Local existence of classical solutions for the Einstein–Euler system using weighted Sobolev spaces of fractional order

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Abstract

We prove the existence of a class of local in time solutions, including static solutions, of the Einstein–Euler system. This result is the relativistic generalisation of a similar result for the Euler–Poisson system obtained by Gamblin [8]. As in his case the initial data of the density do not have compact support but fall off at infinity in an appropriate manner. An essential tool in our approach is the construction and use of weighted Sobolev spaces of fractional order. Moreover, these new spaces allow us to improve the regularity conditions for the solutions of evolution equations. The details of this construction, the properties of these spaces and results on elliptic and hyperbolic equations will be presented in a forthcoming article.

1 The initial value problem for the Euler–Einstein system

We consider the Einstein-Euler system describing a relativistic self-gravitating perfect fluid. The unknowns in the equations are functions of t and x^a, where x^a $(a = 1, 2, 3)$ are Cartesian coordinates of $\mathbb{R}^3$. The alternative notation $x^0 = t$ will also be used and Greek indices will take the values $0, 1, 2, 3$ in the following. The evolution of the gravitational field is described by the Einstein equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{1}$$

where $G_{\alpha\beta}$ is the Einstein tensor of the spacetime metric $g_{\alpha\beta}$ and $T_{\alpha\beta}$ is the energy-momentum tensor of the matter. In the case of a perfect fluid the latter takes the form

$$T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta} \tag{2}$$

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where ϵ is the energy density, p is the pressure and u^α is the four-velocity. The quantity u^α is required to satisfy the normalisation condition

$$g_{\alpha\beta}u^\alpha u^\beta = -1. \tag{3}$$

The Euler equations describing the evolution of the fluid take the form

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{4}$$

To get a determined system of equations it is necessary to specify a relation between ϵ and p (equation of state). The choice we make here is one which has been used for astrophysical problems. It is an analogue of the well known polytropic equation of state of the non-relativistic theory given by:

$$p = f(\epsilon) = K\epsilon^\gamma \qquad K, \gamma \in \mathbb{R}^+ \qquad 1 < \gamma. \tag{5}$$

The new matter variable $w = M(\epsilon)$ which is needed to regularise Euler equations even for $\epsilon = 0$, is given by the expression (10).
In this setting Rendall [16] proved a local in time existence theorem for initial data with compact support for the density generalising a result obtained by Makino [12] for the non relativistic Euler Poisson system. Rendall however worked with C^∞ data and did therefore, as well shall see below, restrict the equation of state.

1.1 The Einstein-Euler equations written as a symmetric hyperbolic system

The initial value problem for the Einstein-Euler system will be treated by writing the equations as a symmetric hyperbolic system in harmonic coordinates. The harmonic condition is that

$$g^{\alpha\beta}g^{\gamma\delta}(\partial_\gamma g_{\beta\delta} - \frac{1}{2}\partial_\delta g_{\beta\gamma}) = 0. \tag{6}$$

When this condition is imposed the Einstein equations imply a system of quasilinear wave equations. To get a symmetric hyperbolic system these are reduced to first order by introducing auxiliary variables $h_{\alpha\beta\gamma} = \partial_\gamma g_{\alpha\beta}$. They can then be written in the following form

$$\begin{array}{rcl} \partial_t g_{\alpha\beta} & = & h_{\alpha\beta 0} \\ g^{ab}\partial_t h_{\gamma\delta a} & = & g^{ab}\partial_a h_{\gamma\delta 0} \\ -g^{00}\partial_t h_{\gamma\delta 0} & = & 2g^{0a}\partial_a h_{\gamma\delta 0} + g^{ab}\partial_a h_{\gamma\delta b} \\ & & +C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\gamma\delta\alpha\beta\rho\sigma} h_{\epsilon\zeta\eta} h_{\kappa\lambda\mu} g^{\alpha\beta} g^{\rho\sigma} - 16\pi T_{\gamma\delta} + 8\pi g^{\rho\sigma} T_{\rho\sigma} g_{\gamma\delta}. \end{array} \tag{7}$$

The Euler system $\nabla_\alpha T^{\alpha\beta} = 0$ is written as a symmetric hyperbolic by decomposing it into two orthogonal components; the first one along u^α is given by $u_\beta \nabla_\nu T^{\nu\beta}$ and the second one is the projection $P_{\alpha\beta}\nabla_\nu T^{\nu\beta}$ on the rest space $\mathcal{O}$ orthogonal to the velocity vector u^α, here $P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$. Since $P_{\alpha\beta}u^\beta = 0$, the Euler equation (4) is transformed into the system

$$\left\{ \begin{array}{ll} u^\nu \nabla_\nu \epsilon + (\epsilon + p)\nabla_\nu u^\nu & = 0 \\ P^\nu_\alpha \nabla_\nu p + (\epsilon + p) u^\nu P_{\alpha\beta} \nabla_\nu u^\beta & = 0 \end{array} \right. . \tag{8}$$

The normalization condition (3) implies $u_\beta \nabla_\nu u^\beta = 0$. So we add $u^\nu u_\beta \nabla_\nu u^\beta$ to the first row, $2u_\alpha u_\beta \nabla_\nu u^\beta$ to the second one and we insert the equation of state (5), then the system (8) is equivalent to

$$\left(\begin{array}{cc} u^\nu & (\epsilon + p) P^\nu{}_\beta \\ \frac{dp}{d\epsilon} P^\nu{}_\alpha & (\epsilon + p) u^\nu \Gamma_{\alpha\beta} \end{array} \right) \nabla_\nu \left(\begin{array}{c} \epsilon \\ u^\beta \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \tag{9}$$

where $\Gamma_{\alpha\beta} = g_{\alpha\beta} + 2u_\alpha u_\beta$ is positive definite. Finally, since we are dealing with a situation in which the density vanishes, we have to regularise the system by introducing a new matter variable, the Makino variable $w = M(\epsilon)$ of the form

$$w = M(\epsilon) = \int_0^\epsilon \frac{1}{g} \frac{1}{\tilde{\epsilon} + p} \sqrt{f'(\tilde{\epsilon})} d\tilde{\epsilon} = \epsilon^{\frac{\gamma-1}{2}}, \qquad g = \frac{\epsilon}{(\epsilon + p)\bar{K}}, \qquad \bar{K} = \frac{\gamma - 1}{2\sqrt{K\gamma}}. \tag{10}$$

These steps result in a system of the form

$$\left(\begin{array}{cc} g^2 u^\nu & w P^\nu{}_\beta \\ w P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu \end{array} \right) \nabla_\nu \left(\begin{array}{c} w \\ u^\beta \end{array} \right) = 0. \tag{11}$$

Note that we have besides the above evolution equations a constraint equation for the velocity u^α, namely $g_{\alpha\beta}u^\alpha u^\beta = -1$. The evolution equations (7) and (11) form a uniform symmetric hyperbolic system, the Einstein–Euler system. Recall that a (uniform) symmetric hyperbolic system is a system of differential equations of the form

$$L[U] = \sum_{\alpha=0}^{3} A^\alpha(U; x, t)\partial_\alpha U + B(U; x, t) = 0 \tag{12}$$

with symmetric matrices A^α and for which the matrix A^0 is uniformly positive definite. Moreover the matrices A^α and B satisfy certain regularity conditions, which will be stated in the local existence theorem below.
For the system introduced above we want to consider a initial value problem for which the initial density falls of at infinity in an appropriate way. The Makino variable and the way the matter variables appear in the Euler and in the Einstein equation provide us with the following complication. The Einstein evolution equations (7) contain the term $T_{\alpha\beta}$ as given by (2), which is a function of ϵ, p and u^α. The pressure p and the energy density ϵ are connected via the equation of state (5). Therefore we have to estimate ϵ and p, in the corresponding norm of our function spaces, by w, which is a algebraic function of ϵ as given by equation (10). This estimate results in an algebraic relation between the order of the functional space k and the coefficient γ of the equation of state $1 < \gamma \leq \frac{2+k}{k}$. We do

not want to restrict γ but instead interpret this inequality as an restriction on k and since we want also to improve the regularity conditions for the solutions of the Einstein Euler system, we are naturally lead to consider weighted Sobolev spaces of fractional order.
Numerically investigations performed by Nilson and Uggla [14] suggest that static spherically solutions of the Einstein–Euler System correspond to values of γ between $1.2 < \gamma \leq 1.29949$. Recently these results have been confirmed analytically by Uggla and Heinzle [9] who also derived the fall of conditions of the density for $r \to \infty$. The solutions given by our existence theorem show such a fall off behaviour as we will discuss in more detail in our forthcoming article [3]

2 New Functionspaces

The weighted Sobolev spaces of integer order below were introduced by Cantor [4] and independently by Nirenberg and Walker [15]. For real δ and nonnegative integer k we define the

$$\left(\|u\|_{k,\delta}^{*}\right)^{2} = \sum_{|\alpha|\leq k} \int |\langle x\rangle^{\delta+|\alpha|}\partial^{\alpha}u|^{2}dx, \tag{13}$$

where $\langle x\rangle = 1 + |x|$.
The space $H_{s,k}$ is the completion of $C_0^{\infty}(\mathbb{R}^3)$ under the norm (13). Triebel [17]extended these spaces to a fractional order:

Definition 1 (Weighted fractional Sobolev spaces: double integral). *For $s \geq 0$ and $-\infty < \delta < \infty$, the Sobolev weighted Space $H_{s,\delta}$ is defined as the completion of $C_0^{\infty}(\mathbb{R}^3)$ under the norm*

$$\left(\|u\|_{s,\delta}^{*}\right)^{2} = \begin{cases} \sum_{|\alpha|\leq k} \int |\langle x\rangle^{\delta+|\alpha|}\partial^{\alpha}u|^{2}dx, & s = k \\ \left.\begin{array}{l} \sum_{|\alpha|\leq k} \int |\langle x\rangle^{\delta+|\alpha|}\partial^{\alpha}u|^{2}dx \\ + \sum_{|\alpha|=k} \int\int \frac{|\langle x\rangle^{k+\lambda+\delta}\partial^{\alpha}u(x) - \langle y\rangle^{k+\lambda+\delta}\partial^{\alpha}u(y)|^{2}}{|x-y|^{3+2\lambda}}dxdy \end{array}\right\} & s = k+\lambda. \end{cases} \tag{14}$$

here k is a nonnegative integer and $0 < \lambda < 1$.

This definition is a natural generalization of the norm (13). However, the double integral causes many difficulties as one turns to prove certain properties (algebra, embedding act.) of the space $H_{s,\delta}$. Therefore we are looking for an equivalent norm.
Let $K_j = \{x : 2^{j-3} \leq |x| \leq 2^{j+2}\}$, $(j = 1, 2, ...)$ and $K_0 = \{x : |x| \leq 4\}$. Let $\{\psi_j\}_{j=0}^{\infty}$ be a sequence of $C_0^{\infty}(\mathbb{R}^3)$ such that $\psi_j(x) = 1$ on K_j, $\mathrm{supp}(\psi_j) \subset \cup_{l=j-4}^{j+3} K_l$, for $j \geq 1$, $\mathrm{supp}(\psi_0) \subset K_0 \cup K_1$ and

$$|\partial^{\alpha}\psi_j(x)| \leq C_{\alpha}2^{-|\alpha|j}. \tag{15}$$

We denote by H^s the Bessel potential spaces with the norm ($p = 2$)

$$\|u\|_{H^s}^2 = c\int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform of u. Also, for a function f, $f_\varepsilon(x) = f(\varepsilon x)$.

Definition 2 (Weighted fractional Sobolev spaces: infinite sum of semi norms). *For $s \geq 0$ and $-\infty < \delta < \infty$,*

$$\left(\|u\|_{H_{s,\delta}}\right)^2 = \sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{(2^j)}\|_{H^s}^2. \tag{16}$$

The space $H_{s,\delta}$ is the set of all temperate distributions with a finite norm given by (16).

The following lemma goes back to Triebel [17].

Lemma 1. *The spaces $H_{s,\delta}$ of Definitions 1 and 2 are equivalent. Moreover*

$$C_1\|u\|_{H_{s,\delta}} \leq \|u\|_{s,\delta}^* \leq C_2\|u\|_{H_{s,\delta}} \tag{17}$$

where C_1 and C_2 depend on s and δ.

Note that the equivalence (17) implies that the norm (16) is independent of the sequence $\{\psi_j\}$ as long as it satisfies inequality (15).

Lemma 2. *The spaces $H_{s,\delta}$ have the following properties:*

1. *(Algebra) For $s_1, s_2 \geq s$, $s_1 + s_2 > s + \frac{3}{2}$ and $\delta_1 + \delta_2 \geq \delta - \frac{3}{2}$,*

$$\|uv\|_{H_{s,\delta}} \leq C\|u\|_{H_{s_1,\delta_1}}\ \|v\|_{H_{s_2,\delta_2}}. \tag{18}$$

2. *(Compact embedding) For $s' < s$ and $\delta' < \delta$ the embedding $H_{s,\delta} \hookrightarrow H_{s',\delta'}$ is compact.*

3. *(Moser's type estimates) Let $F : \mathbb{R}^1 \to \mathbb{R}^1$ be C^∞ such that $F(0) = 0$. Then*

$$\|F(u)\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty})\|u\|_{H_{s,\delta}}. \tag{19}$$

4. *For $1 \leq \gamma$, $s < \gamma + \frac{1}{2}$ and $u \geq 0$,*

$$\|u^\gamma\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty})\|u\|_{H_{s,\delta}}. \tag{20}$$

   This inequality has been proven for the H^s spaces by Kateb [10].

3 The principal result

Our principal results are the solution to the constraint equations and the evolution equations as given in the following two theorems.

Theorem 1 (Main result).

1. ***Solution of the constrains equations (32) and (33):*** *Let* $2 \leq s$ *and* $-\frac{3}{2} < \delta < -\frac{1}{2}$. *Given free initial data* $\bar{h}_{ab} - \delta_{ab} \in H_{s,\delta}(\mathbb{R}^3)$, $\bar{A}_*^{ab} \in H_{s-1,\delta+1}(\mathbb{R}^3)$, $\bar{y}(\epsilon) \in H_{s-2,\delta+2}(\mathbb{R}^3)$, $v^b(u^\alpha) \in H_{s-2,\delta+2}(\mathbb{R}^3)$. *Then there exists an unique solution* ϕ, K_{ab} *of the constraint equations (32) and (33) such that* $\phi - 1 \in H_{s,\delta}(\mathbb{R}^3)$, $K_{ab} \in H_{s-1,\delta+1}(\mathbb{R}^3)$.

2. ***Solution of the evolution equations (7) and (11)****: Let* $s, \delta \in \mathbb{R}$, $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, *and* $-\frac{3}{2} < \delta < -\frac{1}{2}$. *Given the solutions of the constraints (32) and (33), assume moreover that* $\bar{y}(\epsilon) \in H_{s-1,\delta+2}(\mathbb{R}^3)$, $v^b(u^\alpha) \in H_{s-1,\delta+2}(\mathbb{R}^3)$. *Then there exists a* $T > 0$ *and a unique solution* $U = (w, u^0 - 1, u^a, g_{\alpha\beta})$ *of the Einstein–Euler system, with*

$$g_{\alpha\beta} - \eta_{\alpha\beta} \in C^0([0,T), H_{s,\delta}(\mathbb{R}^3)) \cap C^1([0,T), H_{s-1,\delta}(\mathbb{R}^3))$$
$$w, u^0 - 1, u^a \in C^0([0,T), H_{s-1,\delta+2}(\mathbb{R}^3)) \cap C^1([0,T), H_{s-2,\delta+2}(\mathbb{R}^3))$$

4 Strategy of the Proof

The proof consists of three parts: First we solve the elliptic constraints applying the established methods introduced by Cantor [5], Christodoulou and O'Murchadha [7] for our spaces. For details we refer to our forthcoming paper. We present these results here in form of Theorems 3 and 4. The next step concerns the construction of the initial data for the fluid equations: Starting with the initial data for the constrain equations, we construct the initial data for the Euler equations by means of Theorem 2. The last step finally refers to the local existence of the symmetric hyperbolic evolution equation given by Theorem 5.

5 The elliptic constraints

The solution of the Einstein equations coupled to matter fields is usually done in two steps. Initial data for the Einstein equations cannot be given freely; there are constraint equations intrinsic to the initial hypersurface which must be satisfied. So the first step is to construct solutions of these constraints. The second step is then to solve the evolution equations (in the present case the symmetric hyperbolic system just described) with these initial data. To define the harmonic coordinates uniquely it is necessary to supplement the condition (6) with some conditions on the initial hypersurface defined by the equation $t = 0$. The standard choice is that on the initial hypersurface $g_{00} = -1$ and $g_{0a} = 0$. To write down the constraint equations it is convenient to introduce the second fundamental form of the initial surface. When the conditions just introduced hold this object is given by

$$-\frac{1}{2}\partial_t g_{ab} = K_{ab}. \tag{21}$$

Let n^α denote the unit normal to the hypersurface, $\delta^\alpha_\beta + n^\alpha n_\beta$ the projection on it and define

$$\begin{aligned} z &= T_{\alpha\beta} n^\alpha n^\beta, & (22)\\ j^\alpha &= (\delta^\alpha_\gamma + n^\alpha n_\gamma) T^{\gamma\beta} n_\beta. & (23)\end{aligned}$$

The vector j^α is tangent to the initial surface and so can be identified with a vector j^a intrinsic to this surface. More explicit expressions for z and j^a can be given using the projection $\bar{u}^\alpha = (\delta^\alpha{}_\beta + n^\alpha n_\beta)u^\beta$ of the velocity onto this surface. Then it can be identified as follow:

$$\begin{aligned} z &= \epsilon(1 + g_{ab}\bar{u}^a\bar{u}^b) + p g_{ab}\bar{u}^a\bar{u}^b, & (24)\\ j^a &= (\epsilon + p)\bar{u}^a(1 + g_{bc}\bar{u}^b\bar{u}^c)^{1/2}. & (25)\end{aligned}$$

If R_{ab} denotes the Ricci tensor of the induced metric on the initial hypersurface, $R = g^{ab}R_{ab}$ is its scalar curvature and ${}^{(3)}\nabla$ its associated covariant derivative, then

$$\begin{aligned} R - K_{ab}K^{ab} + (g^{ab}K_{ab})^2 &= 16\pi z, & (26)\\ {}^{(3)}\nabla_b K^{ab} - {}^{(3)}\nabla^b(g^{bc}K_{bc}) &= -8\pi j^a. & (27)\end{aligned}$$

We turn now to the conformal method which allows us to construct the solutions of the constraint equations (26) and (27). Before entering into details we have to discuss the relation between the initial data for the Einstein-Euler fluid system (11) and (7) and the initial fluid data given by the solutions of constraint equations. As it turns out this relation is by no means trivial, and indeed it will force us to modify the conformal method.

5.1 The compatibility problem of the initial data for the fluid and the gravitational field

The matter initial data for the Einstein–Euler system are on the one hand $w(\epsilon)$ and u^α for the Euler equations (11). On the other hand $z = F(w(\epsilon), u^\alpha)$ and $j^a = H(w(\epsilon), u^\alpha)$ appear as sources in the constraint equations (26) and (27). There we have the possibility of either to consider w and u^α as the fundamental quantities and construct then z and j^a or, vice versa, to consider z and j^a as the fundamental quantities and construct then w and u^α.
The first possibility does not work, because the geometric quantities which occur on the left hand side of the constraint equations are supposed to scale with some power of ϕ.

So z and j^a, which are the sources in the constraint equations, must also scale with some definite power of ϕ. If ϵ scaled with some power of ϕ then so would p. But in the expression for z a sum of different powers of ϵ occurs. Thus the power with which p scaled would have to be zero and ϵ would be left unchanged by the rescaling. Similarly it can be seen that $\bar{u}^a$ would remain unchanged. So in fact z would be unchanged and this is inconsistent with the scalings used in the conformal method.
Instead of constructing w, u^a from z, j^a it turns out that is more useful to introduce some auxiliary quantities. So we start with

$$w = \epsilon^{\frac{\gamma-1}{2}} \qquad \text{and} \qquad y = z^{\frac{\gamma-1}{2}}. \tag{28}$$

Now we consider the following map

$$(w, u^a) \rightarrow \left(y, \frac{j^a}{z}\right) \tag{29}$$

which is given by (30). As we will show below, this map is in fact C^∞ and a local diffeormophism if $\frac{\|j\|}{z} < 1$.

Theorem 2 (Reconstruction theorem for the initial data). *Let Φ be the mapping from $\mathbb{R}^4$ to $\mathbb{R}^4$ defined by*

$$\Phi(w, \bar{u}^a) = \left(w[(1 + g_{ab}\bar{u}^a\bar{u}^b) + Kw^2 g_{ab}\bar{u}^a\bar{u}^b]^{\frac{\gamma-1}{2}}, \frac{(1+Kw^2)(1+g_{bc}\bar{u}^b\bar{u}^c)^{\frac{1}{2}}\bar{u}^a}{(1+g_{bc}\bar{u}^b\bar{u}^c) + Kw^2 g_{bc}\bar{u}^a\bar{u}^b} \right) \tag{30}$$

Then under the dominate energy condition the map Φ is an C^∞ diffeomorphism from a closed half-space of $\mathbb{R}^4$ onto the region $G = \{(y, x^a) : 0 \le y,\ \delta_{ab}x^a x^b < 1\}$. Here K is the constant of the state equation (5).

It can be seen immediately that Φ is a C^∞ map. The rest of the proof is based Hadamard's Lemma which asserts that a map $f : X \rightarrow Y$ between simply connected manifolds X and Y which is C^∞, proper and locally one to one, is a global bijection. For details we refer to [3].

5.2 The existence of solutions for the constraint equations

This method has been discussed in detail in the literature, see for example [2], [6], [5] [7] and reference therein. So we will just briefly outline the procedure with the necessary modifications we have to perform.
Parts of the data (the so-called free data) are chosen, and the constraints imply four elliptic equations for the remaining parts.
The free initial data are:

$$\left(\bar{h}_{ab}, \bar{A}^{ab}_{*}, \bar{y}, \bar{v}^b\right). \tag{31}$$

where $v^b = \frac{j^b}{z}$ and y is given by (28). Here we have performed a conformal transformation of the metric: $h_{ab} = \phi^4 \bar{h}_{ab}$.

We assume the maximal slice condition for K_{ab}, that is $h^{ab}K_{ab} = 0$.
Let $\bar{A}^{ab}_{*}$ be any smooth symmetric tensor which has zero trace with respect to $\bar{h}_{ab}$. We are looking for solutions using $\bar{A}^{ab} = \bar{A}^{ab}_{*} + \bar{D}^a W^b + \bar{D}^b W^a - \frac{2}{3}\bar{h}^{ab}\bar{D}_k W^k$ and $K^{ab} = \phi^{-10}\bar{A}^{ab}$. Furthermore we have $y = \phi^{4(\gamma-1)}\bar{y}$ $v^a = \phi^{-2}\bar{v}^a$. The transformed constraints are to be solved for the scalar function ϕ and the vector field W^b

$$\bar{\Delta}\phi - \frac{1}{8}\bar{R}\phi + \frac{1}{8}\left(\bar{A}_{ab}\bar{A}^{ab}\right)\phi^{-7} = -2\phi^{-3}\bar{y}, \tag{32}$$

$$\left(\Delta_L W\right)^b = \bar{v}^b, \tag{33}$$

where $\left(\Delta_L W\right)^b = \left(\bar{\Delta}W\right)^b + \frac{1}{3}\bar{D}^b\left(\bar{D}_a W^a\right) + \bar{R}^b_a W^a$, here $\bar{\Delta}$ denotes the Laplace–Beltrami operator with respect to the metric $\bar{h}$.
Once the solution $\left(\phi, W^b\right)$ are constructed the full initial data can be obtained by inverting the above process.
In order to obtain such a solution $\left(\phi, W^b\right)$, we proceed as follows: First, for given $\bar{y} \in H_{s-2,\delta+2}$, we know by the extended Katab result as given in Lemma 2 that $\bar{z} \in H_{s-2,\delta+2}$, moreover by assumption we have $\bar{v}^b \in H_{s-2,\delta+2}$ and by the multiplication property 2, we conclude that $j^b \in H_{s-2,\delta+2}$.
Therefore the momentum constraint (33) is solved for W^b using Theorem 4 which results in $W^b \in H_{s,\delta}$.
Secondly using W^b and the free initial data A^{ab}_{*}, A^{ab} is constructed using the composition mentioned above. Finally with z and A^{ab} given the Lichnerowicz equation (32) is solved for ϕ, see Theorem 3.
We summarise the results in the following theorems:

Theorem 3 (Existence and uniqueness for the solutions of the Lichnerowicz equation). *Let* $2 \leq s$, $-\frac{3}{2} < \delta < -\frac{1}{2}$ *and* $\bar{h}$ *be a Riemann metric and* $\bar{A}_{ab}$ *be a tensor field in* $\mathbb{R}^3$ *such that* $\bar{h}_{ab} - \delta_{ab} \in H_{s,\delta}(\mathbb{R}^3)$ *and* $\bar{A}_{ab} \in H_{s-1,\delta+1}$ *Let* $\bar{y} \in H_{s-2,\delta+2}$. *Then there exists a unique solution* ϕ *of the equation (32) such that* $\phi - 1 \in H_{s,\delta}$.

Theorem 4 (Existence and uniqueness of solutions for the York equation). *Let* $2 \leq s$, $-\frac{3}{2} < \delta < -\frac{1}{2}$ *and* $\bar{v}^b \in H_{s-2,\delta+2}$. *Then there exists a unique solution* $W^b \in H_{s,\delta}$ *of equation (33), where* $\left(\Delta_L W\right)^b := \left(\bar{\Delta}W\right)^b + \frac{1}{3}\bar{D}^b\left(\bar{D}_a W^a\right) + \bar{R}^b_a W^a$.

Remark 1. *During our work we found out that D. Maxwell [13] studied the vacuum Einstein constraint equations using fractional weighted Sobolev spaces. He obtained solutions for the constrain equations under the condition* $\frac{3}{2} < s$, *improving earlier result obtained by Bartnik [1]. We recall that our principal motivation is to adjust the regularity of the solution to the Einstein-Euler system (1), (2) and (4) for each parameter* γ *of the state equation (5).*

6 Symmetric hyperbolic systems and local existence theorems

A principal tool is the following existence theorem which is a generalisation of the well known existence theorem for the H^s spaces, see for example [11], [?], [?].

Theorem 5 (Local existence for quasilinear symmetric–hyperbolic systems). *Let $A^0, A^k \in C^\infty(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R}^{l\times l})$, $B \in C^\infty(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R}^{l\times l})$ be coefficients which define the quasilinear symmetric–hyperbolic system*

$$A^0_{\alpha\beta}(U;x,t)\frac{\partial U^\beta}{\partial t} + \sum_{k=1}^{3} A^k_{\alpha\beta}(U,x,t)\frac{\partial U^\beta}{\partial x^k} + B_{\alpha\beta}(U;x,t) = 0. \tag{34}$$

Let $U(x,0) \in H_{s,\delta}(\mathbb{R}^3)$ and let the initial conditions be chosen such that the condition

$$C\delta_{\alpha\beta}U^\alpha U^\beta \leq A^0_{\alpha\beta}U^\alpha U^\beta \leq C^{-1}\delta_{\alpha\beta}U^\alpha U^\beta, \qquad C \in \mathbb{R}^+ \tag{35}$$

is satisfied. Let $\frac{5}{2} < s$ and $-\frac{3}{2} \leq \delta$.
Then there exists a $T > 0$ which depends on the $H_{s,\delta}$ norm of the initial data and there exists a unique solution

$$U(x,t) \in C^0\left([0,T), H_{s,\delta}\right) \cap C^1\left([0,T), H_{s-1,\delta}\right). \tag{36}$$

Again for a proof of this theorem we refer to [3].
The C^∞ differentiability condition can be weakened by C^N where N depends on s.
In order to obtain the final result, Theorem 1, we take the initial data of the gravitational field, as given by Theorems 3 and 4, and the initial data for the fluid equations as given by the reconstruction Theorem 2 and apply the above existence theorem. Note that in our main result we have demanded a bound from above on the differentiability on the initial data namely $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$. The reason for this is that w which appears in the evolution equations (11) is a function of ϵ in (28).

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