Another Approach to the Stability of Linear Functional Operators

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1 Introduction.

In this paper we deal with compact supported Banach-valued functions F of a single variable satisfying the general linear functional equation

$$\mathcal{P}F := \sum_{j=1}^{n+1} c_j(x)F\big(a_j(x)\big) = H(x)$$

for all points x in a bounded domain $D \subset \mathbb{R}^n,\ n \geq 2$. The problem of its stability goes back to Ulam (see [U]) and since then was considered from various points of view in innumerable papers mainly related to the operator $\mathcal{P}$ with constant coefficients c_j and a_j being linear functions. To be more exact almost all these papers dealt with the Cauchy operator

$$\mathfrak{C} : F \to F(x+y) - F(x) - F(y), \quad (x,y) \in \mathbb{R}^2;$$

the simplest form of the Jensen operator

$$\mathfrak{J} : F \to F\Big(\frac{x+y}{2}\Big) - \frac{1}{2}\Big(F(x) + F(y)\Big), \quad (x,y) \in \mathbb{R}^2;$$

the quadratic operator

$$\mathfrak{Q} : F \to F(x+y) + F(x-y) - 2F(x) - 2F(y), \quad (x,y) \in \mathbb{R}^2,$$

and with slight modifications of them. The original Hyer's idea (see [HIR]) of constructing the desired solution to the corresponding homogeneous equation $\mathcal{P}F = 0$ continues to play a crucial role in almost all these works. Our approach to the stability problem is novel in two essential ways. The first is that, under quite general conditions, the stability problem for $\mathcal{P}$ as formulated in [U] is overdetermined: as will be shown, the smallness of H only on a one-dimensional submanifold $\Gamma \subset \overline{D}$ (but not on the whole $\overline{D}$) implies the nearness of F to a solution of the above homogeneous problem.

The second is a functional analytic point of view. We consider the linear operator $\mathcal{P}_\Gamma$ - the restriction of $\mathcal{P}$ to Γ - between appropriate function spaces and give conditions of its surjectivity. The stability then follows from functional analytic consideration.

The realization of this program in a general form for an arbitrary operator $\mathcal{P}$ seems to be an unrealistic problem at present. That's why in this publication we restrict ourselves to a subclass of operators $\mathcal{P}$ that on the one hand is sufficiently large (and have never been considered earlier), and on the other hand, makes it possible to demonstrate all advantages of the new approach. We also ignore here the noncompact case mainly because the compact case from different reasons have never been studied appropriately even for the triple of the above model operators.

All needed notions and notations are introduced in the next section. In the same place we introduce and shortly discuss some new types of stability. The main results of this work are formulated in Sec. 3, and we prove all of them in Sec. 4.

2 The main notations and notions.

In the course of the work we denote by D an arbitrary connected domain in $\mathbb{R}^n$ consisting of points x. All considered operators L are supposed to be linear. The domain, the kernel and the range of L are denoted by $\mathcal{D}(L)$, ker$L$ and $\mathcal{R}(L)$, respectively. We denote by I_T the interval $\{t \mid 0 \le t \le T\}$, using the notation I in the case of $T = 1$. Given a Banach space B with the norm $|\cdot|_B$, we denote by $C(I_T, B)$ the space of all continuous functions $F : I_T \to B$ with the norm $|F|_B^{\langle 0 \rangle} = \sup\limits_{t \in I_T} |F(t)|_B$. The following spaces play a crucial role in the following. By definition,

$$C_{\langle 1 \rangle}(I_T, B) = \{f \mid f(t) = a_0 + t\varphi(t)\}$$

with $a_0 \in B$ and φ being continuous function: $I_T \to B$. With the norm

$$|f|_B^{\langle 1 \rangle} = |a_0|_B + |\varphi|_B^{\langle 0 \rangle}$$

the space $C_{\langle 1 \rangle}(I_T, B)$ becomes a Banach space. Also, if $0 < r < 1$, then

$$C_{\langle 1+r \rangle}(I_T, B) = \{f \mid f(t) = a_0 + a_1 t + t^{1+r}\varphi(t)\}$$

with the same φ as above. The space $C_{\langle 1+r \rangle}(I_T, B)$ endowed by the norm

$$|f|_B^{\langle 1+r \rangle} = |a_0|_B + |a_1|_B + |\varphi(t)|_B^{\langle 0 \rangle}$$

is also a Banach space.

The space $C_{\langle 1+r \rangle}(D, B)$ is defined analogously: a function $f : D \to B$ is an element of the above space if

$$f(x) = a_0 + a_1 x_1 + \ldots + a_n x_n + |x|^{1+r}\varphi(x).$$

Here all a_j are elements of the space B and φ is a continuous function: $D \to B$.

Remark. Alternatively (and intuitively more clear) the space $C_{\langle 1+r \rangle}(I_T, B)$ can be defined as $\{f \mid (\frac{f(t)-a_0}{t} - a_1)/t^r = \varphi(t)\}$ with some elements a_0, a_1 from B and a continuous function φ from $C(I_T, B)$. Such a definition clarifies the nature of this space: its elements are all B- continuous functions on I_T differentiable at the point $t = 0$ whose derivatives satisfy the Hölder condition of order r.

It can be directly verified that if all the functions $c_j(x)$ and $a_j(x)$ lie in some space $C_{\langle 1+r \rangle}(D, I_T)$, then *the operator $\mathcal{P}$ maps continuously this space into itself.*

Let us fix some space $\mathcal{B} = C_{\langle 1+r \rangle}(I_T, B)$ and consider $\mathcal{P}$ as the operator: $\mathcal{B} \to \mathcal{B}$. The following definition is due to S.Ulam [U].

Definition 1. The operator $\mathcal{P}$ is called $\mathcal{B}$ - *stable* if the relation

$$|\mathcal{P}F|_{\mathcal{B}} < \delta \tag{1}$$

implies the relation

$$|F - f|_{\mathcal{B}} < \varepsilon \tag{2}$$

for an arbitrary positive ε and positive $\delta = \delta(\varepsilon)$ with some $f \in \ker \mathcal{P}$.

By misuse of language, this definition admits the following formulation: every approximate solution to the equation $\mathcal{P}F = 0$ is a near solution of the same equation. As will be seen, condition (1) is extremely restrictive for the inequality (2) to be fulfilled.

The following definitions relate with a weakening condition (1). Let Γ be a one-dimensional submanifold (curve) in $\overline{D}$, and let a_Γ and $\mathcal{P}_\Gamma$ denote the restrictions to Γ of an arbitrary function a on D and the operator $\mathcal{P}$, respectively.

Definition 2. Given an above Γ, the operator $\mathcal{P}$ is called *strongly* $\mathcal{B}$- *stable* (along Γ) if for an arbitrary $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that the relation

$$|\mathcal{P}_\Gamma F|_{\mathcal{B}} < \delta, \quad F \in \mathcal{B},$$

implies the relation

$$|F - f|_{\mathcal{B}} < \varepsilon$$

for an arbitrary $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ with some $f \in \ker \mathcal{P}$.

It is clear that the condition of the stability is considerably more restrictive than that of the strong stability. In turn, the latter stability is more restrictive then the *weak stability* defined as follows.

Definition 3. An operator $\mathcal{P}$ is called *weakly* $\mathcal{B}$ - *stable* (*along* Γ) if it satisfies Definition 2 with $f \in \ker \mathcal{P}_\Gamma$.

The distinction between strong and weak stability is determined by the fact that $\ker\mathcal{P} \subseteq \ker \mathcal{P}_\Gamma$ for any curve Γ.

3 Statement of results

We begin with a functional analytic proposition forming an operator basis in our approach to all the above stability problems. Let $L : E_1 \to E_2$ be a closed linear operator, and $\mathcal{K} = \ker L$.

Proposition 1. *If the range $\mathcal{R}(L)$ is closed, then there is a positive constant c such that the a priori estimate*

$$\inf_{\varphi \in \mathcal{K}} |F - \varphi|_{E_1} < c|LF|_{E_2} \tag{3}$$

holds for all elements $F \subset \mathcal{D}(L)$.

Moreover, there is an element $\Phi \in \mathcal{K}$ for which

$$|F - \Phi|_{E_1} < c|LF|_{E_2}.$$

In the presence of estimate (3) establishing stability of any operator L is reduced to two routine procedures in the framework of classical functional analysis (each, possibly, requires a very serious work):

1) proving the closedness of the range $\mathcal{R}(L)$;
2) a description of the kernel $\mathcal{K}(L)$.

In this work we follow precisely this scheme.

As an overwhelming majority of results related to the Ulam stability grouped around operators $\mathfrak{C}$, $\mathfrak{J}$, and $\mathfrak{Q}$, no classification of the general linear functional operators $\mathcal{P}$ has been considered until now . The following class of such operators $\mathcal{P}$ is studied in the present work.

Definition 4. Let

$$\mathcal{P}F = F\big(a(x)\big) - \sum_{j=1}^{N} F\big(a_j(x)\big), \quad x \in D \subset \mathbb{R}^n, \tag{4}$$

with $F \in C(I_T, B)$ and $a, a_j \in C(D, I_T)$. We call $\mathcal{P}$ *Cauchy type operator*, if

$$a = \sum_{j=1}^{N} a_j \quad \text{everywhere in } D.$$

If for a subset $\Gamma \subset D$ the condition

$$a_\Gamma = \sum_{j=1}^{N} a_{j\Gamma} \tag{5}$$

holds, the operator $\mathcal{P}$ is called *weak Cauchy type operator* (*along* Γ). It is clear that the operator $\mathfrak{C}$ is the simplest model for both Cauchy type and weak Cauchy type operators.

Before formulating the main result of this work, Theorem 2, we describe those curves $\Gamma \subset D$ playing an essential role in questions of stability.

Definition 5. Given an operator $\mathcal{P}$ of form (4), a curve $\Gamma \subset D$ is called $\mathcal{P}$ - *admissible* if the function a_Γ maps Γ one-to-one onto I_T, and the inverse function lies in $C_{\langle 1+r\rangle}(I_T)$.

Theorem 2. *Let $D \subset \mathbb{R}^n$ be a connected bounded domain and $\Gamma \subset \overline{D}$ a nonsingular $\mathcal{P}$ - admissible $C_{\langle 1+r\rangle}$ - curve with $0 < r < 1$. Assume that all the functions a and a_j lie in the space $C_{\langle 1+r\rangle}(D, I_T)$ and satisfy the condition*

$$\sum_{k \neq j} \frac{\partial}{\partial \Gamma'} a_j(x_0) \frac{\partial}{\partial \Gamma'} a_k(x_0) \neq 0 \quad \textit{if } a(x_0) = 0. \tag{6}$$

If $\mathcal{P}$ is a Cauchy type operator (along Γ), then it is strongly stable (along Γ), and if $\mathcal{P}$ is a weak Cauchy type operator (along Γ), then it is weakly stable (along Γ).

To prove this theorem we have to prove (following to the above scheme) first the closedness of the range of the operator $\mathcal{P}_\Gamma$ in $C_{\langle 1+r\rangle}(I_T, B)$. This is a direct corollary of the following theorem.

Theorem 3. *Let $\mathcal{P}$, D, Γ and r are the same as in Theorem 2. Then the operator*

$$\mathcal{P}_\Gamma : C_{\langle 1+r\rangle}(I_T, B) \to C_{\langle 1+r\rangle}(I_T, B)$$

is surjective.

The second step when proving Theorem 2 consists in describing the kernel of the above operator $\mathcal{P}_\Gamma$. It is done in the following theorem.

Theorem 4. *Let $\mathcal{P}$, D, Γ are the same as in Theorem 2 but $r = 0$. Then the kernels $ker\mathcal{P}$ and $ker\mathcal{P}_\Gamma$ consist of only additive functions $F(z) = zF(1)$, $z \in I$.*

This result generalizes considerably the results of the work [P3] devoted to the *overdeterminedness* of several classical functional equations. The term "overdeterminedness" should be understood in the sense that to reconstruct a solution F of the homogeneous equation $\mathcal{P}F = 0$ it suffices that the equality $\mathcal{P}F(x) = 0$ is fulfilled not in the whole domain $D \subset \mathbb{R}^n$ but only at points x of some submanifold $D' \subset D$ of positive codimension.

In [P3] the overdeterminedness of equation $(\mathfrak{C})$, $(\mathfrak{J})$, $(\mathfrak{Q})$, and generalized Cauchy equation

$$F(x_1 + \ldots + x_n) - F(x_2 + \ldots + x_n) - \ldots - F(x_1 + \ldots + x_{n-1}) + \\ + F(x_3 + \ldots + x_n) + \ldots + (-1)^{n+1} \sum_{j=1}^{n} F(x_j) = 0$$

has been established.

Remark. It is trivial that $\ker \mathcal{P} \subseteq \ker \mathcal{P}_\Gamma$ for an arbitrary set Γ. However, the relation $\ker \mathcal{P} = \ker \mathcal{P}_\Gamma$ is far from being true.

Example. Let $\mathcal{P}$ be a weak Cauchy operator, so that $a_\Gamma = \sum a_{j\Gamma}$ but $a \not\equiv a_1 + \ldots + a_n$. Then, by Theorem 2, $\ker \mathcal{P}_\Gamma = \{\lambda z\}_{\lambda \in \mathbb{R}}$, but no function $F = \lambda z$, $\lambda \neq 0$ solves the $F(a) - \sum F(a_j) = 0$.

The concluding theorem is a trivial corollary of the main Theorem 2, and, therefore, it does not require any proof. We formulate this corollary as an independent theorem with the only goal: to make possible for the reader to compare the information supplied by Theorem 2 with the initial Ulam's question [U]. This question having completely analytic nature and being answered by Hyers [HIR], in approximately 30 years suddenly has risen up to a level of a significant problem. Hundreds of papers concerning various modifications (not only analytic but also algebraic) of the Ulam question were published until now and majority of these papers deal with the operator $\mathfrak{C}$ and continue to make use the Hyers formula (see [HIR]).

Theorem 5. *Let $\mathcal{P}$ be a weakly Cauchy operator along Γ being an admissible $C_{\langle 1+r\rangle}$ - curve in D. Given a function $H(x) \in C_{\langle 1+r\rangle}(D, B)$, assume a function $F \in C_{\langle 1+r\rangle}(I_T, B)$ to be a solution of the equation $\mathcal{P}F = H$. If the function H satisfies the inequality*

$$|H_\Gamma|_B^{\langle 1+r\rangle} < \varepsilon,$$

then, for an additive function $f : I_T \to B$, the estimate

$$|F - f|_B^{\langle 1+r\rangle} < c\varepsilon,$$

holds with a constant $c > 0$ not depending on F or ε.

Remark. The value of the constant c can be easily estimated from above with the help of the explicit formula (14).

Remark. It is worth noting that, by this theorem, neither the function f nor the constant c depend on behavior of the right-hand side H outside Γ. This emphasizes again that the Ulam's restriction on the smallness of H *everywhere* is not required when establishing the stability of $\mathcal{P}$.

4 Proofs of the main results.

We begin with the proof of the fact which has probably the nature of folklore.

4.1 Proof of Proposition 1

Along with the operator $L : E_1 \to E_2$ consider the operator

$$\widehat{L} : E_1/\mathcal{K} \to E_2$$

associated an arbitrary element $\{F\}$ of the factor space generated by $F \in E_1$ with the element LF from E_2. It is clear that this operator is injective and closed. Thus by the Banach closed graph theorem, the inverse operator $\widehat{L}^{-1}$ is bounded. This leads to the a priori estimate

$$|\{F\}|_{E_1/\mathcal{K}} < c|LF|_{E_2}, \quad F \in E_1,$$

where c is a constant. To complete the proof of relation (3) it suffices to note that

$$|\{F\}|_{E_1/\mathcal{K}} = \inf_{\varphi \in \mathcal{K}} |F - \varphi|_{B_1}.$$

To prove the concluding assertion of Proposition 1 denote

$$m(\varphi) = |F - \varphi|_{E_1},$$

and let

$$\mu = \inf_{\varphi \in \mathcal{K}} m(\varphi)$$

If $\mu = m(\Phi)$ for some $\Phi \in \mathcal{K}$, then the proof is completed. Assume that $\mu < m(\varphi)$ for all elements $\varphi \in \mathcal{K}$. Let

$$\mu = c|LF|_{E_2} - \varepsilon$$

for some positive ε. Then, by definition of μ, there is an element $\Phi \in \mathcal{K}$ such that

$$\mu < m(\Phi) < \mu + \varepsilon = c|LF|_{E_2}$$

This completes the proof of Proposition 1.

4.2 Proof of Theorem 3.

Without loss of generality we will consider the following situation: the origin $0 \in \mathbb{R}^n$ lies in D and is one of the boundary points of the curve Γ, so that

$$a(0) = 0 \quad \text{and} \quad a_j(0) = 0, \;\; j = 1, \ldots, n. \tag{7}$$

The curve Γ admits a parametric representation of the form

$$\Gamma = \{x \in D \mid x_j = \alpha_j(t),\ j = 1, \ldots, n,\ t \in I\} = \{x \mid x = \alpha(t)\},$$

where all the functions $\alpha_j(t)$ belong to the space $C_{\langle 1+r \rangle}(I)$ and $\alpha_j(0) = 0,\ j = 1, \ldots, n$. In this case the relation $\mathcal{P}_\Gamma F = H_\Gamma$ takes the following form:

$$F\Big(a\big(\delta(t)\big)\Big) - \sum_{j=1}^{N} F\Big(a_j\big(\delta(t)\big)\Big) = H\big(\delta(t)\big), \quad t \in I, \tag{8}$$

where, by hypothesis (5),

$$a\big(\delta(t)\big) - \sum_{j=1}^{N} a_j\big(\delta(t)\big) = 0, \quad t \in I. \tag{9}$$

By conditions of the theorem, the function $a\big(\delta(t)\big)$ is invertible. Let $\beta : I_T \to I$ be the inverse function to the $a \circ \delta$, so that

$$(a \circ \delta)\big(\beta(z)\big) = z, \quad z \in I_T.$$

Then relation (8) becomes

$$F(z) - \sum_{j=1}^{N} F\big(\rho_j(z)\big) = h(z), \quad z \in I_T, \tag{10}$$

where

$$\rho_j(z) = (a_j \circ \delta \circ \beta)(z), \quad h(z) = H(\delta \circ \beta)(z).$$

By (9), we find that

$$\sum_{j=1}^{N} \rho_j(z) = z \tag{11}$$

and, due to (6),

$$\sum_{j \neq k} \rho_j'(0)\rho_k'(0) \neq 0. \tag{12}$$

We are now going to solve equation (10) for an arbitrary function $h(z) \in C_{\langle 1+r \rangle}(I_T)$, and, thus, to complete the proof of Theorem 3. To this end, note first that differentiating relation (10) at the point $z = 0$ and combining the result with the relation

$$\sum_{j=1}^{N} \rho_j'(z) = 1,$$

following (11), we arrive at the necessary condition

$$h'(0) = 0.$$

On the other hand, by (10) and (7),

$$h(0) = (1 - N)F(0).$$

Therefore, the problem is reduced to the solvability of the equation

$$\widetilde{F}(z) - \sum_{j=1}^{N} \widetilde{F}\big(\rho_j(z)\big) = \widetilde{h}(z), \quad z \in I_T, \tag{13}$$

where $F(z) = F(z) - F(0)$ and the right-hand side $\widetilde{h}$ satisfies the conditions

$$\widetilde{h}(0) = 0, \quad \widetilde{h}'(0) = 0.$$

But this makes it possible to apply Theorem 1 from [P2], taking into account that condition (5) in the cited paper is nothing but the above relation (12). This leads to the relation

$$F(z) = \frac{1}{1-N} h(0) + \sum_{|J|=0}^{\infty} \widetilde{h}\big(\rho_J(z)\big), \tag{14}$$

where $J_n = (j_1, \ldots, j_n)$ is a multi-index of the length $|J_n| = n,\ j_k = 1, \ldots, N$ for an arbitrary $k = 1, 2, \ldots, n$, and $\rho_{J_n} = \rho_{j_n} \circ \ldots \circ \rho_{j_1}$. The series in the right-hand side converges in the topology of $C_{\langle 1+r \rangle}(I_T)$, and, therefore, belongs to $C_{\langle 1+r \rangle}(I_T)$.

This completes the proof of Theorem 3.

4.3 Proof of Theorem 4

Let us return to equation (10) with $h = 0$. Dividing both parts of the equation obtained by $a\big(\delta(t)\big)$ we arrive at the equivalent functional equation

$$\Phi\Big(a\big(\delta(t)\big)\Big) - \sum_{j=1}^{N} \rho_j(t) \Phi\Big(a_j\big(\delta(t)\big)\Big) = 0, \quad t \in I, \tag{15}$$

where

$$\Phi(z) = \frac{F(z)}{z}, \quad \rho_j(t) = \frac{a_j\big(\delta(t)\big)}{a\big(\delta(t)\big)}, \quad t \in I \setminus \{0\}$$
$$\Phi(0) = F'(0), \quad \rho_j(0) = \frac{a_j'(0)}{a'(0)}, \quad j = 1, \ldots, N,$$

and all functions $a_j \circ \delta, \rho_j, \Phi$ are continuous on the interval I. Show that any continuous solution Φ of equation (15) is a constant. Indeed, let $\mathcal{M} = \max_I \Phi$ and let $t_0 = \inf\{ t \mid \Phi\Big(a\big(\delta(t)\big)\Big) = \mathcal{M}\}$. Assume that $t_0 > 0$. Then, by continuity, $\Phi\Big(a\big(\delta(t_0)\big)\Big) = \mathcal{M}$, and, therefore, the relation

$$\Phi\Big(a_1\big(\delta(t_1)\big)\Big) = \Phi\Big(a_2\big(\delta(t_1)\big)\Big) = \mathcal{M}$$

is valid for an arbitrary point t_1 from the nonempty subset $\{t \mid a\big(\delta(t)\big) = t_0\}$, as $\rho_1(t) + \rho_2(t) = 1$ for all values $t \in I$. This, however, contradicts to the definition of the point t_0, as $a_1\big(\delta(t_1)\big) < t_0$ (by Definition 5). Thus, we have proved that $t_0 = 0$, and hence, $\Phi(0) = \mathcal{M}$. Repeating literally these arguments with respect to the point of minimum of the function Φ, we arrive at the relation $\Phi(0) = \min_I \Phi$. This proves that $\Phi(z)$ is a constant, and therefore F is additive.

Now everything is ready for proving the main Theorem 2.

4.4 Proof of Theorem 2.

By Theorem 3 and by Proposition 1 (applied to the operator $\mathcal{P}_\Gamma$ in the space $\mathcal{B} = C_{\langle 1+r\rangle}(I_T, B))$, the a priori estimate

$$|F - \Phi|_{\mathcal{B}} < c|\mathcal{P}_\Gamma F|_{\mathcal{B}}, \quad F \in \mathcal{B}, \tag{16}$$

is valid with a function Φ for which $\mathcal{P}_\Gamma \Phi = 0$. By Theorem 4, this function is additive, and hence, it satisfies the condition $\mathcal{P}\Phi = 0$, if $\mathcal{P}$ is a Cauchy type operator. Therefore, in this case the operator $\mathcal{P}$ is strongly stable (along Γ). If $\mathcal{P}$ is a weak Cauchy type operator, then the only additive function satisfying the equation $\mathcal{P}\Phi = 0$ is $\Phi = 0$ (which means, by the way, that $\mathcal{P}$ is an invertible operator). However, there is no possibility to establish, whether the function Φ in (16) coincides with zero. Thus, in the case of a weakly Cauchy type operator $\mathcal{P}$, Theorem 2 guarantees only the additivity of the function Φ. This completes the proof of Theorem 2.

References

[HIR] Hyers, D., Isac, G., and Rassias ,Th., *Stability of Functional Equations in Several Variables*, Birkhouser, Boston - Basel - Berlin, 1998

[P1] Paneah, B., On the Overdeterminedness of Some Functional Equations, *Discret and Continuous Dynamical Systems*, 10 (2004), 497 - 505.

[P2] Paneah, B., Dynamical Systems and Functional Equations Related to Boundary Problems for Hyperbolic Differential Operators, *Doklady Mathematics*, 73 (2005), 949 - 953.

[U] Ulam, S., *A Collection of Mathematical Problems*, Publ. New York, 1960.