2}+\delta+1)j$

for $\delta \geq -\frac{3}{2}$, then we apply the Hölder inequality with $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$ and get

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta+1)2j}\left(\|(\psi_k h)_{2^k}\|_{H^{s+1}}\left\|(\psi_j^2 f)_{2^j}\right\|_{H^s}\left\|(\psi_j^2 g)_{2^j}\right\|_{H^s}\right)\right)\\
&\leq \sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4}\left(2^{(\frac{3}{2}+\delta)j}\|(\psi_k h)_{2^k}\|_{H^{s+1}}\right)\left(2^{(\frac{3}{2}+\delta+1)j}\left\|(\psi_j^2 f)_{2^j}\right\|_{H^s}\right)\left(2^{(\frac{3}{2}+\delta+1)j}\left\|(\psi_j^2 g)_{2^j}\right\|_{H^s}\right)\\
&\leq \left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4}\left(2^{(\frac{3}{2}+\delta)j}\|(\psi_k h)_{2^k}\|_{H^{s+1}}\right)^2\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4}\left(2^{(\frac{3}{2}+\delta+1)j}\left\|(\psi_j^2 f)_{2^j}\right\|_{H^s}\right)^4\right)^{\frac{1}{4}}\\
&\times\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4}\left(2^{(\frac{3}{2}+\delta+1)j}\left\|(\psi_j^2 g)_{2^j}\right\|_{H^s}\right)^4\right)^{\frac{1}{4}}\\
&\leq 2^{(\frac{3}{2}+\delta)3}\sqrt{7}\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4}\left(2^{(\frac{3}{2}+\delta)2k}\|(\psi_k h)_{2^k}\|^2_{H^{s+1}}\right)\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\infty}\left(2^{(\frac{3}{2}+\delta+1)2j}\left\|(\psi_j^2 f)_{2^j}\right\|^2_{H^s}\right)\right)^{\frac{1}{2}}\\
&\times\left(\sum_{j=0}^{\infty}\left(2^{(\frac{3}{2}+\delta+1)2j}\left\|(\psi_j^2 g)_{2^j}\right\|^2_{H^s}\right)\right)^{\frac{1}{2}}\\
&\leq 2^{(\frac{3}{2}+\delta)3}7^{\frac{3}{2}}\left(\sum_{k=0}^{\infty}\left(2^{(\frac{3}{2}+\delta)2k}\|(\psi_k h)_{2^k}\|^2_{H^{s+1}}\right)\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\infty}\left(2^{(\frac{3}{2}+\delta+1)2j}\left\|(\psi_j^2 f)_{2^j}\right\|^2_{H^s}\right)\right)^{\frac{1}{2}}\\
&\times\left(\sum_{j=0}^{\infty}\left(2^{(\frac{3}{2}+\delta+1)2j}\left\|(\psi_j^2 g)_{2^j}\right\|^2_{H^s}\right)\right)^{\frac{1}{2}}\\
&\lesssim \|h\|_{H_{s+1,\delta}}\|f\|_{H_{s,\delta+1,2}}\|g\|_{H_{s,\delta+1,2}} \leq \|h\|_{H_{s+1,\delta}}\left(\|f\|^2_{H_{s,\delta+1,2}}+\|g\|^2_{H_{s,\delta+1,2}}\right)
\end{aligned}
$$

The estimations of Types 4, 5 and 6 are similar to the last one.

We may conclude now that

$$
\begin{aligned}
&\left\langle \begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}, \partial_t\begin{pmatrix}\partial_t u\\ \partial_x u\end{pmatrix}\right\rangle_{s,\delta+1,\mathbf{a}^0_{33}}\\
&=\sum_{j=0}^{\infty}\left(\sum_{k=j-3}^{j+4}2^{(\frac{3}{2}+\delta+1)2j}\left(E_{\partial_t}(j,k)+E_{\partial_x}(j,k)\right)\right)\\
&\leq C\left(\|\partial_t u\|^2_{H_{s,\delta+1,2}}+\|\partial_x u\|^2_{H_{s,\delta+1,2}}+1\right)\leq C\left(\|U\|^2_{X_{s,\delta}}+1\right),
\end{aligned}
\tag{4.58}
$$

where the constant C depends on $\|(\mathcal{A}^0-\mathbf{e})\|_{H_{s+1,\delta}}$, $\|\mathcal{A}^a\|_{H_{s+1,\delta}}$, $\|\widetilde{\mathcal{B}}\|_{H_{s,\delta+1}}$, $\|\mathcal{F}\|_{H_{s,\delta+1}}$, $\|\mathcal{A}^\alpha\|_{L^\infty}$, $\|\partial_x\mathcal{A}^\alpha\|_{L^\infty}$, s and δ. By the embeddings Proposition 2.2:6 and 2.2:3, we may replace $\|\mathcal{A}^\alpha\|_{L^\infty}$ and $\|\partial_x\mathcal{A}^\alpha\|_{L^\infty}$ by their corresponding $H_{s,\delta}$ norm. Thus combining inequalities (4.14), (4.15), (4.58) with Corollary 4.2 we get that

$$\frac{d}{dt}\frac{1}{2}\langle U(t),U(t)\rangle_{X_{s,\delta,\mathcal{A}^0}} \leq C\left(\|U\|^2_{X_{s,\delta}}+1\right) \leq Cc_0\left(\|U\|^2_{X_{s,\delta,\mathcal{A}^0}}+1\right), \tag{4.59}$$

here c_0 is the constant of the equivalence (4.3b) and in addition C also depends on $\|\partial_t \mathbf{a}^0_{33}\|_{L^\infty} = \|\partial_t\mathcal{A}^0\|_{L^\infty}$. This completes the proof of the energy estimates. □

4.2 L^2_δ - energy estimates

The L^2_δ space is the closure of all continuous functions with respect to the norm

$$\|u\|^2_{L^2_\delta} = \int (1+|x|)^{2\delta}|u(x)|^2 dx. \tag{4.60}$$

Similarly to Definition 2.3, we set $Y_\delta = L^2_\delta \times L^2_{\delta+1}\times L^2_{\delta+1}$ and the norm of $V=(v_1,v_2,v_3)\in Y_\delta$ is denoted by

$$\|V\|^2_{Y_\delta} = \|v_1\|^2_{L^2_\delta} + \|v_2\|^2_{L^2_{\delta+1}} + \|v_3\|^2_{L^2_{\delta+1}}. \tag{4.61}$$

The equivalence of norms $\|V\|_{X_{0,\delta}} \simeq \|V\|_{Y_\delta}$ follows from Proposition 2.2:2.

In analogous to Definition 4.1, we define an inner-product which is appropriate to the system (4.1). So let $\mathbf{a}^0_{33}$ be a positive definite matrix and $V,\Phi\in Y_\delta$ be two vector valued functions. We define an inner-product:

$$\begin{aligned}\langle V,\Phi\rangle_{Y_\delta,\mathbf{a}^0_{33}} &= \int (1+|x|)^{2\delta} v_1^T\phi_1 dx \\ &+ \int (1+|x|)^{2\delta+2}\left[\left(v_2^T,v_3^T\right)\begin{pmatrix}\mathbf{e} & \mathbf{0}\\ \mathbf{0} & \mathbf{a}^0_{33}\end{pmatrix}\begin{pmatrix}\phi_2\\ \phi_3\end{pmatrix}\right]dx,\end{aligned} \tag{4.62}$$

and the norm which is associated with this product: $\|V\|^2_{Y_\delta,\mathbf{a}^0_{33}} = \langle V,V\rangle_{L^2_\delta,\mathbf{a}^0_{33}}$. If $\mathbf{a}^0_{33}$ satisfies (4.3b), then

$$\frac{1}{c_0}\|V\|^2_{Y_\delta,\mathbf{a}^0_{33}} \leq \|V\|^2_{Y_\delta} \leq c_0\|V\|^2_{Y_\delta,\mathbf{a}^0_{33}}. \tag{4.63}$$

Lemma 4.5 *(L^2_δ energy estimates) Assume the coefficients of (4.1) satisfy conditions (4.3a), (4.3b), (4.3e), and (4.3h). If $U(t,\cdot)=(u(t,\cdot),\partial_t u(t,\cdot),\partial_x u(t,\cdot))\in X_{1,\delta}$ is a solution to the linear system (4.1), then*

$$\frac{d}{dt}\|U(t)\|^2_{Y_\delta,\mathbf{a}^0_{33}} \leq Cc_0\left(\|U(t)\|^2_{Y_\delta,\mathbf{a}^0_{33}} + \|\mathcal{F}\|^2_{L_{\delta+1}}\right), \tag{4.64}$$

where the constant C depends on the L^∞-norm of $\mathcal{A}^\alpha$, $\partial_\alpha\mathcal{A}^\alpha$ and $\mathcal{B}$.

Proof (Lemma 4.5)**.** Taking the derivative of $\langle U(t),U(t)\rangle_{L^2_\delta,\mathbf{a}^0_{33}}$ yields,

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2_{Y_\delta,\mathbf{a}^0_{33}} = \langle U(t),\partial_t U(t)\rangle_{Y_\delta,\mathbf{a}^0_{33}} + \frac{1}{2}\int (1+|x|)^{2\delta+2}(\partial_x u)^T\partial_t\mathbf{a}^0_{33}(\partial_x u)dx. \tag{4.65}$$

By the Cauchy Schwarz inequality, the second term of the right hand side of (4.65) is less than

$$\sqrt{3N}\|\partial_t\mathbf{a}^0_{33}\|_{L^\infty}\|\partial_x u\|^2_{L_{\delta+1}}. \tag{4.66}$$

Let $\widetilde{\mathcal{A}}^a$ and $\widetilde{\mathcal{C}}^a$ be the matrices which is defined in (4.37), since $U(t)$ satisfies system (4.1) we have

$$\begin{aligned}
&\langle U(t), \partial_t U(t)\rangle_{Y_\delta,\mathbf{a}_{33}^0} = \int (1+|x|)^{2\delta} u^T(\partial_t u)dx \\
&+ \sum_{a=1}^{3} \int (1+|x|)^{2\delta+2} \left[\left((\partial_t u)^T, (\partial_x u)^T\right) \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a\right) \partial_a \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right] dx \\
&+ \int (1+|x|)^{2\delta+2} \left[\left((\partial_t u)^T, (\partial_x u)^T\right) \begin{pmatrix} \mathbf{b}_{22} & \mathbf{b}_{23} \\ \mathbf{b}_{32} & \mathbf{b}_{33} \end{pmatrix} \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right] dx \\
&+ \int (1+|x|)^{2\delta+2} \left[(\partial_t u)^T f_2 + (\partial_x u)^T f_3 \right] dx \\
&=: L_1 + \sum_{a=1}^{3} L_{2,a} + L_3 + L_4.
\end{aligned} \tag{4.67}$$

We are estimating each term separately:

$$|L_1| \le \|u\|_{L^2_\delta} \|\partial_t u\|_{L^2_\delta} \le \frac{1}{2}\left(\|u\|^2_{L^2_\delta} + \|\partial_t u\|^2_{L^2_{\delta+1}}\right), \tag{4.68}$$

$$|2L_{2,a}| \le 2\sqrt{4N}\left(|2\delta+2| \|\widetilde{\mathcal{A}}^a\|_{L^\infty} + \|\partial_a \widetilde{\mathcal{A}}^a\|_{L^\infty} + 1\right)\left(\|\partial_t u\|^2_{L^2_{\delta+1}} + \|\partial_x u\|^2_{L^2_{\delta+1}}\right), \tag{4.69}$$

$$|L_3| \le 4N\|B\|_{L^\infty}\left(\|\partial_t u\|^2_{L^2_{\delta+1}} + \|\partial_x u\|^2_{L^2_{\delta+1}}\right) \tag{4.70}$$

and

$$|L_4| \le \frac{1}{2}\left(\|\partial_t u\|^2_{L^2_{\delta+1}} + \|f_2\|^2_{L^2_{\delta+1}} + \|\partial_x u\|^2_{L^2_{\delta+1}} + \|f_3\|^2_{L^2_{\delta+1}}\right). \tag{4.71}$$

In (4.69) we have used the identity

$$\begin{aligned}
0 &= \int \partial_a \left\{ (1+|x|)^{2\delta+2} \left[\left((\partial_t u)^T, (\partial_x u)^T\right) \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a\right) \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right] \right\} dx \\
&= \int (2\delta+2)(1+|x|)^{2\delta+1} \frac{x_a}{|x|} \left[\left((\partial_t u)^T, (\partial_x u)^T\right) \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a\right) \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right] dx \\
&+ \int (1+|x|)^{2\delta+2} \left[\partial_a \left((\partial_t u)^T, (\partial_x u)^T\right) \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a\right) \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right] dx \\
&+ \int (1+|x|)^{2\delta+2} \left[\left((\partial_t u)^T, (\partial_x u)^T\right) \left(\widetilde{\mathcal{A}}^a + \widetilde{\mathcal{C}}^a\right) \partial_a \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right] dx \\
&+ \int (1+|x|)^{2\delta+2} \left[\left((\partial_t u)^T, (\partial_x u)^T\right) \partial_a \widetilde{\mathcal{A}}^a \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} \right] dx.
\end{aligned}$$

and exploited the symmetry of $\widetilde{\mathcal{A}}^a$ and $\widetilde{\mathcal{C}}^a$.

Summing the inequalities (4.66), (4.68), (4.69), (4.70) and (4.71) and taking into account the equivalence (4.63), we get inequality (4.64). □

5 Local Existence of Quasi-linear Hyperbolic Systems

Let $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^N$ and set $U = (u, \partial_t u, \partial_x u)$, we consider a quasi-linear first order hyperbolic system

$$\mathcal{A}^0(u)\partial_t U = \sum_{a=1}^{3} \left(\mathcal{A}^a(u) + \mathcal{C}^a\right) \partial_a U + \mathcal{B}(U)U \tag{5.1}$$

under the following conditions:

Assumptions 5.1 *All the matrices are smooth function of their arguments and*

1. $\mathcal{A}^0(u)$, $\mathcal{A}^a(u)$ *and* $\mathcal{C}^\alpha$ *are symmetric matrices;*
2. $\mathcal{A}^0(u) = (\mathbf{a}^0_{ij}(u))_{ij=1,2,3}$ *is a block matrix such that* $\mathbf{a}^0_{ij}(u) = \mathbf{0}$ *for* $i \neq j$ *and* $\mathbf{a}^0_{ii}(u) = \mathbf{e}$ *for* $i = 1, 2$;
3. $\mathcal{A}^a(u) = (\mathbf{a}^a_{ij}(u))_{ij=1,2,3}$ *are block matrices such that* $\mathbf{a}^a_{1j}(u) = \mathbf{0}$ *for* $j, a = 1, 2, 3$;
4. $\mathcal{C}^a = (\mathbf{c}^a_{ij})_{ij=1,2,3}$ *are constant block matrices such that* $\mathbf{c}^a_{1j} = \mathbf{0}$ *for* $j, a = 1, 2, 3$;
5. $\mathcal{B}(U) = (\mathbf{b}_{ij}(U))_{ij=1,2,3}$ *is a block matrix such that* $\mathbf{b}_{i1}(U) = \mathbf{0}$ *and* $\mathbf{b}_{1,j}(U)$ *are constant,* $i, j = 1, 2, 3$.

The sizes of the blocks are ruled according to (4.2).

Clearly the system (3.1) satisfies these assumptions. The main result of this section is the well-posedness of the system (5.1) in $X_{s,\delta}$-spaces.

Theorem 5.2 ***(Well-posedness of quasi-linear hyperbolic symmetric systems)*** *Let* $s > \frac{3}{2}$, $\delta > -\frac{3}{2}$, $(f, g) \in H_{s+1,\delta} \times H_{s,\delta+1}$ *and suppose*

$$\frac{1}{\mu} v^T v \leq v^T \mathbf{a}^0_{33}(f) v \leq \mu v^T v, \qquad \forall v \in \mathbb{R}^{3N} \quad \textit{and some } \mu \in \mathbb{R}^+. \tag{5.2}$$

Then under Assumptions 5.1 there exits a positive T *a unique* $U(t) = (u(t), \partial_t u(t), \partial_x u(t))$ *a solution to (5.1) such that* $U(0, x) = (f(x), g(x), \partial_x f(x))$ *and*

$$U \in C([0, T], X_{s,\delta}). \tag{5.3}$$

Remark 5.3 *We may conclude by Mixed norm estimate 10 of Proposition 2.2 and (5.3) that*

$$u \in C([0, T], H_{s+1,\delta}) \cap C^1([0, T], H_{s,\delta+1}). \tag{5.4}$$

We adopt Majda's method and construct the solution through an iteration procedure [21]. Similar approach was carry out in [4], [3] for $s > \frac{5}{2}$. Here we will examine how the special assumptions of (5.1) enable us to improve the regularity.

5.1 Construction of the iteration scheme

We first note that the Embedding 6 of Proposition 2.2 implies that the initial data $(f, g, \partial_x f)$ are continuous, hence there is a constant $c_0 \geq 1$ and a bounded domain $G_2 \subset \mathbb{R}^N$ containing f such that

$$\frac{1}{c_0} v^T v \leq v^T \mathbf{a}_{33}^0(u) v \leq c_0 v^T v \quad \text{for } u \in G_2. \tag{5.5}$$

According to the density properties of $H_{s,\delta}$ (Proposition 2.2:9), there are sequences $\{f^k\}_{k=0}^\infty, \{g^k\}_{k=0}^\infty \subset C_0^\infty$ and a positive constant R such that

$$\|(f^0, g^0, \partial_x f^0)\|_{X_{s+1,\delta}} \leq C\|(f, g, \partial_x f)\|_{X_{s,\delta}}, \tag{5.6}$$

$$\|u - f^0\|_{H_{s,\delta+1,2}} \leq R \Rightarrow u \in G_2, \tag{5.7}$$

and

$$\|(f^k, g^k, \partial_x f^k) - (f, g, \partial_x f)\|_{X_{s,\delta}} \leq 2^{-k} \frac{R}{4c_0}. \tag{5.8}$$

The iteration scheme is defined as follows: Let $U^0(t,x) = (f^0, g^0, \partial_x f^0)$ and $U^{k+1}(t,x) = (u^{k+1}(t,x), \partial_t u^{k+1}(t,x), \partial_x u^{k+1}(t,x))$ be a solution to the linear initial value problem

$$\begin{cases} \mathcal{A}^0(u^k)\partial_t U^{k+1} = \sum_{a=1}^3 (\mathcal{A}^a + \mathcal{C}^a)(u^k)\partial_x U^{k+1} + \mathcal{B}(U^k)U^{k+1} \\ U^{k+1}(0,x) = (f^k(x), g^k(x), \partial_x f^k(x)) \end{cases}. \tag{5.9}$$

The linear theory of first order symmetric hyperbolic systems (see e.g. [18]) guarantees the existence of a sequence $\{U^k(t)\} \subset C_0^\infty(\mathbb{R}^3)$. Therefore for each k

$$T_k = \sup\{T : \sup_{0<t<T} \|U^k(t) - (f^0, g^0, \partial_x f^0)\|_{X_{s,\delta}} \leq R\} > 0. \tag{5.10}$$

We claim that there is $T^* > 0$ such that $T_k \geq T^*$ for all k.

5.2 Boundedness in the $X_{s,\delta}$-norm

Lemma 5.4 ***(Boundedness in the norm)*** *There is a positive constant T^* such that*

$$\sup\{T : \sup_{0<t<T} \|U^k(t) - (f^0, g^0, \partial_x f^0)\|_{X_{s,\delta}} \leq R\} \geq T^* \quad \textit{for all } k. \tag{5.11}$$

Proof (of Lemma 5.4)**.** Let $V^{k+1} = U^{k+1} - U^0$, then it satisfies the linear system

$$\mathcal{A}^0(u^k)\partial_t V^{k+1} = \sum_{a=1}^3 \left(\mathcal{A}^a(u^k) + \mathcal{C}^a\right) \partial_a V^{k+1} + \mathcal{B}(U^k)V^{k+1} + \mathcal{F}^k, \tag{5.12}$$

where

$$\mathcal{F}^k = \sum_{a=1}^3 \left(\mathcal{A}^a(u^k) + \mathcal{C}^a\right) \partial_a U^0 + \mathcal{B}(U^k)U^0$$

and $V^{k+1}(0,x) = (f^{k+1}(x), g^{k+1}(x), \partial_x f^{k+1}(x)) - (f^0(x), g^0(x), \partial_x f^0(x))$. At this stage we need to verify that the linear system (5.12) meets all the requirements of the energy estimates Lemma 4.3. Clearly the matrices $\mathcal{A}^0(u^k)$, $\mathcal{A}^a(u^k)$ and $\mathcal{B}(U^k)$ satisfy conditions (4.3a), (4.3b), (4.3e) and (4.3h).

We check now that rest of the conditions of (4.3). From the induction hypothesis (5.10), we have that $\|u^k - f^0\|^2_{H_{s,\delta,2}} + \|\partial_x u^k - \partial_x f^0\|^2_{H_{s,\delta+1,2}} \leq R^2$, therefore by Proposition 2.2:10, $\|u^k - f^0\|_{H_{s+1,\delta,2}} \leq CR$. Applying the equivalence (2.3) and Moser type estimates (Proposition 2.2:7), we have

$$\begin{aligned}\|\mathcal{A}^0(u^k) - \mathbf{e}\|_{H_{s+1,\delta}} \leq C_1\|u^k\|_{H_{s+1,\delta,2}} &\leq C_1\left(\|u^k - f^0\|_{H_{s+1,\delta,2}} + \|f^0\|_{H_{s+1,\delta,2}}\right)\\ &\leq C_1\left(CR + \|f^0\|_{H_{s+1,\delta,2}}\right).\end{aligned}$$

Similarly we get $\|\mathcal{A}^a(u^k)\|_{H_{s+1,\delta,1}} \leq C_2\left(CR + \|f^0\|_{H_{s+1,\delta,2}}\right)$. Here the constants C_1 and C_2 depend on $\|u^k\|_{L^\infty}$, and $\|\mathcal{A}^0 - \mathbf{e}\|_{C^m(G_2)}$, $\|\mathcal{A}^a\|_{C^m(G_2)}$ respectively, which implies that conditions (4.3c) and (4.3f) hold. Having shown (4.3c) and (4.3f), we conclude from Proposition 2.2:6 that $\mathcal{A}^\alpha(u^k) \in C^1_\beta$ ($\beta \geq 0$). Combing it with inequalities (5.7) and (5.10) we get

$$\begin{aligned}\|\partial_t\mathcal{A}^0(u^k)\|_{L^\infty} \leq \sup_{\bar{G}_2}|\frac{\partial\mathcal{A}^0}{\partial u}(u)|\|\partial_t u^k\|_{L^\infty} \leq C\|\partial_t u^k\|_{H_{s,\delta+1,2}}\\ \leq C(\|\partial_t u^k - g^0\|_{H_{s,\delta+1,2}} + \|g^0\|_{H_{s,\delta+1,2}}) \leq C(R + \|g^0\|_{H_{s,\delta+1,2}}),\end{aligned}$$

this gives condition (4.3d). In order to verify condition (4.3i), we denote by $\widetilde{\mathcal{B}}(U^k)$ the non-constant blocks of $\mathcal{B}(U^k)$. Then we apply again Moser type estimates 7 and Algebra 4 of Proposition 2.2, together with induction hypothesis (5.10) and the structure of the matrix $\mathcal{B}$ yield

$$\|\widetilde{\mathcal{B}}(U^k)\|_{H_{s,\delta+1}} \leq C_3\|U^k\|_{X_{s,\delta}} \leq C_3\left(R + \|(f^0, g^0, \partial_x f^0)\|_{X_{s,\delta}}\right).$$

The constant C_3 depends on C^m norm of $\widetilde{\mathcal{B}}(U)$ taking in a bounded region of $\mathbb{R}^{5N}$ and $\|U^k\|_{L^\infty}$. Finally, the $H_{s,\delta}$ estimates of $\mathcal{A}^\alpha(u^k)$ and $\widetilde{\mathcal{B}}(U^k)$ with Proposition 2.2:4 provide an upper bound for $\|\mathcal{F}^k\|_{s,\delta+1}$. Thus we have verified all the conditions (4.3).

We conclude that the constant C of the energy estimate (4.11) depends only on R and the initial data $\|(f^0, g^0, \partial_x f^0)\|_{X_{s,\delta}}$, hence

$$\frac{d}{dt}\langle V^k(t), V^k(t)\rangle_{X_{s,\delta,\mathcal{A}^0}} \leq Cc_0\left\{\langle V^k(t), V^k(t)\rangle_{X_{s,\delta,\mathcal{A}^0}} + 1\right\}, \tag{5.13}$$

and the constant C of (5.13) is independent of k. By Gronwall's inequality

$$\|V^k(t)\|^2_{X_{s,\delta,\mathcal{A}^0}} \leq e^{Cc_0 t}\left(\|V^k(0)\|^2_{X_{s,\delta,\mathcal{A}^0}} + Cc_0 t\right). \tag{5.14}$$

Taking into account condition (5.8) and Corollary 4.2, we get from (5.14) that

$$\begin{aligned}&\sup_{0\leq t\leq T}\|V^k(t)\|^2_{X_{s,\delta}} \leq\\ &e^{Cc_0T}\left\{c_0^2\left(\|(f^k, g^k, \partial_x f^k) - (f, g, \partial_x f)\|^2_{X_{s,\delta}} + \|(f^0, g^0, \partial_x f^0) - (f, g, \partial_x f)\|^2_{X_{s,\delta}}\right) + Cc_0T\right\}\\ &\leq e^{Cc_0T}\left(\frac{R^2}{8} + Cc_0T\right) \leq R^2\end{aligned}$$

provided that $T \le T^* := \sup\{t : e^{Cc_0t}\left(\frac{R^2}{8} + Cc_0t\right) \le R^2\}$. □

Having shown the boundedness of $\{U^k\}$ we may conclude by the Compact embedding, Proposition 2.2:5, that $U^k \to U$ in the $X_{s',\delta'}$-norm for any $s' < s$ and $\delta' < \delta$. By the Mixed norm estimate 10, $u^k \to u$ in $H_{s'+1,\delta'}$ and if we chose $\frac{3}{2} < s' < s$, $-\frac{3}{2} < \delta' < \delta$, then the Embedding into the continuous 6 implies that

$$u^k(t) \to u(t) \qquad \text{in} \;\; C^1(\mathbb{R}^3),$$

$$\partial_t u^k(t) \to \partial_t u(t), \;\; \partial_x u^k(t) \to \partial_x u(t) \qquad \text{in} \;\; C(\mathbb{R}^3).$$

Therefore $U(t) = (u(t), \partial_t u(t), \partial_x u(t))$ is a solution to system (5.1) for $0 \le t \le T^*$.

5.3 Weak convergence

Here we show the weak converges of $\{U^k\}$ in $X_{s,\delta}$. We chose the simplest inner-product on $X_{s,\delta}$, that is, for $V = (v_1, v_2, v_3), \Phi = (\phi_1, \phi_2, \phi_3) \in X_{s,\delta}$, we set

$$\begin{aligned}\langle V, \Phi\rangle_{X_{s,\delta}} &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\langle \Lambda^s \left(\psi_j^2 v_1\right)_{2^j}, \Lambda^s \left(\psi_j^2 \phi_2\right)_{2^j} \right\rangle_{L^2} \\ &+ \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+1)2j} \left\langle \Lambda^s \left(\psi_j^2 v_2\right)_{2^j}, \Lambda^s \left(\psi_j^2 \phi_2\right)_{2^j} \right\rangle_{L^2} \\ &+ \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+1)2j} \left\langle \Lambda^s \left(\psi_j^2 v_3\right)_{2^j}, \Lambda^s \left(\psi_j^2 \phi_3\right)_{2^j} \right\rangle_{L^2}.\end{aligned} \tag{5.15}$$

This definition coincides with (4.7) in the case where $\mathcal{A}^0$ is the identity matrix.

Proposition 5.5 *Given $0 < s \le \frac{s'+s''}{2}$, $\delta \le \frac{\delta'+\delta''}{2}$, $V \in X_{s',\delta'}$ and $\Phi \in X_{s'',\delta''}$, then*

$$|\langle V, \Phi\rangle_{X_{s,\delta}}| \le \|V\|_{X_{s',\delta'}} \|\Phi\|_{X_{s'',\delta''}}. \tag{5.16}$$

The proof of Proposition 5.14 appears in [4], [3] with $\delta' = \delta'' = \delta$. Only a slight modification of this proof is needed in order to include it to (5.16).Therefore we leave it to the reader.

Lemma 5.6 ***(Weak Convergence)*** *For any $\Phi \in X_{s,\delta}$,*

$$\lim_k \left\langle U^k(t), \Phi \right\rangle_{X_{s,\delta}} = \langle U(t), \Phi\rangle_{X_{s,\delta}} \tag{5.17}$$

uniformly for $0 \le t \le T^*$. Consequently

$$\|U(t)\|_{X_{s,\delta}} \le \liminf_k \|U^k(t)\|_{X_{s,\delta}} \tag{5.18}$$

and hence the solution $U(t)$ of the initial value problem (5.1) belongs to $C_w([0, T^*], X_{s,\delta})$, where C_w denotes the space of functions which are continuous in the weak topology.

Proof (of Lemma 5.6)**.** We recall that $\|U^k(t) - U(t)\|_{H_{s',\delta'}} \to 0$ for $s' < s$ and $\delta' < \delta$. We can pick now s'' and δ'' such that $s < s''$, $s < \frac{s'+s''}{2}$, $\delta < \delta''$ and $\delta < \frac{\delta'+\delta''}{2}$. Given $\Phi \in X_{s,\delta}$ and $\epsilon > 0$, we may find, by Proposition 2.2:9, $\Phi_\epsilon \in X_{s'',\delta''}$ such that

$$\|\Phi - \Phi_\epsilon\|_{X_{s,\delta}} \leq \frac{\epsilon}{2R} \quad \text{and} \quad \|\Phi_\epsilon\|_{X_{s'',\delta''}} \leq C(\epsilon)\|\Phi\|_{X_{s,\delta}}, \tag{5.19}$$

where R is the constant of (5.11). Writing

$$\begin{aligned} \langle U^k(t) - U(t), \Phi\rangle_{X_{s,\delta}} &= \langle U^k(t) - U(t), \Phi_\epsilon\rangle_{X_{s,\delta}} \\ &\quad + \langle U^k(t) - U(t), (\Phi - \Phi_\epsilon)\,\rangle_{X_{s,\delta}} =: I_k + II_k, \end{aligned} \tag{5.20}$$

we have by Proposition 5.5 and (5.19) that

$$|I_k| \leq \|U^k(t) - U(t)\|_{X_{s',\delta'}} C(\epsilon)\|\Phi\|_{X_{s,\delta}} \to 0.$$

As to the second term of (5.20), since $\|U^k(t) - U(t)\|_{X_{s,\delta}} \leq 2R$ by (5.11), we get from the Cauchy-Schwarz inequality and (5.19) that

$$|II_k| \leq \|U^k(t) - U(t)\|_{X_{s,\delta}}\|\Phi - \Phi_\epsilon\|_{X_{s,\delta}} \leq \frac{2R\epsilon}{2R} = \epsilon.$$

Thus,

$$\limsup_k |\langle U^k(t) - U(t), \Phi\rangle_{X_{s,\delta}}| \leq \epsilon$$

and this completes the proof of Lemma 5.6. □

5.4 Uniqueness

Lemma 5.7 ***(Uniqueness)*** *Suppose $U(t), V(t) \in X_{s,\delta}$ are solutions to the first order symmetric hyperbolic system (5.1) with initial data (f, g) which satisfy (5.2), then $U(t) \equiv V(t)$.*

Proof (of Lemma 5.7)**.** Put $W(t) = U(t) - V(t)$, then it satisfies the linear equation

$$\begin{cases} \mathcal{A}^0(u)\partial_t W = \sum_{a=1}^3 (\mathcal{A}^a(u) + \mathcal{C}^a)\,\partial_a W + \mathcal{B}(U)W + \mathcal{F} \\ W(0, x) = 0 \end{cases}, \tag{5.21}$$

where

$$\mathcal{F} = \left(\mathcal{A}^0(u) - \mathcal{A}^0(v)\right)\partial_t V + \sum_{a=1}^3 (\mathcal{A}^a(u) - \mathcal{A}^a(v))\,\partial_a V + (\mathcal{B}(U) - \mathcal{B}(V))\,V. \tag{5.22}$$

Since $U \in X_{s,\delta}$, $\mathcal{A}^\alpha(u)$, $\partial_\beta \mathcal{A}^\alpha(u)$ and $\mathcal{B}(U)$ are bounded, we can apply Lemma 4.5 and obtain

$$\frac{d}{dt}\|W(t)\|^2_{Y^2_\delta, \mathbf{a}^0_{33}(u)} \leq Cc_0\left(\|W(t)\|^2_{Y^2_\delta, \mathbf{a}^0_{33}(u)} + \|\mathcal{F}\|^2_{L^2_{\delta+1}}\right) \tag{5.23}$$

(See (4.62) for the definition of the norm $\|W\|^2_{Y^2_\delta, \mathbf{a}^0_{33}(u)}$).

We turn now to the estimation of $\|\mathcal{F}\|^2_{L_{\delta+1}}$ in terms of the difference $\|U-V\|_{Y,\delta,\mathbf{a}^0_{33}(u)}$. From the structure of the matrices $\mathcal{A}^\alpha(u)$ in Assumptions 5.1, we see that $\left(\mathcal{A}^0(u)-\mathcal{A}^0(v)\right)\partial_t V = \left(\mathbf{a}^0_{33}(u)-\mathbf{a}^0_{33}(v)\right)\partial_t\partial_x v$ and

$$(\mathcal{A}^a(u)-\mathcal{A}^a(v))\,\partial_a V = \left(\widetilde{\mathcal{A}}^a(u)-\widetilde{\mathcal{A}}^a(v)\right)\partial_a\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix},$$

where

$$\widetilde{\mathcal{A}}^a(p)=\begin{pmatrix}\mathbf{a}^a_{22}(p) & \mathbf{a}^a_{23}(p)\\ \mathbf{a}^a_{32}(p) & \mathbf{a}^a_{33}(p)\end{pmatrix},\qquad a=1,2,3.$$

Our idea is to use inequality (2.6) with $s=0$, $s_1=1$ and $s_2=s-1$ and then to apply the Difference estimate 8 of Proposition 2.2. We also note that by Proposition 2.2:2, $\|u-v\|^2_{H_{1,\delta}}\simeq \|u-v\|^2_{L^2_\delta}+\|\partial_x(u-v)\|^2_{L^2_{\delta+1}}\le\|U-V\|^2_{Y_\delta}$. These yield the following estimations:

$$\begin{aligned}
&\left\|\left(\mathbf{a}^0_{33}(u)-\mathbf{a}^0_{33}(v)\right)\partial_t\partial_x v\right\|^2_{L^2_{\delta+1}}\simeq\left\|\left(\mathbf{a}^0_{33}(u)-\mathbf{a}^0_{33}(v)\right)\partial_t\partial_x v\right\|^2_{H_{0,\delta+1}}\\
&\le C\left\|\left(\mathbf{a}^0_{33}(u)-\mathbf{a}^0_{33}(v)\right)\right\|^2_{H_{1,\delta}}\|\partial_t\partial_x v\|^2_{H_{s-1,\delta+2}}\\
&\le C(\|u\|_{H_{s+1,\delta}},\|v\|_{H_{s+1,\delta}})\,\|u-v\|^2_{H_{1,\delta}}\,\|\partial_t v\|^2_{H_{s,\delta+1}}\\
&\le C(\|u\|_{H_{s+1,\delta}},\|v\|_{H_{s+1,\delta}})\,\|V\|^2_{X_{s,\delta}}\,\|U-V\|^2_{Y_\delta}.
\end{aligned}\tag{5.24}$$

Similarly,

$$\begin{aligned}
&\left\|\left(\widetilde{\mathcal{A}}^a(u)-\widetilde{\mathcal{A}}^a(v)\right)\partial_a\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right\|_{L^2_{\delta+1}}\\
&\le C(\|u\|_{H_{s+1,\delta}},\|v\|_{H_{s+1,\delta}})\|u-v\|^2_{H_{1,\delta}}\left(\|\partial_t v\|^2_{H_{s,\delta+1}}+\|\partial_x v\|^2_{H_{s,\delta+1}}\right)\\
&\le C(\|u\|_{H_{s+1,\delta}},\|v\|_{H_{s+1,\delta}})\,\|V\|^2_{X_{s,\delta}}\,\|U-V\|^2_{Y_\delta}.
\end{aligned}\tag{5.25}$$

Writing $\mathcal{B}=\mathcal{B}(p,q,r)$, then by Assumptions 5.1:5 we have

$$(\mathcal{B}(U)-\mathcal{B}(V))\,V=\partial_t(u-v)\cdot\nabla_q\mathcal{B}V+\partial_x(u-v)\cdot\nabla_r\mathcal{B}V.$$

Hence the simple weighted L^2 estimate gives

$$\|\partial_t(u-v)\nabla_q\mathcal{B}V\|^2_{L^2_{\delta+1}}\le\|\nabla_q\mathcal{B}\|^2_{L^\infty}\,\|\partial_t(u-v)\|^2_{L^2_{\delta+1}}\,\|V\|^2_{L^\infty}\tag{5.26}$$

and

$$\|\partial_x(u-v)\nabla_r\mathcal{B}V\|^2_{L^2_{\delta+1}}\le\|\nabla_r\mathcal{B}\|^2_{L^\infty}\,\|\partial_x(u-v)\|^2_{L^2_{\delta+1}}\,\|V\|^2_{L^\infty}.\tag{5.27}$$

Thus, inequalities (5.24)-(5.27) with the equivalence (4.63) show that

$$\|\mathcal{F}\|_{L^2_{\delta+1}}\le C\,\|V\|_{X_{s,\delta}}\,\|U-V\|_{Y_\delta}\le C\,\|V\|_{X_{s,\delta}}\,\|U-V\|_{Y_\delta,\mathbf{a}^0_{33}(u)}.\tag{5.28}$$

Inserting (5.28) in (5.23) and using Gronwall's inequality we get that

$$\|W(t)\|^2_{Y_{\delta,\mathbf{a}^0_{33}(u)}}\le e^{Cc_0t}\,\|W(0)\|^2_{Y_{\delta,\mathbf{a}^0_{33}(u)}}$$

and since $W(0)=0$, it implies that $W(t)\equiv 0$. □

5.5 Continuation in the norm

Lemma 5.8 ***(Continuation in the norm)*** *Let $U(t)$ be a solutions to the first order symmetric hyperbolic system (5.1) with initial data (f, g) which satisfy (5.2), then (5.3) holds.*

Proof (of Lemma 5.8)**.** Since $X_{s,\delta}$ is a Hilbert space it suffices to show that

$$\limsup_{t\to 0^+} \|U(t)\|_{X_{s,\delta,\mathcal{A}^0(f)}} \le \|U(0)\|_{X_{s,\delta,\mathcal{A}^0(f)}}.$$

Having proved the uniqueness, we may assume that U is the limit of the iteration sequence U^k. Furthermore, since $u^k(t) \to u(t)$ uniformly in $[0, T^*]$ and the matrix $\mathcal{A}^0$ depends solely on u, we see from the inner product (4.7) that for a given $\epsilon > 0$ there is a positive integer k_0 such that

$$\|V\|_{X_{s,\delta,\mathcal{A}^0(u(t))}} \le (1+\epsilon)\|V\|_{X_{s,\delta,\mathcal{A}^0(u^k(t))}}, \qquad k \ge k_0, \quad V \in X_{s,\delta}. \tag{5.29}$$

Using the fact that $u(t,\cdot) \to f(\cdot)$ uniformly as $t \to 0$, Lemmas (5.6) and (4.3), and (5.8) we get

$$\begin{aligned}
\limsup_{t\to 0^+} \|U(t)\|^2_{X_{s,\delta,\mathcal{A}^0(f)}} &= \limsup_{t\to 0^+} \|U(t)\|^2_{X_{s,\delta,\mathcal{A}^0(u(t))}} \\
&\le \limsup_{t\to 0^+}\left(\liminf_k \left\|U^{k+1}(t)\right\|^2_{X_{s,\delta,\mathcal{A}^0(u(t))}}\right) \\
&\le (1+\epsilon)^2 \limsup_{t\to 0^+}\left(\liminf_k \left\|U^{k+1}(t)\right\|^2_{X_{s,\delta,\mathcal{A}^0(u^k(t))}}\right) \\
&\le (1+\epsilon)^2 \limsup_{t\to 0^+}\left(\liminf_k e^{Cc_0 t}\left(\left\|U^{k+1}(0)\right\|^2_{X_{s,\delta,\mathcal{A}^0(u^k(0))}} + Cc_0 t\right)\right) \\
&\le (1+\epsilon)^2 \|U(0)\|^2_{X_{s,\delta,\mathcal{A}^0(f)}}.
\end{aligned}$$

This completes the proof of the Lemma and thereby of Theorem 5.2. □

6 Proof of the main result

The solution of the constraint equations (1.5) in the weighted Sobolev spaces of fractional order $H_{s,\delta}$ has been proved by Maxwell [22] for $s > \frac{1}{2}$ and Brauer and Karp for ≥ 1 [3] (see also [2]). Thus for a given set of free data $(\bar{\mathbf{h}}_{ab}, \bar{\mathbf{K}}_{ab})$ such that $(\bar{\mathbf{h}}_{ab} - \mathbf{e}_{ab}, \bar{\mathbf{K}}_{ab}) \in H_{s+1.\delta} \times H_{s,\delta+1}$, there is conformally equivalent data $(\mathbf{h}_{ab}, \mathbf{K}_{ab})$ which satisfies the constraint equations (1.5). Moreover, there is a constant C such that

$$\|(\mathbf{h}_{ab} - \mathbf{e}_{ab}, \mathbf{K}_{ab})\|_{H_{s+1,\delta}\times H_{s,\delta+1}} \le C\left\|(\bar{\mathbf{h}}_{ab} - \mathbf{e}_{ab}, \bar{\mathbf{K}}_{ab})\right\|_{H_{s+1,\delta}\times H_{s,\delta+1}}. \tag{6.1}$$

We apply now Theorem 5.2 to $(\mathbf{g}_{\alpha\beta} - \mathbf{m}_{\alpha\beta}, \partial_t \mathbf{g}_{\alpha\beta}, \partial_x \mathbf{g}_{\alpha\beta})$ with initial data (1.2) and where the pair $(\mathbf{h}_{ab}, \mathbf{K}_{ab})$ satisfies the constraint equations (1.5). Then $\mathbf{g}_{\alpha\beta}(t)$ is the unique solution to the reduced Einstein equation (1.4) and (1.13) holds by Remark 5.3. Inequality (1.14) follows from

(6.1) since for $t \in [0,T]$ the bounds of $\|\mathbf{g}_{\alpha\beta}(t) - \mathbf{m}_{\alpha\beta}\|_{H_{s+1,\delta}}$ and $\|\partial_t \mathbf{g}_{\alpha\beta}(t)\|_{H_{s,\delta+1}}$ depend solely on the initial data $(\mathbf{h}_{ab}, \mathbf{K}_{ab})$.

In order to assure that $\mathbf{g}_{\alpha\beta}(t)$ satisfies the vacuum Einstein equation (1.1) we need to establish the harmonic condition (1.3). Recalling that F^μ satisfies the linear wave equation

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta F^\mu - \Gamma^\nu_{\alpha\beta}\mathbf{g}^{\alpha\beta}\partial_\nu F^\nu = 0 \tag{6.2}$$

(see e.g. [10], [27]), it thus suffices to show that $F^\mu(0,x) = \partial_t F^\mu(0,x) = 0$. Hence by the uniqueness of linear hyperbolic systems, it follows that $F^\mu \equiv 0$. Note that $\mathbf{g}^{\alpha\beta} - \mathbf{e}^{\alpha\beta} \in H_{s+1,\delta}$, $\Gamma^\nu_{\alpha\beta}\mathbf{g}^{\alpha\beta} \in H_{s,\delta+1}$ and $s > \frac{3}{2}$, therefore these facts allow us to use known uniqueness results for linear hyperbolic symmetric systems with coefficients in H^s [15], [19], or alternatively, we apply the L^2_δ-energy estimate Lemma 4.5, combined with Gronwall's inequality. We can now use the free data $\partial_t \mathbf{g}_{0\alpha}$ to get the condition $F^\mu(0,x) = 0$. Then exploiting the fact that $(\mathbf{h}_{ab}, \mathbf{K}_{ab})$ satisfies the constraint equations (1.5) leads to the second condition $\partial_t F^\mu(0,x) = 0$, see e.g. [1], [27].

Acknowledgement: I wish to acknowledge Uwe Brauer for many long and helpful discussions and to Greg Galloway who enlightened the uniqueness issue for me. Part of this project was done during the author's visit at the Department of Mathematics at Potsdam Universität and I would like to thank Professor B.-W. Schulze for his support and kind hospitality.

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