ELLIPTIC QUASICOMPLEXES IN BOUTET DE MONVEL ALGEBRA

K. KRUPCHYK, N. TARKHANOV, AND J. TUOMELA

Abstract. We consider quasicomplexes of Boutet de Monvel operators in Sobolev spaces on a smooth compact manifold with boundary. To each quasicomplex we associate two complexes of symbols. One complex is defined on the cotangent bundle of the manifold and the other on that of the boundary. The quasicomplex is elliptic if these symbol complexes are exact away from the zero sections. We prove that elliptic quasicomplexes are Fredholm. As a consequence of this result we deduce that a compatibility complex for an overdetermined elliptic boundary problem operator is also Fredholm. Moreover, we introduce the Euler characteristic for elliptic quasicomplexes of Boutet de Monvel operators.

Contents

Introduction

When studying the well-posedness of elliptic boundary value problems on a smooth compact manifold with boundary it is reasonable to relax the requirement of existence and uniqueness, and allow boundary problem operators to be Fredholm. The ellipticity of a boundary problem operator consists of both ellipticity of the given differential operator on the manifold and ellipticity of the boundary conditions. The latter is called the Shapiro-Lopatinskii condition. It is known [Agr97] that in the case of square systems (as many equations as unknowns) the ellipticity of a boundary problem is equivalent to the Fredholm property of them in appropriate Sobolev spaces.

To study the solvability of boundary problems for overdetermined systems one has to consider compatibility complexes for them. The formal theory of overdetermined boundary value problems is analogous to that for overdetermined systems of differential equations, as was shown by Samborski in the 1980s, see [DS96]. For any regular boundary problem operator, the compatibility complex may be constructed

2000 *Mathematics Subject Classification.* Primary 58J10; Secondary 35Nxx, 58J20.
Key words and phrases. Elliptic complexes, Fredholm complexes, Hodge theory.

in finitely many steps (within the framework of differentiation of equations and Gröbner bases computations). Then a boundary problem for an overdetermined system is said to be well posed if the cohomology of the compatibility complex is finite dimensional. Such complexes are called *Fredholm*. The natural question arises under which condition the compatibility complex is Fredholm.

The differentials of compatibility complexes for overdetermined boundary value problems are given by triangle (2×2)-matrices, with zero at the upper right corner, cf. [DS96]. They belong to the algebra of pseudodifferential boundary value problems due to [dM71]. The advantage of this algebra lies in the fact that it survives under taking adjoint operators and contains parametrices of elliptic boundary problems. Hence we may as well consider a more general problem, i.e., to find conditions under which a complex of Boutet de Monvel operators is Fredholm in appropriate Sobolev spaces.

Complexes of operators in Boutet de Monvel's algebra were studied in [PS80]. This paper raised the question whether any exact sequence of principal symbols can be extended to a complex of boundary value problems. This latter is then automatically elliptic.

We go further and we observe that from the point of view of analysis, instead of complexes, it is much more natural to consider sequences of operators such that the composition of two consecutive operators is small in some reasonable sense, e.g., a compact operator. Such sequences are called *quasicomplexes*. Indeed, perturbation of a single Fredholm operator by a compact operator leads to a Fredholm operator. However, most perturbations of complexes lead out of the class of complexes, but it turns out that Fredholm quasicomplexes are stable under compact perturbations. In Section 2 we briefly sketch the concept of a quasicomplex.

Our paper deals with elliptic quasicomplexes of boundary value problems in appropriate Sobolev spaces. To this end, in Section 1 we have compiled some basic facts on Boutet de Monvel's algebra [dM71].

In Section 4 we prove that ellipticity of quasicomplexes of boundary problems (i.e., the exactness of both interior and boundary symbol sequences) implies Fredholm property. As but one consequence, we show in Section 7 that a compatibility complex for an overdetermined boundary value problem is Fredholm in suitable function spaces.

In Section 5 we construct a special parametrix for elliptic quasicomplexes on a manifold with boundary. In the case of complexes we derive in this way a complete Hodge theory for elliptic complexes of boundary value problems.

Boundary value problems for complexes of pseudodifferential operators were first considered by Dynin [Dyn72]. To introduce them he invoked the construction of the cone of a cochain mapping from homological algebra, which was very natural in this context. In Section 6 we specify our main results for cones of quasicochain mappings.

For elliptic quasicomplexes of boundary value problems, the topological index is well defined while the analytical index is not, for no cohomology is available. One thus arrives at the question whether, given an elliptic quasicomplex, there is a complex whose sequence of principal symbols coincides with that of the quasicomplex. The answer is by no means obvious because the problem has been open even for quasicomplexes of pseudodifferential operators on compact closed manifolds. In Section 8 we answer affirmatively to this question. This allows us to define the Euler characteristic for elliptic quasicomplexes, thus giving rise to the index theory of such quasicomplexes.

A standard example of an elliptic complex on a compact manifold with boundary is the de Rham complex without any boundary conditions. Any compact perturbation of this within Boutet de Monvel's algebra yields an elliptic quasicomplex of boundary value problems. A less banal example is given by the connection sequence associated with some smooth vector bundle, cf. Section 9. The connection square yields a curvature of the bundle which is a smooth bundle homomorphism and thus a compact operator. Once again this quasicomplex requires no boundary conditions to be elliptic in Boutet de Monvel's algebra.

1. Boutet de Monvel calculus

Let X be a smooth compact manifold with boundary Y. In this section we briefly present a calculus of operators

$$\mathcal{A} = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G} : \begin{array}{c} C^\infty(X,V) \\ \oplus \\ C^\infty(Y,W) \end{array} \to \begin{array}{c} C^\infty(X,\tilde{V}) \\ \oplus \\ C^\infty(Y,\tilde{W}) \end{array}, \tag{1.1}$$

in the spaces of smooth sections of $V, \tilde{V} \in \mathsf{Vect}(X)$ and $W, \tilde{W} \in \mathsf{Vect}(Y)$ introduced by Boutet de Monvel [dM71], cf. also books [RS82] and [Gru96]. Here $\mathsf{Vect}(X)$ is the collection of all smooth complex vector bundles on X and we will denote by k, $\tilde{k}$ and ℓ, $\tilde{\ell}$ the fibre dimensions of vector bundles V, $\tilde{V}$ and W, $\tilde{W}$.

To define operators (1.1), we have to introduce pseudodifferential operators with operator-valued symbols. Let us denote by $\mathcal{L}(F,\tilde{F})$ the space of all continuous linear maps between Banach spaces F and $\tilde{F}$. We first discuss spaces of operator-valued symbols. A strongly continuous group action on a Banach space F is a family $\kappa = \{\kappa_\lambda : \lambda \in \mathbb{R}_+\}$ of isomorphisms in $\mathcal{L}(F,\tilde{F})$, such that $\kappa_\lambda \kappa_\mu = \kappa_{\lambda\mu}$ and the map $\lambda \to \kappa_\lambda f$ is continuous for every $f \in F$. We will need only two group actions. If F is a space of functions on $\mathbb{R}_+$ or $\overline{\mathbb{R}}_+$, we use the group action defined by $(\kappa_\lambda u)(x) = \lambda^{1/2} u(\lambda x)$. And if $F = \mathbb{C}^\ell$, the group action is trivial, i.e. $\kappa_\lambda = \mathrm{Id}$ for any λ.

Let F, $\tilde{F}$ be Banach spaces with strongly continuous group actions κ and $\tilde{\kappa}$, respectively. Suppose $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(F,\tilde{F}))$ and $\mu \in \mathbb{R}$. We write $a \in \mathcal{S}^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(F,\tilde{F}))$ provided that for all multi-indices α, β there is a constant $c = c(\alpha,\beta)$ with

$$\|\tilde{\kappa}_{\langle\xi\rangle^{-1}} D_\xi^\alpha D_x^\beta a(x,\xi) \kappa_{\langle\xi\rangle}\|_{\mathcal{L}(F,\tilde{F})} \le c\, \langle\xi\rangle^{\mu-|\alpha|},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. For $F = \tilde{F} = \mathbb{C}$ we recover the definition of the symbol class $\mathcal{S}^\mu_{1,0}(\mathbb{R}\times\mathbb{R})$. The definition of symbol spaces may be extended to the case of a Fréchet space $F = \mathcal{S}(\overline{\mathbb{R}}_+)$ which is the restriction of the Schwartz space $\mathcal{S}(\mathbb{R})$ to the half-axis $\overline{\mathbb{R}}_+$.

A symbol $a \in \mathcal{S}^\mu(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(F,\tilde{F}))$ is said to be *classical*, if it has an asymptotic expansion

$$a \sim \sum_{j=0}^\infty a_j \tag{1.2}$$

with $a_j \in \mathcal{S}^{\mu-j}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(F,\tilde{F}))$ satisfying the homogeneity relation

$$a_j(x,\lambda\xi) = \lambda^{\mu-j}\, \tilde{\kappa}_\lambda a_j(x,\xi) \kappa_{\lambda^{-1}}$$

for all $\lambda \ge 1$ and $|\xi| \ge R$ with a suitable constant R. The equivalence relation $\sim$ is defined by requiring

$$a - \sum_{j=0}^N a_j \in \mathcal{S}^{\mu-N-1}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(F,\tilde{F}))$$

for every N. We write $a \in \mathcal{S}^\mu_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(F,\tilde{F}))$. For $F = \mathbb{C}$ and $\tilde{F} = \mathbb{C}$ we recover the standard notation.

Let us now define the operator P in (1.1). We first require P to be a classical pseudodifferential operators of order μ on a larger smooth manifold containing X. We introduce local coordinates $x = (x', x_n) \in \Omega \times \overline{\mathbb{R}}_+$, with Ω an open subset of

$\mathbb{R}^{n-1}$, such that Y is represented by $x_n = 0$. Next we need e^+ which is the operator extending functions on $\Omega \times \mathbb{R}_+$ by zero to $\Omega \times \mathbb{R}$, and r^+ which is the restriction operator from $\Omega \times \mathbb{R}$ to $\Omega \times \mathbb{R}_+$. Finally, let $p \in \mathcal{S}^\mu_{\mathrm{cl}}((\Omega \times \overline{\mathbb{R}}_+) \times \mathbb{R}^n, \mathcal{L}(\mathbb{C}^k, \mathbb{C}^{\tilde{k}}))$ and set $\mathrm{op}_x(p) = F^{-1}_{\xi \mapsto x} p(x,\xi) F_{x \mapsto \xi}$, where F is the Fourier transform. Then in these coordinates we can write $P = r^+ \mathrm{op}_x(p) e^+$. In general, the function $e^+ u$ fails to be C^∞ in $\Omega \times \mathbb{R}$ for $u \in C^\infty_{\mathrm{comp}}(\Omega \times \overline{\mathbb{R}}_+)$, since it may have a discontinuity along $x_n = 0$. The symbol p is said to have *transmission property* with respect to $x_n = 0$ when $r^+ \mathrm{op}_x(p) e^+$ preserves the smoothness up to $x_n = 0$, i.e., it maps $C^\infty_{\mathrm{comp}}(\Omega \times \overline{\mathbb{R}}_+)$ to $C^\infty(\Omega \times \overline{\mathbb{R}}_+)$.

Lemma 1.1. *Let $p \in \mathcal{S}^\mu_{cl}(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}(\mathbb{C}^k, \mathbb{C}^{\tilde{k}}))$ be a symbol with asymptotic expansion* (1.2)*. Then p has the transmission property with respect to $x_n = 0$ if and only if any p_j in* (1.2) *satisfies the symmetry condition*

$$D^\beta_x D^\alpha_\xi p_j(x', 0, 0, \xi_n) = e^{\imath \pi (j - |\alpha|)} D^\beta_x D^\alpha_\xi p_j(x', 0, 0, -\xi_n), \tag{1.3}$$

for $|\xi_n| \geq 1$ and all indices α, β, j.

See [Gru96] for the proof. Note that the transmission property is a local property and it is invariant with respect to coordinate changes. Globally, P has *transmission property* with respect to Y when P maps functions which are smooth up to the boundary to functions with the same property. As examples of pseudodifferential operators with transmission property one may consider pseudodifferential operators whose symbols are rational functions in ξ. In particular, all differential operators have the transmission property. If a pseudodifferential operator is elliptic and has transmission property, then every its parametrix is known to have the transmission property.

For $V, \tilde{V} \in \mathsf{Vect}(X)$, let us write $\Psi^\mu_{\mathrm{tp}}(X; V, \tilde{V})$ for the space of all classical pseudodifferential operators of order μ and any type $V \to \tilde{V}$ on X having the transmission property with respect to Y.

To describe $\mathcal{G}$ in (1.1) we need some preliminary notions. As usual, we denote by $\mathsf{Diff}^m(X; V, \tilde{V})$ the space of all linear differential operators of order m on X with coefficients smooth up to the boundary Y, acting in the corresponding spaces of sections.

The integral operators $C^\infty(X, V) \oplus C^\infty(Y, W) \to C^\infty(X, \tilde{V}) \oplus C^\infty(Y, \tilde{W})$ whose kernels are smooth up to the boundary are called smoothing operators of type 0. The space of such operators is denoted by $\mathcal{B}^{-\infty,0}(X; v)$, with vector space data $v = (V, \tilde{V}; W, \tilde{W})$.

Next let us consider operators of the form

$$\mathcal{S} = \mathcal{S}_0 + \sum_{j=1}^{d} \mathcal{S}_j \begin{pmatrix} D^j & 0 \\ 0 & 0 \end{pmatrix}$$

where $D^j \in \mathsf{Diff}^j(X; V)$ and $\mathcal{S}_j \in \mathcal{B}^{-\infty,0}(X; v)$. They are called smoothing operators of type d and the space of such operators is denoted by $\mathcal{B}^{-\infty,d}(X; v)$.

We now introduce operators $\mathcal{G}_\nu$ which are smoothing in $x \in \mathring{X}$ and $y \in Y$ and have in local coordinates $(x', x_n) \in \Omega \times \overline{\mathbb{R}}_+$ near Y the form of pseudodifferential operators

$$\mathrm{op}_{x'}(g) : C^\infty_{\mathrm{comp}}\Big(\Omega, \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \mathbb{C}^k \\ \oplus \\ \mathbb{C}^\ell \end{array}\Big) \longrightarrow C^\infty\Big(\Omega, \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \mathbb{C}^{\tilde{k}} \\ \oplus \\ \mathbb{C}^{\tilde{\ell}} \end{array}\Big) \tag{1.4}$$

along Ω with operator-valued symbols

$$g(x',\xi') \in \mathcal{S}^{\nu}_{\mathrm{cl}}\Big(\Omega\times\mathbb{R}^{n-1},\mathcal{L}\Big(\begin{array}{c} L^2(\mathbb{R}_+)\otimes\mathbb{C}^k\\ \oplus\\ \mathbb{C}^{\ell}\end{array},\begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+)\otimes\mathbb{C}^{\tilde{k}}\\ \oplus\\ \mathbb{C}^{\tilde{\ell}}\end{array}\Big)\Big),$$

$$g^*(x',\xi') \in \mathcal{S}^{\nu}_{\mathrm{cl}}\Big(\Omega\times\mathbb{R}^{n-1},\mathcal{L}\Big(\begin{array}{c} L^2(\mathbb{R}_+)\otimes\mathbb{C}^{\tilde{k}}\\ \oplus\\ \mathbb{C}^{\tilde{\ell}}\end{array},\begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+)\otimes\mathbb{C}^{k}\\ \oplus\\ \mathbb{C}^{\ell}\end{array}\Big)\Big).$$

Further we denote by $\mathcal{B}^{\nu,0}_G(X;v)$ the set of all $\mathcal{G}_\nu+\mathcal{S}$, where $\mathcal{G}_\nu$ is locally given by (1.4) and $\mathcal{S}\in\mathcal{B}^{-\infty,0}(X;v)$. Then the space $\mathcal{B}^{\mu,d}_G(X;v)$ is defined to consist of all operators

$$\mathcal{G}=\mathcal{G}_\mu+\sum_{j=1}^{d}\mathcal{G}_{\mu-j}\begin{pmatrix} D^j & 0\\ 0 & 0\end{pmatrix}+\mathcal{S}, \tag{1.5}$$

with ingredients $\mathcal{G}_{\mu-j}\in\mathcal{B}^{\mu-j,0}_G(X;v)$ for all $0\le j\le d$, $D^j\in\mathsf{Diff}^j(X;V)$, and $\mathcal{S}\in\mathcal{B}^{-\infty,d}(X;v)$.

Finally, $\mathcal{B}^{\mu,d}(X;v)$ stands for the space of all operators (1.1) where μ is an integer, $d=0,1,\ldots$, and $P\in\Psi^{\mu}_{\mathrm{tp}}(X;V,\tilde{V})$, $\mathcal{G}\in\mathcal{B}^{\mu,d}_G(X;v)$. We also write $\mathcal{B}^{-\infty}(X;v)$ for the union of $\mathcal{B}^{-\infty,d}(X;v)$ over all $d=0,1,\ldots$, and denote by $\mathcal{B}(X)$ the collection of all spaces $\mathcal{B}^{\mu,d}(X;v)$. The entries G_{11}, G_{12} and G_{21} of $\mathcal{G}$ are usually called *(singular) Green*, *Poisson* and *trace* operators, respectively, while G_{22} is a standard pseudodifferential operator on the boundary. Note that only Green and trace operators have types. The following mapping properties of operators (1.1) are important for us.

Lemma 1.2. *An operator $\mathcal{A}\in\mathcal{B}^{\mu,d}(X;v)$ extends to a continuous map*

$$\mathcal{A}:\begin{array}{c} H^s(X,V)\\ \oplus\\ H^s(Y,W)\end{array}\longrightarrow\begin{array}{c} H^{s-\mu}(X,\tilde{V})\\ \oplus\\ H^{s-\mu}(Y,\tilde{W})\end{array} \tag{1.6}$$

for all $s\in\mathbb{R}$ satisfying $s-d>-1/2$.

Each operator $P\in\Psi^{\mu}_{\mathrm{tp}}(X;V,\tilde{V})$ has a principal homogeneous interior symbol $\sigma_\psi(P)$. Locally, for each $(x,\xi)\in T^*X$, this is a map $\sigma_\psi(P)(x,\xi):\mathbb{C}^k\to\mathbb{C}^{\tilde{k}}$ actually given by the first term in the expansion (1.2). Globally, the principal homogeneous symbol is specified as a map $\sigma_\psi(P):\pi_X^*V\to\pi_X^*\tilde{V}$ where $\pi_X:T^*X\to X$ is the canonical projection of the cotangent bundle of X, and π_X^*V is the pull-back bundle of V under π_X.

Further, there is a principal homogeneous boundary symbol of P which is locally for $(x',\xi')\in T^*\Omega$ of the form

$$\sigma_\partial(P)(x',\xi')=r^+\sigma_\psi(P)(x',0,\xi',D_{x_n})e^+:\mathcal{S}(\overline{\mathbb{R}}_+)\otimes\mathbb{C}^k\to\mathcal{S}(\overline{\mathbb{R}}_+)\otimes\mathbb{C}^{\tilde{k}}, \tag{1.7}$$

with $\sigma_\psi(P)(x',0,\xi',D_{x_n})u(x_n)=F^{-1}_{\xi_n\mapsto x_n}\sigma_\psi(P)(x',0,\xi',\xi_n)F_{x_n\mapsto\xi_n}u$. Globally on Y (1.7) represents a homomorphism $\sigma_\partial(P):\pi_Y^*\,\mathcal{S}(\overline{\mathbb{R}}_+)\otimes V'\to\pi_Y^*\,\mathcal{S}(\overline{\mathbb{R}}_+)\otimes\tilde{V}'$, where V' and $\tilde{V}'$ are the restrictions of V and $\tilde{V}'$ to Y, respectively, and $\pi_Y:T^*Y\to Y$ the canonical projection.

We next go to define the principal boundary symbol for operators $\mathcal{G}\in\mathcal{B}^{\mu,d}_G(X;v)$. Each $\mathcal{G}$ is given by (1.5), with $\mathcal{G}_{\mu-j}\in\mathcal{B}^{\mu-j,0}_G(X;v)$ possessing asymptotic expansions which determine their principal homogeneous symbols. It is worth pointing out that the homogeneity always refers to relevant group actions. These principal symbols are denoted by $\sigma_\partial(\mathcal{G}_{\mu-j})$. Then we can define the boundary symbol of arbitrary

$\mathcal{G} \in \mathcal{B}_G^{\mu,d}(X;v)$ by the formula

$$\sigma_\partial(\mathcal{G})(x',\xi') = \sigma_\partial(\mathcal{G}_\mu)(x',\xi') + \sum_{j=1}^{d} \sigma_\partial(\mathcal{G}_{\mu-j})(x',\xi') \begin{pmatrix} \sigma_\psi(D^j)(x',0,\xi',D_{x_n}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Definition 1.1. For an operator $\mathcal{A}$ as in (1.1), we set

$$\begin{aligned} \sigma_\psi(\mathcal{A}) &= \sigma_\psi(P), \\ \sigma_\partial(\mathcal{A}) &= \begin{pmatrix} \sigma_\partial(P) & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\partial(\mathcal{G}), \end{aligned}$$

$\sigma_\psi(\mathcal{A})$ being the *principal interior symbol* and $\sigma_\partial(\mathcal{A})$ the *principal boundary symbol* of $\mathcal{A}$.

The pair $\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$ is said to be the whole principal symbol of the operator $\mathcal{A}$.

Globally, the principal boundary symbol $\sigma_\partial(\mathcal{A})$ of $\mathcal{A}$ is a bundle map

$$\sigma_\partial(\mathcal{A}) : \ \pi_Y^* \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \otimes V' \\ \oplus \\ W \end{array} \longrightarrow \pi_Y^* \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \tilde{V}' \\ \oplus \\ \tilde{W} \end{array} \tag{1.8}$$

away from the zero section of T^*Y. For sufficiently large s, the principal boundary symbol is also well defined as a bundle map

$$\sigma_\partial(\mathcal{A}) : \ \pi_Y^* \begin{array}{c} H^s(\mathbb{R}_+) \otimes V' \\ \oplus \\ W \end{array} \longrightarrow \pi_Y^* \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \otimes \tilde{V}' \\ \oplus \\ \tilde{W} \end{array}, \tag{1.9}$$

the fibres now being Hilbert spaces.

Let us now formulate the basic properties of $\mathcal{B}^{\mu,d}(X;v)$. For the details and proofs, see [dM71], [RS82] and [Gru96]. The following theorem gives a multiplicative property of the principal symbols of operators in $\mathcal{B}^{\mu,d}(X;v)$. As is clear, the composition of two operators $\mathcal{BA}$ in the calculus is defined if and only if the vector space data of the operators agree. More precisely, if $v_\mathcal{A} = (V_\mathcal{A}, \tilde{V}_\mathcal{A}; W_\mathcal{A}, \tilde{W}_\mathcal{A})$ and $v_\mathcal{B} = (V_\mathcal{B}, \tilde{V}_\mathcal{B}; W_\mathcal{B}, \tilde{W}_\mathcal{B})$ then $V_\mathcal{B} = \tilde{V}_\mathcal{A}$ and $W_\mathcal{B} = \tilde{W}_\mathcal{A}$ should hold. In this case we set $v_\mathcal{B} \circ v_\mathcal{A} = (V_\mathcal{A}, \tilde{V}_\mathcal{B}; W_\mathcal{A}, \tilde{W}_\mathcal{B})$.

Theorem 1.3. *Let $\mathcal{A} \in \mathcal{B}^{\mu_\mathcal{A},d_\mathcal{A}}(X;v_\mathcal{A})$ and $\mathcal{B} \in \mathcal{B}^{\mu_\mathcal{B},d_\mathcal{B}}(X;v_\mathcal{B})$, the composition $v_\mathcal{B} \circ v_\mathcal{A}$ being defined. Then, $\mathcal{BA} \in \mathcal{B}^{\mu_\mathcal{A}+\mu_\mathcal{B},d}(X;v_\mathcal{B}\circ v_\mathcal{A})$ for $d = \max\{d_\mathcal{A}, d_\mathcal{B}+\mu_\mathcal{A}\}$, and*

$$\sigma(\mathcal{BA}) = \sigma(\mathcal{B})\sigma(\mathcal{A})$$

is formed by componentwise multiplication. In particular, $\mathcal{B}^{0,0}(X;v)$ is an algebra, where $v = (V,V;W,W)$.

Hence, the space of operators $B^{0,0}(X;v)$ is the "best" from the theoretical point of view. However, the operators that one wants to study are rarely in this class. Fortunately we can reduce many problems to this class because of the following result, cf. [Gru84].

Theorem 1.4. *Suppose $V \in \mathsf{Vect}(X)$ and $W \in \mathsf{Vect}(Y)$. Then, for every integer μ, there exists an element $\mathcal{R}^\mu_{V,W} = \operatorname{diag}(R^\mu_V, R^\mu_W)$ in $\mathcal{B}^{\mu,0}(X;v)$, which induces isomorphisms*

$$\mathcal{R}^\mu_{V,W} : \begin{array}{c} H^s(X,V) \\ \oplus \\ H^s(Y,W) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(X,V) \\ \oplus \\ H^{s-\mu}(Y,W) \end{array}$$

for all $s \in \mathbb{R}$, where $(\mathcal{R}^\mu_{V,W})^{-1} \in \mathcal{B}^{-\mu,0}(X;v)$.

The following theorem states in particular that smoothing operators form an ideal in $\mathcal{B}^{0,0}(X;v)$.

Theorem 1.5. *Suppose that* $\mathcal{A} \in \mathcal{B}^{\mu_{\mathcal{A}},d_{\mathcal{A}}}(X;v_{\mathcal{A}})$, $\mathcal{S} \in \mathcal{B}^{\infty,d_{\mathcal{S}}}(X;v_{\mathcal{S}})$ *and the composition* $v_{\mathcal{S}} \circ v_{\mathcal{A}}$ *is defined. Then,* $\mathcal{S}\mathcal{A} \in \mathcal{B}^{-\infty,d}(X;v_{\mathcal{S}} \circ v_{\mathcal{A}})$*. Analogously, if* $\mathcal{A} \in \mathcal{B}^{\mu_{\mathcal{A}},d_{\mathcal{A}}}(X;v_{\mathcal{A}})$, $\mathcal{S} \in \mathcal{B}^{\infty,d_{\mathcal{S}}}(X;v_{\mathcal{S}})$ *and the composition* $v_{\mathcal{A}} \circ v_{\mathcal{S}}$ *is defined, then* $\mathcal{A}\mathcal{S} \in \mathcal{B}^{-\infty,d_{\mathcal{S}}}(X;v_{\mathcal{A}} \circ v_{\mathcal{S}})$.

The calculus of [dM71] allows one to control the formal adjoint operator in most cases. This is especially important for combining the explicit algebra approach with abstract methods of functional analysis.

Theorem 1.6. *Assume that* $\mathcal{A} \in \mathcal{B}^{\mu,0}(X;v)$*, where* $\mu \le 0$*. Then the formal adjoint* $\mathcal{A}^*$ *of* $\mathcal{A}$ *belongs to the space* $\mathcal{B}^{\mu,0}(X;v^{-1})$ *for* $v^{-1} = (\tilde{V}, V; \tilde{W}, W)$*, and it fulfills* $\sigma(\mathcal{A}^*) = \sigma(A)^*$*, the adjoints are understood to be taken in the corresponding symbol spaces.*

The principal symbol of a pseudodifferential operator on a compact closed manifold actually specifies its order relative to the scale of Sobolev spaces. If the principal symbol of an operator vanish, then the operator is compact in the relevant Sobolev spaces, which is due to Rellich's theorem. Hence, the principal symbol map is an explicit substitute for the quotient map in the Calkin algebra. In fact this property of principal symbols is of general character and extends to Boutet de Monvel's calculus.

Theorem 1.7. *Suppose that* $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}^{\mu,d}(X;v)$ *satisfy* $\sigma(\mathcal{A}_1) = \sigma(\mathcal{A}_1)$*. Then* $\mathcal{C} = \mathcal{A}_1 - \mathcal{A}_2$ *is compact as operator*

$$\mathcal{C} : \begin{array}{c} H^s(X,V) \\ \oplus \\ H^s(Y,W) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}(X,\tilde{V}) \\ \oplus \\ H^{s-\mu}(Y,\tilde{W}) \end{array}$$

for every $s > d - 1/2$.

Corollary 1.8. *Each operator* $\mathcal{C} \in \mathcal{B}^{-\infty,d}(X;v)$ *is compact in appropriate Sobolev spaces for all* $s > d - 1/2$*, since* $\sigma(\mathcal{C}) = 0$.

Theorem 1.7 gives rise to a purely algebraic description of Fredholm boundary value problems in the calculus $\mathcal{B}(X)$. The Fredholm property proves to be equivalent to the pointwise invertibility of principal symbols away from zero sections of the corresponding cotangent bundles. The boundary value problems bearing this property are said to be elliptic.

Definition 1.2. An operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;v)$, for $\mu \in \mathbb{Z}$ and $d = 0, 1, \ldots$, is called *elliptic* if the principal interior symbol map $\sigma_\psi(\mathcal{A}) : \pi_X^* V \to \pi_X^* \tilde{V}$ is an isomorphism away from the zero section of T^*X, and the principal boundary symbol $\sigma_\partial(\mathcal{A})$ induces an isomorphism in (1.8) away from the zero section of T^*Y.

Note that for an operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;v)$ with invertible principal interior symbol the mapping (1.8) is an isomorphism away from the zero section of T^*Y if and only if so is the mapping (1.9) for any $s > \max\{\mu, d\} - 1/2$.

Definition 1.3. Let $\mathcal{A} \in \mathcal{B}^{\mu,d_{\mathcal{A}}}(X;v)$. An operator $\mathcal{P} \in \mathcal{B}^{-\mu,d_{\mathcal{P}}}(X;v^{-1})$, with $d_{\mathcal{P}} = 0, 1, \ldots$, is called a *parametrix* for $\mathcal{A}$ if

$$\begin{array}{rcl} \mathcal{P}\mathcal{A} - \mathcal{I} & \in & \mathcal{B}^{-\infty,d_l}(X, v^{-1} \circ v), \\ \mathcal{A}\mathcal{P} - \mathcal{I} & \in & \mathcal{B}^{-\infty,d_r}(X, v \circ v^{-1}) \end{array} \tag{1.10}$$

for certain $d_l, d_r \in \{0\} \cup \mathbb{N}$.

From this it follows that if $\mathcal{P}$ is a parametrix for $\mathcal{A}$ then $\sigma(\mathcal{P}) = \sigma(\mathcal{A})^{-1}$ where the inverse is taken componentwise.

Theorem 1.9. *Suppose that* $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;v)$*. Then the following statements are equivalent:*

1) *$\mathcal{A}$ is elliptic.*
2) *The mapping* (1.6) *is Fredholm for all $s - d > -1/2$.*
3) *$\mathcal{A}$ has a parametrix $\mathcal{P} \in \mathcal{B}^{-\mu, d_{\mathcal{P}}}(X; v^{-1})$, for $d_{\mathcal{P}} = \max\{0, d-\mu\}$, such that* (1.10) *is fulfilled with $d_l = \max\{\mu, d\}$ and $d_r = \max\{0, d-\mu\}$.*

The following theorem was first proved by Schulze [Sch89] in the general context of operator algebras with symbolic structures. When reasonably organised, such algebras prove to be *spectral invariant*, i.e., the inverse operator always belongs to the algebra. For boundary value problems on non-compact manifolds the Fréchet algebra techniques is further developed by Schrohe [Sch99], cf. in particular Theorem 3.1 there.

Theorem 1.10. *If the operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X, v)$ in* (1.6) *is bijective, then its inverse in the Hilbert spaces is an element of $\mathcal{B}^{-\mu, \max\{0, d-\mu\}}(X; v^{-1})$.*

2. Quasicomplexes

In this section we recall some basic facts about complexes and quasicomplexes in Hilbert spaces. In subsequent sections we discuss quasicomplexes where the relevant operators are in $\mathcal{B}(X)$. For the theory of quasicomplexes of Banach spaces we refer to [AV95].[1]

Let us consider the sequence

$$(H^{\cdot}, d): \quad 0 \longrightarrow H^0 \xrightarrow{d^0} H^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{N-1}} H^N \longrightarrow 0$$

where H^i are Hilbert spaces and d^i are continuous linear maps. The sequence $(H^{\cdot}, d)$ is called a *complex* if $d^i d^{i-1} = 0$ for all $i = 1, \dots, N$. The elements of the spaces $Z^i(H^{\cdot}, d) = \ker d^i$ and $B^i(H^{\cdot}, d) = \operatorname{im} d^{i-1}$ are called cocycles and coboundaries, respectively. The quotient space $H^i(H^{\cdot}, d) = \ker d^i / \operatorname{im} d^{i-1}$ is the cohomology of the complex $(H^{\cdot}, d)$ at step i. The complex $(H^{\cdot}, d)$ is said to be *Fredholm* if its cohomology is finite dimensional at each step $i = 0, \dots, N$.

It is well known that "small" perturbations of Fredholm operators do not affect the Fredholm property. For example, perturbing a single Fredholm operator by a compact operator gives us a Fredholm operator. It would be natural to have the same property for Fredholm complexes. However, a "small" perturbation of a Fredholm complex need not be even a complex anymore, i.e., the operators no longer satisfy $d^i d^{i-1} = 0$.

Note that perturbing an elliptic complex by lower order terms does not change the complex of principal symbols which remains to be exact. Hence, instead of complexes it is natural to consider sequences $(H^{\cdot}, d)$ with the property that the compositions $d^i d^{i-1}$ are "small" in some sense. By "small" operators one usually means compact operators. Let us denote by $\mathcal{K}(H, \tilde{H})$ the subspace of $\mathcal{L}(H, \tilde{H})$ consisting of compact operators.

Definition 2.1. A sequence $(H^{\cdot}, d)$ of Hilbert spaces H^i and continuous linear maps d^i is a *quasicomplex* if $d^i d^{i-1} \in \mathcal{K}(H^{i-1}, H^{i+1})$ for all $i = 1, \dots, N$.

For $d^1, d^2 \in \mathcal{L}(H, \tilde{H})$, we write $d^1 \sim d^2$ if $d^1 - d^2 \in \mathcal{K}(H, \tilde{H})$. It is known that an operator $d \in \mathcal{L}(H, \tilde{H})$ is Fredholm if and only if its image in the Calkin algebra $\mathcal{L}(H, \tilde{H})/\mathcal{K}(H, \tilde{H})$ is invertible. Hence, the idea of Fredholm quasicomplexes is to pass in a given quasicomplex to quotients modulo spaces of compact operators and require exactness. To make the definition precise we introduce a functor ϕ_Σ studied by Putinar [Put82] (see also [ST98]).

Taking an arbitrary Hilbert space Σ, we set $\phi_\Sigma(H^i) = \mathcal{L}(\Sigma, H^i)/\mathcal{K}(\Sigma, H^i)$ for each Hilbert space H^i. Then, for any map $d^i \in \mathcal{L}(H^i, H^{i+1})$, we introduce a map

[1] In [AV95] quasicomplexes are called *essential complexes.*

$\phi_\Sigma(d^i) \in \mathcal{L}(\phi_\Sigma(H^i), \phi_\Sigma(H^{i+1}))$ by

$$\phi_\Sigma(d^i)(A + \mathcal{K}(\Sigma, H^i)) = d^i A + \mathcal{K}(\Sigma, H^{i+1})$$

for all $A \in \mathcal{L}(\Sigma, H^i)$. Obviously, this operator is well defined. It is easily seen that $\phi_\Sigma(d^i d^{i-1}) = \phi_\Sigma(d^i)\phi_\Sigma(d^{i-1})$ and that ϕ_Σ vanishes on compact operators for every Hilbert space Σ. Hence, if $(H^\cdot, d)$ is a quasicomplex then $(\phi_\Sigma(H^\cdot), \phi_\Sigma(d))$ is a complex for each Hilbert space Σ.

Definition 2.2. A quasicomplex $(H^\cdot, d)$ is *Fredholm* if the complex $(\phi_\Sigma(H^\cdot), \phi_\Sigma(d))$ is exact for each Hilbert space Σ.

Let $(H^\cdot, d)$ and $(H^\cdot, \tilde{d})$ be two quasicomplexes of Hilbert spaces, such that $d^i \sim \tilde{d}^i$ for any $i = 0, 1, \ldots, N$. Then the complexes $(\phi_\Sigma(H^\cdot), \phi_\Sigma(d))$ and $(\phi_\Sigma(H^\cdot), \phi_\Sigma(\tilde{d}))$ coincide. Hence, the quasicomplexes $(H^\cdot, d)$ and $(H^\cdot, \tilde{d})$ are Fredholm simultaneously. Thus, any compact perturbation of a Fredholm quasicomplex is a Fredholm quasicomplex.

Definition 2.3. A sequence

$$(H^\cdot, \pi): \quad 0 \longleftarrow H^0 \xleftarrow{\pi^1} H^1 \xleftarrow{\pi^2} \cdots \xleftarrow{\pi^N} H^N \longleftarrow 0$$

with $\pi^i \in \mathcal{L}(H^i, H^{i-1})$ is called a *parametrix* of the quasicomplex $(H^\cdot, d)$, if

$$d^{i-1}\pi^i + \pi^{i+1}d^i = I_{H^i} - \kappa^i$$

for all $i = 0, 1, \ldots, N$, where $\kappa^i \in \mathcal{K}(H^i)$.

It is well known that a complex of Hilbert spaces is Fredholm if and only if it has a parametrix. The same property is also true for quasicomplexes, see [ST98].

Theorem 2.1. *A quasicomplex $(H^\cdot, d)$ is Fredholm if and only if it possesses a parametrix.*

Obviously, if a parametrix $(H^\cdot, \pi)$ of a quasicomplex $(H^\cdot, d)$ is a quasicomplex itself, then $(H^\cdot, d)$ is in turn a parametrix of $(H^\cdot, \pi)$.

As it is proved in [Tar06], every quasicomplex can actually be transformed into a complex.

Theorem 2.2. *For any quasicomplex $(H^\cdot, d)$ there are operators $D^i \in \mathcal{L}(H^i, H^{i+1})$ satisfying $D^i \sim d^i$ and $D^i D^{i-1} = 0$ for all i.*

3. Elliptic quasicomplexes

In Section 1 we assumed for simplicity that the orders and types of all components of operators in $\mathcal{B}(X)$ are the same, which causes inconvenience in applications. In this section we consider operators

$$\mathcal{A} = \begin{pmatrix} P + G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \in \mathcal{B}^{\mu,d}(X; v) \qquad (3.1)$$

where μ is a (2×2)-matrix which gives the orders of the corresponding entries of $\mathcal{A}$. For given $\alpha \in \mathbb{Z}$ and $\lambda, \gamma \in \mathbb{R}$ we set

$$\mu = \begin{pmatrix} \alpha & \beta \\ \gamma & \beta - \alpha + \gamma \end{pmatrix}$$

Anyway we assume that the types $d = 0, 1, \ldots$ of all entries of $\mathcal{A}$ are the same. The following is easily verified, cf. [Gru96].

Lemma 3.1. *Any operator* (3.1) *extends to a continuous map*

$$\mathcal{A}: \begin{array}{c} H^s(X, V) \\ \oplus \\ H^{s-\alpha+\beta}(Y, W) \end{array} \longrightarrow \begin{array}{c} H^{s-\alpha}(X, \tilde{V}) \\ \oplus \\ H^{s-\gamma}(Y, \tilde{W}) \end{array}$$

for $s > d - 1/2$.

We want to define a composition of two operators of the above type. So let us consider $\mathcal{A}^i$ and $\mathcal{A}^{i+1}$ and assume that their types are the same. The orders are given by matrices

$$\mu_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \beta_i - \alpha_i + \gamma_i \end{pmatrix},$$

the left upper corner being integer. Now in order that the composition $\mathcal{A}^{i+1}\mathcal{A}^i$ be well defined we have to require that

$$\beta_{i+1} = \alpha_{i+1} + \alpha_i - \gamma_i. \tag{3.2}$$

Supposing this condition to be satisfied for all i we consider the sequence of boundary value problems

$$0 \longrightarrow \begin{array}{c} C^\infty(X, V^0) \\ \oplus \\ C^\infty(Y, W^0) \end{array} \xrightarrow{\mathcal{A}^0} \cdots \xrightarrow{\mathcal{A}^{N-1}} \begin{array}{c} C^\infty(X, V^N) \\ \oplus \\ C^\infty(Y, W^N) \end{array} \longrightarrow 0.$$

Then we pick

$$\begin{array}{rcl} s_{i+1} & = & s_i - \alpha_i, \\ t_{i+1} & = & s_i - \gamma_i \end{array}$$

for $i = 0, 1, \ldots, N-1$. If we choose $s \in \mathbb{N}$ sufficiently large and set $s_0 = s$, $t_0 = s - \alpha_0 + \beta_0$, then we arrive at the sequence of Hilbert spaces and their continuous maps

$$(H^\cdot, \mathcal{A}) : \ 0 \longrightarrow \begin{array}{c} H^{s_0}(X, V^0) \\ \oplus \\ H^{t_0}(Y, W^0) \end{array} \xrightarrow{\mathcal{A}^0} \cdots \xrightarrow{\mathcal{A}^{N-1}} \begin{array}{c} H^{s_N}(X, V^N) \\ \oplus \\ H^{t_N}(Y, W^N) \end{array} \longrightarrow 0. \tag{3.3}$$

Recall that the operators of $\mathcal{B}^{-\infty}(X)$ are regarded to be "small" operators in the calculus $\mathcal{B}(X)$.

The sequence $(H^\cdot, \mathcal{A})$ is actually a *quasicomplex*, for the composition $\mathcal{A}^i\mathcal{A}^{i-1}$ is "small" for all $i = 1, \ldots, N$, i.e.,

$$\mathcal{A}^i\mathcal{A}^{i-1} \in \mathcal{B}^{-\infty}(X; v_i \circ v_{i-1}), \tag{3.4}$$

where $v_i = (V^i, V^{i+1}; W^i, W^{i+1})$. Since s_0 is sufficiently large, the operators $\mathcal{A}^i\mathcal{A}^{i-1}$ are compact by Corollary 1.8.

The calculus of [dM71] yields two principal symbol sequences for the quasicomplex $(H^\cdot, \mathcal{A})$, namely, the sequence of principal interior symbols

$$\sigma_\psi(H^\cdot, \mathcal{A}) : \ \cdots \longrightarrow \pi_X^* V^{i-1} \xrightarrow{\sigma_\psi(\mathcal{A}^{i-1})} \pi_X^* V^i \xrightarrow{\sigma_\psi(\mathcal{A}^i)} \cdots, \tag{3.5}$$

and the sequence of boundary symbols $\sigma_\partial(H^\cdot, \mathcal{A})$:

$$\cdots \longrightarrow \pi_Y^* \begin{array}{c} H^{s_{i-1}}(\mathbb{R}_+) \otimes V^{i-1\prime} \\ \oplus \\ W^{i-1} \end{array} \xrightarrow{\sigma_\partial(\mathcal{A}^{i-1})} \pi_Y^* \begin{array}{c} H^{s_i}(\mathbb{R}_+) \otimes V^{i\prime} \\ \oplus \\ W^i \end{array} \xrightarrow{\sigma_\partial(\mathcal{A}^i)} \cdots. \tag{3.6}$$

The fact that both (3.5) and (3.6) are complexes is a consequence of (3.4) and Theorem 1.3.

Definition 3.1. A quasicomplex $(H^\cdot, \mathcal{A})$ is called *elliptic* if the complex $\sigma_\psi(H^\cdot, \mathcal{A})$ is exact away from the zero section of T^*X and the complex $\sigma_\partial(H^\cdot, \mathcal{A})$ is exact away from the zero section of T^*Y, for any one (and hence for all) sufficiently large s_0.

Let us now apply the order reduction procedure to reduce the orders of operators in the quasicomplex $(H^\cdot, \mathcal{A})$. A slight modification of Theorem 1.4 shows that for every $V^i \in \mathsf{Vect}(X)$, $W^i \in \mathsf{Vect}(Y)$ and $s_i \in \{0\} \cup \mathbb{N}$, $t_i \in \mathbb{R}$ there exists a boundary value problem

$$\begin{array}{rcl} \mathcal{R}_i & = & \operatorname{diag}\left(R^{s_i}_{V^i}, R^{t_i}_{W^i}\right) \\ & \in & \mathcal{B}^{\operatorname{diag}(s_i,t_i),0}(X; V^i; W^i) \end{array}$$

which induces isomorphisms

$$\mathcal{R}_i : \begin{array}{c} H^{s_i}(X, V^i) \\ \oplus \\ H^{t_i}(Y, W^i) \end{array} \longrightarrow \begin{array}{c} H^0(X, V^i) \\ \oplus \\ H^0(Y, W^i) \end{array},$$

the inverse $\mathcal{R}_i^{-1} = \operatorname{diag}\left((R^{s_i}_{V^i})^{-1}, (R^{t_i}_{W^i})^{-1}\right)$ being in $\mathcal{B}^{\operatorname{diag}(-s_i,-t_i),0}(X; V^i; W^i)$. As usual, $\operatorname{diag}(a, b)$ stands for diagonal (block) matrix whose diagonal elements are a and b.

Set $\mathcal{B}^i = \mathcal{R}_{i+1}\mathcal{A}^i\mathcal{R}_i^{-1}$. Since we start with sufficiently large s_0, it follows that $\mathcal{B}^i \in \mathcal{B}^{0,0}(X; v_i)$. Thus, we arrive at the following commutative diagram

$$\begin{array}{cccccccc}
(H^\cdot, \mathcal{A}) : & \cdots \longrightarrow & \begin{array}{c} H^{s_{i-1}}(V^{i-1}) \\ \oplus \\ H^{t_{i-1}}(W^{i-1}) \end{array} & \xrightarrow{\mathcal{A}^{i-1}} & \begin{array}{c} H^{s_i}(V^i) \\ \oplus \\ H^{t_i}(W^i) \end{array} & \xrightarrow{\mathcal{A}^i} \cdots \\
\mathcal{R} \downarrow \uparrow \mathcal{R}^{-1} & & \mathcal{R}_{i-1} \downarrow \uparrow \mathcal{R}_{i-1}^{-1} & & \mathcal{R}_i \downarrow \uparrow \mathcal{R}_i^{-1} & \\
(\tilde{H}^\cdot, \mathcal{B}) : & \cdots \longrightarrow & \begin{array}{c} H^0(V^{i-1}) \\ \oplus \\ H^0(W^{i-1}) \end{array} & \xrightarrow{\mathcal{B}^{i-1}} & \begin{array}{c} H^0(V^i) \\ \oplus \\ H^0(W^i) \end{array} & \xrightarrow{\mathcal{B}^i} \cdots
\end{array} \tag{3.7}$$

whose maps are continuous because the types of $\mathcal{B}^i$ are zero. From this it follows that $(\tilde{H}^\cdot, \mathcal{B})$ is a quasicomplex, for

$$\mathcal{B}^i\mathcal{B}^{i-1} = \mathcal{R}_{i+1}\mathcal{A}^i\mathcal{A}^{i-1}\mathcal{R}_{i-1}^{-1} = 0$$

modulo $\mathcal{B}^{-\infty,0}(X; V^{i-1}, V^{i+1}; W^{i-1}, W^{i+1})$.

Theorem 3.2. *The quasicomplex $(\tilde{H}^\cdot, \mathcal{B})$ is elliptic if and only if so is the quasicomplex $(H^\cdot, \mathcal{A})$.*

Proof. The diagram (3.7) induces a commutative diagram for the principal symbols. This readily yields our claim. □

4. Fredholm property

In order to define the parametrix in the context of Boutet de Monvel operators we must modify our previous Definition 2.3 a little.

Let $(H^\cdot, \mathcal{A})$ be a quasicomplex of Boutet de Monvel operators. A sequence of operators

$$0 \longleftarrow \begin{array}{c} H^{s_0}(X, V^0) \\ \oplus \\ H^{t_0}(Y, W^0) \end{array} \xleftarrow{\mathcal{P}^1} \cdots \xleftarrow{\mathcal{P}^N} \begin{array}{c} H^{s_N}(X, V^N) \\ \oplus \\ H^{t_N}(Y, W^N) \end{array} \longleftarrow 0$$

is said to be a *parametrix* for $(H^\cdot, \mathcal{A})$ if $\mathcal{A}^{i-1}\mathcal{P}^i + \mathcal{P}^{i+1}\mathcal{A}^i = \mathcal{I} - \mathcal{S}^i$ is fulfilled for all $i = 0, 1, \ldots, N$, where $\mathcal{S}^i \in \mathcal{B}^{-\infty}(X; V^i; W^i)$.

The next result gives a connection between the parametrices of the first and second rows in (3.7).

Lemma 4.1. *Let* $\mathcal{Q}^i = \mathcal{R}_{i-1}\mathcal{P}^i\mathcal{R}_i^{-1}$. *Then* $\{\mathcal{P}^i\}_{i=1}^N$ *is a parametrix of* $(H^\cdot, \mathcal{A})$ *if and only if* $\{\mathcal{Q}^i\}_{i=1}^N$ *is a parametrix of* $(\tilde{H}^\cdot, \mathcal{B})$.

Proof. Indeed,

$$\begin{array}{rcl} \mathcal{B}^{i-1}\mathcal{Q}^i + \mathcal{Q}^{i+1}\mathcal{B}^i & = & \mathcal{R}_i\mathcal{A}^{i-1}\mathcal{R}_{i-1}^{-1}\mathcal{R}_{i-1}\mathcal{P}^i\mathcal{R}_i^{-1} + \mathcal{R}_i\mathcal{P}^{i+1}\mathcal{R}_{i+1}^{-1}\mathcal{R}_{i+1}\mathcal{A}^i\mathcal{R}_i^{-1} \\ & = & \mathcal{R}_i\left(\mathcal{A}^{i-1}\mathcal{P}^i + \mathcal{P}^{i+1}\mathcal{A}^i\right)\mathcal{R}_i^{-1}. \end{array}$$

Then Theorem 1.5 implies the desired statement. □

We are thus left with the problem of constructing parametrices for elliptic quasicomplexes whose operators are in $\mathcal{B}^{0,0}(X)$. The advantage of using such a reduction is that $\mathcal{B}^{0,0}(X)$ is an algebra. Hence, the adjoints are available within the algebra and we can reduce the matter to single elliptic operators, namely, the Laplacians of quasicomplexes.

Lemma 4.2. *A quasicomplex* $(\tilde{H}^\cdot, \mathcal{B})$ *of order zero is elliptic if and only if all the Laplacians*

$$\Delta^i = \mathcal{B}^{i-1}\mathcal{B}^{i-1^*} + \mathcal{B}^{i^*}\mathcal{B}^i \in \mathcal{B}^{0,0}(X; V^i; W^i) \tag{4.1}$$

are elliptic.

Proof. First we have

$$\sigma(\Delta^i) = \sigma(\mathcal{B}^{i-1})(\sigma(\mathcal{B}^{i-1}))^* + (\sigma(\mathcal{B}^i))^*\sigma(\mathcal{B}^i).$$

Since, for elliptic quasicomplexes, the principal interior and boundary symbol sequences are exact complexes of Hilbert spaces, the statement of the lemma is a consequence of a familiar construction of homological algebra. Namely, a complex of Hilbert spaces

$$\cdots \longrightarrow H^{i-1} \xrightarrow{d^{i-1}} H^i \xrightarrow{d^i} \cdots$$

is exact at step i if and only if the Laplacian $\Delta^i = d^{i-1}d^{i-1^*} + d^{i^*}d^i$ is an isomorphism of H^i. □

The following theorem is a key result on elliptic complexes of operators in Boutet de Monvel's algebra. For brevity we write $\mathcal{A} \approx \mathcal{B}$ if the operators differ by an element of $\mathcal{B}^{-\infty,0}(X)$.

Theorem 4.3. *Each elliptic quasicomplex* $(H^\cdot, \mathcal{A})$ *with operators in* $\mathcal{B}(X)$ *has a parametrix.*

Proof. By Theorem 3.2, we can reduce the quasicomplex $(H^\cdot, \mathcal{A})$ to an elliptic quasicomplex $(\tilde{H}^\cdot, \mathcal{B})$ whose operators are in $\mathcal{B}^{0,0}(X)$. Then we conclude by Lemma 4.2 that all the Laplacians (4.1) are elliptic. Thus, by Theorem 1.9 we can find a parametrix $\mathcal{G}^i \in \mathcal{B}^{0,0}(X; V^i; W^i)$ for Δ^i, such that

$$\begin{array}{rcl} \mathcal{G}^i\Delta^i & \approx & \mathcal{I}, \\ \Delta^i\mathcal{G}^i & \approx & \mathcal{I}. \end{array}$$

Since $\mathcal{B}^i\mathcal{B}^{i-1} \approx 0$, we conclude that

$$\mathcal{B}^i\Delta^i \approx \Delta^{i+1}\mathcal{B}^i. \tag{4.2}$$

Multiplying (4.2) from left by $\mathcal{G}^{i+1}$ and from right by $\mathcal{G}^i$, we get

$$\mathcal{G}^{i+1}\mathcal{B}^i \approx \mathcal{B}^i\mathcal{G}^i. \tag{4.3}$$

Now we claim that $\mathcal{Q}^i = \mathcal{G}^{i-1}\mathcal{B}^{i-1*}$ is a parametrix for the quasicomplex $(\tilde{H}^\cdot, \mathcal{B})$. Indeed, using (4.3) yields

$$\begin{aligned}\mathcal{B}^{i-1}\mathcal{Q}^i + \mathcal{Q}^{i+1}\mathcal{B}^i &= \mathcal{B}^{i-1}\mathcal{G}^{i-1}\mathcal{B}^{i-1*} + \mathcal{G}^i\mathcal{B}^{i*}\mathcal{B}^i \\ &\approx \mathcal{G}^i\left(\mathcal{B}^{i-1}\mathcal{B}^{i-1*} + \mathcal{B}^{i*}\mathcal{B}^i\right) \\ &= \mathcal{G}^i\Delta^i \\ &\approx \mathcal{I}.\end{aligned}$$

To complete the proof it suffices to make use of Lemma 4.1 according to which $\mathcal{P}^i = \mathcal{R}_{i-1}^{-1}\mathcal{Q}^i\mathcal{R}_i$ is a parametrix of the quasicomplex $(H^\cdot, \mathcal{A})$. □

Thus, elliptic quasicomplexes of boundary value problems possess parametrices not only in the sense of Hilbert spaces but also in the sense of operator calculus $\mathcal{B}(X)$.

Corollary 4.4. *The parametrix of an elliptic quasicomplex $(H^\cdot, \mathcal{A})$ constructed in Theorem 4.3 is a quasicomplex.*

Proof. From $\mathcal{B}^i\mathcal{B}^{i-1} \approx 0$ it follows that $\mathcal{B}^{i-1*}\mathcal{B}^{i*} = (\mathcal{B}^i\mathcal{B}^{i-1})^* \approx 0$. Hence in the same way as in the proof of Theorem 4.3 we get $\mathcal{B}^{i-1*}\Delta^i \approx \Delta^{i-1}\mathcal{B}^{i-1*}$. Multiplying this from left by $\mathcal{G}^{i-1}$ and from right by $\mathcal{G}^i$, we deduce readily that

$$\mathcal{G}^{i-1}\mathcal{B}^{i-1*} \approx \mathcal{B}^{i-1*}\mathcal{G}^i. \tag{4.4}$$

Using (4.4) we get

$$\begin{aligned}\mathcal{P}^i\mathcal{P}^{i+1} &= \mathcal{R}_{i-1}^{-1}\left(\mathcal{G}^{i-1}\mathcal{B}^{i-1*}\right)\left(\mathcal{G}^i\mathcal{B}^{i*}\right)\mathcal{R}_{i+1} \\ &\approx \mathcal{R}_{i-1}^{-1}\mathcal{G}^{i-1}\left(\mathcal{G}^{i-1}\mathcal{B}^{i-1*}\right)\mathcal{B}^{i*}\mathcal{R}_{i+1} \\ &= \mathcal{R}_{i-1}^{-1}\mathcal{G}^{i-1}\mathcal{G}^{i-1}\left(\mathcal{B}^{i-1*}\mathcal{B}^{i*}\right)\mathcal{R}_{i+1} \\ &\approx 0.\end{aligned}$$

Hence, $(H^\cdot, \mathcal{P})$ is a quasicomplex. □

Let us now formulate the main result.

Theorem 4.5. *Let $(H^\cdot, \mathcal{A})$ be an elliptic quasicomplex of operators in $\mathcal{B}(X)$. Then $(H^\cdot, \mathcal{A})$ is Fredholm for a sufficiently large s_0.*

Proof. Theorem 4.3 provides us with an explicit parametrix $\{\mathcal{P}^i\}_{j=1}^N$, such that $\mathcal{A}^{i-1}\mathcal{P}^i + \mathcal{P}^{i+1}\mathcal{A}^i = \mathcal{I} - \mathcal{S}^i$ with $\mathcal{S}^i \in \mathcal{B}^{-\infty}(X; V^i; W^i)$. Since s_0 is assumed to be large enough, Corollary 1.8 implies that $\mathcal{S}^i \in \mathcal{K}(H^{s_i}(X, V^i) \oplus H^{t_i}(Y, W^i))$ for $i = 0, 1, \ldots, N$. Hence, by Theorem 2.1 the quasicomplex $(H^\cdot, \mathcal{A})$ is Fredholm, as desired. □

More generally, by a quasicomplex of operators in Boutet de Monvel's algebra one might mean any sequence (3.3) with the property that the principal symbols of the composition $\mathcal{A}^i\mathcal{A}^{i-1}$ are zero for all i. The definition of an elliptic quasicomplex applies to such quasicomplexes as well. The proof of Theorem 4.3 still goes through in this case and we construct a parametrix for any elliptic quasicomplex modulo remainders $\mathcal{S}^i$ whose principal symbols are zero. By Theorem 1.7, such operators $\mathcal{S}^i$ are compact in the corresponding Sobolev spaces, i.e., we obtain in this way a proper parametrix for (3.3).

Remark 4.1. This proves that any elliptic quasicomplex (with respect to zero principal symbol compositions) is Fredholm in appropriate Sobolev spaces for sufficiently large s.

5. Hodge theory for elliptic quasicomplexes

The Hodge theory for elliptic complexes on compact manifolds with smooth edges is developed in [ST98]. While smooth compact manifolds with boundary constitute a subclass of compact manifolds with smooth edges, the known pseudodifferential calculi on them are different from each other. Here we construct the Hodge theory first for elliptic quasicomplexes of Hilbert spaces and then for elliptic quasicomplexes of Boutet de Monvel operators.

Let

$$(H^\cdot, d): \quad \cdots \longrightarrow H^{i-1} \xrightarrow{d^{i-1}} H^i \xrightarrow{d^i} \cdots$$

be a quasicomplex of Hilbert spaces, i.e., $d^i d^{i-1} \in \mathcal{K}(H^{i-1}, H^{i+1})$. Along with this we consider the adjoint quasicomplex

$$(H^\cdot, d^*): \quad \cdots \longleftarrow H^{i-1} \xleftarrow{d^{i-1*}} H^i \xleftarrow{d^{i*}} \cdots.$$

Lemma 5.1. *A quasicomplex $(H^\cdot, d)$ of Hilbert spaces is Fredholm if and only if all Laplacians $\Delta^i = d^{i-1}d^{i-1^*} + d^{i^*}d^i$ are Fredholm.*

Proof. By Theorem 2.2 there exists a complex of operators $D^i \in \mathcal{L}(H^i, H^{i+1})$, such that $D^i \sim d^i$. The quasicomplex $(H^\cdot, d)$ is Fredholm if and only if the complex $(H^\cdot, D)$ is Fredholm. The complex $(H^\cdot, D)$ is in turn Fredholm if and only if all Laplacians of this complex are Fredholm. Since the Laplacians of the complex $(H^\cdot, D)$ and the quasicomplex $(H^\cdot, d)$ actually differ by compact operators, the lemma follows. □

Note that each Laplacian $\Delta^i : H^i \to H^i$ is a selfadjoint operator. So it is both injective and surjective or possesses neither of these properties.

Theorem 5.2. *Let $(H^\cdot, d)$ be a Fredholm quasicomplex. Then, for $i = 0, 1, \ldots, N$, the null-space of Δ^i is finite dimensional and there is an operator $g^i \in \mathcal{L}(H^i)$, such that the decomposition*

$$I_{H^i} = h^i + d^{i-1}d^{i-1^*}g^i + d^{i^*}d^i g^i \tag{5.1}$$

holds with an orthogonal projection $h^i : H^i \to \ker \Delta^i$.

Proof. Fix some $i = 0, 1, \ldots, N$. By Lemma 5.1, the Laplacian Δ^i is Fredholm, and hence its null-space $\ker \Delta^i$ is finite dimensional. Denote by $(\ker \Delta^i)^\perp$ the orthogonal complement of $\ker \Delta^i$ in H^i. Since $\Delta^i : H^i \to H^i$ is a selfadjoint Fredholm operator, the restriction $\Delta^i : (\ker \Delta^i)^\perp \to (\ker \Delta^i)^\perp$ is a topological isomorphism. Then we set

$$g^i = (\Delta^i|_{(\ker \Delta^i)^\perp})^{-1}(I_{H^i} - h^i).$$

This is a bounded operator in H^i satisfying $\Delta^i g^i = I_{H^i} - h^i$. The latter is precisely (5.1). □

In the case of complexes the decomposition (5.1) is orthogonal, as is easily checked. In the case of quasicomplexes $h^i u$ is orthogonal to $\Delta^i g^i u$ for any $u \in H^i$. However, $d^{i-1}d^{i-1^*}g^i u$ and $d^{i^*}d^i g^i u$ may be not orthogonal.

Lemma 5.3. *The operators g^i constructed above satisfy $d^i g^i \sim g^{i+1}d^i$. Moreover, the operators $p^i = d^{i-1^*}g^i$ define a parametrix for the quasicomplex $(H^\cdot, d)$.*

Proof. We first observe that $d^i \Delta^i \sim \Delta^{i+1} d^i$. Multiplying this from left and from right by g^{i+1} and g^i, respectively, we get $g^{i+1}d^i \Delta^i g^i \sim g^{i+1}\Delta^{i+1}d^i g^i$. Then (5.1) implies $g^{i+1}d^i - g^{i+1}d^i h^i \sim d^i g^i - h^{i+1}d^i g^i$. Since the operator h^i is of finite rank and, therefore, compact, we get $d^i g^i \sim g^{i+1}d^i$, as desired. Then (5.1) yields

$$d^{i-1}\left(d^{i-1^*}g^i\right) + \left(d^{i^*}g^{i+1}\right)d^i = I_{\mathcal{H}^i} - h^i - d^{i^*}c^i$$

where $c^i = d^i g^i - g^{i+1} d^i$ is compact. The operator $r^i = h^i + {d^i}^* c^i$ is compact, too, and we get

$$d^{i-1} p^i + p^{i+1} d^i = I_{\mathcal{H}^i} - r^i,$$

as desired. □

Let us now construct a special parametrix for an elliptic quasicomplex of operators in the calculus $\mathcal{B}(X)$

$$(H^{\cdot}, \mathcal{A}) : \ 0 \longrightarrow \begin{array}{c} H^{s_0}(X, V^0) \\ \oplus \\ H^{t_0}(Y, W^0) \end{array} \xrightarrow{\mathcal{A}^0} \cdots \xrightarrow{\mathcal{A}^{N-1}} \begin{array}{c} H^{s_N}(X, V^N) \\ \oplus \\ H^{t_N}(Y, W^N) \end{array} \longrightarrow 0,$$

cf. (3.3). This parametrix is analogous to the parametrix used in Hodge theory for elliptic complexes.

Suppose s_0 is sufficiently large. First we reduce the quasicomplex $(H^{\cdot}, \mathcal{A})$ to a quasicomplex $(\tilde{H}^{\cdot}, \mathcal{B})$ with differentials $\mathcal{B}^i = \mathcal{R}_{i+1} \mathcal{A}^i \mathcal{R}_i^{-1} \in \mathcal{B}^{0,0}(X; v_i)$. This allows us to use the Laplacians Δ^i.

Theorem 5.4. *Let $(H^{\cdot}, \mathcal{A})$ be an elliptic quasicomplex with differential in $\mathcal{B}(X)$. Then there are operators $\mathcal{G}^i \in \mathcal{B}^{0,0}(X; V^i; W^i)$, such that*

$$\mathcal{P}^i = \mathcal{R}_{i-1}^{-1} \left(\mathcal{R}_i \mathcal{A}^{i-1} \mathcal{R}_{i-1}^{-1} \right)^* \mathcal{G}^i \mathcal{R}_i,$$

$i = 1, \ldots, N$, is a parametrix of $(H^{\cdot}, \mathcal{A})$.

The operators $\mathcal{R}_i^{-1} \mathcal{G}^i \mathcal{R}_i$ obtained by conjugating $\mathcal{G}^i$ through order reduction are sometimes called Green operators.

Proof. From Theorem 3.2 and the ellipticity of $(H^{\cdot}, \mathcal{A})$ it follows that the quasicomplex $(\tilde{H}^{\cdot}, \mathcal{B})$ is elliptic. By Lemma 4.2, all Laplacians $\Delta^i = \mathcal{B}^{i-1} {\mathcal{B}^{i-1}}^* + {\mathcal{B}^i}^* \mathcal{B}^i$ are elliptic operators in $\mathcal{B}^{0,0}(X; V^i; W^i)$. Then Theorem 1.9 implies that Δ^i induces a Fredholm operator on $H^0(V^i) \oplus H^0(W^i)$. Hence, its null-space $\ker \Delta^i$ is finite dimensional.

Note that Δ^i is a selfadjoint operator. Let us write $(\ker \Delta^i)^{\perp}$ for the orthogonal complement of $\ker \Delta^i$ in $H^0(V^i) \oplus H^0(W^i)$. The operator Δ^i restricts to an isomorphism $(\ker \Delta^i)^{\perp} \to (\ker \Delta^i)^{\perp}$. Denote by $\mathcal{H}^i$ the orthogonal projection of $H^0(V^i) \oplus H^0(W^i)$ onto $\ker \Delta^i$. A familiar argument of functional analysis shows that

$$\mathcal{G}^i = (\Delta^i|_{(\ker \Delta^i)^{\perp}})^{-1} (\mathcal{I} - \mathcal{H}^i) \tag{5.2}$$

is a bounded operator on $H^0(V^i) \oplus H^0(W^i)$. It is clear from the very definition that

$$\begin{array}{rcl} \mathcal{I} - \mathcal{H}^i & = & \Delta^i \mathcal{G}^i \\ & = & \mathcal{G}^i \Delta^i \end{array} \tag{5.3}$$

on $H^0(V^i) \oplus H^0(W^i)$.

We claim that operator $\mathcal{G}^i$ defined by (5.2) belongs to the calculus $\mathcal{B}(X)$. To show this, choose an orthogonal basis $\{u_\nu^{(i)}\}$ in $\ker \Delta^i$. Then $\mathcal{H}^i$ is an integral operator with the kernel $\sum_\nu u_\nu^{(i)} \otimes * u_\nu^{(i)}$, where $*$ is a Hodge star operator naturally associated with the scalar product in $H^0(V^i) \oplus H^0(W^i)$. Thus, the kernel of $\mathcal{H}^i$ is smooth whence

$$\mathcal{H}^i \in \mathcal{B}^{-\infty,0}(X; V^i; W^i). \tag{5.4}$$

Let us consider an operator $\mathcal{L}^i \in \mathcal{L}(H^0(V^i) \oplus H^0(W^i))$ defined by

$$\mathcal{L}^i u = \mathcal{H}^i u + \Delta^i \left(\mathcal{I} - \mathcal{H}^i \right) u.$$

It is easily seen that $\mathcal{L}^i \in \mathcal{B}^{0,0}(X; V^i; W^i)$. The inverse of the operator $\mathcal{L}^i$ is given by

$$(\mathcal{L}^i)^{-1} f = \mathcal{H}^i f + (\Delta^i|_{(\ker \Delta^i)^{\perp}})^{-1} \left(\mathcal{I} - \mathcal{H}^i \right) f.$$

The spectral invariance of $\mathcal{B}(X)$ yields $(\mathcal{L}^i)^{-1} \in \mathcal{B}^{0,0}(X; V^i; W^i)$, cf. Theorem 1.10. Since $\mathcal{G}^i = (\mathcal{L}^i)^{-1} \left(\mathcal{I} - \mathcal{H}^i\right)$, we conclude immediately that $\mathcal{G}^i \in \mathcal{B}^{0,0}(X; V^i; W^i)$, as desired.

By Theorem 4.5, the quasicomplex $(\tilde{H}^\cdot, \mathcal{B})$ is Fredholm. Hence, Theorem 5.2 specifies to

$$\mathcal{I} = \mathcal{H}^i + \mathcal{B}^{i-1}\mathcal{B}^{i-1^*}\mathcal{G}^i + \mathcal{B}^{i^*}\mathcal{B}^i\mathcal{G}^i, \tag{5.5}$$

cf. (5.3). We claim that $\mathcal{B}^i\mathcal{G}^i \approx \mathcal{G}^{i+1}\mathcal{B}^i$. Indeed, since $(\tilde{H}^\cdot, \mathcal{B})$ is a quasicomplex, we have $\mathcal{B}^i\Delta^i \approx \Delta^{i+1}\mathcal{B}^i$. Multiplying this from left by $\mathcal{G}^{i+1}$ and from right by $\mathcal{G}^i$ and applying (5.3), we get $\mathcal{G}^{i+1}\mathcal{B}^i(\mathcal{I} - \mathcal{H}^i) \approx (\mathcal{I} - \mathcal{H}^{i+1})\mathcal{B}^i\mathcal{G}^i$. Hence, (5.4) yields the claim. Thus, (5.5) implies $\mathcal{B}^{i-1}\mathcal{Q}^i + \mathcal{Q}^{i+1}\mathcal{B}^i \approx \mathcal{I} - \mathcal{H}^i$, where $\mathcal{Q}^i = \mathcal{B}^{i-1^*}\mathcal{G}^i$, or $\mathcal{B}^{i-1}\mathcal{Q}^i + \mathcal{Q}^{i+1}\mathcal{B}^i \approx \mathcal{I}$.

Multiplying this equality from left and from right by $\mathcal{R}_i^{-1}$ and $\mathcal{R}_i$, respectively, and substituting $\mathcal{B}^i = \mathcal{R}_{i+1}\mathcal{A}^i\mathcal{R}_i^{-1}$, we readily get $\mathcal{A}^{i-1}\mathcal{P}^i + \mathcal{P}^{i+1}\mathcal{A}^i \approx \mathcal{I}$ where

$$\begin{aligned} \mathcal{P}^i &= \mathcal{R}_{i-1}^{-1}\mathcal{B}^{i-1^*}\mathcal{G}^i\mathcal{R}_i \\ &= \mathcal{R}_{i-1}^{-1}\left(\mathcal{R}_i\mathcal{A}^{i-1}\mathcal{R}_{i-1}^{-1}\right)^*\mathcal{G}^i\mathcal{R}_i, \end{aligned}$$

as desired. □

6. Cone of quasicochain mappings

We may construct examples of elliptic quasicomplexes of pseudodifferential operators on a manifold with boundary by realising elliptic quasicomplexes as cones of quasicochain mappings. Let us first discuss the construction of a cone for arbitrary Hilbert spaces.

Definition 6.1. Let $(L^\cdot, a)$ and $(M^\cdot, b)$ be two quasicomplexes. By a *quasicochain mapping* of these quasicomplexes is meant a collection of operators $t^i \in \mathcal{L}(L^i, M^i)$, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^0 & \xrightarrow{a^0} & L^1 & \xrightarrow{a^1} \cdots \xrightarrow{a^{N-1}} & L^N & \longrightarrow & 0 \\ & & \downarrow t^0 & & \downarrow t^1 & & \downarrow t^N & & \\ 0 & \longrightarrow & M^0 & \xrightarrow{b^0} & M^1 & \xrightarrow{b^1} \cdots \xrightarrow{b^{N-1}} & M^N & \longrightarrow & 0 \end{array} \tag{6.1}$$

commutes modulo compact operators, i.e., $t^{i+1}a^i - b^it^i \in \mathcal{K}(L^i, M^{i+1})$ holds for all $i = 0, 1, \ldots, N$.

To any quasicochain mapping $t = \{t^i\}$ we may associate a new quasicomplex

$$0 \longrightarrow \begin{array}{c} L^0 \\ \oplus \\ 0 \end{array} \xrightarrow{d^0} \begin{array}{c} L^1 \\ \oplus \\ M^0 \end{array} \xrightarrow{d^1} \cdots \xrightarrow{d^{N-1}} \begin{array}{c} L^N \\ \oplus \\ M^{N-1} \end{array} \xrightarrow{d^N} \begin{array}{c} 0 \\ \oplus \\ M^N \end{array} \longrightarrow 0$$

where

$$d^i = \begin{pmatrix} -a^i & 0 \\ t^i & b^{q-1} \end{pmatrix}.$$

Indeed, all compositions

$$d^id^{i-1} = \begin{pmatrix} a^ia^{i-1} & 0 \\ -t^ia^{i-1} + b^{i-1}t^i & b^{i-1}b^{i-2} \end{pmatrix}$$

are compact.

It is called the *cone* of the quasicochain mapping t and denoted by $\mathcal{C}(t)$, cf. [Spa66] and elsewhere.

We now turn to quasicomplexes of pseudodifferential operators. Consider a quasicomplex of pseudodifferential operators P^i of type $V^i \to V^{i+1}$ with the transmission property on X, such that P^iP^{i-1} is a smoothing operator in Boutet de Monvel's calculus on X, and a quasicomplex of pseudodifferential operators Q^i of type $W^i \to W^{i+1}$ on Y, such that $Q^i Q^{i-1}$ is a smoothing operator in the standard calculus on Y.

Choose a quasicochain mapping $T^i : C^\infty(X, V^i) \to C^\infty(X, W^i)$ between these quasicomplexes, each T^i being a singular trace operator in Boutet de Monvel's calculus on X. We require the types of T^i to be the same and $T^{i+1}P^i - Q^i T^i$ to be smoothing singular trace operators for all i. Fix a sufficiently large $s \in \mathbb{N}$ and set $s_0 = s$, $t_0 = s - \operatorname{ord} T^0$, and

$$\begin{array}{rcl} s_i & = & s_0 - \operatorname{ord} P^0 - \ldots - \operatorname{ord} P^{i-1}, \\ t_i & = & t_0 - \operatorname{ord} Q^0 - \ldots - \operatorname{ord} Q^{i-1} \end{array}$$

for $i = 1, \ldots, N$. The diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^{s_{i-1}}(X, V^{i-1}) & \xrightarrow{P^{i-1}} & H^{s_i}(X, V^i) & \xrightarrow{P^i} & \cdots \\ & \downarrow{\scriptstyle T^{i-1}} & & \downarrow{\scriptstyle T^i} & & \\ \cdots \longrightarrow & H^{t_{i-1}}(Y, W^{i-1}) & \xrightarrow{Q^{i-1}} & H^{t_i}(X, W^i) & \xrightarrow{Q^i} & \cdots \end{array} \qquad (6.2)$$

commutes modulo compact operators, and the cone of the quasicochain mapping (6.2) is

$$(H^\cdot, \mathcal{A}) : \quad \cdots \longrightarrow \begin{array}{c} H^{s_{i-1}}(X, V^{i-1}) \\ \oplus \\ H^{t_{i-2}}(Y, W^{i-2}) \end{array} \xrightarrow{\mathcal{A}^{i-1}} \begin{array}{c} H^{s_i}(X, V^i) \\ \oplus \\ H^{t_{i-1}}(Y, W^{i-1}) \end{array} \xrightarrow{\mathcal{A}^i} \cdots, \qquad (6.3)$$

where

$$\mathcal{A}^i = \begin{pmatrix} -P^i & 0 \\ T^i & Q^{i-1} \end{pmatrix}$$

for $i = 0, 1, \ldots, N$.

Suppose that the quasicomplex $(H^\cdot, \mathcal{A})$ is elliptic. The exactness of the principal interior symbol sequence $\sigma_\psi(H^\cdot, \mathcal{A})$ away from the zero section of T^*X is equivalent to the ellipticity of the first quasicomplex in (6.2) in the usual sense. Further, the principal boundary symbol sequence $\sigma_\partial(H^\cdot, \mathcal{A})$ is the cone of the cochain mapping $\sigma_\partial(T)$. Hence, it is exact away from the zero section of T^*Y if and only if both complexes

$$\ker \sigma_\partial(T) : 0 \longrightarrow \ker \sigma_\partial(T^0) \xrightarrow{\sigma_\partial(P^0)} \ker \sigma_\partial(T^1) \xrightarrow{\sigma_\partial(P^1)} \cdots,$$

$$\operatorname{coker} \sigma_\partial(T) : 0 \longrightarrow \operatorname{coker} \sigma_\partial(T^0) \xrightarrow{\sigma_\partial(Q^0)} \operatorname{coker} \sigma_\partial(T^1) \xrightarrow{\sigma_\partial(Q^1)} \cdots$$

are exact away from the zero section of T^*Y. Hence, as but one consequence of Theorem 4.5 we obtain

Corollary 6.1. *The quasicomplex* (6.3) *is Fredholm for a sufficiently large s_0, if the first quasicomplex in* (6.2) *is elliptic and both complexes* $\ker \sigma_\partial(T)$ *and* $\operatorname{coker} \sigma_\partial(T)$ *are exact away from zero section of T^*Y.*

It was Dynin [Dyn72] who first studied cones of cochain mappings of the form (6.3). He called them boundary problems for elliptic complexes of pseudodifferential operators on X. To the best of our knowledge, no proofs of these results have ever been published.

Note that any quasicomplex $(H^\cdot, \mathcal{A})$ of the form (3.3) whose differential is given by lower triangle block operator matrices is actually the cone of a quasicochain map between two quasicomplexes, the first of the two being over X and the second being over Y.

7. Compatibility complexes of overdetermined boundary problems

Let X be a compact C^∞ manifold with boundary Y. Consider an elliptic differential operator $A : C^\infty(X,V) \to C^\infty(X,\tilde{V})$ where V and $\tilde{V}$ are smooth vector bundles over X. A boundary value problem for A is classically regarded as an operator

$$\mathcal{A}^0 = \begin{pmatrix} A \\ T \end{pmatrix} : \ C^\infty(X,V) \to \begin{array}{c} C^\infty(X,\tilde{V}) \\ \oplus \\ C^\infty(Y,W), \end{array} \tag{7.1}$$

where W is a smooth vector bundle over W and $T : C^\infty(X,V) \to C^\infty(Y,W)$ is a trace operator which is a differential operator on X followed by restriction to the boundary Y. In general, the solvability of a boundary problem $Au = f$, $Tu = g$ requires compatibility conditions of the form $\mathcal{A}^1(f,g) = 0$, where $\mathcal{A}^1$ is a lower triangle block matrix whose diagonal entries are differential operators on X and Y, respectively, and the lower left entry is a trace operator. More precisely, such an operator $\mathcal{A}^1$ is called a *compatibility operator* for $\mathcal{A}^0$, if $\mathcal{A}^1\mathcal{A}^0 = 0$ and for any operator $\tilde{\mathcal{A}}^1$ satisfying the condition $\tilde{\mathcal{A}}^1\mathcal{A}^0 = 0$ there is an operator $\mathcal{B}$, such that $\tilde{\mathcal{A}}^1 = \mathcal{B}\mathcal{A}^1$. A complex of classical boundary problems is said to be a *compatibility complex* for an operator $\mathcal{A}^0$, if every operator $\mathcal{A}^i$ is a compatibility operator for $\mathcal{A}^{i-1}$, $i \geq 1$.

For any boundary problem operator $\mathcal{A}^0$ satisfying the condition of "non-degeneracy of the coefficients," there exists a compatibility complex. When evaluated at Sobolev spaces, it is given by

$$0 \to \ H^{s_0}(X,V^0) \xrightarrow{\mathcal{A}^0} \begin{array}{c} H^{s_1}(X,V^1) \\ \oplus \\ H^{t_1}(Y,W^1) \end{array} \xrightarrow{\mathcal{A}^1} \cdots \xrightarrow{\mathcal{A}^{N-1}} \begin{array}{c} H^{s_N}(X,V^N) \\ \oplus \\ H^{t_N}(Y,W^N) \end{array} \to 0 \tag{7.2}$$

with

$$\mathcal{A}^i = \begin{pmatrix} A^i & 0 \\ T^i & Q^i \end{pmatrix}$$

for $i = 0, 1, \ldots, N-1$, cf. [DS96]. Here, $A^0 = A$, $T^0 = T$, $Q^0 = 0$, and A^i, Q^i are differential operators on X and on Y, respectively, T^i are trace operators. Furthermore, $s_0 = s$ is sufficiently large, and α_i is the order of A^i, γ_i is the order of T^i, δ_i is the order of Q^i. We set

$$\begin{array}{rcl} s_{i+1} & = & s_i - \alpha_i, \\ t_{i+1} & = & \max\{s_i - \gamma_i, t_i - \delta_i\}, \end{array}$$

with $t_0 = 0$. By the very construction, the complex

$$0 \longrightarrow H^{s_0}(X,V^0) \xrightarrow{A^0} \cdots \xrightarrow{A^{N-1}} H^{s_N}(X,V^N) \longrightarrow 0 \tag{7.3}$$

is a compatibility complex for the operator A itself.

Theorem 4.5 applies to give us conditions for a compatibility complex for a boundary problem operator to be Fredholm.

Corollary 7.1. *The compatibility complex* (7.2) *for a boundary problem operator* $\mathcal{A} = (A,T)^T$ *is Fredholm for a sufficiently large* s_0*, if the compatibility complex* (7.3) *for* A *is elliptic and the principal boundary symbol complex* $\sigma_\partial(H^\cdot,\mathcal{A})$ *is exact away from the zero section of* T^*Y*.*

It is interesting to mention that if a differential operator A is of "normal" form then the ellipticity of A implies the ellipticity of the compatibility complex (7.3) for A, cf. [Tar95].

Let us now show that the compatibility complex (7.2) for a boundary problem operator $\mathcal{A}$ is the cone of the cochain mapping

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^{s_{i-1}}(X, V^{i-1}) & \xrightarrow{-A^{i-1}} & H^{s_i}(X, V^i) & \xrightarrow{-A^i} \cdots \\ & \downarrow T^{i-1} & & \downarrow T^i & \\ \cdots \longrightarrow & H^{t_i}(Y, W^i) & \xrightarrow{Q^i} & H^{t_{i+1}}(X, W^{i+1}) & \xrightarrow{Q^{i+1}} \cdots \end{array}$$

Indeed, since (7.2) is a complex, we get

$$\begin{aligned} 0 &= \mathcal{A}^i \mathcal{A}^{i-1} \\ &= \begin{pmatrix} A^i A^{i-1} & 0 \\ T^i A^{i-1} + Q^i T^{i-1} & Q^i Q^{i-1} \end{pmatrix}, \end{aligned}$$

proving our claim.

Thus, Corollary 6.1 yields the following condition for a compatibility complex to be Fredholm.

Corollary 7.2. *The compatibility complex* (7.2) *for the boundary problem operator* $\mathcal{A} = (A, T)^T$ *is Fredholm for a sufficiently large* s_0*, if the compatibility complex* (7.3) *for* A *is elliptic and both complexes* $\ker \sigma_\partial(T)$ *and* $\operatorname{coker} \sigma_\partial(A_T)$ *are exact away from the zero section of* T^*Y*.*

8. Euler characteristic of elliptic quasicomplexes

In order to show how to introduce the Euler characteristic for elliptic quasicomplexes we will first discuss an auxiliary problem. Namely, take two exact symbol sequences

$$0 \longrightarrow \pi_X^* V^0 \xrightarrow{\sigma_\psi^0} \pi_X^* V^1 \xrightarrow{\sigma_\psi^1} \cdots \xrightarrow{\sigma_\psi^{N-1}} \pi_X^* V^N \longrightarrow 0\ ,$$

$$0 \longrightarrow \pi_Y^* F^0 \xrightarrow{\sigma_\partial^0} \pi_Y^* F^1 \xrightarrow{\sigma_\partial^1} \cdots \xrightarrow{\sigma_\partial^{N-1}} \pi_Y^* F^N \longrightarrow 0$$

over the cotangent bundles of X and Y, respectively, where

$$F^i = \begin{array}{c} H^{s_i}(\mathbb{R}_+) \otimes V^{i\prime} \\ \oplus \\ W^i \end{array},$$

such that σ_ψ^i and σ_∂^i have the structure of principal interior and boundary symbols of operators in the calculus $\mathcal{B}(X)$. Here, $s_0 \in \mathbb{Z}_+$ is sufficiently large, and we assume that

$$\operatorname{ord} \sigma_\partial^i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \beta_i - \alpha_i + \gamma_i \end{pmatrix}$$

with $\alpha_i \in \mathbb{Z}$, $\lambda_i, \gamma_i \in \mathbb{R}$, and $s_{i+1} = s_i - \alpha_i$.

We suppose that the orders of symbols σ_∂^i are well defined with respect to the compositions $\sigma_\partial^{i+1} \sigma_\partial^i$. This means that $\beta_{i+1} = \alpha_{i+1} + \alpha_i - \gamma_i$ is fulfilled for all $i = 0, 1, \ldots, N-1$.

Define

$$\begin{aligned} t_i &= s_{i+1} + \beta_i \\ &= s_{i-1} - \gamma_{i-1}, \end{aligned}$$

for $i = 0, 1, \ldots, N$, and set $\sigma^i = (\sigma_\psi^i, \sigma_\partial^i)$. Then the question arises whether there is a *complex* of operators $\mathcal{D}^i \in \mathcal{B}^{\mu_i}(X; v_i)$

$$0 \longrightarrow \begin{array}{c} H^{s_0}(X, V^0) \\ \oplus \\ H^{t_0}(Y, W^0) \end{array} \xrightarrow{\mathcal{D}^0} \cdots \xrightarrow{\mathcal{D}^{N-1}} \begin{array}{c} H^{s_N}(X, V^N) \\ \oplus \\ H^{t_N}(Y, W^N) \end{array} \longrightarrow 0\ , \qquad (8.1)$$

cf. (3.3), such that $\sigma(\mathcal{D}^i) = \sigma^i$ for all $i = 0, 1, \dots, N-1$.

This problem goes back at least as far as [PS80]. The study given in [PS80] falls short of providing complete arguments. The proof given there for pseudodifferential operators on a compact closed manifold is wrong. To the best of our knowledge, this question has been far from being solved. The following theorem gives the affirmative solution.

Theorem 8.1. *Given any exact sequence of symbols $\{\sigma^i\}_{i=0}^{N-1}$ which are of the form of principal symbols of operators in $\mathcal{B}(X)$, there is a complex of operators $\mathcal{D}^i \in \mathcal{B}^{\mu_i}(X; v_i)$ satisfying $\sigma(\mathcal{D}^i) = \sigma^i$ for all $i = 0, 1, \dots, N-1$.*

Proof. By the surjectivity of the principal symbol map, there is a sequence of operators $\mathcal{A}^i \in \mathcal{B}^{\mu_i, d_i}(X; v_i)$, such that $\sigma(\mathcal{A}^i) = \sigma^i$ for all $i = 0, 1, \dots, N-1$. Using order reduction operators one can assume without loss of generality that the order of each $\mathcal{A}^i$ is zero. We thus get

$$0 \longrightarrow \begin{matrix} H^0(X, V^0) \\ \oplus \\ H^0(Y, W^0) \end{matrix} \xrightarrow{\mathcal{A}^0} \begin{matrix} H^0(X, V^1) \\ \oplus \\ H^0(Y, W^1) \end{matrix} \xrightarrow{\mathcal{A}^1} \cdots \xrightarrow{\mathcal{A}^{N-1}} \begin{matrix} H^0(X, V^N) \\ \oplus \\ H^0(Y, W^N) \end{matrix} \longrightarrow 0.$$

Since σ^i forms a complex, the principal symbol $\sigma(\mathcal{A}^i\mathcal{A}^{i-1}) = \sigma(\mathcal{A}^i)\sigma(\mathcal{A}^{i-1})$ vanishes, and hence $\mathcal{A}^i\mathcal{A}^{i-1}$ is a compact operator for all i. We are going to modify the quasicomplex into a complex with the same principal symbol complexes by starting from the end of the quasicomplex.

First set $\mathcal{D}^{N-1} = \mathcal{A}^{N-1}$. Since $\sigma(\mathcal{A}^{N-1})$ is surjective, it follows that the Laplacian $\Delta^N = \mathcal{D}^{N-1}\mathcal{D}^{N-1*}$ is elliptic. By the Hodge theory for complexes, cf. (5.5), there is an operator $\mathcal{G}^N \in \mathcal{B}^{0,0}(X; V^N; W^N)$ satisfying

$$\mathcal{I} = \mathcal{H}^N + \Delta^N \mathcal{G}^N = \mathcal{H}^N + \mathcal{D}^{N-1}\varPhi^N,$$

where $\mathcal{H}^N$ stands for the orthogonal projection onto the finite-dimensional space $\ker \Delta^N = \ker \mathcal{D}^{N-1*}$ and $\varPhi^N = \mathcal{D}^{N-1*}\mathcal{G}^N$. We set $\varPi^{N-1} = \mathcal{I} - \varPhi^N\mathcal{D}^{N-1}$ and we claim that $\varPi^{N-1}$ is a projection onto $\ker \mathcal{D}^{N-1}$. Indeed, $\varPi^{N-1} = \mathcal{I}$ is valid on $\ker \mathcal{D}^{N-1}$ and

$$\begin{aligned} \varPi^{N-1}\varPi^{N-1} &= (\mathcal{I} - \varPhi^N\mathcal{D}^{N-1})(\mathcal{I} - \varPhi^N\mathcal{D}^{N-1}) \\ &= \mathcal{I} - 2\,\varPhi^N\mathcal{D}^{N-1} + \mathcal{D}^{N-1*}\mathcal{G}^N\mathcal{D}^{N-1}\mathcal{D}^{N-1*}\mathcal{G}^N\mathcal{D}^{N-1} \\ &= \mathcal{I} - 2\,\varPhi^N\mathcal{D}^{N-1} + \mathcal{D}^{N-1*}\mathcal{G}^N(\mathcal{I} - \mathcal{H}^N)\mathcal{D}^{N-1} \\ &= \varPi^{N-1}, \end{aligned}$$

since $\mathcal{H}^N\mathcal{D}^{N-1} = (\mathcal{D}^{N-1*}\mathcal{H}^N)^* = 0$.

Next we set $\mathcal{D}^{N-2} = \varPi^{N-1}\mathcal{A}^{N-2}$. Then $\mathcal{D}^{N-1}\mathcal{D}^{N-2} = 0$, for $\varPi^{N-1}$ is a projection onto $\ker \mathcal{D}^{N-1}$. For symbols, we get

$$\begin{aligned} \sigma(\mathcal{D}^{N-2}) &= \sigma(\mathcal{A}^{N-2}) - \sigma(\varPhi^N)\sigma(\mathcal{D}^{N-1})\sigma(\mathcal{A}^{N-2}) \\ &= \sigma^{N-2} \end{aligned}$$

because $\sigma(\mathcal{D}^{N-1})\sigma(\mathcal{A}^{N-2})$ vanishes.

Consider now a slightly modified quasicomplex

$$0 \to \begin{matrix} H^0(X, V^0) \\ \oplus \\ H^0(Y, W^0) \end{matrix} \xrightarrow{\mathcal{A}^0} \cdots \xrightarrow{\mathcal{D}^{N-2}} \begin{matrix} H^0(X, V^{N-1}) \\ \oplus \\ H^0(Y, W^{N-1}) \end{matrix} \xrightarrow{\mathcal{D}^{N-1}} \begin{matrix} H^0(X, V^N) \\ \oplus \\ H^0(Y, W^N) \end{matrix} \to 0.$$

Since the principal symbol complex of the above quasicomplex is exact, the Laplacian $\Delta^{N-1} = \mathcal{D}^{N-2}\mathcal{D}^{N-2*} + \mathcal{D}^{N-1*}\mathcal{D}^{N-1}$ is elliptic. Using the Hodge theory for complexes, we deduce that there is an operator $\mathcal{G}^{N-1} \in \mathcal{B}^{0,0}(X; V^{N-1}; W^{N-1})$, such that

$$\begin{aligned} \mathcal{I} &= \mathcal{H}^{N-1} + \mathcal{D}^{N-2}\mathcal{D}^{N-2*}\mathcal{G}^{N-1} + \mathcal{D}^{N-1*}\mathcal{G}^{N}\mathcal{D}^{N-1} \\ &= \mathcal{H}^{N-1} + \mathcal{D}^{N-2}\Phi^{N-1} + \Phi^{N}\mathcal{D}^{N-1} \end{aligned}$$

where $\mathcal{H}^{N-1}$ is the orthogonal projection onto the null-space of $\ker \Delta^{N-1}$ which is $\ker \mathcal{D}^{N-2*} \cap \ker \mathcal{D}^{N-1}$, and $\Phi^{N-1} = \mathcal{D}^{N-2*}\mathcal{G}^{N-1}$. Then, we claim that

$$\Pi^{N-2} = \mathcal{I} - \Phi^{N-1}\mathcal{D}^{N-2}$$

is the orthogonal projection onto $\ker \mathcal{D}^{N-2}$. Indeed, $\mathcal{P}^{N-2}$ is the identity operator on $\ker \mathcal{D}^{N-2}$. Moreover,

$$\begin{aligned} (\Pi^{N-2})^2 &= \Pi^{N-2} - \Phi^{N-1}\mathcal{D}^{N-2} + \Phi^{N-1}(\mathcal{D}^{N-2}\Phi^{N-1})\mathcal{D}^{N-2} \\ &= \Pi^{N-2} - \Phi^{N-1}\mathcal{D}^{N-2} + \Phi^{N-1}(\mathcal{I} - \mathcal{H}^{N-1} - \Phi^{N}\mathcal{D}^{N-1})\mathcal{D}^{N-2} \\ &= \Pi^{N-2} - \Phi^{N-1}\mathcal{H}^{N-1}\mathcal{D}^{N-2} \\ &= \Pi^{N-2}, \end{aligned}$$

since $\mathcal{H}^{N-1}\mathcal{D}^{N-2} = (\mathcal{D}^{N-2*}\mathcal{H}^{N-1})^*$ vanishes. Introducing $\mathcal{D}^{N-3} = \Pi^{N-2}\mathcal{A}^{N-3}$ we thus obtain $\mathcal{D}^{N-2}\mathcal{D}^{N-3} = 0$ and

$$\begin{aligned} \sigma(\mathcal{D}^{N-3}) &= \sigma(\mathcal{A}^{N-3}) - \sigma(\Phi^{N-1})\sigma(\mathcal{D}^{N-2})\sigma(\mathcal{A}^{N-3}) \\ &= \sigma^{N-3}, \end{aligned}$$

for $\sigma(\mathcal{D}^{N-2})\sigma(\mathcal{A}^{N-3})$ vanishes.

Continuing in this fashion, in a finite number of steps we obtain a complex of operators $\mathcal{D}^i \in \mathcal{B}^{0,0}(X; v_i)$, such that $\sigma(\mathcal{D}^i) = \sigma^i$ for all $i = 0, 1, \ldots, N-1$. □

It is worth pointing out that the desired complex (8.1) is constructed within the pseudodifferential calculus $\mathcal{B}(X)$. I.e., $\mathcal{D}^i$ are pseudodifferential operators even in the case if the initial sequences of symbols stem from differential boundary value problems.

Let us now consider an elliptic quasicomplex with differential in Boutet de Monvel's calculus,

$$(H^\cdot, \mathcal{A}): \quad 0 \longrightarrow \begin{matrix} H^{s_0}(X, V^0) \\ \oplus \\ H^{t_0}(Y, W^0) \end{matrix} \xrightarrow{\mathcal{A}^0} \cdots \xrightarrow{\mathcal{A}^{N-1}} \begin{matrix} H^{s_N}(X, V^N) \\ \oplus \\ H^{t_N}(Y, W^N) \end{matrix} \longrightarrow 0\,,$$

cf. (3.3). By Theorem 8.1, there is a complex $(H^\cdot, \mathcal{D})$, such that $\sigma(\mathcal{D}^i) = \sigma(\mathcal{A}^i)$ for all $i = 0, 1, \ldots, N-1$. Hence, the complex $(H^\cdot, \mathcal{D})$ is elliptic as well, and thus, by Theorem 4.5, it is Fredholm. Note that the difference $\mathcal{A}^i - \mathcal{D}^i$ is a compact operator for all i, since $\sigma(\mathcal{D}^i) = \sigma(\mathcal{A}^i)$.

For the Fredholm complex $(H^\cdot, \mathcal{D})$, the Euler characteristic $\chi(H^\cdot, \mathcal{D})$ (index) is defined by

$$\chi(H^\cdot, \mathcal{D}) = \sum_{i=0}^{N} (-1)^i \dim H^i(H^\cdot, \mathcal{D}),$$

where $H^i(H^\cdot, \mathcal{D})$ is the cohomology of the complex at step i.

Definition 8.1. By the *Euler characteristic* of elliptic quasicomplex (3.3) is meant the Euler characteristic of the corresponding Fredholm complex $(H^\cdot, \mathcal{D})$.

Next we have to prove that this definition is independent of the particular choice of complex $(H^\cdot, \mathcal{D})$.

Theorem 8.2. *Suppose that $(H^\cdot, \mathcal{D}_1)$ and $(H^\cdot, \mathcal{D}_2)$ are two complexes with the property that $\sigma(\mathcal{D}_1^i) = \sigma(\mathcal{D}_2^i) = \sigma(\mathcal{A}^i)$. Then $\chi(H^\cdot, \mathcal{D}_1) = \chi(H^\cdot, \mathcal{D}_2)$.*

Proof. To prove this theorem we recall a familiar construction from the theory of abstract Fredholm complexes, cf. [RS82] and elsewhere. Let $(H^\cdot, d)$ be a Fredholm complex of Hilbert spaces

$$0 \longrightarrow H^0 \xrightarrow{d^0} H^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{N-1}} H^N \longrightarrow 0\ ,$$

and let $(H^\cdot, \pi)$ be a parametrix of $(H^\cdot, d)$. Then the block operator

$$(d+\pi)_e = \begin{pmatrix} d^0 & p^2 & 0 & \dots & 0 \\ 0 & d^2 & p^4 & \dots & 0 \\ 0 & 0 & d^4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d^{N-1} \end{pmatrix} : \oplus H^{2i} \to \oplus H^{2i+1}$$

is Fredholm, and for the Euler characteristic of the complex $(H^\cdot, d)$ and operator $(d+\pi)_e$ we have $\chi(H^\cdot, d) = \mathrm{ind}(d+\pi)_e$. Without loss of generality we can assume here that N is odd, otherwise the length of $(H^\cdot, d)$ can be modified by adding a segment $\to 0$ to the complex.

Now note that the operators $\mathcal{D}_1^i$ and $\mathcal{D}_2^i$ differ from each other by compact operators. Hence it follows that if $(H^\cdot, \mathcal{P})$ is a parametrix for the complex $(H^\cdot, \mathcal{D}_1)$, then it is also a parametrix for the complex $(H^\cdot, \mathcal{D}_2)$. Using the same parametrix, we construct Fredholm operators $(\mathcal{D}_1 + \mathcal{P})_e$ and $(\mathcal{D}_2 + \mathcal{P})_e$ which differ from each other by a compact operator. Hence, their indices coincide. □

9. Connection quasicomplex

Let V be a vector bundle over a compact manifold X with boundary, both X and V being C^∞. Choose a connection ∂ for V, i.e., a first order differential operator of type $V \to V \otimes \Lambda T^*X$ on X satisfying the Leibniz rule $\partial(fu) = df\, u + f\, \partial u$ for all $u \in C^\infty(X, V)$ and $f \in C^\infty(X)$. It is just the Leibniz rule that allows one to naturally extend any connection ∂ for V to differential forms of degree i with coefficients in V on X. We write $\partial^i : C^\infty(X, V^i) \to C^\infty(X, V^{i+1})$ for it, where $V^i = V \otimes \Lambda^i T^*X$ for $i = 0, 1, \dots, n = \dim X$, so that $V^0 = V$ and $\partial^0 = \partial$. The composition $\partial^i \partial^{i-1}$ is known to be a smooth bundle homomorphism $V^{i-1} \to V^{i+1}$ called the curvature of V. In fact, this is a $(k \times k)$-matrix whose entries are smooth differential forms of degree 2 on X, k being the rank of V. We thus arrive at a quasicomplex of Hilbert spaces

$$0 \longrightarrow H^{s_0}(X, V^0) \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{n-1}} H^{s_n}(X, V^n) \longrightarrow 0\ , \qquad (9.1)$$

with $s_0 = s \geq n$ and $s_i = s_0 - i$ for $i \geq 1$. We specify sequence (9.1) within the calculus $\mathcal{B}(X)$ by identifying ∂^i with a (2×2)-matrix whose upper left corner is ∂^i and the other entries are zero.

The principal symbol complexes for (9.1) are

$$0 \longrightarrow \pi_X^* V^0 \xrightarrow{\sigma_\psi(\partial^0)} \pi_X^* V^1 \xrightarrow{\sigma_\psi(\partial^1)} \cdots \xrightarrow{\sigma_\psi(\partial^{N-1})} \pi_X^* V^N \longrightarrow 0\ ,$$

$$0 \longrightarrow \pi_Y^* F^0 \xrightarrow{\sigma_\partial(\partial^0)} \pi_Y^* F^1 \xrightarrow{\sigma_\partial(\partial^1)} \cdots \xrightarrow{\sigma_\partial(\partial^{N-1})} \pi_Y^* F^n \longrightarrow 0$$

over the cotangent bundles of X and Y, respectively, where $F^i = \mathcal{S}(\overline{\mathbb{R}}_+) \otimes V^{i\prime}$. The first of the two is locally on X the direct sum of k copies of the principal interior symbol sequence for the de Rham complex on X. Analogously, the second sequence is locally on Y the direct sum of k copies of the principal boundary symbol sequence for the de Rham complex on X. Hence, the calculations of [BS91] actually show that both symbol sequences are exact away from the zero sections of T^*X and T^*Y, respectively. It follows that the quasicomplex (9.1) is elliptic in Boutet de Monvel's calculus.

By Remark 4.1, the quasicomplex (9.1) is Fredholm and it possesses a parametrix within the calculus.

Suppose S is a smooth submanifold of codimension $\gamma > 0$ in X, and $\iota : S \hookrightarrow X$ the embedding map. The vector bundle V restricts naturally to S, the restriction being the induced bundle $V' = \iota^* V$. Fix an arbitrary connection ∂' for V'. Setting $W^i = V' \otimes \Lambda^i T^* S$ for $i = 0, 1, \ldots, n - \gamma$, we get similarly a quasicomplex of Hilbert spaces

$$0 \longrightarrow H^{t_0}(S, W^0) \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{n-\gamma-1}} H^{t_{n-\gamma}}(S, W^{n-\gamma}) \longrightarrow 0 \, , \tag{9.2}$$

with $t_0 = s_0 - \gamma/2$ and $t_i = t_0 - i$ for $i \geq 1$.

By the Sobolev embedding theorem, there is a map T of (9.1) to (9.2) given by $T^i u = \iota^* u$ for $u \in H^{s_i}(X, V^i)$. Here, ι^* stands for the pull-back operator under the embedding ι.

Since any two connections for the vector bundle V' differ by a global smooth one-form on S with coefficients in $\mathrm{Hom}(V')$, it follows that $\iota^* \partial^{i-1} = \partial'^{i-1} \iota^*$ modulo compact operators from $H^{s_{i-1}}(X, V^{i-1})$ to $H^{t_i}(S, W^i)$. This just amounts to saying that $T = \{T^i\}$ is a quasicochain mapping of quasicomplexes. The cone of this mapping is

$$(H^\cdot, \mathcal{A}) : \quad \cdots \longrightarrow \begin{array}{c} H^{s_{i-1}}(X, V^{i-1}) \\ \oplus \\ H^{t_{i-2}}(S, W^{i-2}) \end{array} \xrightarrow{\mathcal{A}^{i-1}} \begin{array}{c} H^{s_i}(X, V^i) \\ \oplus \\ H^{t_{i-1}}(S, W^{i-1}) \end{array} \xrightarrow{\mathcal{A}^i} \cdots \, , \tag{9.3}$$

where

$$\mathcal{A}^i = \begin{pmatrix} -\partial^i & 0 \\ T^i & \partial'^{i-1} \end{pmatrix}$$

for $i = 0, 1, \ldots, n$.

In this general setting we have no calculus structure but the ellipticity of both quasicomplexes (9.1) and (9.2) in Boutet de Monvel's calculus on smooth manifolds with boundary. When combined with an easy computation, Theorem 4.5 still implies the following result.

Corollary 9.1. *For any $s \geq 0$, quasicomplex* (9.3) *is Fredholm. If $\{\pi^i\}$ and $\{\pi'^i\}$ are parametrices for* (9.1) *and* (9.2)*, respectively, then a parametrix for* (9.3) *is given by*

$$\mathcal{P}^i = \begin{pmatrix} -\pi^i & 0 \\ \pi'^{i-1} T^{i-1} \pi^i & \pi'^{i-1} \end{pmatrix}.$$

One may conjecture that the Euler characteristic of (9.3) is equal to $k\, \chi(X, S)$, where $\chi(X, S)$ is the Euler characteristic of the pair (X, S). However, this topic exceeds the scope of this paper.

Acknowledgements The first author gratefully acknowledges the financial support of the Academy of Finland, grant 108394.

References

[Agr97] M.S. Agranovich, *Elliptic boundary problems*, Encyclopaedia of Mathematical Sciences 79, Springer-Verlag, Berlin/Heidelberg, 1997, pp. 1–144.

[AV95] C.-G. Ambrozie and F.-H. Vasilescu, *Banach space complexes*, Mathematics and its Applications, Vol. 334, Kluwer Academic Publishers, Dordrecht, NL, 1995.

[BS91] A. V. Brenner and M. A. Shubin, *The Atiyah-Bott-Lefschetz formula for elliptic complexes on manifolds with boundary*, Current Problems of Mathematics. Fundamental Directions. Vol. 38, VINITI, Moscow, 1991, pp. 119–183.

[dM71] L. Boutet de Monvel, *Boundary problems for pseudodifferential operators*, Acta Math. **126** (1971), 11–51.

[DS96] P.I. Dudnikov and S.N. Samborski, *Linear overdetermined systems of partial differential equations. Initial and initial-boundary value problems*, Encyclopaedia of Mathematical Sciences 65, Springer-Verlag, Berlin/Heidelberg, 1996, pp. 1–86.

[Dyn72] A. Dynin, *Elliptic boundary problems for pseudo-differential complexes*, Funct. Anal. Appl. **6** (1972), no. 1, 75–76.

[Gru84] G. Grubb, *Singular Green operators and their spectral asymptotics*, Duke Math. J. **51** (1984), 477–528.

[Gru96] G. Grubb, *Functional calculus of pseudodifferential boundary problems*, second ed., Progress in Mathematics, vol. 65, Birkhäuser Boston Inc., Boston, MA, 1996.

[PS80] U. Pillat and B.-W. Schulze, *Elliptische Randwertprobleme für Komplexe von Pseudodifferentialoperatoren*, Math. Nachr. **94** (1980), 173–210.

[Put82] M. Putinar, *Some invariants for semi-Fredholm systems of essentially commuting operators*, J. Operator Theory **8** (1982), 65–90.

[RS82] S. Rempel and B.-W. Schulze, *Index theory of elliptic boundary problems*, Akademie-Verlag, Berlin, 1982.

[Sch99] E. Schrohe, *Fréchet algebra techniques for boundary value problems on noncompact manifolds: Fredholm criteria and functional calculus via spectral invariance*, Math. Nachr. **199** (1999), 145–185.

[Sch89] B.-W. Schulze, *Topologies and invertibility in operator spaces with symbolic structures*, In: Problems and Methods in Mathematical Physics, Teubner-Texte zur Mathematik, Vol. 111, BSB Teubner, Leipzig, 1989, 257–270.

[ST98] B.-W. Schulze and N. Tarkhanov, *Elliptic complexes of pseudodifferential operators on manifolds with edges*, Preprint 98/14, ISSN 1437-739X, Institut für Mathematik, Universität Potsdam, 1998.

[Spa66] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[Tar95] N. Tarkhanov, *Complexes of differential operators*, Mathematics and its Applications, vol. 340, Kluwer Academic Publishers, Dordrecht, NL, 1995, Translated from the 1990 Russian original by P. M. Gauthier and revised by the author.

[Tar06] N. Tarkhanov, *Euler characteristic of Fredholm quasicomplexes*, 2006, to appear in Funct. Anal. Appl.

(Katsiaryna Krupchyk) University of Joensuu, Department of Mathematics, P.O. Box 111, FI-80101 Joensuu, Finland

E-mail address: katya.krupchyk@joensuu.fi

(Nikolai Tarkhanov) Universität Potsdam, Institut für Mathematik, Postfach 60 15 53, 14415 Potsdam, Germany

E-mail address: tarkhanov@math.uni-potsdam.de

(Jukka Tuomela) University of Joensuu, Department of Mathematics, P.O. Box 111, FI-80101 Joensuu, Finland

E-mail address: jukka.tuomela@joensuu.fi