The existence and regularity of multiple solutions for a class of infinitely degenerate elliptic equations *

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Abstract

Let $X = (X_1, ..., X_m)$ be an infinitely degenerate system of vector fields, we study the existence and regularity of multiple solutions of Dirichelt problem for a class of semi-linear infinitely degenerate elliptic operators associated with the sum of square operator $\Delta_X = \sum_{j=1}^m X_j^* X_j$.

Keywords: degenerate elliptic equations, Logarithmic Sobolev inequality.

1 Introduction

In this paper, we study the existence and regularity of solutions for a class of semi-linear infinitely degenerate elliptic operators. Consider a system of vector fields $X = (X_1, ..., X_m)$ defined on an open domain $\widetilde{\Omega} \subset \mathbb{R}^d$. We suppose that this system satisfies the following Logarithmic regularity estimates,

$$\|(\log \Lambda)^s u\|_{L^2(\Omega)}^2 \leq C \left\{ \sum_{j=1}^m \|X_j u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right\}, \forall\ u \in C_0^\infty(\widetilde{\Omega}), \tag{1.1}$$

where $\Lambda = (e^2 + |D|^2)^{1/2} = \langle D \rangle$. The results of [4, 6, 7, 8, 9] gave some sufficient conditions for the estimates (1.1). We remark that if $s > 1$, the estimate (1.1) implies the hypoellipticity of the infinitely degenerate elliptic operator $\Delta_X = \sum_{j=1}^m X_j^* X_j$, where X_j^* is the formal adjoint of X_j.

Definition 1.1. If Γ is a smooth surfaces of $\widetilde{\Omega}$, we say that Γ is non characteristic for the system of vector fields X, if for any point $x_0 \in \Gamma$ there exists at least one vector field in $X = (X_1, ...X_m)$ which is transversal to Γ at x_0.

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Definition 1.2. Let now $\Gamma = \bigcup_{j\in J} \Gamma_j$ be the union of a family of smooth surface in $\widetilde{\Omega}$. We say that Γ is non characteristic for X, if for any point $x_0 \in \Gamma$, there exists at least one vector field of $X_1, ..., X_m$ which is transversal to Γ_j at x_0 for all j in which $x_0 \in \Gamma_j$.

We say that the vector fields $X = (X_1, ..., X_m)$ satisfies the finite type of Hörmander's condition on an open domain $\omega \subset \widetilde{\Omega}$ in $\mathbb{R}^d$ if the rank of the Lie algebra generated by the vector fields $X = (X_1, ..., X_m)$ and its finite times commutators is equal to the space dimension d at every point in ω.

A typical example is the vector fields in $\mathbb{R}^3$, i.e. $X_1 = \partial_{x_1}$, $X_2 = \partial_{x_2}$, $X_3 = \exp(-|x_1|^{-1/s})\partial_{x_3}$ with $s > 0$. The operator Δ_X in this example is degenerate infinitely on $\Gamma_0 = \{x_1 = 0\}$, and the vector fields $X = (X_1, X_2, X_3)$ satisfies the finite type of Hörmander's condition in $\mathbb{R}^3 \diagdown \Gamma_0$.

The example with infinitely degeneracy on a union of surfaces $\Gamma = \bigcup_{j\in J} \Gamma_j$ is the system in $\mathbb{R}^2$ such that $X_1 = \partial_{x_1}, X_2 = \exp(-(x_1^2 \sin^2(\frac{\pi}{x_1}))^{\frac{-1}{2s}})\partial_{x_2}$, we have $\Gamma_j = \{x_1 = \frac{1}{j}\}$ for $j \in \mathbb{Z}\diagdown\{0\}, \Gamma_0 = \{x_1 = 0\}$, then X_1 is transverse to all $\Gamma_j, j \in \mathbb{Z}$, and X_2 vanishes infinitely on $\Gamma = \bigcup_{j\in\mathbb{Z}} \Gamma_j$. The vector fields $X = (X_1, X_2)$ satisfies the finite type of Hörmander's condition in $\mathbb{R}^2 \diagdown \Gamma$.

Related to the systems of vector fields $X = (X_1, ...X_m)$, Morimoto and Xu introduce the following function space (cf.[10]),

$$H^1_X(\widetilde{\Omega}) = \left\{ u \in L^2(\widetilde{\Omega}), X_j u \in L^2(\widetilde{\Omega}), j = 1, ..., m \right\},$$

which is a Hilbert space with norm $\|u\|^2_{H^1_X} = \|u\|^2_{L^2} + \|Xu\|^2_{L^2}$, and $\|Xu\|^2_{L^2} = \sum_{j=1}^{m} \|X_j u\|^2_{L^2}$. Take $\Omega \subset\subset \widetilde{\Omega}$ as a bounded open subset and suppose that $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X, Morimoto and Xu define the space $H^1_{X,0}(\Omega)$ as a closure of $C^\infty_0(\Omega)$ in $H^1_X(\Omega)$, which is also a Hilbert space.

If the system of vector fields X satisfies the estimates (1.1), we have the following Logarithmic Sobolev inequality;

Proposition 1.1. *(cf.[10]) Suppose that the system of vector fields* $X = (X_1, ..., X_m)$ *verifies the estimates* (1.1) *for some* $s > 1/2$. *Then there exists* $C_0 > 0$ *such that*

$$\int_\Omega |v|^2 \left| \log(\frac{|v|}{\|v\|_{L^2(\Omega)}}) \right|^{2s-1} \leq C_0 \left\{ \sum_{j=1}^{m} \|X_j v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \right\}, \qquad (1.2)$$

for all $v \in H^1_{X,0}(\Omega)$.

Using the Logarithmic Sobolev inequality above, Morimoto and Xu [10] have studied the following semi-linear Dirichlet problems,

$$\triangle_X u = au \log |u| + bu, u|_{\partial\Omega} = 0, \tag{1.3}$$

where constant coefficients $a, b \in \mathbb{R}$. They have obtained,

Proposition 1.2. (*cf.*[*10*]) *We suppose that the system of vector fields* $X = (X_1, ...X_m)$ *satisfies the following conditions*:
$\widetilde{H}$*-1)* $\partial\Omega$ *is* C^∞ *and non characteristic for the system of vector fields* X;
$\widetilde{H}$*-2) the system of vector fields* X *satisfies the finite type of Hörmander's condition on* $\widetilde{\Omega}$ *except an union of smooth surfaces* Γ *which are non characteristic for* X;
$\widetilde{H}$*-3) the system of vector fields* X *satisfies the estimate* (1.1) *for* $s > 3/2$.

Suppose $a \neq 0$ *in* (1.3)*, then the semi-linear Dirichlet problem (1.3) posses at least one non trivial weak solution* $u \in H^1_{X,0}(\Omega) \cap L^\infty(\Omega)$*. Moreover, if* $a > 0$*, we have* $u \in C^\infty(\Omega \setminus \Gamma) \bigcap C^0(\overline{\Omega} \setminus \Gamma)$ *and* $u > 0$ *for all* $x \in \Omega \setminus \Gamma$.

Next, it will be useful for us to introduce following Poincaré's inequality,

Proposition 1.3. (*cf.*[*10*]) *Under the hypotheses* $\widetilde{H}$*-1),* $\widetilde{H}$*-2) and* $\widetilde{H}$*-3), the first eigenvalue* λ_1 *of the operator* $\triangle_X$ *is strictly positive, which is equivalent to following Poincaré's inequality*

$$\|\varphi\|^2_{L^2} \leq \frac{1}{\lambda_1} \|X\varphi\|^2_{L^2}, \quad \forall\, \varphi \in H^1_{X,0}(\Omega). \tag{1.4}$$

In this paper, we shall study the following semi-linear Dirichlet problem

$$-\triangle_X u = a(x)u \log |u| + b(x)u + g(x), \qquad \text{in } \Omega, \tag{1.5}$$

$$u|_{\partial\Omega} = 0. \tag{1.6}$$

Our main result is as follows.

Theorem 1.4. *Suppose that the system of vector fields* $X = (X_1, ...X_m)$ *satisfies the following conditions*:
H-1) $\partial\Omega$ *is* C^∞ *and non characteristic for the system of vector fields* X;
H-2) *the system of vector fields* X *satisfies the finite type of Hörmander's condition on* $\widetilde{\Omega}$ *except an union of smooth surfaces* Γ *which are non characteristic for* X;
H-3) *the system of vector fields* X *satisfies the estimate* (1.1) *for* $s \geq 5/2$;
H-4) $a(x), b(x) \in L^\infty(\Omega)$*, and there exist* $a_0, b_0 \in \mathbb{R}_+$*, such that* $a(x) \geq a_0$*, and* $b(x) \geq b_0$*, a.e. in* Ω*. Then*

1) *there exists* $C > 0$ *such that the problem (1.5) and (1.6) has at least two solutions in* $H^1_{X,0}(\Omega)$*, for any* $g \not\equiv 0$ *satisfying* $\|g\|_{L^2(\Omega))} < C$;
2) *the problem (1.5) and (1.6) has at least one non-negative solution* $u \in H^1_{X,0}(\Omega)$; *furthermore, if* $g(x) \in L^\infty(\Omega)$*, then the non-negative solution* $u(x) \in L^\infty(\Omega)$.
3) *If* $a(x), b(x), g(x) \in C^\infty(\Omega)$*, and there exists* $g_0 > 0$ *such that* $g(x) \geq g_0$*, then we have* $u \in C^\infty(\Omega \backslash \Gamma) \bigcap C^0(\bar{\Omega} \backslash \Gamma)$ *and* $u(x) > 0$ *for all* $x \in \Omega \backslash \Gamma$.

The proof of Theorem 1.4 relies essentially on the Ekeland Variational Principle (cf.[5]) and on the Mountain Pass Theorem without the Palais-Smale condition, established by Brezis-Nirenberg [3], namely

Proposition 1.5. $(cf.[5])$ *Let* V *be a complete metric space, and* $F : V \to \mathbb{R} \cup \{+\infty\}$ *a lower semicontinuous function,* $\not\equiv +\infty$*, bounded from below. For any* $\epsilon > 0$*, there is some point* $v \in V$ *with*

$$F(v) \leq \inf_V F + \epsilon. \tag{1.7}$$

$$\forall\, w \in V,\ F(w) \geq F(v) - \epsilon d(v, w). \tag{1.8}$$

Proposition 1.6. $(cf.[3])$ *Let* Φ *be a* C^1 *function on a Banach space* E*. Suppose there exists a neighborhood* U *of 0 in* E *and a constant* ρ *such that* $\Phi(u) \geq \rho$ *for every* u *in the boundary of* U*,*

$$\Phi(0) < \rho, \quad \text{and} \quad \Phi(v) < \rho \quad \text{for some} \quad v \notin U.$$

Set

$$c = \inf_{\mathbb{P} \in M} \max_{W \in \mathbb{P}} \Phi(w) \geq \rho,$$

where M *denotes the class of paths joining 0 to* v.

Conclusion: there is a sequence $\{u_i\}$ *in* E *such that*

$$\Phi(u_i) \to c \quad \text{and} \quad \Phi'(u_i) \to 0 \quad \text{in } E^*.$$

2 Auxiliary results

Definition 2.1. We say that $u \in H^1_{X,0}(\Omega)$ is a weak solution of (1.5) and (1.6) if

$$\int_\Omega \sum_{j=1}^m X_j u X_j v dx - \int_\Omega a(x) uv \log|u| dx - \int_\Omega b(x) uv dx - \int_\Omega g(x) v dx = 0,$$

for all $v \in C_0^\infty(\Omega)$.

We define the function $J_\eta, H^1_{X,0}(\Omega) \to \mathbb{R}$, $0 \le \eta < 1$ by

$$\begin{aligned} J_\eta(u) &= \int_\Omega \sum_{j=1}^m (X_j u)^2 dx - \int_\Omega a(x) u^2 \log(|u|+\eta) dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u|+\eta)} \\ &- \int_\Omega b(x) u^2 dx - 2\int_\Omega g(x) u dx. \end{aligned}$$

A simple calculation shows that as $0 < \eta < 1$, $J_\eta \in C^1(H^1_{X,0}(\Omega), \mathbb{R})$ and it's derivative is given by,

$$\begin{aligned} \langle J'_\eta(u), v\rangle &= 2\int_\Omega \sum_{j=1}^m (X_j u)(X_j v) - 2\int_\Omega a(x) uv \log(|u|+\eta) dx \\ &+ \int_\Omega \frac{a(x)u|u|v\eta}{2(|u|+\eta)^2} dx - 2\int_\Omega b(x) uv dx - 2\int_\Omega g(x) v dx, \end{aligned}$$

for all $u, v \in H^1_{X,0}(\Omega)$.

We have denoted by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1_{X,0}(\Omega)$ and $H^{-1}_{X,0}(\Omega)$, and $H^{-1}_{X,0}(\Omega)$ is the dual space of $H^1_{X,0}(\Omega)$, i.e. $H^{-1}_{X,0}(\Omega) = (H^1_{X,0}(\Omega))^*$. We use the notation $\rightharpoonup$ as the weak convergence and the notation $\to$ as the strong convergence in Banach space.

Definition 2.2. If F is a C^1 functional on some Banach space E and c is a real number, we say that a sequence $\{u_n\}$ in E is a $(PS)_c$ sequence of F if $F(u_n) \to c$ and $F'(u_n) \to 0$ in E^*.

Remark: If $\{u_n\}$ is a bounded sequence in $H^1_{X,0}(\Omega)$, then there exists a subsequence (denote still by $\{u_n\}$) such that $u_n \rightharpoonup u_0$ in $H^1_{X,0}(\Omega)$, $u_n \to u_0$ in $L^2(\Omega)$.

Lemma 2.1. *Let $M > 0$ and let $\{v_j, j \in \mathbb{N}\}$ be a sequence in $H^1_{X,0}(\Omega)$, satisfying*

$$\|v_j\|^2_{H^1_{X,0}(\Omega)} \le M.$$

Then $\{|v_j||\log|v_j||\}$ is uniformly integrable.

Proof.

$$\begin{aligned} &\int_\Omega |v_j||\log|v_j||^2 \le \frac{1}{2}|\Omega| + \frac{1}{2}\int_\Omega v_j^2 |\log|v_j||^4 dx \\ =\ & \frac{1}{2}|\Omega| + \frac{1}{2}\int_\Omega v_j^2 |\log \frac{|v_j|}{\|v_j\|_{L^2}} + \log\|v_j\|_{L^2}|^4 dx \\ \le\ & \frac{1}{2}|\Omega| + 4\int_\Omega v_j^2 \log^4 \frac{|v_j|}{\|v_j\|_{L^2}} + 4|\log\|v_j\|_{L^2}|^4 \|v_j\|^2_{L^2} \\ \le\ & \frac{1}{2}|\Omega| + 4C_0(\|Xv_j\|^2_{L^2} + \|v_j\|^2_{L^2}) + 4|\log\|v_j\|_{L^2}|^4 \|v_j\|^2_{L^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}|\Omega| + 4C_0(\|Xv_j\|_{L^2}^2 + \|v_j\|_{L^2}^2) + \frac{4}{2^4}|\log \|v_j\|_{L^2}^2|^4 \|v_j\|_{L^2}^2 \\
&\le \frac{1}{2}|\Omega| + 4C_0M + \frac{4}{2^4}[(4e^{-1})^4 + (\log M)^4 M] \\
&= \tilde{M},
\end{aligned}
$$

where $C_0 > 0$ is a positive constant given by Proposition 1.1. We use the fact $t(\log t)^4 \le l \log^4 l + (4e^{-1})^4$ for any $0 \le t \le l$.

Now, we prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $A \subset \Omega$, the measure of A, $\mu(A) < \delta$, then

$$\int_A |v_j||\log|v_j|| < \epsilon, \quad \forall \ j.$$

But for any $\epsilon > 0$, there exists $t_0 > e^2$ such that

$$\frac{1}{\log t} < \epsilon, \quad \forall \ t \ge t_0.$$

Take now $\delta = \epsilon(t_0 \log t_0)^{-1}, \mu(A) < \delta$ and

$$A_j = A \cap \{|v_j| \le t_0\}, \quad B_j = A \cap \{|v_j| > t_0\},$$

then we have,

$$\int_{A_j} |v_j||\log|v_j|| \le t_0 \log t_0 \mu(A_j) < \epsilon,$$

$$\int_{B_j} |v_j||\log|v_j|| \le \epsilon \int_{B_j} |v_j||\log|v_j||^2 < \epsilon\tilde{M}.$$

The proof of Lemma 2.1 is complete.

Lemma 2.2. *If $a(x) \in L^\infty(\Omega), \zeta \in C_0^\infty(\Omega), \|u_n\|_{H^1_{X,0}(\Omega)} < M, M$ is a positive constant independent of n, then there exists a convergent subsequence (denote still by $\{u_n\}$)such that $u_n \rightharpoonup u_0$ in $H^1_{X,0}(\Omega)$ and*

$$\lim_{n\to\infty} \int_\Omega a(x)u_n\zeta \log(|u_n| + 1/2^n)dx = \int_\Omega a(x)u_0\zeta \log(|u_0|)dx.$$

Proof. We have

$$
\begin{aligned}
&\int_\Omega |a(x)u_n\zeta||\log(|u_n| + 2^{-n})|^2 dx \le C\int_\Omega |u_n||\log(|u_n| + 2^{-n})|^2 dx \\
&\le C\int_{\{x:\ |u_n|+2^{-n}\le 1\}} |u_n||\log(|u_n| + 2^{-n})|^2 dx \\
&+ C\int_{\{x:\ |u_n|+2^{-n}\ge 1\}} |u_n||\log(|u_n| + 2^{-n})|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_{\{x:\ |u_n|+2^{-n}\le 1\}} |u_n||\log(|u_n|)|^2dx\\
&+ C\int_{\{x:\ |u_n|+2^{-n}\ge 1\}} |u_n||\log(2|u_n|)|^2dx\\
&\le C\int_{\{x:\ |u_n|+2^{-n}\le 1\}} |u_n||\log(|u_n|)|^2dx\\
&+ C\int_{\{x:\ |u_n|+2^{-n}\ge 1\}} |u_n|(\log^2 2+|\log(|u_n|)|^2)dx\\
&\le C\int_\Omega |u_n||\log(|u_n|)|^2dx + C(\int_\Omega |u_n|^2dx+|\Omega|),
\end{aligned}
$$

since $a(x)\in L^\infty(\Omega), \zeta\in C_0^\infty(\Omega)$. By the proof of Lemma 2.1, we know there exists $\tilde{M}$, such that

$$
\int_\Omega |a(x)u_n\xi||\log(|u_n|+2^{-n})|^2dx\le \tilde{M}.
$$

Next, we prove that for any $\epsilon>0$, there exists $\delta>0$ such that if $A\subset\Omega, \mu(A)<\delta$, then

$$
\int_A |a(x)u_n\zeta||\log(|u_n|+2^{-n})|dx<\epsilon, \quad \forall\ n.
$$

But for any $\epsilon>0$, there exists $t_0>e^2$ such that

$$
\frac{1}{\log t}<\epsilon, \quad \forall\ \ t\ge t_0.
$$

Take now $\delta=\epsilon\{a_\infty \max_{x\in\Omega}|\zeta(x)|[(t_0+2^{-1})^2+e^{-1}]\}^{-1}, \mu(A)<\delta, a_\infty=\|a(x)\|_{L^\infty(\Omega)}$ and

$$
A_n=A\cap\{|u_n|\le t_0\}, \quad B_n=A\cap\{|u_n|>t_0\},
$$

then we have,

$$
\begin{aligned}
&\int_{A_n} |a(x)u_n\zeta||\log(|u_n|+2^{-n})|dx\\
\le\ & a_\infty \max_{x\in\Omega}|\zeta(x)|\int_{A_n} |u_n||\log(|u_n|+2^{-n})|dx\\
\le\ & a_\infty \max_{x\in\Omega}|\zeta(x)|\int_{A_n} [(|u_n|+2^{-n})^2+e^{-1}]\\
\le\ & a_\infty \max_{x\in\Omega}|\zeta(x)|[(|t_0|+2^{-1})^2+e^{-1}]\mu(A_n)\\
<\ & \epsilon,
\end{aligned}
$$

$$
\int_{B_n} |a(x)u_n\zeta||\log(|u_n|+2^{-n})|dx<\epsilon\int_{B_n} |a(x)u_n\zeta||\log(|u_n|+2^{-n})|^2dx<\epsilon\tilde{M}.
$$

Similarly, we can prove that

Lemma 2.3. *For any fixed* $0 < \eta << 1, a(x) \in L^\infty(\Omega), \zeta \in C_0^\infty(\Omega), \|u_n\|_{H^1_{X,0}(\Omega)} < M, M$ *is a positive constant independent of* n*, there exists a convergent subsequence (denote still by* $\{u_n\}$*) such that* $u_n \rightharpoonup u_0$ *in* $H^1_{X,0}(\Omega)$*, and*

$$\lim_{n\to\infty}\int_\Omega a(x)u_n\zeta\log(|u_n|+\eta)dx = \int_\Omega a(x)u_0\zeta\log(|u_0|+\eta)dx.$$

Lemma 2.4. *For any fixed* $0 < \eta << 1, a(x) \in L^\infty(\Omega), u(x) \in H^1_{X,0}(\Omega), u_n \in C_0^\infty(\Omega)$ *and* $\|u_n - u\|_{H^1_{X,0}(\Omega)} \to 0$*, we have*

$$\lim_{n\to\infty}\int_\Omega a(x)uu_n\log(|u_n|+\eta)dx = \int_\Omega a(x)u^2\log(|u|+\eta)dx.$$

Lemma 2.5. *If* $a(x) \in L^\infty(\Omega), u(x) \in H^1_{X,0}(\Omega), u_n \in C_0^\infty(\Omega)$ *and* $\|u_n - u\|_{H^1_{X,0}(\Omega)} \to 0$*, we have*

$$\lim_{n\to\infty}\int_\Omega a(x)uu_n\log(|u_n|)dx = \int_\Omega a(x)u^2\log(|u|)dx.$$

Similar to Lemma 2.1, we have

Lemma 2.6. *Let* $M > 0$ *and let* $\{v_j, j \in \mathbb{N}\}$ *be a sequence in* $H^1_{X,0}(\Omega)$ *satisfying*

$$\|v_j\|^2_{H^1_{X,0}(\Omega)} \le M.$$

Then there exists a convergent sub-sequence $\{v_{j_k}\}$ *such that* $v_{j_k} \rightharpoonup v_0 \in H^1_{X,0}(\Omega)$ *and*

$$\lim_{j_k\to\infty}\int_\Omega |v_{j_k}|^2|\log|v_{j_k}|| = \int_\Omega |v_0|^2|\log|v_0||,$$

and

$$\int_\Omega |v_0|^2|\log|v_0|| \le CM,$$

where C *is a positive constant independent of* j*.*

Proof. Using the fact $|t\log t| \le t^2 + e^{-1}$, for $\forall\ t > 0$, we have

$$\begin{aligned}
\int_\Omega |v_j|^2|\log|v_j||^2 &= \int_\Omega |v_j|^2|\log\frac{|v_j|}{\|v_j\|_{L^2}} + \log\|v_j\|_{L^2}|^2 \\
&\le 2\int_\Omega |v_j|^2|\log\frac{|v_j|}{\|v_j\|_{L^2}}|^2 + 2\|v_j\|^2_{L^2(\Omega)}|\log\|v_j\|_{L^2}|^2 \\
&\le 2C_0(\|Xv_j\|^2_{L^2} + \|v_j\|^2_{L^2}) + 2(M+e^{-1})^2 \\
&\le 2C_0M + 4(M^2 + e^{-2}) \\
&= \tilde{M},
\end{aligned}$$

C_0 is a positive constant given by Proposition 1.1. The rest of the proof is similar to the proof of Lemma 2.1.

Next, we can prove that for any $\epsilon > 0$, there exists $\delta > 0$, such that if $A \subset \Omega,\ \mu(A) < \delta$, then

$$\int_A |v_j|^2 |\log |v_j|| dx < \epsilon, \quad \forall\ j.$$

Actually for any $\epsilon > 0$, there exists $t_0 > e^2$, such that

$$\frac{1}{\log t} < \epsilon, \quad \forall\ \ t \geq t_0.$$

Take now $\delta = \epsilon(t_0^2 \log t_0 + \frac{1}{2}e^{-1})^{-1}, \mu(A) < \delta$ and

$$A_j = A \cap \{|v_j| \leq t_0\}, \quad B_j = A \cap \{|v_j| > t_0\},$$

then we have,

$$\int_{A_j} |v_j|^2 |\log |v_j|| dx \leq \int_{A_j} (t_0^2 \log t_0 + \frac{1}{2}e^{-1}) < (t_0^2 \log t_0 + \frac{1}{2}e^{-1})\mu(A_j) < \epsilon,$$

$$\int_{B_j} |v_j|^2 |\log |v_j|| dx \leq \epsilon \int_{B_j} |v_j|^2 |\log |v_j||^2 dx < \epsilon \tilde{M}.$$

Thus we have

Lemma 2.7. *For any fixed* $0 < \eta << 1, a(x) \in L^\infty(\Omega), u_n \in H^1_{X,0}(\Omega)$ *and* $\|u_n\|_{H^1_{X,0}(\Omega)} < M$, ($M$ *is a positive constant independent of* n) *there exists a convergent subsequence (denote still by* $\{u_n\}$*)such that* $u_n \rightharpoonup u_0$ *in* $H^1_{X,0}(\Omega)$ *and*

$$\lim_{n\to\infty} \int_\Omega a(x) u_n u_0 \log(|u_n| + \eta) dx = \int_\Omega a(x) u_0^2 \log(|u_0| + \eta) dx.$$

Lemma 2.8. *If* $a(x) \in L^\infty(\Omega), \|u_n\|_{H^1_{X,0}(\Omega)} < M, M$ *is a positive constant independent of* n*, then there exists a convergent subsequence* (*denote still by* $\{u_n\}$)*such that* $u_n \rightharpoonup u_0$ *in* $H^1_{X,0}(\Omega)$ *and*

$$\lim_{n\to\infty} \int_\Omega a(x) u_n^2 \log(|u_n| + 1/2^n) dx = \int_\Omega a(x) u_0^2 \log(|u_0|) dx.$$

3 The existence of solutions

For any fixed $0 < \epsilon < 1$, $0 < \eta << 1$ and $u \in H^1_{X,0}(\Omega)$, by using Young's inequality, Proposition 1.1 and Proposition 1.3, we have,

$$
\begin{aligned}
J_\eta(u) &= \|Xu\|^2_{L^2(\Omega)} - \int_\Omega a(x)u^2\log(|u|+\eta)dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u|+\eta)} - \int_\Omega b(x)u^2dx \\
&- 2\int_\Omega g(x)udx \\
&= \|Xu\|^2_{L^2(\Omega)} - \int_{|u|>\eta} a(x)u^2\log(|u|+\eta)dx - \int_{|u|\le\eta} a(x)u^2\log(|u|+\eta)dx \\
&+ \int_\Omega \frac{a(x)u^2|u|}{2(|u|+\eta)} - \int_\Omega b(x)u^2dx - 2\int_\Omega g(x)udx \\
&\ge \|Xu\|^2_{L^2(\Omega)} - \int_{|u|>\eta} a(x)u^2\log 2|u|dx - \log 2\eta\int_{|u|\le\eta} a(x)u^2dx
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_\Omega \frac{a(x)u^2|u|}{2(|u|+\eta)} - \int_\Omega b(x)u^2dx - 2\int_\Omega g(x)udx \\
&\ge \|Xu\|^2_{L^2(\Omega)} - \log 2\int_{|u|>\eta} a(x)u^2dx - \int_{|u|>\eta} a(x)u^2(\log\frac{|u|}{\|u\|_{L^2}} + \log\|u\|_{L^2})dx \\
&- a_0\log 2\eta\int_{|u|\le\eta} u^2dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u|+\eta)} - \int_\Omega b(x)u^2dx - 2\int_\Omega g(x)udx \\
&> \|Xu\|^2_{L^2(\Omega)} - a_\infty\log 2\int_\Omega u^2dx - \frac{\epsilon}{C_0}\int_\Omega u^2\log^2\frac{|u|}{\|u\|_{L^2}} - \frac{C_0}{4\epsilon}\int_\Omega a^2(x)u^2 \\
&- \log\|u\|_{L^2}\int_{|u|>\eta} a(x)u^2 - a_0\log 2\eta\int_{|u|\le\eta} u^2dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u|+\eta)} \\
&- b_\infty\int_\Omega u^2dx - \int_\Omega g^2(x)dx - \int_\Omega u^2(x)dx \\
&> \|Xu\|^2_{L^2(\Omega)} - a_\infty\log 2\int_\Omega u^2dx - \epsilon(\|Xu\|^2_{L^2} + \|u\|^2_{L^2}) - \frac{C_0a^2_\infty}{4\epsilon}\int_\Omega u^2 \\
&- \log\|u\|_{L^2}\int_{|u|>\eta} a(x)u^2 - a_0\log 2\eta\int_{|u|\le\eta} u^2dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u|+\eta)} \\
&- b_\infty\int_\Omega u^2dx - \int_\Omega g^2(x)dx - \int_\Omega u^2(x)dx \\
&> (1-\epsilon)\frac{\lambda_1}{1+\lambda_1}\|u\|^2_{H^1_{X,0}(\Omega)} - C_1\|u\|^2_{L^2(\Omega)} - \log\|u\|_{L^2}\int_{|u|>\eta} a(x)u^2dx \\
&- a_0\log 2\eta\int_{|u|\le\eta} u^2dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u|+\eta)} - \|g\|^2_{L^2(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
> & \ (1-\epsilon)\frac{\lambda_1}{1+\lambda_1}\|u\|^2_{H^1_{X,0}(\Omega)} - C_1\|u\|^2_{L^2(\Omega)} - \log\|u\|_{H^1_{X,0}(\Omega)}\int_{|u|>\eta} a(x)u^2dx \\
- & \ a_0\log 2\eta\int_{|u|\leq\eta} u^2dx + \int_\Omega \frac{a(x)u^2|u|}{2(|u|+\eta)} - \|g\|^2_{L^2(\Omega)},
\end{aligned}
$$

where $C_1 = a_\infty \log 2 + \epsilon + \frac{C_0}{4\epsilon}a_\infty^2 + b_\infty + 1$, $C_0 > 0$ is a positive constant given by Proposition 1.1, $a_\infty = \|a\|_{L^\infty}$, $b_\infty = \|b\|_{L^\infty}$.

If we set $B_R = \{u \in H^1_{X,0}(\Omega), \|u\|_{H^1_{X,0}(\Omega)} < R\}$, the estimate above shows that, as η is small enough, there exist $R = R(\epsilon)$ >0, and $\delta = \delta(R) > 0$ such that $J_\eta(u)|_{\partial B_R} \geq \delta > 0$ for all g with $\|g\|_{L^2(\Omega)} \leq C$. For example, we can take,

$$
R(\epsilon) = \exp\{\frac{C_1}{-a_0}\}, \qquad C = C(\epsilon) = \frac{R}{2}\sqrt{\frac{\lambda_1(1-\epsilon)}{1+\lambda_1}},
$$

$$
\delta(R) = \frac{\lambda_1(1-\epsilon)}{8(1+\lambda_1)}R^2(\epsilon), \qquad \eta < \frac{1}{2}\exp\{\frac{C_1}{-a_0}\}.
$$

Define $c_\eta = c_\eta(R) = \inf_{u\in\bar{B}_R} J_\eta(u)$, then $c_\eta \leq J_\eta(0) = 0$. The set $\bar{B}_R$ becomes a complete metric space with respect to the distance,

$$
dist(u,v) = \|u-v\|_{H^1_{X,0}(\Omega)} \text{ for any } u,v \in \bar{B}_R.
$$

On the other hand, J_η is lower semi-continuous and bounded from below on $\bar{B}_R$. So, by Proposition 1.5 (cf. [5] Theorem 1.1), for any positive integer n there exists $\{u_{\eta,n}\}$, satisfying

$$
c_\eta \leq J_\eta(u_{\eta,n}) \leq c_\eta + \frac{1}{n} \tag{3.1}
$$

$$
J_\eta(w) \geq J_\eta(u_{\eta,n}) - \frac{1}{n}\|u_{\eta,n} - w\|_{H^1_{X,0}(\Omega)} \text{ for all } w \in \bar{B}_R. \tag{3.2}
$$

We claim that $0 < \|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} < R$ for any n large enough. Indeed, if $\|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} = R$ for infinitely many n, we may assume, without loss of generality, that $\|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} = R$ for all $n \geq 1$. It follows that $J_\eta(u_{\eta,n}) \geq \delta > 0$. Combining this with (3.1) and letting $n \to \infty$, we have $0 \geq c_\eta \geq \delta > 0$ which is a contradiction.

We now prove that $J'_\eta(u_{\eta,n}) \to 0$ as $n \to \infty$ in $H^{-1}_{X,0}(\Omega)$. Indeed, for any $u \in H^{-1}_{X,0}(\Omega)$ with $\|u\|_{H^1_{X,0}(\Omega)} = 1$, let $w_n = u_{\eta,n}+tu$. For a fixed n, we have $\|w_n\|_{H^1_{X,0}(\Omega)} \leq \|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} + t < R$, where $t > 0$ is small enough. From (3.2) we obtain

$$
J_\eta(u_{\eta,n} + tu) \geq J_\eta(u_{\eta,n}) - \frac{t}{n}\|u\|_{H^1_{X,0}(\Omega)},
$$

that is

$$
\frac{J_\eta(u_{\eta,n} + tu) - J_\eta(u_{\eta,n})}{t} \geq -\frac{1}{n}\|u\|_{H^1_{X,0}(\Omega)} = -\frac{1}{n}.
$$

Letting $t \searrow 0$, we deduce that $\langle J'_\eta(u_{\eta,n}), u\rangle \geq -1/n$ and a similar argument for $t \nearrow 0$ produces $|\langle J'_\eta(u_{\eta,n}), u\rangle| \leq 1/n$ for any $u \in H^1_{X,0}(\Omega)$ with $\|u\|_{H^1_{X,0}(\Omega)} = 1$. So

$$\|J'_\eta(u_{\eta,n})\|_{-1} = \sup_{\substack{u\in H^1_{X,0}(\Omega)\\ \|u\|_{H^1_{X,0}(\Omega)}=1}} |\langle J'_\eta(u_{\eta,n}), u\rangle| \leq \frac{1}{n} \to 0 \text{ as } n \to \infty. \tag{3.3}$$

Thus, $\{u_{\eta,n}\}$ is a $(PS)_{c_\eta}$ sequence in $H^1_{X,0}(\Omega)$, i.e.

$$J_\eta(u_{\eta,n}) \to c_\eta, \text{ and } J'_\eta(u_{\eta,n}) \rightharpoonup 0 \text{ in } H^{-1}_{X,0}(\Omega). \tag{3.4}$$

Since $\|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} \leq R$, $\{u_{\eta,n}\}$ is a bounded sequence in $H^1_{X,0}(\Omega)$, and passing to a subsequence (denote still by $\{u_{\eta,n}\}$), we may assume that $u_{\eta,n} \rightharpoonup u_{\eta,0}$ in $H^1_{X,0}(\Omega)$ for some $u_{\eta,0} \in H^1_{X,0}(\Omega)$. So, by Lemma 2.3, we know that $J'_\eta(u_{\eta,0}) = 0$, i.e.

$$\begin{aligned} 2\int_\Omega \sum_{j=1}^m (X_j u_{\eta,0})(X_j v) &- 2\int_\Omega a(x)u_{\eta,0}v\log(|u_{\eta,0}|+\eta)dx + \int_\Omega \frac{a(x)u^2_{\eta,0}v\eta}{2(|u_{\eta,0}|+\eta)^2}dx \\ &- 2\int_\Omega b(x)u_{\eta,0}vdx - 2\int_\Omega g(x)vdx = 0, \end{aligned}$$

for all $v \in C_0^\infty(\Omega)$.

We know $\{u_{\eta,0}\}$ is also bounded in $H^1_{X,0}(\Omega)$. For $\eta = \eta_i = \frac{1}{2^i}$, $\frac{1}{2^i} < \frac{1}{2}\exp\{\frac{C_1}{-a_0}\}$, passing to a subsequence (denote still by $\{u_{\eta,n}\}$), we may assume that $u_{\eta_i,0} \rightharpoonup u_0$ in $H^1_{X,0}(\Omega)$ as $i \to \infty$. Now by Lemma 2.2, we have,

$$\int_\Omega \sum_{j=1}^m (X_j u_0)(X_j v) - \int_\Omega a(x)u_0 v\log|u_0|dx - \int_\Omega b(x)u_0vdx - \int_\Omega g(x)vdx = 0, \tag{3.5}$$

u_0 is a weak solution of (1.5) and (1.6).

We can prove that $J_0(u_0) = c_0$. Actually, we have

$$\begin{aligned} &J_\eta(u_{\eta,n}) + \frac{1}{2}\|J'_\eta(u_{\eta,n})\|_{-1}\|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} \geq J_\eta(u_{\eta,n}) - \frac{1}{2} < J'_\eta(u_{\eta,n}), u_{\eta,n} > \\ = &\int_\Omega \frac{a(x)u^2_{\eta,n}|u_{\eta,n}|}{2(|u_{\eta,n}|+\eta)} - \int_\Omega \frac{a(x)u^2_{\eta,n}|u_{\eta,n}|\eta}{4(|u_{\eta,n}|+\eta)^2} - \int_\Omega gu_{\eta,n}. \end{aligned}$$

Letting $n \to \infty$, we know

$$c_\eta \geq \int_\Omega \frac{a(x)u^2_{\eta,0}|u_{\eta,0}|}{2(|u_{\eta,0}|+\eta)} - \int_\Omega \frac{a(x)u^2_{\eta,0}|u_{\eta,0}|\eta}{4(|u_{\eta,0}|+\eta)^2} - \int_\Omega gu_{\eta,0}. \tag{3.6}$$

By Lemma 2.7, we have

$$\begin{aligned} 0 &= \langle J'_{\eta_i}(u_{\eta_i,0}), u_{\eta_i,0}\rangle = 2\|Xu_{\eta_i,0}\|^2_{L^2} - 2\int_\Omega a(x)u^2_{\eta_i,0}\log(|u_{\eta_i,0}|+\eta_i)dx \\ &+ \int_\Omega \frac{a(x)u^2_{\eta_i,0}|u_{\eta_i,0}|\eta_i}{2(|u_{\eta_i,0}|+\eta_i)^2}dx - 2\int_\Omega b(x)u^2_{\eta_i,0}dx - 2\int_\Omega g(x)u_{\eta_i,0}dx. \end{aligned}$$

Therefore

$$J_{\eta_i}(u_{\eta_i,0}) = \int_\Omega \frac{a(x)u^2_{\eta_i,0}|u_{\eta_i,0}|}{2(|u_{\eta_i,0}|+\eta_i)}dx - \int_\Omega \frac{a(x)u^2_{\eta_i,0}|u_{\eta_i,0}|\eta_i}{4(|u_{\eta_i,0}|+\eta_i)^2}dx - \int_\Omega g(x)u_{\eta_i,0}dx. \tag{3.7}$$

By (3.5), (3.6) and (3.7), we have:

$$\begin{aligned} 0 \geq c_0 &= \inf_{u\in \bar{B}_R} J_0(u) \geq \lim_{i\to\infty} \inf_{u\in\bar{B}_R} J_{\eta_i}(u) = \lim_{i\to\infty} c_{\eta_i} \\ &\geq \frac{1}{2}\int_\Omega a(x)u_0^2 dx - \int_\Omega g(x)u_0 dx = J_0(u_0). \end{aligned}$$

Since $u_0 \in \bar{B}_R$, it follows that $J_0(u_0) = c_0$.

On the other hand, letting $\tilde{u} \in H^1_{X,0}(\Omega)$, $\|\tilde{u}\|_{H^1_{X,0}(\Omega)} = R$, and $t > 0$, we have

$$\begin{aligned} J_\eta(t\tilde{u}) &< J_0(t\tilde{u}) = t^2\Big[\|X\tilde{u}\|^2_{L^2(\Omega)} - \log t\int_\Omega a(x)\tilde{u}^2 - \int_\Omega a(x)\tilde{u}^2\log|\tilde{u}| \\ &+ \frac{1}{2}\int_\Omega a(x)\tilde{u}^2 - \int_\Omega b(x)\tilde{u}^2 - 2\int_\Omega g(x)\tilde{u}/t\Big] \\ &< t^2\Big[\|X\tilde{u}\|^2_{L^2(\Omega)} - \log t\int_\Omega a(x)\tilde{u}^2 - \int_\Omega a(x)\tilde{u}^2\log|\tilde{u}| \\ &+ \frac{1}{2}\int_\Omega a(x)\tilde{u}^2 - \int_\Omega b(x)\tilde{u}^2 + \frac{1}{t}(\int_\Omega g^2(x) + \int_\Omega \tilde{u}^2)\Big]. \end{aligned}$$

We can find $\bar{t} >> 1$, such that $J_\eta(t\tilde{u}) < J_0(t\tilde{u}) < 0$ for all $t \geq \bar{t}$. Letting $\bar{u} = \bar{t}\tilde{u}$, then we have $\|\bar{u}\|_{H^1_{X,0}(\Omega)} > R$ and $J_\eta(\bar{u}) < 0$.

We put

$$\varrho = \{\gamma \in C([0,1], H^1_{X,0}(\Omega)) : \gamma(0) = 0, \gamma(1) = \bar{t}\tilde{u}, \}, \tag{3.8}$$

$$\bar{c}_\eta = \inf_{\gamma\in\varrho}\sup_{u\in\gamma} J_\eta(u). \tag{3.9}$$

For $\gamma_0 = \{t\bar{t}\tilde{u} :\ 0 \leq t \leq 1\}$, we have

$$\begin{aligned} &\sup_{u\in\gamma_0} J_\eta(u) \leq \sup_{u\in\gamma_0} J_0(u) = \sup_{0\leq t\leq 1}\ [(t\bar{t})^2\|X\tilde{u}\|^2_{L^2(\Omega)} - (t\bar{t})^2\log(t\bar{t})\int_\Omega a(x)\tilde{u}^2 \\ &- (t\bar{t})^2\int_\Omega a(x)\tilde{u}^2\log|\tilde{u}| + \frac{(t\bar{t})^2}{2}\int_\Omega a(x)\tilde{u}^2 - (t\bar{t})^2\int_\Omega b(x)\tilde{u}^2 - 2(t\bar{t})\int_\Omega g(x)\tilde{u}] \\ &\leq \bar{t}^2\|X\tilde{u}\|^2_{L^2(\Omega)} + \frac{1}{2e}\int_\Omega a(x)\tilde{u}^2 + \bar{t}^2\int_\Omega a(x)\tilde{u}^2|\log|\tilde{u}|| + \frac{\bar{t}^2}{2}\int_\Omega a(x)\tilde{u}^2 \\ &+ \bar{t}\int_\Omega g^2 + \bar{t}\int_\Omega \tilde{u}^2. \end{aligned}$$

So there exists a positive constant B (which is independent of η), satisfying

$$\bar{c}_\eta \leq B. \tag{3.10}$$

It follows from the Proposition 1.6 (cf. [3] Theorem 2.2) that there is a $(PS)_{c_\eta}$ sequence $\{u_{\eta,n}\}$ of $J_\eta(u)$ such that

$$J_\eta(u_{\eta,n}) = \bar{c}_\eta + o(1) \text{ and } J'_\eta(u_{\eta,n}) \ \to 0 \qquad \text{in} \qquad H^{-1}_{X,0}(\Omega).$$

We have

$$\begin{aligned}
& J_\eta(u) - \frac{1}{2} < J'_\eta(u),\ u >= \int_\Omega \frac{a(x)u^2|u|}{2(|u|+\eta)}dx - \int_\Omega \frac{a(x)u^2|u|\eta}{4(|u|+\eta)^2}dx - \int_\Omega g(x)udx \\
> & \int_\Omega \frac{a(x)u^2|u|}{4(|u|+\eta)}dx - \frac{a_0}{16}\int_\Omega u^2dx - \frac{4}{a_0}\int_\Omega g^2dx \\
= & \int_{|u|>\eta} \frac{a(x)u^2|u|}{4(|u|+\eta)}dx + \int_{|u|\leq\eta} \frac{a(x)u^2|u|}{4(|u|+\eta)}dx - \frac{a_0}{16}\int_{|u|>\eta} u^2dx - \frac{a_0}{16}\int_{|u|\leq\eta} u^2dx \\
- & \frac{4}{a_0}\int_\Omega g^2dx \\
> & \frac{1}{4}\int_{|u|>\eta} \frac{a(x)u^2|u|}{2|u|}dx - \frac{a_0}{16}\int_{|u|>\eta} u^2dx - \frac{a_0}{16}\int_{|u|\leq\eta} u^2dx - \frac{4}{a_0}\int_\Omega g^2dx \\
> & \frac{a_0}{16}\int_{|u|>\eta} u^2dx - \frac{a_0\eta^2|\Omega|}{16} - \frac{4}{a_0}\|g\|^2_{L^2}.
\end{aligned}$$

So, we have

$$\begin{aligned}
& \bar{c}_\eta + o(1) + \frac{1}{2}\|J'_\eta(u_{\eta,n})\|_{-1}\|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} + \frac{a_0|\Omega|}{16} + \frac{4}{a_0}\|g\|^2_{L^2} \\
\geq & \ J_\eta(u_{\eta,n}) - \frac{1}{2}\langle J'_\eta(u_{\eta,n}), u_{\eta,n}\rangle + \frac{a_0|\Omega|}{16} + \frac{4}{a_0}\|g\|^2_{L^2} \\
> & \frac{a_0}{16}\int_{|u|>\eta} u^2_{\eta,n}dx.
\end{aligned}$$

By (3.10), we have

$$\begin{aligned}
& \int_\Omega |u_{\eta,n}|^2dx = \int_{|u|>\eta} |u_{\eta,n}|^2dx + \int_{u\leq\eta} |u_{\eta,n}|^2dx \\
< & \frac{16}{a_0}\left[\bar{c}_\eta + o(1) + \frac{1}{2}\|J'_\eta(u_{\eta,n})\|_{-1}\|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} + + \frac{a_0|\Omega|}{16} + \frac{4}{a_0}\|g\|^2_{L^2}\right] + \eta^2|\Omega| \\
< & \frac{16}{a_0}\left[B + o(1) + \frac{1}{2}\|J'_\eta(u_{\eta,n})\|_{-1}\|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} + \frac{a_0|\Omega|}{16} + \frac{4}{a_0}\|g\|^2_{L^2}\right] + |\Omega| \\
< & \ C + C\|J'_\eta(u_{\eta,n})\|_{-1}\|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} + o(1),
\end{aligned} \tag{3.11}$$

where C is a positive constant which is independent of η and n, and dependent of $|\Omega|$, $\|g\|_{L^2}^2$, a_0, and B. Similar to the estimate of $J_\eta(u)$ at the beginning of this section, we have (if taking $\epsilon = \frac{1}{2}$)

$$\begin{aligned} B + o(1) &> \bar{c}_\eta + o(1) = J_\eta(u_{\eta,n}) \geq \frac{\lambda_1}{2(1+\lambda_1)} \|u_{\eta,n}\|_{H^1_{X,0}}^2 - C_1 \|u_{\eta,n}\|_{L^2}^2 \\ &- a_\infty \|u_{\eta,n}\|_{L^2}^2 |\log \|u_{\eta,n}\|_{L^2}| - \|g\|_{L^2(\Omega)}^2, \end{aligned}$$

where $C_1 = a_\infty \log 2 + \frac{C_0}{2} a_\infty^2 + b_\infty + \frac{3}{2}$, to be independent of η and n, and C_0 and λ_1 are given by Proposition 1.1 and Proposition 1.3 respectively.

Furthermore, using the fact $|t \log t| \leq t^2 + e^{-1}$ for $t \geq 0$, we have

$$\begin{aligned} \frac{\lambda_1}{2(1+\lambda_1)} \|u_{\eta,n}\|_{H^1_{X,0}}^2 &\leq B + o(1) + C_1 \|u_{\eta,n}\|_{L^2}^2 + a_\infty |\|u_{\eta,n}\|_{L^2}^2 | \log \|u_{\eta,n}\|_{L^2}| + \|g\|_{L^2(\Omega)}^2 \\ &\leq B + o(1) + C_1 \|u_{\eta,n}\|_{L^2}^2 + \frac{1}{2} a_\infty (\|u_{\eta,n}\|_{L^2}^4 | + e^{-1}) + \|g\|_{L^2(\Omega)}^2 \\ &< C + o(1) + C \|u_{\eta,n}\|_{L^2}^2 + C \|u_{\eta,n}\|_{L^2}^4, \end{aligned}$$

where C is independent of η and n.

By (3.11), we have

$$\|u_{\eta,n}\|_{H^1_{X,0}}^2 \leq C + o(1) + C \|J'_\eta(u_{\eta,n})\|_{-1} \|u_{\eta,n}\|_{H^1_{X,0}(\Omega)} + C \|J'_\eta(u_{\eta,n})\|_{-1}^2 \|u_{\eta,n}\|_{H^1_{X,0}(\Omega)}^2.$$

Since $J'_\eta(u_{\eta,n}) \to 0$ in $H^{-1}_{X,0}(\Omega)$, thus there exists $N_0 > 0$ such that $\|u_{\eta,n}\|_{H^1_{X,0}}^2 \leq M$, if $n > N_0$, where M is a constant, independent of η and n. That means $\{u_{\eta,\ N_0+j}\}_{j \in N}$ is a bounded sequence in $H^1_{X,0}(\Omega)$. Hence there exists a subsequence (we still denote by $\{u_{\eta,n}\}$), such that $u_{\eta,n} \rightharpoonup u_{\eta,0}$ in $H^1_{X,0}(\Omega)$ for some $u_{\eta,0} \in H^1_{X,0}(\Omega)$. By Lemma 2.3, we have $J'_\eta(u_{\eta,0}) = 0$, that is

$$\begin{aligned} &2 \int_\Omega \sum_{j=1}^m (X_j u_{\eta,0})(X_j v) - 2 \int_\Omega a(x) u_{\eta,0} v \log(|u_{\eta,0}| + \eta) dx \\ &+ \int_\Omega \frac{a(x) u_{\eta,0} |u_{\eta,0}| v \eta}{2(u_{\eta,0} + \eta)^2} dx - 2 \int_\Omega b(x) u_{\eta,0} v dx - 2 \int_\Omega g(x) v dx = 0, \end{aligned} \tag{3.12}$$

for any $v \in C_0^\infty(\Omega)$.

For $\eta = \eta_i = \frac{1}{2^i}$, $\frac{1}{2^i} < \frac{1}{2} \exp\{\frac{C_1}{-a_0}\}$, we know $\{u_{\eta_i,0}\}$ is also bounded in $H^1_{X,0}(\Omega)$. Passing to a subsequence, we may assume that $u_{\eta_i,0} \rightharpoonup u_1$ in $H^1_{X,0}(\Omega)$ as $i \to \infty$. Now by Lemma 2.2, we have,

$$\begin{aligned} &\int_\Omega \sum_{j=1}^m (X_j u_1)(X_j v) - \int_\Omega a(x) u_1 v \log |u_1| dx - \int_\Omega b(x) u_1 v dx \\ &- \int_\Omega g(x) v dx = 0 \end{aligned} \tag{3.13}$$

for all $v \in C_0^\infty(\Omega)$. That means u_1 is a weak solution of problem (1.5) and (1.6).

Next, we prove $u_{\eta_i,0} \to u_1$ in $H^1_{X,0}(\Omega)$. In fact, $C_0^\infty(\Omega)$ is dense in $H^1_{X,0}(\Omega)$, thus from Lemma 2.4 and Lemma 2.5, we know that (3.12) and (3.13) are also true for any $v \in H^1_{X,0}(\Omega)$.

Especially, we have

$$\begin{aligned} & 2\int_\Omega \sum_{j=1}^m (X_j u_{\eta_i,0})^2 - 2\int_\Omega a(x) u_{\eta_i,0}^2 \log(|u_{\eta_i,0}| + \eta_i) dx \\ + & \int_\Omega \frac{a(x) u_{\eta_i,0}^2 |u_{\eta_i,0}| \eta_i}{2(|u_{\eta_i,0}| + \eta_i)^2} dx - 2\int_\Omega b(x) u_{\eta_i,0}^2 dx - 2\int_\Omega g(x) u_{\eta_i,0} dx = 0, \end{aligned} \tag{3.14}$$

$$\int_\Omega \sum_{j=1}^m (X_j u_1)^2 - \int_\Omega a(x) u_1^2 \log|u_1| dx - \int_\Omega b(x) u_1^2 dx - \int_\Omega g(x) u_1 dx = 0. \tag{3.15}$$

Letting $i \to \infty$ in (3.14), and from Lemma 2.8 and (3.15), we have

$$\|X_j u_{\eta_i,0}\|_{L^2(\Omega)} \to \|X_j u_1\|_{L^2(\Omega)}, \quad i \to \infty,$$

which means $u_{\eta_i,0} \to u_1$ in $H^1_{X,0}(\Omega)$.
Now by Proposition 1.6 ([3]), we have

$$J_0(u_1) = \lim_{i\to\infty} J_{\eta_i}(u_{\eta_i,0}) = \bar{c}_0 > 0 \geq J_0(u_0),$$

that means the problem (1.5) and (1.6) has at least two solutions in $H^1_{X,0}(\Omega)$.

If we replace, at the beginning, B_R by $B_R^+ = \{u \in H^1_{X,0}(\Omega),\ \|u\|_{H^1_{X,0}(\Omega)} < R,\ u \geq 0\}$, thus it is similar to the proof of existence of the solution u_0, we can deduce that the problem (1.5) and (1.6) has a non-negative solution in $H^1_{X,0}(\Omega)$.

4 Boundedness and regularity of weak solutions

Similar to the proof of [10], we can deduce the boundedness and regularity of weak solutions.

By using the interpolation inequality, the condition H-3) and the Logarithmic Sobolev inequality (1.2) give that, for any $N \geq 1$, there exists C_N such that,

$$\int_\Omega v^2 \log^2\left(\frac{|v|}{\|v\|_{L^2}}\right) \leq \frac{1}{N}\|Xv\|_{L^2}^2 + C_N \|v\|_{L^2}^2, \tag{4.1}$$

for all $v \in H^1_{X,0}(\Omega)$.

In order to prove that the solution $u \in L^\infty(\Omega)$, it suffices to show that, under the assumptions of Theorem 1.4, there exists $\overline{A} > 0$ such that the estimate

$$\|u\|_{L^p} \le \overline{A} \tag{4.2}$$

holds for any $p \ge 2$. In fact, for $\epsilon > 0$, $\Omega_\epsilon = \{x \in \Omega; |u(x)| \ge \overline{A} + \epsilon\}$, it follows from (4.2) that $|\Omega_\epsilon| \le (\frac{\overline{A}}{\overline{A}+\epsilon})^p \to 0$ (as $p \to \infty$) and hence we have $\|u\|_{L^\infty} \le \overline{A}$.

We prove the estimate (4.2) by the following three steps. First, for any $p \ge 1$, $m \in \mathbf{N}$, we shall use u^{2p-1} or $u^{2p-1}\log^{2m}(u^p)$ as test function for the equation (1.5). Since we do not know if $u^{2p-1}\log^{2m}(u^p) \in H^1_{X,0}(\Omega)$, so we replace the function u by $u_{(k)}$, where $k > 1$ and $u_{(k)}(x) = u(x)$ if $x \in \{x \in \Omega; |u(x)| < k\}$ and $u_{(k)}(x) = k$ if $x \in \{x \in \Omega; |u(x)| \ge k\}$. Then it is easy to check (see [6] and [7, Theorem 7 and Theorem 8]) that $u^{2p-1}_{(k)}\log^{2m}(u^p_{(k)}) \in H^1_{X,0}(\Omega)$ for all $p > 1$, $m \in \mathbb{N}$. In the case of $p = 1$, we use $u(\log^m u)^2_{(k)} \in H^1_{X,0}(\Omega)$ as the test function. To simplify the notation, we shall drop the subscript and use $u^{2p-1}\log^{2m}(u^p)$ as the test function. We have

Proposition 4.1. *Under the hypotheses H-1), H-2), H-3), H-4) of Theorem 1.4, and $g(x) \in L^\infty(\Omega)$, $u \in H^1_{X,0}(\Omega), u \ge 0, \|u\|_{L^2(\Omega)} \ne 0$ be a weak solution of the equation* (1.5). *Suppose that for some $p_0 \ge 1$, there exists A_0, A_1 such that*

$$0 < A_1 \le \|u\|_{L^{2p_0}} \le A_0.$$

Then

$$\begin{aligned} &\int_\Omega |X(\bar{u})^{p_0}|^2 + \int_\Omega (\bar{u})^{2p_0}\log^2(\bar{u}^{p_0}) \\ \le\ & 2C_2 + a_\infty^2 + 2p_0[b_\infty + a_\infty|\log A_0| + (1+|\Omega|)g_\infty/A_1], \end{aligned} \tag{4.3}$$

where $a_\infty = \|a\|_{L^\infty},\ b_\infty = \|b\|_{L^\infty},\ g_\infty = \|g\|_{L^\infty}$ and the constant C_2 is given by (4.1) *and $\bar{u} = u/\|u\|_{L^{2p_0}}$.*

Proof. We have $\bar{u} \in H^1_{X,0}(\Omega), \|\bar{u}\|_{L^{2p_0}} = 1$, and $\bar{u}$ is a weak solution of equation

$$-\triangle_X \bar{u} = a(x)\bar{u}\log\bar{u} + (a(x)\log\|u\|_{L^{2p_0}} + b(x))\bar{u} + \frac{g(x)}{\|u\|_{L^{2p_0}}}. \tag{4.4}$$

Take $\bar{u}^{2p_0-1}$ as the test function, we have

$$\begin{aligned} &\frac{2p_0-1}{p_0^2}\int_\Omega |X\bar{u}^{p_0}|^2 = \frac{1}{p_0}\int_\Omega a(x)\bar{u}^{2p_0}\log\bar{u}^{p_0} \\ +\ &\int_\Omega (a(x)\log\|u\|_{L^{2p_0}} + b(x))\bar{u}^{2p_0} + \frac{1}{\|u\|_{L^{2p_0}}}\int_\Omega g(x)\bar{u}^{2p_0-1}, \end{aligned}$$

where

$$\frac{1}{\|u\|_{L^{2p_0}}}\int_\Omega g(x)\bar{u}^{2p_0-1} \le \frac{g_\infty}{A_1}[\int_{\bar{u}>1}|\bar{u}^{2p_0-1}| + \int_{\bar{u}\le 1}|\bar{u}^{2p_0-1}|]$$
$$\le \frac{g_\infty}{A_1}(\int_{\bar{u}>1}\bar{u}^{2p_0} + |\Omega|) \le \frac{g_\infty}{A_1}(\int_\Omega \bar{u}^{2p_0} + |\Omega|) = \frac{(1+|\Omega|)g_\infty}{A_1}.$$

Furthermore

$$\int_\Omega |X\bar{u}^{p_0}|^2 \le \frac{1}{2}\int_\Omega \bar{u}^{2p_0}\log^2(\bar{u}^{p_0}) + \frac{1}{2}a_\infty^2 + p_0 a_\infty|\log A_0| + p_0 b_\infty + \frac{(1+|\Omega|)p_0 g_\infty}{A_1}. \quad (4.5)$$

On the other hand, the Logarithmic Sobolev inequality (4.1) gives

$$\int_\Omega (u^{p_0})^2\log^2(\frac{|u^{p_0}|}{\|u^{p_0}\|_{L^2}}) \le \frac{1}{2}\|X(u^{p_0})\|_{L^2}^2 + C_2\|u^{p_0}\|_{L^2}^2.$$

Note that $\|u^{p_0}\|_{L^2} = \|u\|_{L^{2p_0}}^{p_0}$ and $\bar{u} = u/\|u\|_{L^{2p_0}}$, we have

$$\int_\Omega \bar{u}^{2p_0}\log^2(\bar{u}^{p_0}) \le \frac{1}{2}\|X(\bar{u}^{p_0})\|_{L^2}^2 + C_2. \quad (4.6)$$

Adding (4.5) and (4.6), we have the desired estimate (4.3).

Proposition 4.2. *We have for any $m \in \mathbb{N}$,*

$$\int_\Omega |X(\bar{u}^{p_0})|^2\log^{2m-2}(\bar{u}^{p_0}) + \int_\Omega \bar{u}^{2p_0}\log^{2m}(\bar{u}^{p_0}) \le M_1^{2m}P(m,p_0)(m!)^2 \quad (4.7)$$

where $P(m,p_0) = p_0^m$ if $m \le \sqrt{p_0}$, $P(m,p_0) = p_0^{\sqrt{p_0}}$ if $m > \sqrt{p_0}$, and

$$M_1 \ge (2|\Omega| + 4C_2 + 2C_4 + 10 + 6a_\infty^2 + 8b_\infty + 8a_\infty|\log A_0| + 4g_\infty(1+|\Omega|)/A_1)^{\frac{1}{2}}.$$

Proof. From the estimate $0 < A_1 \le \|u\|_{L^{2p_0}} \le A_0$, we have the estimate (4.7) for $m = 1$. By induction, we suppose that (4.7) is also hold for $m \in \mathbb{N}$, then we need to prove that (4.7) is hold for $m+1$. Here we simplify the notation again, i.e. $\bar{u}$ and p_0 would be replaced by u and p in the equation (4.4). We take $u^{2p-1}\log^{2m}(u^p)$ as the test function in (4.4) , then

$$\frac{2p-1}{p^2}\int_\Omega |Xu^p|^2\log^{2m}(u^p) + \frac{2m}{p}\int_\Omega |Xu^p|^2\log^{2m-1}(u^p)$$
$$= \frac{1}{p}\int_\Omega a(x)u^{2p}\log^{2m+1}(u^p) + \int_\Omega (a(x)\log\|u\|_{L^{2p}} + b(x))u^{2p}\log^{2m}(u^p)$$
$$+ \int_\Omega \frac{g(x)}{\|u\|_{L^{2p}}}u^{2p-1}\log^{2m}(u^p).$$

That is

$$\int_\Omega |Xu^p|^2\log^{2m}(u^p)\le \frac{1}{2}\int_\Omega |Xu^p|^2\log^{2m}(u^p)+2m^2\int_\Omega |Xu^p|^2\log^{2m-2}(u^p)$$
$$+\quad \frac{1}{4}\int_\Omega u^{2p}\log^{2m+2}(u^p)+(a_\infty^2+pa_\infty\log A_0+pb_\infty)\int_\Omega u^{2p}\log^{2m}(u^p)$$
$$+\quad \frac{pg_\infty}{A_1}\int_\Omega u^{2p-1}\log^{2m}(u^p).$$

Using the fact $l^l\le e^l l!$, we have

$$\int_\Omega u^{2p-1}\log^{2m}(u^p)=\int_{|u|<1} u^{2p-1}\log^{2m}(u^p)+\int_{|u|\ge 1} u^{2p-1}\log^{2m}(u^p)$$
$$\le\quad 2^{2m}(m!)^2|\Omega|+\int_\Omega u^{2p}\log^{2m}(u^p)<(1+|\Omega|)M_1^{2m}P(m,p)(m!)^2,$$

so that

$$\int_\Omega |Xu^p|^2\log^{2m}(u^p)\le \frac{1}{2}\int_\Omega (u^p)^2\log^{2m+2}(u^p)+[4m^2+2a_\infty^2+$$
$$2(pa_\infty|\log A_0|+pb_\infty+pg_\infty(1+|\Omega|)/A_1)]M_1^{2m}P(m,p)(m!)^2. \qquad (4.8)$$

We study now the term $\int_\Omega u^{2p}\log^{2m+2}(u^p)$. Set $\Omega=\Omega_1\bigcup\Omega_2^+\bigcup\Omega_2^-$ with $\Omega_1=\{x\in\Omega; u(x)\le 1\}$ and

$$\Omega_2^+=\{x\in\Omega;\quad u(x)>1,\quad |\log^m(u^p)|\le\|u^p\log^m(u^p)\|_{L^2}\},$$
$$\Omega_2^-=\{x\in\Omega;\quad u(x)>1,\quad |\log^m(u^p)|>\|u^p\log^m(u^p)\|_{L^2}\}.$$

Then

$$\int_{\Omega_1} u^{2p}\log^{2m+2}(u^p)\le|\Omega|((m+1)!)^2.$$

For the second part, (4.3) gives

$$\int_{\Omega_2^+} u^{2p}\log^{2m+2}(u^p)\le\|u^p\log^m(u^p)\|_{L^2}^2\int_\Omega u^{2p}\log^2(u^p)$$
$$\le\quad (2C_2+a_\infty^2+2pb_\infty+2pa_\infty|\log A_0|+(1+|\Omega|)g_\infty/A_1)M_1^{2m}P(m,p)(m!)^2.$$

Next, for the third part, we use the Logarithmic Sobolev inequality (4.1) for $N=4$,

$$\begin{aligned}\int_{\Omega_2^-} u^{2p}\log^{2m+2}(u^p) &\le \int_{\Omega_2^-}(u^p\log^m u^p)^2\log^2\Big(\frac{u^p\log^m(u^p)}{\|u^p\log^m(u^p)\|_{L^2}}\Big)\\ &\le \frac{1}{4}\|X(u^p\log^m u^p)\|_{L^2}^2+C_4\|u^p\log^m u^p\|_{L^2}^2\\ &\le \frac{1}{2}\int_\Omega |X(u^p)|^2\log^{2m}(u^p)+m^2\int_\Omega |X(u^p)|^2\log^{2m-2}(u^p)\\ &+ C_4\int_\Omega u^{2p}\log^{2m}(u^p)\\ &\le \frac{1}{2}\int_\Omega |X(u^p)|^2\log^{2m}(u^p)+(C_4+m^2)M_1^{2m}P(m,p)(m!)^2.\end{aligned}$$

Sum up the three parts above, we get

$$\int_\Omega u^{2p}\log^{2m+2}(u^p) \le \frac{1}{2}\int_\Omega |X(u^p)|^2\log^{2m}(u^p) + |\Omega|((m+1)!)^2$$
$$+[2C_2 + C_4 + m^2 + a_\infty^2 + 2pb_\infty + 2pa_\infty|\log A_0|$$
$$+(1+|\Omega|)g_\infty/A_1]M_1^{2m}P(m,p)(m!)^2.$$

which implies by (4.8),

$$\int_\Omega u^{2p}\log^{2m+2}(u^p) + \int_\Omega |X(u^p)|^2\log^{2m}(u^p) \le [2\Omega + 4C_2 + 2C_4 + 10 \tag{4.9}$$
$$+6a_\infty^2 + 8b_\infty + 8a_\infty|\log A_0| + 2g_\infty(1+|\Omega|)/A_1]M_1^{2m}P(m+1,p)((m+1)!)^2.$$

Proposition 4.2 is proved.

Proposition 4.3. *Under the hypotheses of Proposition* 4.1*, if for some* $p_0 \ge 1$ *and* $A_0 \ge e^{12}$ *we have*

$$\|u\|_{L^{2p_0}} \le A_0,$$

then for

$$M_1 \ge [2|\Omega| + 4C_2 + 2C_4 + 10 + 6a_\infty^2 + 8b_\infty + 8a_\infty\log A_0 + 2g_\infty(1+|\Omega|)/A_1]^{\frac{1}{2}},$$

and $\delta = 1/2M_1$*, we have*

$$\int_\Omega u^{2p_0(1+\delta)} \le A_0^{2p_0(1+\delta)(1+(\frac{1}{p_0(1+\delta)})^{\frac{1}{3}})} \tag{4.10}$$

Proof. For any $\delta > 0$, the estimate (4.7) gives that

$$\left(\int_\Omega |\bar{u}^{p_0(1+\delta)}|^2 dx\right)^{\frac{1}{2}} = \left(\int_\Omega |\bar{u}^{p_0}\bar{u}^{\delta p_0}|^2 dx\right)^{\frac{1}{2}} = \left(\int_\Omega |\bar{u}^{p_0}e^{\delta\log(\bar{u}^{p_0})}|^2 dx\right)^{\frac{1}{2}}$$
$$= \left(\int_\Omega |\bar{u}^{p_0}\sum_{m=0}^\infty \frac{(\delta\log(\bar{u}^{p_0}))^m}{m!}|^2 dx\right)^{\frac{1}{2}} \le \sum_{m=0}^\infty \left(\int_\Omega \bar{u}^{2p_0}\frac{(\delta\log(\bar{u}^{p_0}))^{2m}}{(m!)^2}dx\right)^{\frac{1}{2}}$$
$$\le \sum_{m=0}^\infty \frac{\delta^m}{m!}\left(\int_\Omega \bar{u}^{2p_0}\log^{2m}(\bar{u}^{p_0})dx\right)^{\frac{1}{2}} \le \sum_{m=0}^\infty \delta^m M_1^m P(m,p_0) \le p_0^{\sqrt{p_0}}\sum_{m=0}^\infty (\delta M_1)^m.$$

For $\delta = 1/2M_1$, we have finally

$$\int_\Omega u^{2p_0(1+\delta)}dx \le 4p_0^{2\sqrt{p_0}}A_0^{2p_0(1+\delta)}.$$

Since for any $p_0 > 1$,

$$4p_0^{2\sqrt{p_0}} = 4e^{2\sqrt{p_0}\log p_0} \le (e^{12})^{2p_0^{\frac{2}{3}}},$$

which implies the estimate (4.10) if $A_0 \geq e^{12}$.

We set now for $k \in \mathbb{N}$,

$$p_k = p_0(1+\delta)^k, \qquad A_k = A_0^{1+p_0^{-1/3}\sum_{j=1}^{k}(\frac{1}{1+\delta})^{j/3}},$$

then Proposition 4.3 implies that

$$\begin{aligned}\int_\Omega u^{2p_0(1+\delta)^{k+1}} &= \int_\Omega u^{2p_k(1+\delta)} \leq A_k^{2p_k(1+\delta)(1+(\frac{1}{p_k(1+\delta)})^{1/3})} \\ &\leq A_0^{2p_0(1+\delta)^{k+1}(1+p_0^{-1/3}\sum_{j=1}^{k+1}(\frac{1}{1+\delta})^{j/3})},\end{aligned}$$

where $\delta = \frac{1}{2}M_1$ and

$$M_1 \geq [2|\Omega| + 4C_2 + 2C_4 + 10 + 6a_\infty^2 + 8b_\infty + 8a_\infty|\log A_k| + 2g_\infty(1+|\Omega|)/A_1]^{1/2}. \quad (4.11)$$

We have now for $\delta = \frac{1}{2}M_1 \leq 1/4$,

$$\begin{aligned}\frac{\log A_k}{\log A_0} &= 1 + p_0^{-1/3}\sum_{j=1}^{k}(\frac{1}{1+\delta})^{j/3} \leq 1 + p_0^{-1/3}\sum_{j=1}^{\infty}(\frac{1}{1+\delta})^{j/3} \\ &= 1 + p_0^{-1/3}\frac{(\frac{1}{1+\delta})^{1/3}}{1-(\frac{1}{1+\delta})^{1/3}} \leq 1 + 4p_0^{-1/3}M_1 \leq 5M_1,\end{aligned}$$

where M_1 is independent of k, thus we have proved for any $k \in \mathbb{N}$,

$$\int_\Omega u^{2p_0(1+\delta)^k} \leq (A_0^{5M_1})^{2p_0(1+\delta)^k}.$$

If we choose $A_0 = e^{12}$, then the estimate (4.2) holds for $\bar{A} = e^{60M_1}$.

The regularity of the solution for the problem (1.5) and (1.6) can be deduced by following result:

Proposition 4.4. *Suppose* $a(x),\ b(x),\ g(x) \in C^\infty(\Omega)$, *and there exist* $a_0,\ b_0,\ g_0 > 0$, *such that* $a(x) \geq a_0,\ b(x) \geq b_0,\ g(x) \geq g_0$ *in* Ω. *Let* $u \in H^1_{X,0}(\Omega),\ u \geq 0,\ \|u\|_{L^2} \neq 0$ *be a weak solution of the problem* (1.5) *and* (1.6), *and* $\partial\Omega$ *is non characteristic. Then* $u \in C^\infty(\Omega\backslash\Gamma)\bigcap C^0(\overline{\Omega}\backslash\Gamma)$, *and* $u(x) > 0$ *for all* $x \in \Omega\backslash\Gamma$.

Proof. Suppose $x_0 \in \Omega\backslash\Gamma$, then there exists a neighborhood $V_0 \subset \Omega\backslash\Gamma$ of x_0, for $\varphi \in C_0^\infty(V_0)$ we shall prove that $v = \varphi u \in C^\infty(V_0)$. It follows from equation (1.5) that,

$$-\Delta_X v = a(x)\varphi u \log u + b(x)\varphi u + g(x)\varphi + \sum_{j=1}^{m}\varphi_j X_j u + \varphi_0 u = f_0 + \sum_{j=1}^{m} X_j f_j,$$

with $\varphi_j \in C^\infty(V_0)$, $f_j \in L^\infty(V_0)$, $j = 0, ..., m$. Since the system of vector fields X satisfies the finitely type Hörmander's condition on V_0, the regularity result of [8] (see also [7, 9]) implies that $u \in C^\epsilon(V_0)$ for some $\varepsilon > 0$. If we have $u(x) \geq \alpha > 0$ for $x \in V_0$, then by $t \log t \in C^\infty(t \geq \alpha)$, we can deduce $u \log u \in C^\varepsilon(V_0)$, thus we can prove by recurrence that $u \in C^\infty(V_0)$. For $x_0 \in \partial\Omega \backslash \Gamma$, we have also $u \in C^\epsilon(V_0 \bigcap \bar{\Omega})$, but we know only $u \log u \in C^0(V_0 \bigcap \bar{\Omega})$, so we can not obtain the C^∞ regularity of u near to the boundary $\partial\Omega$. Therefore the Proposition 4.4 will be deduced by the following Lemma directly.

Lemma 4.5. *Suppose $a(x), b(x), g(x)$ satisfy the conditions of Proposition 4.4, and $u \in C^0(\Omega_1)$, $u \geq 0$ is a non trivial weak solution of the equation* (1.5) *on an open set $\Omega_1 \subset \Omega$, then $u(x) > 0$ for all $x \in \Omega_1$.*

Proof. Suppose that $u(x_0) = 0$ for some $x_0 \in \Omega_1$, then for any $\epsilon > 0$, there exists a small neighborhood $U_0 \subset \Omega_1$ of x_0 such that $0 \leq u(x) \leq \epsilon$ on $\bar{U}_0$. Since $g(x)$ is continuous on $\bar{U}_0$, there exists $\alpha > 0$ such that $g(x) \geq \alpha$ on $\bar{U}_0$.

Choosing ϵ small enough such that in U_0, we have

$$a(x)u \log u + b(x)u < 0,$$

and

$$a(x)u \log u + b(x)u + g(x) \geq 0.$$

That is $\Delta_X u \leq 0$ in U_0. But x_0 is a minimum point of u, the maximum principle of Bony [10] implies that $u \equiv 0$ in U_0. That means u is a trivial solution by continuous of u in Ω_1.

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