GEVREY HYPOELLIPTICITY FOR LINEAR AND NON-LINEAR FOKKER-PLANCK EQUATIONS

Hua Chen[1] and Wei-Xi Li[1] and Chao-Jiang Xu[1,2]

[1]School of Mathematics and Statistics, Wuhan University,
Wuhan 430072, China
email: chenhua@whu.edu.cn
weixi.li@yahoo.com

[2]Universitéde Rouen, UMR 6085-CNRS, Mathématiques
Avenue de l'Université, BR.12,76801 Saint Etienne du Rouvray, France
email: Chao-Jiang.Xu@univ-rouen.fr

Abstract. This paper studies the Gevrey regularity of weak solutions of a class of linear and semilinear Fokker-Planck equations.

1. Introduction

Much attention has been paid to the study of the spatially homogeneous Boltzmann equation without the angular cut-offs in recent years (see [2, 3, 8, 22] and references therein). These studies demonstrate that the singularity of the collision cross-section improves the regularity on weak solutions for the Cauchy problem. For instance, one can obtain, from these studies, the C^∞ regularity of weak solutions for the spatially homogeneous Boltzmann operator when there are no angular cut-offs. In the local setting, the Gevrey regularity has been first proved in [21] for the initial data that has the same Gevrey regularity. A more general result on the Gevrey regularity is obtained in [17] for the spatially homogeneous linear Boltzmann equation with any initial Cauchy data. Hence, one sees a similar smoothness effect for the homogeneous Boltzmann equations as in the case of the heat equation.

The consideration for the inhomogeneous equation seems to be a relatively open field. There is no general result in this study yet. A recent work in [1] investigated a kinetic equation with the diffusion coefficient as a nonlinear function of the velocity variable. In [1], making us of the uncertainty principle and microlocal analysis, a C^∞ regularity result was obtained when there is no angular cut-off in the linear spatially inhomogeneous Boltzmann equation.

In this paper, we study the Gevrey regularity of the weak solutions for the following Fokker-Planck operator in $\mathbb{R}^{2n+1}$

$$\mathcal{L} = \partial_t + v\cdot\partial_x - a(t,x,v)\triangle_v, \tag{1.1}$$

Research partially supported by the NSFC.

where $\triangle_v$ is the Laplace operator in the velocity variables v and $a(t,x,v)$ is a strictly positive function in $\mathbb{R}^{2n+1}$.

The motivation of studying such an equation is dependent on the study of inhomogenous Boltzmann equation without angular cutoffs, Landau equation (see [15]) and a non-linear Vlasov-Fokker-Planck equation (see [11, 12]).

To state our main results, we first recall the definition of Gevrey class functions. Let U be an open subset of $\mathbb{R}^N$ and f be a real function defined in U. We say $f \in G^s(U)(s \geq 1)$ if $f \in C^\infty(U)$ and for any compact subset K of U, there exists a constant $C = C_K$, depending only on K, such that for all multi-indices $\alpha \in \mathbb{N}^N$ and for all $x \in K$

$$|\partial^\alpha f(x)| \leq C_K^{|\alpha|+1}(|\alpha|!)^s. \tag{1.2}$$

Denote by $\bar{U}$ the closure of U in $\mathbb{R}^N$. we say $f \in G^s(\bar{U})$ if $f \in G^s(W)$ for some open neighborhood W of $\bar{U}$. The estimate (1.2) for $x \in K$ is equivalent to the following L^2-estimate (See, for instance, Chen-Rodino[5, 6] or Rodino[18]):

$$\|\partial^\alpha f\|_{L^2(K)} \leq C_K^{|\alpha|+1}(|\alpha|)^{s|\alpha|}.$$

In what follows, we shall use the definition based on the above L^2 estimate for the Gevrey functions.

We say that an operator P is G^s-hypoelliptic in U if for any $u \in \mathcal{D}'$ and $Pu \in G^s(U)$ it then holds that $u \in G^s(U)$. Likewise, we say an operator P is C^∞ hypoelliptic in U if for any $u \in \mathcal{D}'$ and $Pu \in C^\infty(U)$ it then holds that $u \in C^\infty(U)$.

When the operator $\mathcal{L}$ satisfies the well-known Hörmander condition, then a famous result of Hörmander [13] says that $\mathcal{L}$ is C^∞ hypoelliptic. In the aspect of the Gevrey class, Derridj-Zuily [7] studied the G^s-hypoellipticity for the second order degenerate operators of Hörmander type, and proved that $\mathcal{L}$ is G^s-hypoelliptic when $s > 6$.

In this paper, we first improve the result in [7] for the Fokker-Planck operator (1.1). In fact, similar to the result of [19], we have obtained the following optimal estimate for Gevrey index $s \geq 3$:

Theorem 1.1. *For any $s \geq 3$, if the positive coefficient $a(t,x,v)$ is in $G^s(\mathbb{R}^{2n+1})$, then the operator $\mathcal{L}$ given in (1.1) is G^s-hypoelliptic in $\mathbb{R}^{2n+1}$.*

Remark: A. Our proof of Theorem 1.1 actually shows that the result in Theorem 1.1 holds even for the following more general operators:

$$\tilde{\mathcal{L}} = \partial_t + A(v) \cdot \partial_x - \sum_{j,k=1}^{n} a_{jk}(t,x,v)\partial^2_{v_j v_k},$$

defined over a domain U in $\mathbb{R}^{2n+1}$. Here, A is a non-singular $n \times n$ constant matrix, $\big(a_{jk}(t,x,v)\big)$ is a positive definite matrix over U with all entries being in the $G^s(U)$-class.

B. Our result in Theorem 1.1 is of the local nature. Namely, if there exists a weak solution in $\mathcal{D}'$, then this solution is in the Gevrey class in the interior of the domain. Hence, interior regularity of a weak solution does not depend much on the regularity of the initial Cauchy date.

Our second result is concerned with the Gevrey regularity of a non-linear version of (1.1). We consider the following semi-linear equation:

$$\mathcal{L}u = \partial_t u + v \cdot \nabla_x u - a(t,x,v)\triangle_v u = F(t,x,v,u,\nabla_v u), \tag{1.3}$$

where $F(t,x,v,w,p)$ is a non-linear function of real variables (t,x,v,w,p). We prove the following:

Theorem 1.2. *Let u be a weak solution of the equation (1.3). Assume that $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ and $\nabla_v u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$. Then*

$$u \in G^s(\mathbb{R}^{2n+1})$$

for any $s \geq 3$, if the positive coefficient $a(t,x,v) \in G^s(\mathbb{R}^{2n+1})$ and the nonlinear function $F(t,x,v,w,p) \in G^s(\mathbb{R}^{2n+2+n})$.

Remark: **C**. If the non-linear term $F(t,x,v,w,p)$ is independent of p or F is of the form: $\nabla_v G(t,x,v,u)$, then it is enough to suppose in Theorem 1.2 that the weak solution $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$.

The paper is organized as follows : In Section 2, we obtain a sharp subelliptic estimate for the Fokker-Planck operator $\mathcal{L}$ via a direct computation. We then prove the Gevrey hypoellipticity of $\mathcal{L}$. In Section 3, we prove the Gevrey regularity for the weak solution of the semi-linear Fokker-Planck equation (1.3).

2. Subelliptic estimates

As usual, we write $\|\cdot\|_\kappa, \kappa \in \mathbb{R}$, for the classical Sobolev norm in $H^\kappa(\mathbb{R}^{2n+1})$, and (h,k) for the inner product of $h,\ k \in L^2(\mathbb{R}^{2n+1})$. For $f,\ g \in C_0^\infty(\mathbb{R}^{2n+1})$, by the Hölder and Young inequality, we have that for any $\varepsilon > 0$,

$$|(f,\ g)| \leq \|h\|_\kappa \|g\|_{-\kappa} \leq \frac{\varepsilon\|h\|_\kappa^2}{2} + \frac{\|g\|_{-\kappa}^2}{2\varepsilon}. \tag{2.1}$$

We also recall the following interpolation inequality in the Sobolev space: For any $\varepsilon > 0$ and $r_1 < r_2 < r_3$, it holds that

$$\|h\|_{r_2} \leq \varepsilon\|h\|_{r_3} + \varepsilon^{-(r_2-r_1)/(r_3-r_2)}\|h\|_{r_1}. \tag{2.2}$$

Let Ω be an open subset of $\mathbb{R}^{2n+1}$. We denote by $S^m = S^m(\Omega), m \in \mathbb{R}$, the symbol space of the classical pseudo-differential operators and $P = P(t,x,v,D_t,D_x,D_v) \in \mathrm{Op}(S^m)$ a pseudo-differential operator of symbol $p(t,x,v;\tau,\xi,\eta) \in S^m$. If $P \in \mathrm{Op}(S^m)$, then P is a continuous operator from $H^\kappa_c(\Omega)$ to $H^{\kappa-m}_{loc}(\Omega)$. Here $H^\kappa_c(\Omega)$ is the subspace of $H^\kappa(R^{2n+1})$ consisting of the distributions having their compact support in Ω, and $H^{\kappa-m}_{loc}(\Omega)$ consists of the distributions h such that $\phi h \in H^{\kappa-m}(\mathbb{R}^{2n+1})$ for any $\phi \in C_0^\infty(\Omega)$. For more properties concerning the pseudo-differential operators, we refer the reader to the book [20]. Observe that if $P_1 \in \mathrm{Op}(S^{m_1})$, $P_2 \in \mathrm{Op}(S^{m_2})$, then $[P_1,\ P_2] \in \mathrm{Op}(S^{m_1+m_2-1})$.

We next prove a sharp sub-elliptic estimate for the operator $\mathcal{L}$. Our proof is based on the work of Bouchut [4] and Morimoto-Xu [15].

Proposition 2.1. *Let K be a compact subset of $\mathbb{R}^{2n+1}$. Then for any $r \geq 0$, there exists a constant $C_{K,r}$, depending only on K and r, such that for any $f \in C_0^\infty(K)$,*

$$\|f\|_r \leq C_{K,r}\{ \ \|\mathcal{L}f\|_{r-2/3} + \|f\|_0 \ \}. \tag{2.3}$$

For brevity, we will write, in this section, C_K for a constant that may be different in a different context. We proceed with the following three lemmas, which establishes the regularity result in the variables v, x and t, respectively.

Lemma 2.2. *For any $r \geq 0$, there exists a constant $C_{K,r}$ such that for any $f \in C_0^\infty(K)$,*

$$\|\nabla_v f\|_r \leq C_{K,r}(\|\mathcal{L}f\|_r + \|f\|_r).$$

Lemma 2.1 indicates the regularity gain of order 1 in the variable v. It can be obtained directly by the positivity of the coefficient a and the compact supported property of f. For the space variable x, we have the following sub-elliptic estimate:

Lemma 2.3. *There exists a constant C_K such that for any $f \in C_0^\infty(K)$,*

$$\|D_x^{2/3} f\|_0 \leq C_K(\|\mathcal{L}f\|_0 + \|f\|_0),$$

where $D_x^{2/3} = (-\triangle_x)^{1/3}$.

This result is due to [4]. It follows from the estimates:

$$\|D_x^{2/3} f\|_0 \leq C_K \ \|\triangle_v f\|_0^{1/3} \|\partial_t f + v \cdot \partial_x f\|_0^{2/3},$$

and

$$\|\triangle_v f\|_0 \leq C_K(\ \|\mathcal{L}f\|_0 + \|f\|_0 \ ).$$

For the time variable t, we have the regularity result of order 2/3, namely, we have the following:

Lemma 2.4. *There exists a constant C_K such that for any $f \in C_0^\infty(K)$,*

$$\|\partial_t f\|_{-1/3} \leq C_K(\|\mathcal{L}f\|_0 + \|f\|_0).$$

In fact, we have

$$\|\partial_t f\|_{-1/3} = \|\Lambda^{-1/3}\partial_t f\|_0 \leq \|\Lambda^{-1/3}(\partial_t + v \cdot \partial_x) f\|_0 + \|\Lambda^{-1/3} v \cdot \partial_x f\|_0,$$

where $\Lambda = (1 + |D_t|^2 + |D_x|^2 + |D_v|^2)^{1/2}$. From Lemma 2.3, we have

$$\|\Lambda^{-1/3} v \cdot \partial_x f\|_0 \leq C_K \|D_x^{2/3} f\|_0 \leq C_K(\|\mathcal{L}f\|_0 + \|f\|_0).$$

The estimate for the term $\|\Lambda^{-1/3}(\partial_t + v \cdot \partial_x) f\|_0$ can be obtained by a direct computation as in [15].

Proof of Proposition 2.1. By Lemma 2.2, Lemma 2.3 and Lemma 2.4, we have

$$\|f\|_{2/3} \ \ \leq \ \ C_K\{ \ \|\mathcal{L}f\|_0 + \|f\|_0 \ \}. \tag{2.4}$$

Moreover, choose a function $\psi \in C_0^\infty(\mathbb{R}^{2n+1})$ with $\psi|_K \equiv 1$ and supp ψ being contained in a neighborhood of K. Then, for any $f \in C_0^\infty(K)$ and $r \geq 0$, we have

$$\|f\|_r \ \ = \ \ \|\psi f\|_r \leq C_K\{ \ \|\psi \Lambda^{r-2/3} f\|_{2/3} + \|[\Lambda^{r-2/3}, \ \psi] f\|_{2/3} \ \}.$$

By virtue of (2.4) and the interpolation inequality (2.2), we have

$$\begin{aligned}\|f\|_r &\leq C_K\{ \|\mathcal{L}\psi\Lambda^{r-2/3}f\|_0 + \|f\|_{r-2/3} \}\\ &\leq C_{\varepsilon,K}\{ \|\mathcal{L}\psi\Lambda^{r-2/3}f\|_0 + \|f\|_0 \} + \varepsilon\|f\|_r.\end{aligned}$$

Letting ε sufficiently small, we get

$$\|f\|_r \leq C_K\{ \|\mathcal{L}f\|_{r-2/3} + \|f\|_0 + \|[\mathcal{L},\ \psi\Lambda^{r-2/3}]f\|_0 \}.$$

Next, a direct calculation yields

$$\begin{aligned}[\mathcal{L},\ \psi\Lambda^{r-2/3}] &= [\partial_t + v\cdot\partial_x,\ \psi\Lambda^{r-2/3}] - \sum_{j=1}^n \{ [a,\ \psi\Lambda^{r-2/3}]\partial_{v_j}^2\\ &\qquad + a[\partial_{v_j},\ [\partial_{v_j}, \psi\Lambda^{r-2/3}]\ ] + 2a[\partial_{v_j},\ \psi\Lambda^{r-2/3}]\partial_{v_j} \}.\end{aligned}$$

From Lemma 2.2, it thus follows that

$$\begin{aligned}\|[\mathcal{L},\ \psi\Lambda^{r-2/3}]f\|_0 &\leq C_K\{ \|f\|_{r-2/3} + \sum_{j=1}^n \|\partial_{v_j}f\|_{r-2/3} \}\\ &\leq C_K\{ \|\mathcal{L}f\|_{r-2/3} + \|f\|_{r-2/3} \}.\end{aligned}$$

From the estimates above, we deduce that

$$\|f\|_r \leq C_K\{ \|\mathcal{L}f\|_{r-2/3} + \|f\|_0 + \|f\|_{r-2/3} \}.$$

Applying the interpolation inequality (2.2) again and making ε small enough, we see the proof of Proposition 2.1.

We next consider the commuting property of $\mathcal{L}$ with differential operators and cut-off functions.

Proposition 2.5. *Let K be a compact subset of $\mathbb{R}^{2n+1}$. Then for any $r \geq 0$, there are constants $C_{K,r}$, $C_{K,r,\varphi}$ such that for any $f \in C_0^\infty(K)$, we have*

$$\|[\mathcal{L},\ D]f\|_r \leq C_{K,r}\{ \|\mathcal{L}f\|_{r+1-2/3} + \|f\|_0 \},$$

and

$$\|[\mathcal{L},\ \varphi]f\|_r \leq C_{K,r,\varphi}\{ \|\mathcal{L}f\|_{r-1/3} + \|f\|_0 \}.$$

Here $\varphi \in C_0^\infty(\mathbb{R}^{2n+1})$ and D is one of the differential operators: ∂_t, ∂_x or ∂_v.

Proof. By using the positivity of the coefficient a, we have

$$\|\triangle_v f\|_r \leq C_K\{ \|\mathcal{L}f\|_r + \|f\|_{r+1} \}.$$

Notice that $[\mathcal{L},\ D] = [\partial_t + v\cdot\partial_x,\ D] - [a,\ D]\triangle_v$. We have

$$\|[\mathcal{L},\ D]f\|_r \leq C_K\{ \|f\|_{r+1} + \|\triangle_v f\|_r \}.$$

The first estimate of Proposition 2.5 is thus deduced by the two inequalities above and the sub-elliptic estimate (2.3).

To treat $\|[\mathcal{L},\ \varphi]f\|_r$, we use the sub-elliptic estimate (2.3), which gives

$$\|\nabla_v f\|_r \leq C_K(\|\mathcal{L}f\|_{r-1/3} + \|f\|_0).$$

Now a simple verification shows that

$$\begin{aligned}\|[\mathcal{L},\ \varphi]f\|_r &\leq C_K\{\ \|f\|_r + \sum_{j=1}^{n} \|\partial_{v_j} f\|_r\ \}\\ &\leq C_{K,r}\{\ \|\mathcal{L}f\|_{r-1/3} + \|f\|_0\ \}.\end{aligned}$$

This completes the proof of Proposition 2.5.

We are now at a position to prove the Gevrey hypoellipticity of $\mathcal{L}$. We need the following result due to M. Durand [9]:

Proposition 2.6. *Let P be a linear differential operator with smooth coefficients in $\mathbb{R}^N_y$ and let ϱ, ς be two fixed positive numbers. If for $r \geq 0$, compact subset $K \subseteq \mathbb{R}^N$ and $\varphi \in C^\infty(\mathbb{R}^N)$, there exist constants $C_{K,r}$ and $C_{K,r}(\varphi)$ such that for all $f \in C_0^\infty(K)$, the following conditions are fulfilled:*

$$(H_1)\qquad \|f\|_r \leq C_{K,r}(\|Pf\|_{r-\varrho} + \|f\|_0),$$

$$(H_2)\qquad \|[P,\ D_j]f\|_r \leq C_{K,r}(\|Pf\|_{r+1-\varsigma} + \|f\|_0),$$

$$(H_3)\qquad \|[P,\ \varphi]f\|_r \leq C_{K,r}(\varphi)(\|Pf\|_{r-\varsigma} + \|f\|_0),$$

where

$$D_j = \frac{1}{i}\frac{\partial}{\partial y_j}, j = 1, 2, \cdots, N.$$

Then for $s \geq \max(1/\varsigma, 2/\varrho)$, P is $G^s(\mathbb{R}^N)$ hypoelliptic, provided that the coefficients of P are in the class of $G^s(\mathbb{R}^N)$.

Proposition 2.1 shows that the operator $\mathcal{L}$ satisfies Condition (H_1) with $\varrho = 2/3$. Proposition 2.5 assures the conditions (H_2) and (H_3) with $\varsigma = 1/3$. Thus, $\mathcal{L}$ is $G^s(\mathbb{R}^{2n+1})$-hypoelliptic for $s \geq 3$. This completes the proof of Theorem 1.1.

3. Gevrey regularity of nonlinear equations

Let $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ be a weak solution of (1.3). We will prove $u \in C^\infty(\mathbb{R}^{2n+1})$. To this aim, we need the following nonlinear composition result (see for example [23]):

Lemma 3.1. *Let $F(t, x, v, w, p) \in C^\infty(\mathbb{R}^{2n+2+n})$ and $r \geq 0$. If $u, \nabla_v u \in L^\infty_{loc}(\mathbb{R}^{2n+1}) \cap H^r_{loc}(\mathbb{R}^{2n+1})$, then $F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big) \in H^r_{loc}(\mathbb{R}^{2n+1})$ with*

$$(3.1)\qquad \big\|\phi_1 F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big)\big\|_r \leq \bar{C}\ \{\ \|\phi_2 u\|_r + \|\phi_2 \nabla_v u\|_r\ \},$$

where $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^{2n+1})$, $\phi_2 = 1$ on the support of ϕ_1, and $\bar{C}$ is a constant depending only on r, ϕ_1, ϕ_2.

Remark: D. If the nonlinear term F is independent of p or in the form of

$$\nabla_v(F(t, x, v, u))$$

and if $u \in L^\infty_{loc}(\mathbb{R}^{2n+1}) \cap H^r_{loc}(\mathbb{R}^{2n+1})$, then it holds that $F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big) \in H^r_{loc}(\mathbb{R}^{2n+1})$.

Lemma 3.2. *Let $u, \nabla_v u \in H^r_{loc}(\mathbb{R}^{2n+1}), r \geq 0$. Then we have*

$$\|\varphi_1 \nabla_v u\|_r \leq C \|\varphi_2 u\|_r \,, \tag{3.2}$$

where $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^{2n+1})$, $\varphi_2 = 1$ on the support of φ_1, and C is a constant depending only on r, φ_1, φ_2.

In fact, we have

$$\|\varphi_1 \nabla_v u\|_r \leq \|[\nabla_v,\ \varphi_1]u\|_r + \|\nabla_v \varphi_1 u\|_r \,.$$

Clearly, the first term on the right is bounded by $C\|\varphi_2 u\|_r$. For the second term , combining the second inequality in Lemma 2.2 with (3.1), we see the desired estimate (3.2). This completes the proof of Lemma 3.2.

Now we are ready to prove

Proposition 3.3. *Let u be a weak solution of (1.3) such that $u, \nabla_v u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$. Then u is in $C^\infty(\mathbb{R}^{2n+1})$.*

In fact, from the subelliptic estimate (2.3) and the fact that $\mathcal{L}u(\cdot) = F(\cdot, u(\cdot), \nabla_v u(\cdot))$, it follows that

$$\|\psi_1 u\|_{r+2/3} \leq \bar{C}\{\ \|\psi_2 F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big)\|_r + \|\psi_2 u\|_0\ \}, \tag{3.3}$$

where $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^{2n+1})$ and $\psi_2 = 1$ on the support of ψ_1. Combining (3.1), (3.2) with (3.3), we have $u \in H^\infty_{loc}(\mathbb{R}^{2n+1})$ by the standard iteration prcedure. This completes the proof of Proposition 3.3.

Now starting from a smooth solution, we prove the Gevrey regularity. It suffices for us to work on the open unit ball

$$\Omega = \{(t, x, v) \in \mathbb{R}^{2n+1} : t^2 + |x|^2 + |v|^2 < 1\}.$$

Set

$$\Omega_\rho = \big\{(t, x, v) \in \Omega : \big(t^2 + |x|^2 + |v|^2\big)^{1/2} < 1 - \rho\big\}, \qquad 0 < \rho < 1.$$

Let U be an open subset of $\mathbb{R}^{2n+1}$. Denote by $H^r(U)$ the space consisting of the functions which are defined in U and can be extended to $H^r(\mathbb{R}^{2n+1})$. Define

$$\|u\|_{H^r(U)} = \inf\big\{\|\tilde{u}\|_{H^s(\mathbb{R}^{n+1})} : \tilde{u} \in H^s(\mathbb{R}^{2n+1}), \tilde{u}|_U = u\big\}.$$

We denote $\|u\|_{r,U} = \|u\|_{H^r(U)}$, and

$$\|D^j u\|_r = \sum_{|\beta|=j} \|D^\beta u\|_r.$$

In order to treat the nonlinear term F on the right hand of (1.3), we need the following two lemmas. The first one (see [23] for example) concerns weak solutions, and the second is an analogue of Lemma 1 in [10]. In the sequel, $C_j > 1$ will be used to denote constants depending only on n or the function F.

Lemma 3.4. *Let $r > (2n+1)/2$ and $u_1, u_2 \in H^r(\mathbb{R}^{2n+1})$. Then $u_1 u_2 \in H^r(\mathbb{R}^{2n+1})$, moreover*

$$\|u_1 u_2\|_r \leq \tilde{C}\|u_1\|_r \|u_2\|_r, \tag{3.4}$$

where $\tilde{C}$ is a constant depending only on n, r.

Lemma 3.5. *Let M_j be a sequence of positive numbers. Assume that for some $B_0 > 0$, the M_j satisfy the monotonicity condition:*

$$\frac{j!}{i!(j-i)!} M_i M_{j-i} \leq B_0 M_j, \ (i = 1, 2, \cdots, j; \ j = 1, 2, \cdots). \tag{3.5}$$

Suppose $F(t, x, v, u, p)$ satisfies that

$$\| \left(D^j_{t,x,v} D^l_u D^m_p F \right) (\cdot, u(\cdot), \nabla_v u(\cdot)) \|_{r+n+1,\Omega} \leq C_1^{j+l+m} M_{j-2} M_{m+l-2}, \quad j, m+l \geq 2, \tag{3.6}$$

where r is a real number satisfying $r + n + 1 > (2n+1)/2$. Then there exist two constants C_2, C_3 such that for any H_0, H_1 satisfying $H_0, H_1 \geq 1$ and $H_1 \geq C_2 H_0$, if $u(t, x, v)$ satisfies the following conditions

$$\|D^j u\|_{r+n+1,\Omega_{\tilde{\rho}}} \leq H_0, \quad 0 \leq j \leq 1, \tag{3.7}$$

$$\|D^j u\|_{r+n+1,\Omega_{\tilde{\rho}}} \leq H_0 H_1^{j-2} M_{j-2}, \quad 2 \leq j \leq N, \tag{3.8}$$

$$\|D_v D^j u\|_{r+n+1,\Omega_{\tilde{\rho}}} \leq H_0 H_1^{j-2} M_{j-2}, \quad 2 \leq j \leq N, \tag{3.9}$$

then for all α with $|\alpha| = N$,

$$\left\| \psi_N D^\alpha \left[F\left(\cdot, u(\cdot), \nabla_v u(\cdot)\right) \right] \right\|_{r+n+1} \leq C_3 H_0 H_1^{N-2} M_{N-2}, \tag{3.10}$$

where $\psi_N \in C_0^\infty(\Omega_{\tilde{\rho}})$ is an arbitrary function.

Proof of Lemma 3.5: Denote $p = (p_1, p_2, \cdots, p_n) = \nabla_v u$ and $k = (k_1, k_2 \cdots, k_n)$. From Faa di Bruno' formula, $\psi_N D^\alpha[F(\cdot, u(\cdot), \nabla_v u(\cdot))]$ is the linear combination of terms of the form

$$\frac{\psi_N \partial^{|\tilde{\alpha}|+l+|k|} F}{\partial^{\tilde{\alpha}}_{t,x,v} \partial u^l \partial p_1^{k_1} \cdots \partial p_n^{k_n}} \prod_{j=1}^l D^{\gamma_j} u \cdot \prod_{i=1}^n \prod_{j_i=1}^{k_i} D^{\beta_{j_i}} (\partial_{v_i} u), \tag{3.11}$$

where $|\tilde{\alpha}| + l + |k| \leq |\alpha|$ and

$$\sum_{j=1}^l \gamma_i + \sum_{i=1}^n \sum_{j_i}^{k_i} \beta_{j_i} = \alpha - \tilde{\alpha}.$$

Choose a function $\tilde{\psi} \in C_0^\infty(\Omega_{\tilde{\rho}})$ such that $\tilde{\psi} = 1$ on Supp ψ_N. Notice that $n + 1 + r > (2n+1)/2$. Applying Lemma 3.4, we have

$$\begin{aligned}
&\left\| \tfrac{\psi_N \partial^{|\tilde{\alpha}|+l+|k|} F}{\partial^{\tilde{\alpha}}_{t,x,v} \partial u^l \partial p_1^{k_1} \cdots \partial p_n^{k_n}} \textstyle\prod_{j=1}^l D^{\gamma_j} u \cdot \prod_{i=1}^n \ \prod_{j_i=1}^{k_i} D^{\beta_{j_i}} (\partial_{v_i} u) \right\|_{r+n+1} \\
= & \left\| \tfrac{\psi_N \partial^{|\tilde{\alpha}|+l+|k|} F}{\partial^{\tilde{\alpha}}_{t,x,v} \partial u^l \partial p_1^{k_1} \cdots \partial p_n^{k_n}} \textstyle\prod_{j=1}^l \tilde{\psi} D^{\gamma_j} u \cdot \prod_{i=1}^n \ \prod_{j_i=1}^{k_i} \tilde{\psi} \partial_{v_i} D^{\beta_{j_i}} u \right\|_{r+n+1} \\
\leq & \ \tilde{C} \left\| \psi_N (\partial^{|\tilde{\alpha}|+l+|k|} F) \right\|_{r+n+1} \cdot \textstyle\prod_{j=1}^l \left\| \tilde{\psi} D^{\gamma_j} u \right\|_{r+n+1} \times \prod_{i=1}^n \ \prod_{j_i=1}^{k_i} \left\| \tilde{\psi} \partial_{v_i} D^{\beta_{j_i}} u \right\|_{r+n+1} \\
\leq & \ C_0 \left\| (\partial^{|\tilde{\alpha}|+l+|k|} F) \right\|_{r+n+1,\Omega} \cdot \textstyle\prod_{j=1}^l \|D^{\gamma_j} u\|_{r+n+1,\Omega_{\tilde{\rho}}} \times \prod_{i=1}^n \ \prod_{j_i=1}^{k_i} \left\| \partial_{v_i} D^{\beta_{j_i}} u \right\|_{r+n+1,\Omega_{\tilde{\rho}}} .
\end{aligned} \tag{3.12}$$

With (3.7)-(3.9) and (3.12) at our disposal, our consideration is now similar to that in [10]. Indeed, the only difference is that we need to replace the Hölder norm $|u|_j$ by $\|D^j u\|_{r+n+1,\Omega_{\tilde\rho}}$ and $\left\|D_v D^j u\right\|_{r+n+1,\Omega_{\tilde\rho}}$. Hence, the same argument as the proof of Lemma 1 in [10] yields (3.10). This completes the proof of Lemma 3.5.

Proposition 3.6. *Let $s \geq 3$. Suppose $u \in C^\infty(\bar\Omega)$ is a solution of (1.3), $a(t,x,v) \in G^s(\mathbb{R}^{2n+1})$, $F(t,x,v,w,p) \in G^s(\mathbb{R}^{2n+2+n})$ and $a \geq c_0 > 0$. Then there is a constant A such that for any $r \in [0,1]$ and any $N \in \mathbb{N}$, $N \geq 3$,*

$$(E)_{r,N} \qquad \begin{aligned} & \|D^\alpha u\|_{r+n+1,\Omega_\rho} + \|D_v D^\alpha u\|_{r-1/3+n+1,\Omega_\rho} \\ \leq\ & \frac{A^{|\alpha|-1}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{sr}, \quad \forall\ |\alpha| = N,\ \forall\ 0<\rho<1. \end{aligned}$$

From $(E)_{r,N}$, we immediately obtain

Proposition 3.7. *Under the same assumption as in Proposition 3.6, we have $u \in G^s(\Omega)$.*

In fact, for any compact sunset K of Ω, we have $K \subset \Omega_{\rho_0}$ for some ρ_0 with $0<\rho_0<1$. For any α with $|\alpha| \geq 3$, letting $r=0$ in $(E)_{r,N}$, we have

$$\|D^\alpha u\|_{L^2(K)} \ \leq\ \|D^\alpha u\|_{n+1,\Omega_{\rho_0}} \leq \tfrac{A^{|\alpha|-1}}{{\rho_0}^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s \leq \big(\tfrac{A}{{\rho_0}^s}\big)^{|\alpha|+1}(|\alpha|!)^s.$$

This completes the proof of Proposition 3.7.

The result of Theorem 1.2 can be directly deduced from Proposition 3.3 and Proposition 3.7.

Proof of Proposition 3.6. We apply an induction argument on N. Assume that $(E)_{r,N-1}$ holds for any r with $0 \leq r \leq 1$. We will show that $(E)_{r,N}$ still holds for any $r \in [0,\ 1]$. For an α with $|\alpha| = N$, and for a ρ with $0<\rho<1$, choose a function $\varphi_{\rho,N} \in C_0^\infty(\Omega_{\frac{(N-1)\rho}{N}})$ such that $\varphi_{\rho,N} = 1$ in Ω_ρ. It is easy to see that

$$\sup|D^\gamma \varphi_{\rho,N}| \leq C_\gamma (\rho/N)^{-|\gamma|} \leq C_\gamma (N/\rho)^{|\gamma|}, \quad \forall\ \gamma.$$

We will verify the estimate in $(E)_{r,N}$ by the following lemmas.

Lemma 3.8. *For $r=0$, we have*

$$\|D^\alpha u\|_{n+1,\Omega_\rho} + \|D_v D^\alpha u\|_{-1/3+n+1,\Omega_\rho} \leq \frac{C_7 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s, \quad \forall\ 0<\rho<1.$$

Proof of Lemma 3.8: Write $|\alpha| = |\beta|+1$. Then $|\beta| = N-1$. Denote $\frac{N-1}{N}\rho$ by $\tilde\rho$. In the sequel, we will often apply the following inequalities:

$$\frac{1}{\rho^{sk}} \leq \frac{1}{\tilde\rho^{sk}} = \frac{1}{\rho^{sk}} \times \big(\frac{N}{N-1}\big)^{sk} \leq \frac{C_4}{\rho^{sk}}, \quad k = 1,2,\cdots,N-3.$$

Notice that $\varphi_{\rho,N} = 1$ in Ω_ρ. Hence

$$\begin{aligned} \|D^\alpha u\|_{n+1,\Omega_\rho} \ &\leq\ \|\varphi_{\rho,N} D^\alpha u\|_{n+1} \leq \|\varphi_{\rho,N} D^\beta u\|_{1+n+1} + \|(D\varphi_{\rho,N}) D^\beta u\|_{n+1} \\ &\leq\ C_5\{\ \|D^\beta u\|_{1+n+1,\Omega_{\tilde\rho}} + (N/\rho)\|D^\beta u\|_{n+1,\Omega_{\tilde\rho}}\ \}. \end{aligned}$$

Since $(E)_{r,N-1}$ holds by assumption for any r with $0 \le r \le 1$, we have immediately

$$\begin{aligned} & \|D^\beta u\|_{1+n+1,\Omega_{\tilde\rho}} + (N/\rho)\|D^\beta u\|_{n+1,\Omega_{\tilde\rho}} \\ \le\ & \frac{A^{|\beta|-1}}{\tilde\rho^{s(|\beta|-3)}}\big((|\beta|-3)!\big)^s (N/\tilde\rho)^s + (N/\rho)\frac{A^{|\beta|-1}}{\tilde\rho^{s(|\beta|-3)}}\big((|\beta|-3)!\big)^s \\ \le\ & \frac{2A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s \big(N/(N-3)\big)^s \\ \le\ & \frac{C_6 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s. \end{aligned}$$

Thus

$$\|D^\alpha u\|_{n+1,\Omega_\rho} \le \frac{C_5 C_6 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s. \tag{3.13}$$

The same argument as above shows that

$$\|D_v D^\alpha u\|_{-1/3+n+1,\Omega_\rho} \le \frac{C_5 C_6 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.$$

This along with (3.13) yields the conclusion.

Lemma 3.9. *For $0 \le r \le 1/3$, we have*

$$\|D^\alpha u\|_{r+n+1,\Omega_\rho} + \|D_v D^\alpha u\|_{r-1/3+n+1,\Omega_\rho} \le \frac{C_{35} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{rs}, \quad \forall\ 0<\rho<1.$$

Proof of Lemma 3.9: We first verify Lemma 3.9 for $r = 1/3$, namely, we first show that

$$\|D^\alpha u\|_{1/3+n+1,\Omega_\rho} + \|D_v D^\alpha u\|_{n+1,\Omega_\rho} \le \tfrac{C_{35} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{s/3}, \quad \forall\ 0<\rho<1.$$

We divide our discussions in the following four steps.

Step 1. We claim that

$$\|[\mathcal{L},\ \varphi_{\rho,N} D^\alpha]u\|_{-1/3+n+1} \le \tfrac{C_{19} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{s/3}. \tag{3.14}$$

In fact, write $\mathcal{L} = X_0 - a\triangle_v$ with $X_0 = \partial_t + v\cdot\partial_x$. Then a direct verification shows that

$$\begin{aligned} \|[\mathcal{L},\ \varphi_{\rho,N} D^\alpha]u\|_{-1/3+n+1} \ \le\ & \|[X_0,\ \varphi_{\rho,N} D^\alpha]u\|_{-1/3+n+1} + \|a[\triangle_v,\ \varphi_{\rho,N} D^\alpha]u\|_{-1/3+n+1} \\ & + \|\varphi_{\rho,N}[a,\ D^\alpha]\triangle_v u\|_{-1/3+n+1} \\ =:\ & (I) + (II) + (III). \end{aligned}$$

Denote $[X_0,\ D^\alpha]$ by D^{α_0}. Then $|\alpha_0| \le |\alpha|$ and

$$\begin{aligned} (I)\ \le\ & \|[X_0,\ \varphi_{\rho,N}]D^\alpha u\|_{n+1} + \|\varphi_{\rho,N} D^{\alpha_0} u\|_{n+1} \\ \le\ & C_8\big\{\ (N/\rho)\|D^\alpha u\|_{n+1,\Omega_{\tilde\rho}} + \|D^{\alpha_0} u\|_{n+1,\Omega_{\tilde\rho}}\ \big\}. \end{aligned}$$

Notice that $s \ge 3$. By Lemma 3.8, we have

$$(I)\ \le\ C_8\big(N/\rho+1\big)\tfrac{C_7 A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s \le \tfrac{C_9 A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{s/3}. \tag{3.15}$$

Next we will estimate (II). It is easy to see that

$$
\begin{aligned}
\|[\triangle_v,\ \varphi_{\rho,N}]D^\alpha u\|_{-1/3+n+1} &\leq 2\|[D_v,\ \varphi_{\rho,N}]D_vD^\alpha u\|_{-1/3+n+1} \\
&\quad +\|[D_v,\ [D_v,\ \varphi_{\rho,N}]\ ]D^\alpha u\|_{-1/3+n+1}.
\end{aligned} \tag{3.16}
$$

We first consider the first term on the right hand side. By Lemma 3.8 again, we have

$$
\begin{aligned}
\|[D_v,\ \varphi_{\rho,N}]D_vD^\alpha u\|_{-1/3+n+1} &\leq (N/\rho)\|D_vD^\alpha u\|_{-1/3+n+1,\Omega_{\tilde\rho}} \\
&\leq (N/\rho)\frac{C_7A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s \\
&\leq \frac{C_{10}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.
\end{aligned} \tag{3.17}
$$

Next we treat $\|[D_v,\ [D_v,\ \varphi_{\rho,N}]\ ]D^\alpha u\|_{-1/3+n+1}$. We compute

$$
\begin{aligned}
&\|[D_v,\ [D_v,\ \varphi_{\rho,N}]\ ]D^\alpha u\|_{-1/3+n+1} \\
\leq\ & \|(D^2\varphi_{\rho,N})D^\beta u\|_{2/3+n+1} + \|(D^3\varphi_{\rho,N})D^\beta u\|_{-1/3+n+1} \\
\leq\ & C_{11}\big\{\ (N/\rho)^2\|D^\beta u\|_{2/3+n+1,\Omega_{\tilde\rho}} + (N/\rho)^3\|D^\beta u\|_{n+1,\Omega_{\tilde\rho}}\ \big\} \\
\leq\ & C_{11}\big\{\ (N/\rho)^2\frac{A^{|\beta|-1}}{\tilde\rho^{s(|\beta|-3)}}\big((|\beta|-3)!\big)^s(N/\tilde\rho)^{2s/3} \\
& +(N/\rho)^3\frac{A^{|\beta|-1}}{\tilde\rho^{s(|\beta|-3)}}\big((|\beta|-3)!\big)^s\ \big\} \\
\leq\ & C_{11}\big\{\ (N/\rho)^2(N/\tilde\rho)^{-s/3}\frac{A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s \\
& +(N/\rho)^3(N/\tilde\rho)^{-s}\frac{A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\ \big\} \\
\leq\ & \frac{C_{12}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.
\end{aligned}
$$

This along with (3.16) and (3.17) shows that

$$
(II) \leq \frac{C_{13}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \tag{3.18}
$$

It remains to treat (III). By Leibniz' formula,

$$
\begin{aligned}
(III) &\leq \sum_{0<|\gamma|\leq|\alpha|}\binom{\alpha}{\gamma}\big\|\varphi_{\rho,N}(D^\gamma a)\triangle_vD^{\alpha-\gamma}u\big\|_{-1/3+n+1} \\
&\leq \sum_{0<|\gamma|\leq|\alpha|}\binom{\alpha}{\gamma}\big\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\triangle_vD^{\alpha-\gamma}u\big\|_{-1/3+n+1}.
\end{aligned}
$$

Since $a\in G^s(\mathbb{R}^{2n+1})$, then

$$
\|D^\gamma a\|_{n+1,\Omega} \leq C_{14}^{|\gamma|-2}\big((|\gamma|-3)!\big)^s, \quad |\gamma|\geq 3,
$$

and

$$\|D^\gamma a\|_{n+1,\Omega} \le C_{14}, \quad |\gamma| = 1,\ 2.$$

Moreover, notice that $|\alpha|-|\gamma|+1 \le N$. Applying Lemma 3.8, we have for any $\gamma,\ |\gamma| \le |\alpha|-2$,

$$\begin{aligned}\|\varphi_{\rho,N}\triangle_v D^{\alpha-\gamma}u\|_{-1/3+n+1} &\le \|D_v D^{\alpha-\gamma+1}u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\\ &\le \frac{C_7 A^{|\alpha|-|\gamma|+1-2}}{\tilde\rho^{s(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s\\ &\le \frac{C_{15} A^{|\alpha|-|\gamma|+1-2}}{\rho^{s(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s.\end{aligned}$$

Consequently, we compute

$$\begin{aligned}&\sum_{2\le|\gamma|\le|\alpha|-2}\binom{\alpha}{\gamma}\big\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\triangle_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1}\\ \le\ &\sum_{2\le|\gamma|\le|\alpha|-2}\binom{\alpha}{\gamma}C_{14}^{|\gamma|-2}\big((|\gamma|-2)!\big)^s\frac{C_{15}A^{|\alpha|-|\gamma|+1-2}}{\rho^{s(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s\\ \le\ &\frac{C_{15}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\sum_{2\le|\gamma|\le|\alpha|-2}\Big(\frac{C_{14}}{A}\Big)^{|\gamma|-1}|\alpha|!\big((|\gamma|-2)!\big)^{s-1}\big((|\alpha|-|\gamma|-2)!\big)^{s-1}\\ \le\ &\frac{C_{15}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\sum_{2\le|\gamma|\le|\alpha|-2}\Big(\frac{C_{14}}{A}\Big)^{|\gamma|-1}|\alpha|\frac{(|\alpha|-1)(|\alpha|-2)}{(|\alpha|-3)^{s-1}}\\ \le\ &\frac{C_{16}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}\sum_{2\le|\gamma|\le|\alpha|-2}\Big(\frac{C_{14}}{A}\Big)^{|\gamma|-1}.\end{aligned}$$

Making A large enough such that $\sum_{2\le|\gamma|\le|\alpha|-2}\big(\frac{C_{14}}{A}\big)^{|\gamma|-1} \le 1$, then we get

$$\sum_{2\le|\gamma|\le|\alpha|-2}\binom{\alpha}{\gamma}\big\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\triangle_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1} \le \frac{C_{16}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

For $|\gamma| = 1,\ |\alpha|-1$ or $|\alpha|$, we can compute directly

$$\binom{\alpha}{\gamma}\big\|D^\gamma a\|_{n+1,\Omega}\cdot\|\varphi_{\rho,N}\triangle_v D^{\alpha-\gamma}u\big\|_{-1/3+n+1} \le \frac{C_{17}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

Combination of the above two inequalities give that

$$(III) \le \frac{C_{18}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

This along with (3.15) and (3.18) yields the conclusion (3.14).

Step 2. We next claim that

$$\|\varphi_{\rho,N}D^{\alpha}[F(\cdot,u(\cdot),\nabla_v u(\cdot))]\|_{-1/3+n+1}\leq \tfrac{C_{21}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \tag{3.19}$$

We first prove F and u satisfy the conditions in (3.7)-(3.9) for some M_j. By Lemma 3.8, we have

$$\|D^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\leq \|D^j u\|_{n+1,\Omega_{\tilde\rho}}\leq \frac{C_7A^{j-2}}{\tilde\rho^{s(j-3)}}\big((j-3)!\big)^s,\quad 3\leq j\leq N, \tag{3.20}$$

$$\|D_vD^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\leq \frac{C_7A^{j-2}}{\tilde\rho^{s(j-3)}}\big((j-3)!\big)^s,\quad 3\leq j\leq N, \tag{3.21}$$

and

$$\|D^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\leq C_7,\quad 0\leq j\leq 2. \tag{3.22}$$

Since $F\in G^s(\mathbb{R}^{2n+1}\times\mathbb{R})$, then

$$\|(D^k_{t,x,v}\partial^l_u D^m_p F)(\cdot,u(\cdot),\nabla_v u(\cdot))\|_{-1/3+n+1,\Omega}\leq C_{20}^{k+l}\big((k-3)!\big)^s\big((l-3)!\big)^s,\quad k,m+l\geq 3. \tag{3.23}$$

Define M_j,H_0,H_1 by setting

$$H_0=C_7,\quad H_1=A,\quad M_0=C_7,\quad M_j=\frac{\big((j-1)!\big)^s}{\tilde\rho^{s(j-1)}},\quad j\geq 1.$$

We can choose A large enough such that $H_1=A\geq C_2H_0$. Then (3.20)-(3.23) can be rewritten as

$$\|D^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\leq H_0,\quad 0\leq j\leq 1, \tag{3.24}$$

$$\|D^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\leq H_0H_1^{j-2}M_{j-2},\quad 2\leq j\leq |\alpha|=N, \tag{3.25}$$

$$\|D_vD^j u\|_{-1/3+n+1,\Omega_{\tilde\rho}}\leq H_0H_1^{j-2}M_{j-2},\quad 2\leq j\leq |\alpha|=N, \tag{3.26}$$

$$\|(D^k_{t,x,v}\partial^l_u D^m_p F)\|_{-1/3+n+1,\Omega}\leq C_{20}^{k+l}M_{k-2}M_{m+l-2},\qquad k,m+l\geq 2. \tag{3.27}$$

For each j, notice that $s\geq 3$. Hence

$$\begin{aligned}\tfrac{j!}{i!(j-i)!}M_iM_{j-i} &= \tfrac{j!}{i(j-i)}\big((i-1)!\big)^{s-1}\big((j-i-1)!\big)^{s-1}\tilde\rho^{-s(i-1)}\tilde\rho^{-s(j-i-1)}\\ &\leq (j!)\big((j-2)!\big)^{s-1}\tilde\rho^{-s(j-1)}\\ &\leq \tfrac{j}{(j-1)^{s-1}}(j-1)!\big((j-1)!\big)^{s-1}\tilde\rho^{-s(j-1)}\\ &\leq M_j.\end{aligned} \tag{3.28}$$

Thus M_j satisfy the monotonicity condition (3.5). In view of (3.24)-(3.28) and making use of Lemma 3.5, we have

$$\begin{aligned}\|\varphi_{\rho,N}D^\alpha[F(\cdot,u(\cdot))]\|_{-1/3+n+1} &\leq C_3H_0H_1^{|\alpha|-2}M_{|\alpha|-2}\\ &\leq \frac{C_3C_7A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\\ &\leq \frac{C_{21}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.\end{aligned}$$

This completes the proof of conclusion (3.19).

Step 3. We verify in this step the following:

$$\|\mathcal{L}\varphi_{\rho,N}D^\alpha u\|_{-1/3+n+1} \leq \frac{C_{23}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \tag{3.29}$$

In fact,

$$\begin{aligned}\|\mathcal{L}\varphi_{\rho,N}D^\alpha u\|_{-1/3+n+1} &\leq C_{22}\{\ \|[\mathcal{L},\ \varphi_{\rho,N}D^\alpha]u\|_{-1/3+n+1}+\|\varphi_{\rho,N}D^\alpha\mathcal{L}u\|_{-1/3+n+1}\ \}\\ &= C_{22}\big\{\|[\mathcal{L},\ \varphi_{\rho,N}D^\alpha]u\|_{-1/3+n+1}\\ &\quad+\big\|\varphi_{\rho,N}D^\alpha[F\big(\cdot,u(\cdot),\nabla_v u(\cdot)\big)]\big\|_{-1/3+n+1}\big\}.\end{aligned}$$

This along with (3.14), (3.19) in step 1 and step 2 yields immediately the conclusion (3.29).

Step 4. We claim that

$$\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}+\|\varphi_{\rho,N}D_vD^\alpha u\|_{1/3-1/3+n+1} \leq \tfrac{C_{31}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \tag{3.30}$$

In fact, applying the subelliptic estimate (2.3), we obtain

$$\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1} \leq C_{24}\{\ \|\mathcal{L}\varphi_{\rho,N}D^\alpha u\|_{-1/3+n+1}+\|\varphi_{\rho,N}D^\alpha u\|_{n+1}\ \}.$$

Combining Lemma 3.8 and (3.29) in Step 3, we have

$$\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1} \leq \frac{C_{25}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}. \tag{3.31}$$

Now it remains to treat $\|\varphi_{\rho,N}D_vD^\alpha u\|_{1/3-1/3+n+1}$, and

$$\|\varphi_{\rho,N}D_vD^\alpha u\|_{1/3-1/3+n+1} \leq \|D_v\varphi_{\rho,N}D^\alpha u\|_{n+1}+\|[D_v,\ \varphi_{\rho,N}]D^\alpha u\|_{n+1}.$$

First, we treat the first term on the right. By a direct calculation, it follows that

$$\begin{aligned}&\|D_v\varphi_{\rho,N}D^\alpha u\|_{n+1}^2\\ =\ &\mathrm{Re}\big(\mathcal{L}\varphi_{\rho,N}D^\alpha u,\ a^{-1}\Lambda^{2n+2}\varphi_{\rho,N}D^\alpha u\big)-\mathrm{Re}\big(X_0\varphi_{\rho,N}D^\alpha u,\ a^{-1}\Lambda^{2n+2}\varphi_{\varepsilon,k\varepsilon}D^\alpha u\big)\\ =\ &\mathrm{Re}\big(\mathcal{L}\varphi_{\rho,N}D^\alpha u,\ a^{-1}\Lambda^{2n+2}\varphi_{\rho,N}D^\alpha u\big)-\frac{1}{2}\big(\varphi_{\rho,N}D^\alpha u,\ [a^{-1}\Lambda^{2n+2},\ X_0]\varphi_{\rho,N}D^\alpha u\big)\\ &-\frac{1}{2}\big(\varphi_{\rho,N}D^\alpha u,\ [\Lambda^{2n+2},\ a^{-1}]X_0\varphi_{\rho,N}D^\alpha u\big)\\ \leq\ &C_{26}\{\ \|\mathcal{L}\varphi_{\rho,N}D^\alpha u\|_{-1/3+n+1}^2+\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}^2\ \}.\end{aligned}$$

This along with (3.29) and (3.31) shows that

$$\|D_v\varphi_{\rho,N}D^\alpha u\|_{r-1/3+n+1} \le \frac{C_{27}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

Moreover Lemma 3.8 yields

$$\begin{aligned}\|[D_v,\ \varphi_{\rho,N}]D^\alpha u\|_{n+1} &\le C_{28}(N/\rho)\|D^\alpha u\|_{n+1,\Omega_{\tilde\rho}}\\ &\le \frac{C_{28}C_7A^{|\alpha|-2}}{\tilde\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}\\ &\le \frac{C_{29}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.\end{aligned}$$

From the above two inequalities, we have

$$\|\varphi_{\rho,N}D_vD^\alpha u\|_{1/3+n+1} \le \frac{C_{30}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}.$$

This completes the proof of Step 4.

It's clear for any $\rho,\ 0<\rho<1$,

$$\|D^\alpha u\|_{1/3+n+1,\Omega_\rho}+\|D_vD^\alpha u\|_{1/3-1/3+n+1,\Omega_\rho} \le \|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}+\|\varphi_{\rho,N}D_vD^\alpha u\|_{1/3-1/3+n+1}.$$

It thus follows from Step 4 that the conclusion in Lemma 3.9 is true for $r=1/3$.

Moreover for any $0<r<1/3$, using the interpolation inequality (2.2), we have

$$\begin{aligned}\|D^\alpha u\|_{r+n+1,\Omega_\rho} &\le \|\varphi_{\rho,N}D^\alpha u\|_{r+n+1}\\ &\le \varepsilon\|\varphi_{\rho,N}D^\alpha u\|_{1/3+n+1}+\varepsilon^{-r/(1/3-r)}\|\varphi_{\rho,N}D^\alpha u\|_{n+1}\\ &\le \varepsilon\frac{C_{31}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{s/3}+\varepsilon^{-r/(1/3-r)}\frac{C_{32}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s,\end{aligned}$$

Taking $\varepsilon=(N/\rho)^{s(r-1/3)}$, then

$$\|D^\alpha u\|_{r+n+1,\Omega_\rho} \le \frac{C_{33}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{rs}.$$

Similarly,

$$\|D_vD^\alpha u\|_{r-1/3+n+1,\Omega_\rho} \le \frac{C_{34}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{rs}.$$

This completes the proof of Lemma 3.9.

Inductively, we have the following

Lemma 3.10. *For any r with $1/3\le r\le 2/3$,*

(3.32)

$$\|D^\alpha u\|_{r+n+1,\Omega_\rho}+\|D_vD^\alpha u\|_{r-1/3+n+1,\Omega_\rho} \le \frac{C_{38}A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{sr},\quad \forall\ 0<\rho<1.$$

Moreover, the above inequality still holds for any r with $2/3\le r\le 1$.

Proof of Lemma 3.10: Repeating the proof of Lemma 3.9, we have (3.32) for $1/3 \leq r \leq 2/3$. When $2/3 \leq r \leq 1$, the consideration is a little different. The conclusion in Step 1 in the above proof still holds for $r = 1$. For the corresponding Step 2, we have to make some modification to prove

$$\left\|\varphi_{\rho,N} D^\alpha[F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big)]\right\|_{1/3+n+1} \leq \tfrac{C_{36} A^{|\alpha|-2}}{\rho^{s(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^s.$$

From (3.32) with $1/3 \leq r \leq 2/3$, it follows that

$$\|D^j u\|_{1/3+n+1,\Omega_{\tilde\rho}} \leq \frac{C_{37} A^{j-2}}{\tilde\rho^{s(j-3)}}\big((j-3)!\big)^s (j/\tilde\rho)^{s/3}, \quad 3 \leq j \leq N,$$

$$\|D_v D^j u\|_{1/3+n+1,\Omega_{\tilde\rho}} \leq \|D_v D^j u\|_{2/3-1/3+n+1,\Omega_{\tilde\rho}} \leq \frac{C_{37} A^{j-2}}{\tilde\rho^{s(j-3)}}\big((j-3)!\big)^s (j/\tilde\rho)^{2s/3}, \quad 3 \leq j \leq N,$$

and

$$\|D^j u\|_{1/3+n+1,\Omega_{\tilde\rho}} \leq C_{37}, \qquad 0 \leq j \leq 2,$$

Hence we need to define a new sequence $\bar M_j$ by setting

$$\bar M_0 = C_{37}, \quad \bar M_j = \frac{\big((j-1)!\big)^s}{\tilde\rho^{s(j-1)}}\big((j+2)/\tilde\rho\big)^{2s/3}, \quad j \geq 1.$$

For each j, notice that $s \geq 3$. Hence a direct computation shows that for $0 < i < j$,

$$\begin{aligned}
\tfrac{j!}{i!(j-i)!}\bar M_i \bar M_{j-i} &= \tfrac{j!}{i(j-i)}\big((i-1)!\big)^{s-1}\big((j-i-1)!\big)^{s-1} \\
&\quad \times (i+2)^{2s/3}(j-i+2)^{2s/3}\tilde\rho^{-s(j-2)}\tilde\rho^{-4s/3} \\
&\leq 4(j!)\big((j-2)!\big)^{s-1}(j+2)^{2s/3-1}(j+1)^{2s/3-1}\tilde\rho^{-s(j-1)}\tilde\rho^{-2s/3}\tilde\rho^{s-2s/3} \\
&\leq \tfrac{4j(j+1)^{2s/3-1}}{(j-1)^{s-1}}(j-1)!\big((j-1)!\big)^{s-1}\tilde\rho^{-s(j-1)}\big((j+2)/\tilde\rho\big)^{2s/3} \\
&\leq C_{39}\bar M_j.
\end{aligned}$$

In the last inequality, we used the fact that $s-1 \geq 2s/3$. Thus $\bar M_j$ satisfy the monotonicity condition (3.5). Now the remaining argument is identical to that in the proof of Lemma 3.9. Thus (3.32) holds for $r = 1$ and thus for $2/3 \leq r \leq 1$ by the interpolation inequality (2.2). This completes the proof of Lemma 3.10.

Recall C_7, C_{35} and C_{35} are the constants appearing in Lemma 3.8, Lemma 3.9 and Lemma 3.10. Now make A sufficiently large such that $A \geq \max\{C_7, C_{35}, C_{38}\}$. Then, by the above three Lemmas, we see that the estimate in $(E)_{r,N}$ holds for any $r \in [0,\ 1]$. This complete the proof of Proposition 3.6.

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