

ON NULL QUADRATURE DOMAINS

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ABSTRACT. The characterization of null quadrature domains in \mathbb{R}^n ($n \geq 3$) has been an open problem throughout the past two and a half decades. A substantial contribution was done by Friedman and Sakai [10]; they showed that if the complement is bounded, then null quadrature domains are exactly the complement of ellipsoids. The first result with unbounded complements appeared in [15], there it is assumed the complement is contained in an infinitely cylinder.

The aim of this paper is to show the relation between null quadrature domains and Newton's theorem on the gravitational force induced by homogeneous homoeoidal ellipsoids. We also succeed to make progress in the classification problem and we show that if the boundary of null quadrature domain is contained in a strip and the complement satisfies a certain capacity condition at infinity, then it must be a half-space or a complement of a strip. In addition, we present a Phragmén-Lindelöf type theorem which seems to be forgotten in the literature.

1. INTRODUCTION

Let E be an ellipsoid in \mathbb{R}^3 centered at the origin. We call $\lambda E \setminus E$, $\lambda > 1$ an *homoeoidal ellipsoid*. A remarkable theorem due to Newton (see e.g. [17], [6]) asserts the gravitational attraction at any internal point of a homogeneous homoeoidal ellipsoid is zero.

Are ellipsoids the exactly bodies having the property that gravitational force induced by a homoeoid is zero at all internal points?

So let $K \subset \mathbb{R}^3$ be a closed bounded set containing a neighborhood of the origin. If the homogeneous homoeoid $\lambda K \setminus K$ ($\lambda > 1$) produces no gravity force in the cavity K , then

$$(1.1) \quad \int_{\lambda K \setminus K} \frac{1}{|x - y|} dy = \text{const.} \quad \text{for all } x \in K.$$

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Noting that for x in a neighborhood of the origin,

$$\int_K \frac{1}{|x-y|} dy = \sum_{m=0}^{\infty} P_m(x),$$

where P_m is a homogeneous polynomial of degree m , we see from (1.1) that

$$\int_{\lambda K \setminus K} \frac{1}{|x-y|} dy = \sum_{m=0}^{\infty} (\lambda^{m-2} - 1) P_m(x) = \text{const.}$$

Thus, the gravitational attraction at any internal point of a homogeneous homoeoid $\lambda K \setminus K$ is zero for all $\lambda > 1$, if and only if

$$\int_K \frac{1}{|x-y|} dy = q(x), \quad x \in K,$$

where q is a quadratic polynomial. In the beginning of 1930's Dive [7] and Nikliborc [21] characterized closed bounded sets in \mathbb{R}^3 for which the Newtonian potential induced by them coincides with a quadratic polynomial inside these sets. Their results give an affirmative answer to the above question. Hölder [12] studied the related problem in the two dimensional plane. Later on, analogous results were proved in arbitrary dimension by DiBenedetto and Friedman in 1986 [3], and by the author in 1995 [14].

In order to discuss the third equivalence we introduce the following definitions.

Definition 1.1. An open set Ω is called a *null quadrature domain* if

$$\int_{\Omega} h dx = 0$$

for all harmonic and integrable functions h in Ω .

Assuming Ω is a null quadrature domain having a bounded complement $K := \mathbb{R}^3 \setminus \Omega$. Then Theorem 2.4 below claims the Newtonian potential of the measure $\chi_K dx$ coincides with a quadratic polynomial in K . Thus in case Ω has a bounded complement, it is a null quadrature domain if and only if the complement of Ω is an ellipsoid. However, there are null quadrature domains with unbounded complement, for example, a half-space, or $\Omega = \{(\frac{x_1}{a})^2 + (\frac{x_2}{b})^2 > 1, x_3 \in \mathbb{R}\}$. In this paper we will discuss the characterization of null quadrature domains in \mathbb{R}^n with unbounded complement.

Null quadrature domains are also relevant for Hele-Shaw flow problems with sink/source located at infinity. They are the self similar solutions to these flows [3], [8], [11] and [27].

The organization of the paper is as follows. In Section 2 we reintroduce the generalized Newton potential and prove a new estimate for it. In Section 3 we present and prove the main result and Section 4 deals Phragmén-Lindelöf type theorem and Newtonian capacity. We use most common notations: For a set $D \subset \mathbb{R}^n$, χ_D is the characteristic function, \bar{D} is the closure and $D^c = \mathbb{R}^n \setminus D$ is the complement of D . The Laplacian will be denoted by Δ . $B_\varrho(x_0)$ is a ball centered at x_0 and radius ϱ , and $B_\varrho = B_\varrho(0)$. The Newtonian capacity of a set F in \mathbb{R}^n is denoted by $\text{Cap}_n(F)$. In this paper n is an integer greater or equal to 3.

2. NULL QUADRATURE DOMAINS AND GENERALIZED NEWTONIAN POTENTIALS

Generalized Newtonian potential was introduced and studied [15]. Here we briefly repeat its definition, study its relation to null quadrature domains and establish an estimate for measures with a support being contained in a region bounded by two hyperplanes.

2.1. Generalized Newtonian Potential. Let

$$J_n(x) = \frac{1}{(n-2)\omega_n} |x|^{n-2}$$

be the Newtonian kernel, here ω_n is the area of the unit sphere in \mathbb{R}^n . The Newtonian potential of a Radon measure μ with compact support is defined by means of the convolution

$$(2.1) \quad V\mu(x) = (J_n * \mu)(x) = \int J_n(x-y) d\mu(y).$$

It satisfies the equation

$$\Delta V\mu = -\mu$$

in a distributional sense.

Let \mathcal{L} be the Banach space of all Radon measures such that

$$(2.2) \quad \|\mu\|_{\mathcal{L}} := \int \frac{d|\mu|(x)}{(1+|x|)^{n+1}}$$

is finite. Note that \mathcal{L} contains all measures of the type: $\mu = f dx$, $f \in L^\infty(\mathbb{R}^n)$. For $\mu \in \mathcal{L}$ and a multi-index α with $|\alpha| = 3$ we define the third order formal derivative of $V\mu$ by

$$(2.3) \quad \langle V^\alpha(\mu), \varphi \rangle = - \int \partial^\alpha (V\varphi)(x) d\mu(x), \quad \varphi \in \mathcal{S},$$

where \mathcal{S} is the Schwartz class of rapidly decreasing functions. The operator $V^\alpha : \mathcal{L} \rightarrow \mathcal{S}'$ is continuous [15]. The *generalized Newtonian*

potential of a measure μ in \mathcal{L} is denoted by $V[\mu]$ and is the set of all solutions to the system

$$(2.4) \quad \begin{cases} \Delta v = -\mu \\ \partial^\alpha v = V^\alpha(\mu), \quad |\alpha| = 3. \end{cases}$$

The existence of solutions to (2.4) were proved in [15], furthermore, two elements of $V[\mu]$ differ by a harmonic polynomial of degree less or equal to two.

2.2. Estimate of the Generalized Newtonian Potential. For measures $\mu = f dx$ with $f \in L^\infty(\mathbb{R}^n)$ we can control the growth of the generalized Newtonian potential as follow: for any $v \in V[f]$

$$(2.5) \quad |v(x)| \leq C \|f\|_{L^\infty} (1 + |x|)^2 \log(2 + |x|)$$

(cf. [13], [25]). If there are additional restrictions on the support of f , then the estimate (2.5) can be improved. For example, if $\text{supp}(f) \subset \{x_{n-1}^2 + x_n^2 \leq \rho^2\}$, then there is $v \in V[f]$ such that

$$|v(x)| \leq C(\rho) \|f\|_{L^\infty} \log(2 + |x|)$$

(see [15] or [26]).

Next, put $\Pi(a, b) = \{x : a \leq x_n \leq b\}$.

Lemma 2.1. *Suppose $f \in L^\infty$ and $\text{supp}(f) \subset \Pi(a, b)$. Then there is $v \in V[f]$ such that*

$$(2.6) \quad |v(x)| \leq C \|f\|_{L^\infty} (b - a)^2 (1 + |x|) \log(2 + (b - a)^{-1} |x|)$$

and

$$(2.7) \quad |\nabla v(x)| \leq C \|f\|_{L^\infty} (b - a)^2 \log(2 + (b - a)^{-1} |x|).$$

Proof. We start the proof with the observations that if $v \in V[f]$, then

$$(b - a)^2 v(x_1, \dots, x_{n-1}, \frac{x_n - \frac{a+b}{2}}{b - a}) \in V[\tilde{f}],$$

where $\tilde{f}(x_1, \dots, x_n) = f(x_1, \dots, x_{n-1}, \frac{x_n - \frac{a+b}{2}}{b - a})$. Since the right hand sides of (2.6) and (2.7) are homogeneous of degree 2 with respect to $(b - a)$, we may assume $b - a = 1$. In addition, by a translation we can prove the Lemma under the assumption $\text{supp}(f) \subset \Pi := \Pi(1, 2)$. Let

$$(2.8) \quad \hat{J}_n(x, y) = J_n(x - y) - J_n(y) - x \cdot \nabla J_n(y)$$

and put

$$(2.9) \quad v(x) = v(f)(x) = \int_{\Pi} \hat{J}_n(x, y) f(y) dy.$$

For $|y| \geq 2|x|$,

$$|\widehat{J}_n(x, y)| \leq C \frac{|x|^2}{|y|^n},$$

therefore

$$\begin{aligned} |v(f)(x)| &\leq \|f\|_{L^\infty} \int_{\Pi \cap \{|y| \leq 2|x|\}} (|J_n(x-y)| + |J_n(y)| + |x| |\nabla J_n(y)|) dy \\ &\quad + C \|f\|_{L^\infty} |x|^2 \int_{\Pi \cap \{|y| \geq 2|x|\}} \frac{1}{|y|^n} dy. \end{aligned}$$

Since $\int_{\Pi} \frac{1}{|y|^n} dy$ converges, we have $|v(f)(x)| \leq C \|f\|_{L^\infty} (1 + |x|)^2$. In Addition, $\Delta v(f) = -f$. Thus $v(f)$ belongs to $V[f]$. Now it suffices to show inequality (2.7) for the gradient of $v(f)$:

$$\begin{aligned} |\nabla v(f)(x)| &= \int_{\Pi} (\nabla J_n(x-y) - \nabla J_n(y)) f(y) dy \\ &= \int_{\Pi \cap \{|y| \leq 2(1+|x|)\}} \nabla J_n(x-y) f(y) dy \\ &\quad + \int_{\Pi \cap \{|y| \leq 2(1+|x|)\}} \nabla J_n(y) f(y) dy \\ &\quad + \int_{\Pi \cap \{|y| \geq 2(1+|x|)\}} (\nabla J_n(x-y) - \nabla J_n(y)) f(y) dy \\ &= A(x) + B(x) + C(x). \end{aligned}$$

We estimate each term separately. So let $(x-y) = (x' - y', x_n - y_n)$, $(x' - y') = \rho\Theta$, where $\Theta \in \mathbb{S}^{n-2}$, the unite sphere in \mathbb{R}^{n-1} and let

$|x_n - y_n| = r$. Then

$$\begin{aligned}
|A(x)| &\leq C\|f\|_{L^\infty} \int_{\Pi \cap \{|x-y| \leq 3(1+|x|)\}} \frac{1}{|x-y|^{n-1}} dy \\
&\leq C\|f\|_{L^\infty} \int_1^2 \left(\int_0^{3(1+|x|)} \frac{\rho^{n-2}}{(r^2 + \rho^2)^{\frac{n-1}{2}}} d\rho \right) dy_n \\
&= C\|f\|_{L^\infty} \int_1^2 \left(\frac{1}{r} \int_0^{3(1+|x|)} \frac{\left(\frac{\rho}{r}\right)^{n-2}}{\left(1 + \left(\frac{\rho}{r}\right)^2\right)^{\frac{n-1}{2}}} d\rho \right) dy_n \\
&= C\|f\|_{L^\infty} \int_1^2 \left(\int_0^{\frac{3(1+|x|)}{r}} \frac{t^{n-2}}{(1+t^2)^{\frac{n-1}{2}}} dt \right) dy_n \\
&\leq C\|f\|_{L^\infty} \int_1^2 \left(\int_0^1 dt + \int_1^{\frac{3(1+|x|)}{r}} \frac{1}{t} dt \right) dy_n \\
&\leq C\|f\|_{L^\infty} \int_1^2 (1 + \log 3 + \log(1+|x|) + |\log(|x_n - y_n|)|) dy_n.
\end{aligned}$$

Now, if $-1 \leq x_n \leq 4$, then $\int_1^2 |\log(|x_n - y_n|)| dy_n \leq 5$, otherwise $\int_1^2 |\log(|x_n - y_n|)| dy_n \leq \log(2 + |x_n|)$. Combine this with the above, we have

$$(2.10) \quad |A(x)| \leq C\|f\|_{L^\infty} \log(2 + |x|).$$

For the next one, let $y = (y', y_n)$ and $y' = \rho\Theta$. Then

$$\begin{aligned}
|B(x)| &\leq C\|f\|_{L^\infty} \int_{\Pi \cap \{|y| \leq 2(1+|x|)\}} \frac{1}{|y|^{n-1}} dy \\
&\leq C\|f\|_{L^\infty} \int_1^2 \left(\int_1^{2(1+|x|)} \frac{\rho^{n-2}}{(y_n^2 + \rho^2)^{\frac{n-1}{2}}} d\rho \right) dy_n \\
(2.11) \quad &\leq C\|f\|_{L^\infty} \int_1^2 \left(\int_1^{2(1+|x|)} \frac{1}{\rho} d\rho \right) dy_n \\
&\leq C\|f\|_{L^\infty} \log(2 + |x|).
\end{aligned}$$

For the last estimate we use the Taylor expansion,

$$\partial_k J_n(x-y) - \partial_k J_n(y) = \sum_m x_m \partial_m \partial_k J_n(tx-y), \quad 0 \leq t \leq 1.$$

Since the second order derivatives of J_n are homogeneous of order $-n$,

$$|\partial_k J_n(x-y) - \partial_k J_n(y)| \leq C \frac{|x|}{|y|^n}, \quad |y| \geq 2(1+|x|).$$

Therefore,

$$\begin{aligned}
 |C(x)| &\leq C\|f\|_{L^\infty} \int_{\Pi \cap \{|y| \geq 2(1+|x|)\}} \frac{|x|}{|y|^n} dy \\
 (2.12) \quad &\leq C\|f\|_{L^\infty} |x| \int_1^2 \left(\int_{2|x|}^\infty \frac{\rho^{n-2}}{(y_n^2 + \rho^2)^{\frac{n}{2}}} d\rho \right) dy_n \\
 &\leq C\|f\|_{L^\infty} |x| \int_1^2 \left(\int_{2|x|}^\infty \frac{1}{\rho^2} d\rho \right) dy_n = \frac{C\|f\|_{L^\infty}}{2}.
 \end{aligned}$$

We complete the proof by combining the estimates (2.10), (2.11) and (2.12). \square

2.3. The Potential of the Complement of a Null Quadrature Domain. Theorem 2.4 below was proved previously in [13] and [15]. Here we reprove it a different manner. We first recall a known fact from the theory of unbounded quadrature domains.

Proposition 2.2. ([13],[15], [22]) *An open set Ω is a null quadrature domain if and only if there is a function u satisfying*

$$(2.13) \quad \begin{cases} \Delta u = \chi_\Omega, & \text{in } \mathbb{R}^n \\ u = |\nabla u| = 0, & \text{on } \Omega^c \\ |u(x)| \leq C(1 + |x|)^2 \log(2 + |x|). \end{cases}$$

Example 2.3. *A half-space $\Omega = \{x : x_n > b\}$ is a null quadrature domain. The function*

$$u(x) = \begin{cases} \frac{1}{2}(x_n - b)^2, & x_n > b \\ 0, & x_n \leq b \end{cases}$$

satisfies equation (2.13). The complement of a strip is a disjoint union of two half-spaces, so it is also a null quadrature domain.

We first recall that for any $\varphi \in \mathcal{S}$ and all multi-indices α with $|\alpha| = 3$ there holds, $|\partial^\alpha V\varphi(x)| \leq C(1 + |x|)^{-(n+1)}$ (see [15]; Proposition 1.1). Now, if Ω is a null quadrature domain and $\text{supp}(\varphi) \subset \Omega^c$, then $\partial^\alpha V\varphi$ is harmonic and integrable in Ω and hence,

$$(2.14) \quad \int_{\Omega} \partial^\alpha (V\varphi)(x) dx = 0.$$

Notice that

$$\int_{\mathbb{R}^n} \partial^\alpha (V\varphi)(x) dx = 0, \quad \varphi \in \mathcal{S},$$

we have

$$\begin{aligned}
 (2.15) \quad \langle V^\alpha(\chi_{\Omega^c}), \varphi \rangle &= - \int_{\Omega^c} \partial^\alpha(V\varphi)(x) dx \\
 &= \int_{\Omega} \partial^\alpha(V\varphi)(x) dx - \int_{\mathbb{R}^n} \partial^\alpha(V\varphi)(x) dx = 0
 \end{aligned}$$

for all $\varphi \in \mathcal{S}$ with $\text{supp}(\varphi) \subset \Omega^c$. Therefore for any $v \in V[\chi_{\Omega^c}]$ the second order derivatives of v are constant on Ω^c . Hence, it coincides with a quadratic polynomial on Ω^c .

Conversely, if for some $v \in V[\chi_{\Omega^c}]$ equals to a quadratic polynomial q on Ω^c , then the function $u = v - q$ satisfies (2.13).

Theorem 2.4. *An open set Ω in \mathbb{R}^n is a null quadrature domain if and only if there is $v \in V[\chi_{\Omega^c}]$ such that v coincides with a quadratic polynomial in Ω^c .*

Remark 2.5. *Theorem 2.4 suggests a generalization of Newton's theorem of no gravitational force in the interior of homoeoidal ellipsoids to unbounded sets. A different extension of this theorem to hyperbolic quadratic surfaces was done by Arnold (see Appendix in [2] and references in there).*

3. THE MAIN RESULT

We first recall that if Ω is a null quadrature domain such that: (i) $\bar{\Omega} = \mathbb{R}^n$ and $\Omega \neq \mathbb{R}^n$, then Ω^c is contained in a hyperplane ([15]; Example 2.20); (ii) $\bar{\Omega} \neq \mathbb{R}^n$, then $\Omega_0 = \text{int}(\bar{\Omega})$, the interior of the closure of Ω , is also a null quadrature domain. Therefore we always assume here $\Omega = \text{int}(\bar{\Omega})$ and $\Omega \neq \mathbb{R}^n$.

Null quadrature domains in the two dimensional plane were completely classified by M. Sakai.

Theorem 3.1. (Sakai [22],[23]) *The open set $\Omega \subset \mathbb{R}^2$ is a null quadrature domain if and only if Ω is one of the following:*

1. *The exterior on an ellipse;*
2. *The complement of a strip;*
3. *A half-plane;*
4. *The non-convex domain bounded by a parabola.*

The author and Margulis [15] stated the following conjecture (see also [24]).

Conjecture 3.2. *The open set $\Omega \subset \mathbb{R}^n$ is a null quadrature domain if and only if Ω is one of the following:*

1. *The exterior on an ellipsoid;*
2. *The complement of a strip;*
3. *A half-space;*
4. *The non-convex domain bounded by a paraboloid;*
5. *A cylinder over 1. or 4.*

We refer to [15] and [24] for the verification that each of the set from the above list is indeed a null quadrature domain. The conjecture was confirmed in the following cases:

- (i) If Ω^c is bounded, then it is an ellipsoid (Friedman and Sakai [10]);
- (ii) If in a certain coordinates system $\partial\Omega \subset \{x : x_{n-1}^2 + x_n^2 < R^2\}$ for some positive R , then Ω^c is an ellipsoid or a cylinder over an ellipsoid ([15]; Theorem 4.13).

Our main result is:

Theorem 3.3. *If $\Omega \subset \mathbb{R}^n$ is a null quadrature domain such that in a certain coordinates system $\partial\Omega \subset \{x : |x_n| \leq R\}$ for some positive R and*

$$(3.1) \quad \liminf_{\varrho \rightarrow \infty} \frac{\text{Cap}_n(\Omega^c \cap B_\varrho)}{\text{Cap}_n(B_\varrho)} > 0,$$

then Ω is a half-space or a complement of a strip.

Here $\text{Cap}_n(F)$ denotes the Newtonian capacity of a compact set F in \mathbb{R}^n .

Remark 3.4. *The combination of Theorem 3.3 with (ii) above almost completes the characterization of null quadrature domains with boundary contained in a strip. Because if $\partial\Omega \subset \{x : x_{n-1}^2 + x_n^2 \leq R^2\}$, then $\Omega^c \subset \{x : x_{n-1}^2 + x_n^2 \leq R^2\}$ (see [15]; Theorem 4.13) which implies*

$$(3.2) \quad \liminf_{\varrho \rightarrow \infty} \frac{\text{Cap}_n(\Omega^c \cap B_\varrho)}{\text{Cap}_n(B_\varrho)} = 0.$$

Yet, there is a small gap between the negations of $\Omega^c \subset \{x : x_{n-1}^2 + x_n^2 \leq R^2\}$ and condition (3.1).

Proof. We write down the proof by following four steps.

Step 1. Reduction to the case $\Omega^c \subset \{x : |x_n| \leq R\}$.

When $\partial\Omega \subset \{x : |x_n| \leq R\}$, then Ω contains either both components $\{x : x_n < -R\}$ and $\{x : x_n > R\}$ or just one of these half-spaces.

Suppose Ω contains just the upper half-space, then we may set $\tilde{\Omega} = \Omega \cup \{x : x_n < -R\}$. Since $\{x : x_n < -R\}$ is a null quadrature domain,

$$\int_{\tilde{\Omega}} h dx = \int_{\Omega} h dx + \int_{\{x: x_n < -R\}} h dx = 0$$

for all harmonic and integrable functions $\tilde{\Omega}$. So $\tilde{\Omega}$ is a null quadrature domain with $(\tilde{\Omega})^c \subset \{x : |x_n| \leq R\}$.

Step 2. Ω^c is convex. Let $v(\chi_{\Omega^c})$ be the potential defined by (2.9). Then according to Theorem 2.4 it coincides with a quadratic polynomial q in Ω^c . So the function $u = v(\chi_{\Omega^c}) - q$ satisfies the overdeterminate system

$$\begin{cases} \Delta u = \chi_{\Omega}, & \text{in } \mathbb{R}^n \\ u = |\nabla u| = 0, & \text{on } \Omega^c \end{cases} .$$

In addition, by Lemma 2.1 $|u(x)| \leq C(1 + |x|)^2$. In this situation we are allowed to employ Theorem II of [5] which gives that $u \geq 0$. This implies by Caffarelli's result in [4] that $\frac{\partial^2 u}{\partial \nu^2} \geq 0$ on $\partial\Omega$ for any direction ν . From the last property it follows that u is a convex function (for details see [16]; Section 4). Now $\Omega^c = \{u = 0\}$ is a convex set.

Step 3. Ω^c (and hence Ω) is a cylindrical set. Since Ω^c is convex and unbounded, it contains an infinity ray. By a rotation and a translation in the variables (x_1, \dots, x_{n-1}) , we may assume this ray is in the positive direction of the x_1 axis. Using Theorem 2.4, we have

$$v(\chi_{\Omega^c})(x) = \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j + \sum_{i=1}^n b_i x_i + c \quad \text{for } x \in \Omega^c.$$

Now $(x_1, 0, \dots, 0) \in \Omega^c$ for all $x_1 \geq 0$, so

$$v(\chi_{\Omega^c})(x_1, 0, \dots, 0) = a_{11} x_1^2 + b_1 x_1 + c$$

for $x_1 \geq 0$. Thus, by the estimate (2.6) we must have $a_{11} = 0$. Similarly, for $i = 2, 3, \dots, n$,

$$\frac{\partial v(\chi_{\Omega^c})}{\partial x_i}(x_1, 0, \dots, 0) = a_{1i} x_1 + b_i \quad \text{for } x_1 \geq 0.$$

Again, the estimate (2.7) yields $a_{1i} = 0$. Setting

$$w(x) = \frac{\partial v(\chi_{\Omega^c})}{\partial x_1}(x) - b_1,$$

then

$$w(x) = 2a_{11}x_1 + \sum_{j=2}^n a_{1j}x_j = 0$$

for $x \in \Omega^c$. The potential $v(\chi_{\Omega^c})$ has Hölder continuous first order derivatives, therefore w vanishes on $\partial\Omega$ and in Ω , $\Delta w = \partial_1(\Delta v(\chi_{\Omega^c})) = \partial_1 1 = 0$. If $w \not\equiv 0$ in Ω , then it follows from (3.1) and Phragmén-Lindelöf Type Theorem 4.2 that

$$(3.3) \quad \liminf_{x \in \Omega, |x| \rightarrow \infty} \frac{|w(x)|}{|x|^\beta} > 0,$$

for some positive β . But this contradicts the estimate (2.7). Therefore $w \equiv 0$ in \mathbb{R}^n , and hence

$$\partial_1 \chi_{\Omega^c} = \partial_1(\Delta v(\chi_{\Omega^c})) = \Delta w \equiv 0 \quad \text{in } \mathbb{R}^n.$$

This means that the characteristic function χ_{Ω^c} (or χ_Ω) does not depend on x_1 . We conclude that $\Omega = \mathbb{R} \times \Omega_1$, where Ω_1 is an open set \mathbb{R}^{n-1} .

Step 4. An induction argument. The function $v(\chi_{\Omega^c})(x) - b_1x_1$ does not depend on x_1 and it coincides with a quadratic polynomial $q_1 = q_1(x_2, \dots, x_n)$ on Ω^c . Hence

$$u(x_2, \dots, x_n) := v(\chi_{\Omega^c})(x) - b_1x_1 - q_1(x_2, \dots, x_n)$$

satisfies

$$\begin{cases} \Delta u = \chi_{\Omega_1}, & \text{in } \mathbb{R}^{n-1} \\ u = |\nabla u| = 0, & \text{on } \Omega_1^c \\ u(x_2, \dots, x_n) \leq C(1 + x_2^2 + \dots + x_n^2) \end{cases}.$$

These means that Ω_1 is a null quadrature domain in \mathbb{R}^{n-1} . Clearly, $\partial\Omega_1 \subset \{(x_2, \dots, x_n) : |x_n| \leq R\}$ and by Proposition 4.4,

$$(3.4) \quad \liminf_{\rho \rightarrow \infty} \frac{\text{Cap}_{n-1}(\Omega_1^c \cap B_\rho)}{\text{Cap}_{n-1}(B_\rho)} > 0.$$

Continuing this process, we get $\Omega, \Omega_1, \dots, \Omega_{n-2}$, where $\Omega_{n-2} \subset \mathbb{R}^2$ is a null quadrature domain with $\partial\Omega_{n-2} \subset \{(x_{n-1}, x_n) : |x_n| \leq R\}$. At this stage we apply Sakai's Theorem 3.1 and get that Ω_{n-2} is a complement of a strip in \mathbb{R}^2 . Finally,

$$\Omega = \mathbb{R} \times \Omega_1 = \mathbb{R} \times (\mathbb{R} \times \Omega_2) = \dots = \mathbb{R}^{n-2} \times \Omega_{n-2}$$

is a complement of a strip in \mathbb{R}^n . □

4. APPENDIX

We discuss here two results which are related to Newtonian Capacity. There are several equivalent ways to define it, we will employ the following one:

$$(4.1) \quad C_n(F) = \inf_{\mu} \{ \text{total mass of } \mu : V\mu \geq 1 \text{ p.p. on } F \},$$

where $V\mu$ is the Newtonian potential defined in (2.1), F is a compact set in of \mathbb{R}^n and *p.p.* means the inequality holds except for a set of zero capacity (see [1], [20]).

4.1. The Obscure Phragmén-Lindelöf Type Theorem. The principle idea of Phragmén-Lindelöf Type Theorem is that the maximum principle holds in unbounded domains provided that the function has a limited growth at infinity. The rate of growth depends upon the "size" of the domain in question. As the "size" increases the rate of growth decreases. A typical Phragmén-Lindelöf type theorem is the following one.

Theorem 4.1. (Essèn, Haliste, Lewis, Shea [9]) *Let u be subharmonic in a domain Ω , continuous on $\bar{\Omega}$ and such that $u \leq 0$ on $\partial\Omega$, the boundary of Ω . Assume the Wiener condition holds at infinity, that is,*

$$(4.2) \quad \int_1^{\infty} \frac{Cap_n(\Omega^c \cap B_{\varrho})}{Cap_n(B_{\varrho})} \frac{d\varrho}{\varrho} = \infty.$$

Then either $u \leq 0$ in Ω or u is unbounded in Ω .

The below the theorem could be extracted in the monograph of Kondrat'ev and Landis [19]. On one hand it strengthens Wiener condition (4.2), while in the other hand it allows higher growth.

Theorem 4.2. *Let u be a subharmonic function in an unbounded domain Ω , continuous on $\bar{\Omega}$ and such that $u \leq 0$ on $\partial\Omega$. Suppose*

$$(4.3) \quad \liminf_{\varrho \rightarrow \infty} \frac{Cap_n(\Omega^c \cap B_{\varrho})}{Cap_n(B_{\varrho})} > 0.$$

Then either $u \leq 0$ in Ω or there is $\beta > 0$ such that

$$(4.4) \quad \liminf_{\varrho \rightarrow \infty} \frac{\sup_{\{\Omega \cap \{|x|=\varrho\}\}} |u(x)|}{\varrho^{\beta}} > 0.$$

The basic tool of the proof is a capacity version of the *growth lemma* which was proved by Mazeya (see [19]).

Lemma 4.3. (Growth Lemma) *Suppose u is subharmonic function in Ω , continuous on $\bar{\Omega}$, $u \leq 0$ on $\partial\Omega \cap B_{4\varrho}(x_0)$ (we assume $\Omega \cap B_{4\varrho}(x_0) \neq \emptyset$) and $\sup_{\Omega \cap B_{4\varrho}(x_0)} u > 0$. Then*

$$(4.5) \quad \sup_{\Omega \cap B_{4\varrho}(x_0)} u > \left(1 + \xi \frac{\text{Cap}_n(\Omega^c \cap B_{\varrho}(x_0))}{\text{Cap}_n(B_{\varrho})}\right) \sup_{\Omega \cap B_{\varrho}(x_0)} u,$$

where $\xi = \text{Cap}_n(B_1)(2^{-(n-2)} - 3^{-(n-2)})$.

Proof of Theorem 4.2. We show that if $u(x_0) > 0$ for some $x_0 \in \Omega$, then (4.4) holds. Let

$$M(\varrho) = \sup_{\Omega \cap \{|x|=\varrho\}} u(x).$$

Condition (4.3) implies that there are positive constants ϱ_0 and γ such that

$$(4.6) \quad \frac{\text{Cap}_n((\mathbb{R}^n \setminus \Omega) \cap B_{\varrho})}{\text{Cap}_n(B_{\varrho})} \geq \gamma > 0$$

for $\varrho \geq \varrho_0$. Pick an integer k_0 such that both ϱ_0 and $|x_0|$ are less or equal to 4^{k_0} . Set $M_0 = M(4^{k_0})$ and $M_k = M(4^{k_0+k})$ for $k \in \mathbb{N}$. Note that $u(x_0) > 0$ implies $M_0 > 0$, so by the growth lemma 4.3 we obtain

$$M_k \geq (1 + \xi\gamma) M_{k-1} \geq (1 + \xi\gamma)^k M_0 = M_0 \exp(k \log(1 + \xi\gamma)).$$

For $4^{k+k_0} \leq \varrho \leq 4^{k+1+k_0}$, we have $k + 1 + k_0 \geq \frac{1}{\log 4} \log \varrho$. Letting $C_0 = \exp((k_0 + 1) \log(1 + \xi\gamma))$ and $\beta = \frac{1}{\log 4} \log(1 + \xi\gamma)$, then $\beta > 0$ and

$$(4.7) \quad M(\varrho) \geq M_k \geq \frac{M_0}{C_0} \exp((k + 1 + k_0) \log(1 + \xi\gamma)) \geq \frac{M_0}{C_0} \varrho^\beta$$

□

4.2. Capacity of Cylindrical Sets. Here we estimate the capacity of a cylindrical set $E = F \times \mathbb{R}$ in term of the capacity of F in the lower dimension space. We denote by Q_n^R be the closed cube centered at the origin and with a edge of length $2R$. In order to emphasize the relation between the potential and the dimension we add lower index n to the notation of the Newtonian potential

$$(2.1) \quad V_n \mu(x) = (J_n * \mu)(x) = \frac{1}{(n-2)\omega_n} \int \frac{1}{|x-y|^{n-2}} d\mu(y).$$

Proposition 4.4. *Let $E = F \times \mathbb{R}$, where $F \subset \mathbb{R}^n$ and $n \geq 3$. Then the inequality*

$$\frac{\text{Cap}_{n+1}(E \cap Q_{n+1}^R)}{\text{Cap}_{n+1}(Q_{n+1}^R)} \geq c_{n+1},$$

implies

$$\frac{\text{Cap}_n(F \cap Q_n^R)}{\text{Cap}_n(Q_n^R)} \geq c_n,$$

where c_{n+1} and c_n are positive constants independent of R .

Proof. Let

$$\Phi(t) = \int_0^t \frac{ds}{(1+s^2)^{\frac{n-1}{2}}}.$$

Then $\Phi(-t) = -\Phi(t)$, Φ is increasing and $\lim_{t \rightarrow \infty} \Phi(t) < \infty$. Let σ_R be the characteristic function of the interval $[-R, R]$. Let γ be the equilibrium measure of the set $F \cap Q_n^R$, that is, $\text{supp}(\gamma) \subset F \cap Q_n^R$, $V_n \gamma = 1$ p.p. on $F \cap Q_n^R$ and total mass of γ is $\text{Cap}_n(F \cap Q_n^R)$. We denote coordinates on \mathbb{R}^{n+1} by (x', x_{n+1}) and define a measure

$$\mu(x', x_{n+1}) = \gamma(x') \sigma_{2R}(x_{n+1}).$$

Then

$$V_{n+1}(\mu)(x', x_{n+1}) = \frac{1}{(n-1)\omega_{n+1}} \int \int \frac{d\gamma(y') \sigma_{2R}(y_{n+1}) dy_{n+1}}{(|x' - y'|^2 + |x_{n+1} - y_{n+1}|^2)^{\frac{n-1}{2}}}.$$

Computing first the integral with respect to y_{n+1} , we have

$$\begin{aligned} & \int \frac{\sigma_{2R}(y_{n+1}) dy_{n+1}}{(|x' - y'|^2 + |x_{n+1} - y_{n+1}|^2)^{\frac{n-1}{2}}} \\ &= \frac{1}{|x' - y'|^{n-1}} \int_{-2R}^{2R} \left(1 + \left(\frac{|x_{n+1} - y_{n+1}|}{|x' - y'|} \right)^2 \right)^{\frac{1-n}{2}} dy_{n+1} \\ &= \frac{1}{|x' - y'|^{n-2}} \left(\Phi \left(\frac{2R - x_{n+1}}{|x' - y'|} \right) - \Phi \left(\frac{-2R - x_{n+1}}{|x' - y'|} \right) \right) \end{aligned}$$

Let $(x', x_{n+1}) \in Q_{n+1}^R$ and $y' \in \text{supp}(\gamma) \subset Q_n^R$, then $|x' - y'| \leq \sqrt{n}2R$, $2R - x_{n+1} \geq R$ and $-2R - x_{n+1} \leq -R$. So the properties of Φ implies

$$\Phi \left(\frac{2R - x_{n+1}}{|x' - y'|} \right) \geq \Phi \left(\frac{R}{\sqrt{n}2R} \right) \geq \Phi \left(\frac{1}{\sqrt{n}2} \right)$$

and

$$-\Phi \left(\frac{-2R - x_{n+1}}{|x' - y'|} \right) = \Phi \left(\frac{2R + x_{n+1}}{|x' - y'|} \right) \geq \Phi \left(\frac{1}{\sqrt{n}2} \right).$$

Setting

$$\tilde{\mu} = \frac{(n-1)w_{n+1}}{(n-2)\omega_n 2\Phi(\frac{1}{\sqrt{n2}})}\mu,$$

and letting $(x', x_{n+1}) \in E \cap Q_{n+1}^R$, then

$$\begin{aligned} V_{n+1}(\tilde{\mu})(x', x_{n+1}) &\geq \frac{1}{(n-2)\omega_n} \int \frac{d\gamma(y')}{|x' - y'|^{n-2}} \\ &= V_n(\gamma)(x') = 1 \text{ p.p. on } E \cap Q_{n+1}^R. \end{aligned}$$

Hence $\tilde{\mu}$ is an admissible measure for capacity of $E \cap Q_{n+1}^R$ and therefore

$$\begin{aligned} \text{Cap}_{n+1}(E \cap Q_{n+1}^R) &\leq \text{total mass of } \tilde{\mu} = \int d|\tilde{\mu}| \\ &= \frac{(n-1)w_{n+1}}{(n-2)\omega_n 2\Phi(\frac{1}{\sqrt{n2}})} \int \int d\gamma(x') \sigma_{2R}(x_{n+1}) dx_{n+1} \\ (4.8) \quad &= \frac{(n-1)w_{n+1}}{(n-2)\omega_n 2\Phi(\frac{1}{\sqrt{n2}})} (4R) \int d\gamma(x') \\ &= \frac{(n-1)w_{n+1}}{(n-2)\omega_n 2\Phi(\frac{1}{\sqrt{n2}})} (4R) \text{Cap}_n(F \cap Q_n^R). \end{aligned}$$

Now $\text{Cap}_{n+1}(Q_{n+1}^R) \simeq R \text{Cap}_n(Q_n^R)$, so dividing both sides of (4.8) by $\text{Cap}_{n+1}(Q_{n+1}^R)$, we get

$$\frac{\text{Cap}_{n+1}(E \cap Q_{n+1}^R)}{\text{Cap}_{n+1}(Q_{n+1}^R)} \leq C \frac{\text{Cap}_n(F \cap Q_n^R)}{\text{Cap}_n(Q_n^R)}.$$

This completes the proof. \square

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REFERENCES

1. H. Aikawa and M.R. Essen, *Potential Theory: Selected Topics*, Lecture note in Mathematics 1633, Springer-Verlag, Berlin Heidelberg New York, 1996. [4](#)
2. V. I. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, New York, 1989. [2.5](#)
3. E. DiBenedetto and A. Friedman, *Bubble growth in porous media*, Indiana Univ. Math. J. **35** (1986), 5730-606. [1](#), [1](#)

4. L. A. Caffarelli, *Compactness methods in free boundary problems*, Comm. Partial Diff. Eq. **5** (1980), 427-448. [3](#)
5. L. A. Caffarelli, L. Karp and H. Shahgholian, *Regularity of a free boundary problem with application to the Pompeiu problem*, Ann. Math. **151**, (2000), 269-292. [3](#)
6. S. Chandrasekhar *Ellipsoidal figures of equilibrium*, Dover, 1987. [1](#)
7. P. Dive, Attraction des ellipsoïdes homogènes et réciproque d'un théorème de Newton *Bull Soc. Math. France*, **59** (1931),128-140. [1](#)
8. V.M. Entov and P.I. Etingof, *Bubble contraction in Hele-Shaw cells*, Quart. J. Mech. Appl. Math. **44** (1991), no. 4, 507-535. [1](#)
9. M. Essèn, K. Haliste, J.L. Lewis, and D.F. Shea, *Harmonic majorization and classical analysis*. J. London Math. Soc. (2) **32** (1985), no. 3, 506-520. [4.1](#)
10. A. Friedman and M. Sakai, *A characterization of null quadrature domains in \mathbb{R}^n* , Indiana Univ. Math. J. **35** (1986), 607-610. ([document](#)), [3](#)
11. S. D. Howison, *Bubble growth in porous media and Hele Shaw flow* , Proc. Roy. Soc.Edinburgh **A102** (1985), 141-148. [1](#)
12. E. Hölder, Über eine Potentialtheoretische Eigenschaft der Ellipse, *Math Z*, **35** (1932), 632-643. [1](#)
13. L. Karp, *Generalized Newtonian potential and its applications*, J. Math. Anal. Appl. **174** (1993), 480-497. [2.2](#), [2.3](#), [2.2](#)
14. L. Karp, *On the Newtonian potential of ellipsoids*, Complex Variables **25** (1995), 367-372. [1](#)
15. L. Karp and A. Margulis, *Newtonian potential theory for unbounded sources and applications to free boundary problems*, J. Analyse Math. **70** (1996), 1-63. ([document](#)), [2](#), [2.1](#), [2.1](#), [2.2](#), [2.3](#), [2.2](#), [2.3](#), [3](#), [3](#), [3](#), [3.4](#)
16. L. Karp and H. Shahgholian, *Regularity of a free boundary problem*, J. Geom. Anal. **9**, (1999), 653-669. [3](#)
17. O.D. Kellogg, *Fundation of Potential Theory*, Julius Springer. Reprint by Dover Publ. New york, 1953. [1](#)
18. D. Khavinson, *Holomorphic Partial Differential Equations and Classical Potential Theory*, Departamento de Análisis Matemático, Universidad de La Laguna, Tenerife, 1996.
19. V.A. Kondratt'ev and E.M. Landis, *Qualitative theory of second order linear partial differential equations*, in Partial Differential Equations III (Yu. V. Egorov and M.A. Shubin eds.) Springer-Verlag, Berlin 1991. [4.1](#), [4.1](#)
20. N.S. Landkof, *Fundation of Modern Potential Theory*, Springer-Velag, Berlin Heidelberg New York, 1972. [4](#)
21. W. Nikliborc, Eine Bemerkung über die Volumpotentiale, *Math Z*, **35** (1932), 625-631. [1](#)
22. M. Sakai, *Null quadrature domains*, J. Analyse Math. **40** (1981), 144-154. [2.2](#), [3.1](#)
23. M. Sakai, *Regularity of boundaries of quadrature domains in two dimensions*, SIAM J. Math. Anal. **24** (1994), 341-364. [3.1](#)
24. H. Shahgholian, *On quadrature domains and the Schwarz Potential*, J. Math. Anal. Appl. **177** (1992), 61-78. [3](#), [3](#)
25. H. S. Shapiro, *A weighted L^1 estimate for the Laplace Operator*, in Recent Advance in Fourier Analysis and its Applications (Il Ciocco 1989) Kluwer Acad Publ. Dordrecht 1990, pp. 563-577. [2.2](#)

26. H. S. Shapiro, *Quasi-balayage and a priori estimates for the Laplace operator, Multivariate approximation* (Witten-Bommerholz 1996), 203-230, 231-254, Math. Res. 101, Akademie Verlag, Berlin, 1997. [2.2](#)
27. A. N. Varchenko, P. I. Etingof, *Why the Boundary of a Round Drop Becomes a Curve of Order Four*, American Mathematical Society AMS University Lecture Series, Volume 3, Providence, Rhode Island 1992. [1](#)

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