On Existence of Solutions for Some Hyperbolic-Parabolic Type Chemotaxis Systems*

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Abstract: In this paper, we discuss the local and global existence of week solutions for some hyperbolic-parabolic systems modelling chemotaxis.

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1 Introduction

The earliest model for chemosensitive movement has been developed by Keller and Segel [1,2,3], which we call it as KS model. Assume that in absence of any external signal the spread of a population $u(t,x)$ is described by the diffusion equation

$$u_t = d\Delta u, \tag{1}$$

where $d > 0$ is the diffusion constant. We define the net flux as $j = -d\nabla u$. If there is some external signal s, we just assume that it results in a chemotactic velocity β. Then the flux is

$$j = -d\nabla u + \beta u. \tag{2}$$

To be more specific, we assume that the chemotactic velocity β has the direction of the gradient ∇s and that the sensitivity χ to the gradient depends on the signal concentration $s(t,x)$, then $\beta = \chi(s)\nabla s$.

We use this modified flux in (2) to obtain the parabolic chemotaxis equation

$$u_t = \nabla(d\nabla u - \chi(s)\nabla s \cdot u). \tag{3}$$

If $\chi(s)$ is positive, which means that the chemotactic velocity is in direction of s, we call it positive bias, whereas $\chi < 0$ is called negative bias.

To our general knowledge, the external signal is produced by the individuals and decays, which is described by a nonlinear function $g(s,u)$. We assume that the spatial spread of the external signal is driven by diffusion. Then the full system for u and s reads

*Research supported by the NSFC

$$u_t = \nabla(d\nabla u - \chi(s)\nabla s \cdot u), \tag{4}$$

$$\tau s_t = d\Delta s + g(s,u), \tag{5}$$

the time constant $0 \leq \tau \leq 1$ indicates that the spatial spread of the organisms u and the signal s are on different time scales. The case $\tau = 0$ corresponds to a quasi-steady state assumption for the signal distribution. When we assume that the spatial spread of external signal is driven by wave motion, then the equation (5) would be replaced by

$$s_{tt} = d\Delta s + g(s,u). \tag{6}$$

The full system for u and s becomes

$$u_t = \nabla(d\nabla u - \chi(s)\nabla s \cdot u), \tag{7}$$

$$s_{tt} = d\Delta s + g(s,u), \tag{8}$$

which is called as hyperbolic-parabolic chemotaxis system.

2 Main Results

Let us consider the following problem:

$$\begin{array}{ll} u_t = \nabla(\nabla u - \chi u \nabla v) & in \quad (0,T)\times\Omega, \\ v_{tt} = \Delta v + g(u,v) & in \quad (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & on \quad (0,T)\times\partial\Omega, \end{array} \tag{9}$$

with initial data

$$u(0,\cdot) = u_0, \qquad v(0,\cdot) = \varphi, \qquad v_t(0,\cdot) = \psi \quad in \quad \Omega,$$

where $\Omega \subset \mathbf{R}^n$, a bounded open domain with smooth boundary $\partial\Omega$, χ is a nonnegative constant.

Choose a constant σ, which satisfies

$$1 < \sigma < 2 \tag{10}$$

and

$$n < 2\sigma < n+2 \tag{11}$$

It is easy to check that (10) and (11) can be simultaneously satisfied in the case of $1 \leq n \leq 3$.

Our main results are

Theorem 4.1. *Under the conditions (10) and (11), if $g(u,v) = -\gamma v + f(u)$ and $f \in C^2(\mathbf{R})$, then for each initial data $u_0 \in H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \ on \ \partial\Omega\}$, $\varphi \in H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \ on \ \partial\Omega\}$, $\psi \in H^1(\Omega)$, the problem (9) has a unique local solution $(u,v) \in X_{t_0} \times Y_{t_0}$ for some $t_0 > 0$.*

Theorem 5.1. *Let $n = 1$ and $\sigma = \frac{5}{4}$, if $g(u,v) = -\gamma v + f(u)$ and $f \in C_0^2(\mathbf{R})$, then for each initial data $u_0 \in H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \;\; on \;\; \partial\Omega\}$ and $u_0 \geq 0$, $\varphi \in H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \;\; on \;\; \partial\Omega\}$ and $\psi \in H^1(\Omega)$, the problem (9) has a unique global solution $(u,v) \in X_\infty \times Y_\infty$.*

Where we define

$$X_{t_0} = C([0,t_0], H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \;\; on \;\; \partial\Omega\})$$

$$Y_{t_0} = C([0,t_0], H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \quad on \;\; \partial\Omega\}) \cap C^1([0,t_0], H^1(\Omega))$$

3 Some Basic Lemmas

For $g(u,v) = -\gamma v + f(u)$, and γ is a constant, $f(x) \in C^2(\mathbf{R})$. We divide the system (9) into two pars:

$$\begin{cases} u_t = \nabla(\nabla u - \chi u \nabla v) \quad in \quad (0,T) \times \Omega \\ \frac{\partial u}{\partial n} = 0 \quad on \quad (0,T) \times \partial\Omega \\ u(0,\cdot) = u_0 \qquad\qquad\qquad in \;\; \Omega, \end{cases} \tag{12}$$

and

$$\begin{cases} v_{tt} = \Delta v - \gamma v + f(u) \quad in \quad (0,T) \times \Omega \\ \frac{\partial v}{\partial n} = 0 \qquad\qquad\qquad on \quad (0,T) \times \partial\Omega \\ v(0,\;\cdot) = \varphi, \quad v_t(0,\;\cdot) = \psi \quad in \quad \Omega. \end{cases} \tag{13}$$

We have

Lemma 3.1. *For any $T > 0$, and*

$$\varphi \in H^2(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \; on \; \partial\Omega\}, \;\; \psi \in H^1(\Omega), \;\; f(u(t,.)) \in C([0,T]; H^1(\Omega)),$$

then (13) has a unique solution v, satisfying

$$v \in C([0,T]; H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \; on \; \partial\Omega\}), \;\; v_t \in C([0,T]; H^1(\Omega)), \;\; v_{tt} \in C([0,T]; L^2(\Omega)),$$

and

$$\begin{aligned} \|v(t,\;\cdot)\|_{H^2(\Omega)} + \|v_t(t,\;\cdot)\|_{H^1(\Omega)} \;\; &\leq e^{cT}(\|\varphi\|_{H^2(\Omega)} + \|\psi\|_{H^1(\Omega)} \\ &+ \textstyle\int_0^T \|f(u(\tau,\;\cdot))\|_{H^1(\Omega)}\, d\tau), \quad \forall\, t \in [0,T], \end{aligned} \tag{14}$$

where $c > 0$ is a constant which is independent of T.

Proof: Set $v_t = w$, we have following system

$$\begin{cases} v_t = w, \\ w_t = \triangle v - \gamma v + f(u). \end{cases} \tag{15}$$

Thus we can write it in a abstract form:

$$\begin{cases} U_t = LU + F(U) \quad in \quad X = H^1(\Omega) \times L^2(\Omega), \\ U_0 = U(0,x) = (\varphi, \psi), \end{cases} \tag{16}$$

where $L(v,w)=(w,\triangle v-v)$ for $(v,w)\in D(L)$, $D(L)=H^2(\Omega)\cap\{\frac{\partial v}{\partial n}=0 \ on\ \partial\Omega\}\times H^1(\Omega)$ and $F(v,w)=(0,(1-\gamma)v+f(u))$.

Define the inner product in X as

$$<(v,w),(v',w')>_X=(v,v')_{H^1}+(w,w')_{L^2},$$

where $(\cdot,\cdot)_{H^1}$ and $(\cdot,\cdot)_{L^2}$ represent the inner products in H^1 and L^2 respectively, then X is a Hilbert space.

For $U=(v,w)\in D(L)$, we have

$$\begin{aligned}&<LU,U>_X=<(w,\triangle v-v),(v,w)>_X\\&=(w,v)_{H^1}+(\triangle v-v,w)_{L^2}\\&=(w,v)_{H^1}+(\triangle v,w)_{L^2}-(v,w)_{L^2}\\&=(w,v)_{H^1}-(\nabla v,\nabla w)_{L^2}-(v,w)_{L^2}\\&=0\end{aligned}\tag{17}$$

Otherwise, for $U=(v,w)\in D(L)$, $U'=(v',w')\in X$,

$$\begin{aligned}&<L(v,w),(v',w')>_X\\&=<(w,\triangle v-v),(v',w')>_X\\&=(w,v')_{H^1}+(\triangle v-v,w')_{L^2}\\&=(w,v')_{H^1}+(\triangle v,w')_{L^2}-(v,w')_{L^2}\end{aligned}\tag{18}$$

If $<L(v,w),(v',w')>_X$ is bounded for each $(v,w)\in D(L)$, then $(w,v')_{H^1}$, $(\triangle v,w')_{L^2}$ and $(v,w')_{L^2}$ are bounded for each $(v,w)\in D(L)$, which means that

$$v'\in\ H^2\cap\{\frac{\partial v}{\partial n}=0\quad on\ \ \partial\Omega\},\quad w'\in\ H^1,\tag{19}$$

that implies $D(L^*)\subset D(L)$. On the other hand, from (17) and the lemma in [6, p9], we know that

$$L^*=-L.$$

Thus we know that L is a generator of a unitary operator group. It is easy to check that for $f(u(t,\cdot))\in C([0,T],H^1(\Omega))$,

$$F:\ X\to X,$$

and

$$\|F(U_1)-F(U_2)\|_X\le c\,\|U_1-U_2\|_X\quad U_i\in X,\ \ i=1,2,$$

where $\|(v,w)\|_X^2=\|v\|_{H^1}^2+\|w\|_{L^2}^2$.

Now we can declare that (16) has a unique solution

$$U\in C^1([0,T],X)\cap C([0,T],D(L))\ \ for\ each\ U_0\in D(L),\tag{20}$$

which means that for each $(\varphi,\psi)\in D(L)$, (13) has a unique solution

$$v\in C([0,T],H^2(\Omega)\cap\{\frac{\partial v}{\partial n}=0\ on\ \partial\Omega\}),\ v_t\in C([0,T],H^1(\Omega))\ and\ v_{tt}\in C([0,T],L^2(\Omega)).$$

Next, we estimate the norm of v. By using the semigroup notation $T(t)=e^{tL}$, we have

$$U=T(t)U_0+\int_0^t T(t-s)F(U)ds.\tag{21}$$

Since $L=-L^{*}$, and in terms of (17), we have that

$$< LU, U >_X = 0 \quad for\ each\ U \in D((L),$$

and

$$< L^{*}U, U >_X = < -LU, U >_X = 0 \ \ for\ each\ U \in D(L).$$

Hence L generates a strongly continuous contractive semigroup on Hilbert space X (cf. [4, 5]), in other words, we have

$$\left\| e^{tL} \right\| = \|T(t)\| \leq 1. \tag{22}$$

So we know that

$$\begin{aligned} &\|U(t)\|_{H^2\times H^1} \leq \|T(t)U_0\|_{H^2\times H^1} + \textstyle\int_0^t \|T(t-s)F(U(s))\|_{H^2\times H^1}\, ds \\ &\leq \|U_0\|_{H^2\times H^1} + \textstyle\int_0^t \|F(U)\|_{H^2\times H^1}\, ds \\ &= \|\varphi\|_{H^2} + \|\psi\|_{H^1} + \textstyle\int_0^t \|(1-\gamma)v + f(u)\|_{H^1} ds \\ &\leq \|\varphi\|_{H^2} + \|\psi\|_{H^1} + c\textstyle\int_0^t \|v\|_{H^1} ds + \int_0^t \|f(u)\|_{H^1} ds \\ &\leq \|\varphi\|_{H^2} + \|\psi\|_{H^1} + c\textstyle\int_0^t \|U\|_{H^2\times H^1} ds + \int_0^T \|f(u)\|_{H^1} ds, \quad 0 \leq t \leq T. \end{aligned} \tag{23}$$

From Gronwall's inequality, we know that

$$\begin{aligned} &\|U\|_{H^2\times H^1} \leq e^{ct}(\|\varphi\|_{H^2} + \|\psi\|_{H^1} + \textstyle\int_0^T \|f(u)\|_{H^1} ds) \\ &\leq e^{cT}(\|\varphi\|_{H^2} + \|\psi\|_{H^1} + \textstyle\int_0^T \|f(u)\|_{H^1} ds), \end{aligned} \quad 0 \leq t \leq T, \tag{24}$$

which implies the estimate (14) and the uniqueness follows.

If Ω is a bounded open domain with smooth boundary, in which we can consider the Neumann boundary condition. As we known that the $e^{t\triangle}$ defines a holomorphic semigroup on the Hilbert space $L^2(\Omega)$, so we have that

$$f \in \ L^2(\Omega) \Rightarrow \left\| e^{t\triangle} f \right\|_{H^2(\Omega)} \leq \frac{c}{t} \|f\|_{L^2(\Omega)}, \tag{25}$$

where $D(\Delta) = \{u \in H^2(\Omega),\ \frac{\partial u}{\partial n} = 0 \ \ on\ \ \partial\Omega\}$.

Applying interpolation to (25), it yields

$$\left\| e^{t\triangle} f \right\|_{H^\sigma(\Omega)} \leq ct^{-\frac{\sigma}{2}} \|f\|_{L^2(\Omega)} \ \ for\ 0 \leq \sigma \leq 2,\ 0 < t \leq 1. \tag{26}$$

Take $Y = H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0\ on\ \partial\Omega\}$ and $Z = L^2(\Omega)$, $\Phi(u) = -\chi\nabla v\nabla u - \chi\Delta v \cdot u$. Then For $v \in Y_{t_0}$, and from the lemma in [4, p273], we can declare that

Lemma 3.2. *For each $u_0 \in Y$ and $v \in Y_{t_0}$, σ and n satisfy the conditions (10) and (11), then the problem (12) has a unique solution*

$$u \in X_{t_0} = C([0,t_0], H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \ \ on\ \ \partial\Omega\}).$$

Proof: If we can show that $\Phi:\ Y \to Z$ is a locally Lipschitz map, then the lemma 3.2 is true. In fact, for arbitrary u_1, $u_2 \in Y$ and $v \in Y_{t_0}$, the difference

$$\Phi(u_1) - \Phi(u_2) = -\chi\nabla v\nabla(u_1 - u_2) - \chi\triangle v \cdot (u_1 - u_2).$$

That is

$$\|\Phi(u_1)-\Phi(u_2)\|_Z = \|\Phi(u_1)-\Phi(u_2)\|_{L^2}$$
$$\leq \|\chi\nabla v\nabla(u_1-u_2)\|_{L^2} + \|\chi\triangle v\cdot(u_1-u_2)\|_{L^2}.$$

By Sobolev imbedding theorems, we have

$$H^1(\Omega)\hookrightarrow L^\infty(\Omega),\quad for\ \ n=1,$$

$$H^1(\Omega)\hookrightarrow L^q(\Omega),\ \ 1<q<\infty,\quad for\ \ n=2,$$

$$H^1(\Omega)\hookrightarrow L^{\frac{2n}{n-2}}(\Omega),\quad for\ \ n=3.$$

Thus in terms of (10) and (11), we know that $H^1(\Omega)\hookrightarrow L^{\frac{n}{\sigma-1}}(\Omega)$ and $H^{\sigma-1}(\Omega)\hookrightarrow L^{\frac{2n}{n-2(\sigma-1)}}(\Omega)$ for $n=2,3$.

Firstly we estimate $\|\chi\nabla v\nabla(u_1-u_2)\|_{L^2}$. If $n=1$, then

$$\|\chi\nabla v\nabla(u_1-u_2)\|_{L^2}$$

$$\leq \chi\|\nabla(u_1-u_2)\|_{L^2}\|\nabla v\|_{L^\infty}$$
$$\leq c\|u_1-u_2\|_{H^1}\|\nabla v\|_{H^1}$$
$$\leq c\|u_1-u_2\|_{H^\sigma}\|v\|_{H^2}.$$

If $n=2,3$, then

$$\|\chi\nabla v\nabla(u_1-u_2)\|_{L^2}$$
$$\leq \chi\|\nabla(u_1-u_2)\|_{L^{\frac{2n}{n-2(\sigma-1)}}}\|\nabla v\|_{L^{\frac{n}{\sigma-1}}}$$
$$\leq c\|u_1-u_2\|_{H^\sigma}\|v\|_{H^2}.$$

Hence for $n=1,2,3$, we have that

$$\|\chi\nabla v\nabla(u_1-u_2)\|_{L^2}\leq c\|u_1-u_2\|_{H^\sigma}\|v\|_{H^2}.$$

Similarly, we have

$$\|\chi\triangle v\cdot(u_1-u_2)\|_{L^2}$$

$$\leq c\|v\|_{H^2}\|u_1-u_2\|_{L^\infty}$$

$$\leq c\|u_1-u_2\|_{H^\sigma}\|v\|_{H^2}.$$

Thus we have proved that

$$\|\Phi(u_1)-\Phi(u_2)\|_Z\leq c\|u_1-u_2\|_Y\|v\|_{H^2},$$

as required.

Lemma 3.3. *Under the conditions (10) and (11), if $u\in X_{t_0}$ is a solution of (12), the there exists a constant c which is independent of t_0, such that*

$$\|u\|_{X_{t_0}}\leq c\|u_0\|_{\sigma,2}+ct_0^{1-\frac{\sigma}{2}}\|v\|_{Y_{t_0}}\cdot\|u\|_{X_{t_0}}, \tag{27}$$

where $\|\cdot\|_{k,p}$ is the norm of Sobolev space $W^{k,p}$.

Proof: Let $T(t)=e^{t\Delta}$, then

$$u(t) = T(t)u_0 - \chi \int_0^t T(t-s)\nabla v \nabla u ds - \chi \int_0^t T(t-s)\Delta v \cdot u ds.$$

By (26), we have $T(t):\ L^2(\Omega) \to H^\sigma(\Omega)$ with norm $c_\sigma t^{-\frac{\sigma}{2}}$. Thus

$$\left\| \int_0^t T(t-s)\nabla v \nabla u ds \right\|_{\sigma,2} \le c_\sigma t^{1-\frac{\sigma}{2}} \sup_{0\le s\le t} \|\nabla v(s,\cdot)\nabla u(s,\cdot)\|_2$$

where we use $\|\cdot\|_p$ as the norm of L^p.

By Sobolev imbedding theorem, $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ for $n=1$, we have

$$\begin{aligned} \|\nabla v \nabla u\|_2 &\le \|\nabla v\|_\infty \cdot \|\nabla u\|_2 \\ &\le c\|v\|_{2,2} \cdot \|u\|_{1,2} \\ &\le c\|v\|_{2,2} \cdot \|u\|_{\sigma,2}. \end{aligned}$$

For $n=2,3$, we have $H^1(\Omega) \hookrightarrow L^{\frac{n}{\sigma-1}}(\Omega)$, $H^{\sigma-1}(\Omega) \hookrightarrow L^{\frac{2n}{n-2(\sigma-1)}}(\Omega)$, thus $f^2 \in L^{\frac{n}{2(\sigma-1)}}$, $g^2 \in L^{\frac{n}{n-2(\sigma-1)}}$ if $f \in H^1$ and $g \in H^{\sigma-1}$. By using Cauchy inequality, we get

$$\left\|f^2 g^2\right\|_1 \le \left\|f^2\right\|_{\frac{n}{2(\sigma-1)}} \cdot \left\|g^2\right\|_{\frac{n}{n-2(\sigma-1)}}$$

which implies $\|fg\|_2 \le \|f\|_{\frac{n}{\sigma-1}} \cdot \|g\|_{\frac{2n}{n-2(\sigma-1)}}$. Thus

$$\begin{aligned} &\|\nabla v \nabla u\|_2 \le \|\nabla v\|_{\frac{n}{\sigma-1}} \cdot \|\nabla u\|_{\frac{2n}{n-2(\sigma-1)}} \\ &\le c\|\nabla v\|_{1,2} \cdot \|\nabla u\|_{\frac{2n}{n-2(\sigma-1)}} \\ &\le c\|v\|_{2,2} \cdot \|\nabla u\|_{\sigma-1,2} \le c\|v\|_{2,2} \cdot \|u\|_{\sigma,2}. \end{aligned}$$

Now we obtain that, for $0 \le t \le t_0$,

$$\begin{aligned} &\left\|\textstyle\int_0^t \tau(t-s)\nabla v \nabla u ds\right\|_{\sigma,2} \le c_\sigma t^{1-\frac{\sigma}{2}} \sup_{0\le s\le t} \|\nabla v \nabla u\|_2 \\ &\le C t^{1-\frac{\sigma}{2}} \sup_{0\le s\le t} \|v\|_{2,2} \cdot \|u\|_{\sigma,2} \le C t_0^{1-\frac{\sigma}{2}} \|u\|_{X_{t_0}} \cdot \|v\|_{Y_{t_0}}. \end{aligned}$$

Meanwhile

$$\begin{aligned} &\left\|\textstyle\int_0^t T(t-s)\Delta v \cdot u\right\|_{\sigma,2} \\ &\le c_\sigma t^{1-\frac{\sigma}{2}} \sup_{0\le s\le t} \|\Delta v \cdot u\|_2 \\ &\le c_\sigma t_0^{1-\frac{\sigma}{2}} \sup_{0\le s\le t_0} \|u\|_{L^\infty} \cdot \|\Delta v\|_{L^2} \\ &\le C t_0^{1-\frac{\sigma}{2}} \sup_{0\le s\le t_0} \|u\|_{\sigma,2} \cdot \sup_{0\le s\le t_0} \|v\|_{2,2} \\ &\le C t_0^{1-\frac{\sigma}{2}} \|u\|_{X_{t_0}} \cdot \|v\|_{Y_{t_0}}. \end{aligned}$$

Finally we can deduce that

$$\begin{aligned} \|u(t)\|_{\sigma,2} &\le \|T(t)u_0\|_{\sigma,2} + \chi \left\|\textstyle\int_0^t T(t-s)\nabla v \nabla u ds\right\|_{\sigma,2} \\ &\quad + \chi \left\|\textstyle\int_0^t T(t-s)\Delta v \cdot u ds\right\|_{\sigma,2} \\ &\le C\|u_0\|_{\sigma,2} + \chi c c_\sigma t_0^{1-\frac{\sigma}{2}} \|u\|_{X_{t_0}} \cdot \|v\|_{Y_{t_0}}, \quad 0 \le t \le t_0, \end{aligned}$$

which implies

$$\|u\|_{X_{t_0}} \le C\|u_0\|_{\sigma,2} + C t_0^{1-\frac{\sigma}{2}} \|u\|_{X_{t_0}} \|v\|_{Y_{t_0}}.$$

Lemma 3.3 is proved.

4 Local Existence of Solutions

In this section, we establish the local solution of the system (9). Our main result is as follows:

Theorem 4.1. *If σ and n satisfy the conditions (10) and (11), $g(u,v) = -\gamma v + f(u)$ and $f \in C^2(\mathbf{R})$, then for each initial data $u_0 \in H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0$ on $\partial\Omega\}$, $\varphi \in H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0$ on $\partial\Omega\}$, $\psi \in H^1(\Omega)$, the problem (9) has a unique local solution $(u,v) \in X_{t_0} \times Y_{t_0}$ for some $t_0 > 0$.*

Proof: Consider $w \in X_{t_0}$, $w(0,x) = u_0(x)$ and let $v = v(w)$ denote the corresponding solution of the equation:

$$\begin{array}{ll} v_{tt} = \Delta v - \gamma v + f(w) \;\; in \;\; (0,t_0) \times \Omega, \\ \frac{\partial v}{\partial n} = 0 \qquad\qquad\qquad\quad on \;\; (0,t_0) \times \partial\Omega, \\ v(0) = \varphi \;\; in \;\; \Omega, \\ v_t(0) = \psi \;\; in \;\; \Omega. \end{array} \tag{28}$$

By Lemma 3.1, we have $v \in Y_{t_0}$, and

$$\begin{aligned} \|v(t)\|_{H^2(\Omega)} \le e^{c_1 t_0}(&\|\varphi\|_{H^2(\Omega)} + \|\psi\|_{H^1(\Omega)} \\ &+ \textstyle\int_0^{t_0} \|f(w(\tau,\cdot))\|_{H^1(\Omega)} d\tau), \quad \forall\, t \in [0,t_0]. \end{aligned} \tag{29}$$

Secondly, for the solution v of (28), we define $u = u(v(w))$ to be the corresponding solution of

$$\begin{array}{ll} u_t = \nabla(\nabla u - \chi u \nabla v) \;\; in \;\; (0,t_0) \times \Omega, \\ \frac{\partial u}{\partial n} = 0 \qquad\qquad\qquad\quad on \;\; (0,t_0) \times \partial\Omega, \\ u(0,x) = u_0(x) = w(0,x) \;\; in \;\; \Omega. \end{array} \tag{30}$$

If we define $Gw = u(v(w))$, then Lemma 3.2 shows that

$$G: \quad X_{t_0} \to X_{t_0}.$$

Take $M = 2c\,\|u_0\|_{\sigma,2}$ and a ball

$$B_M = \left\{ w \in X_{t_0} \mid w(0,x) = u_0(x),\; \|w(t,\cdot)\|_{\sigma,2} \le M,\; 0 \le t \le t_0 \right\},$$

where the constant $c \ge 1$ is given by (27). Then we combine the estimates (27) and (29) to obtain

$$\begin{aligned} \|Gw\|_{X_{t_0}} &\le c\,\|u_0\|_{\sigma,2} + ct_0^{1-\frac{\sigma}{2}}\,\|v\|_{Y_{t_0}} \cdot \|Gw\|_{X_{t_0}} \\ &\le c\,\|u_0\|_{\sigma,2} + ct_0^{1-\frac{\sigma}{2}} e^{c_1 t_0}(\|\varphi\|_{H^2} + \|\psi\|_{H^1} \\ &\quad + \textstyle\int_0^{t_0} \|f(w(\tau,\cdot))\|_{H^1} d\tau) \cdot \|Gw\|_{X_{t_0}}. \end{aligned}$$

Since $\|w\|_{1,2} \le \|w\|_{\sigma,2} \le M$, and $f \in C^2(\mathbf{R})$, we can deduce that

$$\|f(w(\tau,\cdot))\|_{1,2} \le \|f\|_{C^2[-M,M]} \cdot M + \|f(0)\|_{L^2},$$

which shows that $\|Gw\|_{X_{t_0}} \le 2c\,\|u_0\|_{\sigma,2}$ for $t_0 > 0$ small enough.

Thus we have proved that, for $t_0 > 0$ small enough, G maps B_M into B_M. Next, we can prove that, for t_0 small enough, G is a contract mapping. In fact, let $w_1,\ w_2 \in X_u$, and $v_1,\ v_2$ denote the corresponding solutions of (28). Then the difference $Gw_1 - Gw_2$ satisfies:

$$
\begin{aligned}
&Gg_1 - Gg_2 = u_1 - u_2 \\
&= -\chi \int_0^t T(t-s)u_1\Delta v_1 ds - \chi \int_0^t T(t-s)\nabla u_1 \nabla v_1 ds \\
&+\chi \int_0^t T(t-s)u_2\nabla v_2 ds + \chi \int_0^t T(t-s)\nabla u_2 \nabla v_2 ds \\
&= -\chi \int_0^t T(t-s)(u_1\Delta v_1 - u_2\Delta v_2) ds - \chi \int_0^t T(t-s)(\nabla u_1\nabla v_1 - \nabla u_2 \nabla v_2) ds.
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
&\left\| \int_0^t T(t-s)(u_1\Delta v_1 - u_2\Delta v_2)ds \right\|_{\sigma,2} \\
&\leq \left\| \int_0^t T(t-s)u_1(\Delta v_1 - \Delta v_2)ds \right\|_{\sigma,2} + \left\| \int_0^t T(t-s)(u_1 - u_2)\Delta v_2 ds \right\|_{\sigma,2}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left\| \int_0^t T(t-s)u_1(\Delta v_1 - \Delta v_2)ds \right\|_{\sigma,2} \\
&\leq ct_0^{1-\frac{\sigma}{2}} \sup_{0\leq t\leq t_0} \|u_1(\Delta v_1 - \Delta v_2)\|_2 \\
&\leq ct_0^{1-\frac{\sigma}{2}} \sup_{0\leq t\leq t_0} \|u_1\|_{L^\infty} \cdot \|\Delta(v_1 - v_2)\|_2 \\
&\leq CMt_0^{1-\frac{\sigma}{2}} \sup_{0\leq t\leq t_0} \|v_1 - v_2\|_{2,2},
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
&\left\| \int_0^t T(t-s)(u_1 - u_2)\Delta v_2 ds \right\|_{\sigma,2} \\
&\leq ct_0^{1-\frac{\sigma}{2}} \sup_{0\leq t\leq t_0} \|(u_1 - u_2)\Delta v_2\|_2 \\
&\leq ct_0^{1-\frac{\sigma}{2}} \sup_{0\leq t\leq t_0} \|v_2\|_{2,2} \cdot \|u_1 - u_2\|_{L^\infty} \\
&\leq ct_0^{1-\frac{\sigma}{2}} \|v_2\|_{Y_{t_0}} \cdot \|u_1 - u_2\|_{X_{t_0}}.
\end{aligned}
\tag{32}
$$

Thus we have that

$$
\begin{aligned}
&\left\| \int_0^t T(t-s)(u_1\Delta v_1 - u_2\Delta v_2)ds \right\|_{\sigma,2} \\
&\leq Ct_0^{1-\frac{\sigma}{2}} \|v_1 - v_2\|_{Y_{t_0}} \\
&+Ct_0^{1-\frac{\sigma}{2}} \|v_2\|_{Y_{t_0}} \cdot \|u_1 - u_2\|_{X_{t_0}}, \quad 0 \leq t \leq t_0.
\end{aligned}
\tag{33}
$$

Similarly, we have

$$
\begin{aligned}
&\left\| \int_0^t T(t-s)(\nabla u_1\nabla v_1 - \nabla u_2\nabla v_2)ds \right\|_{\sigma,2} \\
&\leq \left\| \int_0^t T(t-s)(\nabla u_1\nabla v_1 - \nabla u_2\nabla v_1)ds \right\|_{\sigma,2} \\
&+ \left\| \int_0^t T(t-s)(\nabla u_2\nabla v_1 - \nabla u_2\nabla v_2)ds \right\|_{\sigma,2}.
\end{aligned}
$$

Here

$$
\begin{aligned}
&\left\| \int_0^t T(t-s)(\nabla u_1\nabla v_1 - \nabla u_2\nabla v_1)ds \right\|_{\sigma,2} \\
&\leq ct_0^{1-\frac{\sigma}{2}} \sup_{0\leq t\leq t_0} \|\nabla v_1 \cdot \nabla(u_1 - u_2)\|_2, \quad 0 \leq t \leq t_0.
\end{aligned}
$$

As we have done in Lemma 3.3, we can deduce that

$$\begin{aligned}&\left\|\int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_1)ds\right\|_{\sigma,2}\\&\le Ct_0^{1-\frac{\sigma}{2}}\left\|v_1\right\|_{Y_{t_0}}\cdot\left\|u_1-u_2\right\|_{X_{t_0}},\quad 0\le t\le t_0.\end{aligned}\tag{34}$$

And we have similarly that

$$\begin{aligned}&\left\|\int_0^t T(t-s)(\nabla u_2\nabla v_1-\nabla u_2\nabla v_2)ds\right\|_{\sigma,2}\\&\le ct_0^{1-\frac{\sigma}{2}}\sup_{0\le t\le t_0}\left\|\nabla u_2\cdot\nabla(v_1-v_2)\right\|_2\\&\le ct_0^{1-\frac{\sigma}{2}}\left\|u_2\right\|_{X_{t_0}}\cdot\left\|v_1-v_2\right\|_{Y_{t_0}}\\&\le cMt_0^{1-\frac{\sigma}{2}}\left\|v_1-v_2\right\|_{Y_{t_0}},\quad 0\le t\le t_0.\end{aligned}\tag{35}$$

Then

$$\begin{aligned}&\left\|\int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_2)ds\right\|_{\sigma,2}\\&\le Ct_0^{1-\frac{\sigma}{2}}\left\|v_1\right\|_{Y_{t_0}}\cdot\left\|u_1-u_2\right\|_{X_{t_0}}+Ct_0^{1-\frac{\sigma}{2}}\left\|v_1-v_2\right\|_{Y_{t_0}},\quad 0\le t\le t_0.\end{aligned}\tag{36}$$

Combining the estimates (33) and (36), we have

$$\begin{aligned}&\left\|Gw_1-Gw_2\right\|_{\sigma,2}=\left\|u_1-u_2\right\|_{\sigma,2}\\&\le Ct_0^{1-\frac{\sigma}{2}}\left\|v_1-v_2\right\|_{Y_{t_0}}+Ct_0^{1-\frac{\sigma}{2}}\left\|v_2\right\|_{Y_{t_0}}\cdot\left\|u_1-u_2\right\|_{X_{t_0}}\\&+Ct_0^{1-\frac{\sigma}{2}}\left\|v_1\right\|_{Y_{t_0}}\cdot\left\|u_1-u_2\right\|_{X_{t_0}}+Ct_0^{1-\frac{\sigma}{2}}\left\|v_1-v_2\right\|_{Y_{t_0}},\end{aligned}$$

which implies

$$\begin{aligned}&\left\|Gw_1-Gw_2\right\|_{X_{t_0}}\\&\le 2Ct_0^{1-\frac{\sigma}{2}}\left\|v_1-v_2\right\|_{Y_{t_0}}+Ct_0^{1-\frac{\sigma}{2}}(\left\|v_2\right\|_{Y_{t_0}}+\left\|v_1\right\|_{Y_{t_0}})\cdot\left\|Gw_1-Gw_2\right\|_{X_{t_0}}.\end{aligned}$$

Also, we have

$$\begin{aligned}\left\|v_1-v_2\right\|_{2,2}&\le e^{c_1t_0}\int_0^{t_0}\left\|f(w_1)-f(w_2)\right\|_{H^1}d\tau\\&\le e^{c_1t_0}\left\|f\right\|_{C^2[-M,M]}\int_0^{t_0}\left\|w_1-w_2\right\|_{H^\sigma}d\tau,\end{aligned}$$

and

$$\begin{aligned}\left\|v_1\right\|_{2,2}&\le e^{c_1t_0}(\left\|\varphi\right\|_{H^2}+\left\|\psi\right\|_{H^1}+\int_0^{t_0}\left\|f(w_1(\tau))\right\|_{H^1}d\tau)\\&\le e^{c_1t_0}(\left\|\varphi\right\|_{H^2}+\left\|\psi\right\|_{H^1}+c\int_0^{t_0}(\left\|w_1(\tau)\right\|_{H^\sigma}+\left\|f(0)\right\|_{H^1})d\tau)\\&\le e^{c_1t_0}(\left\|\varphi\right\|_{H^2}+\left\|\psi\right\|_{H^1}+ct_0(M+\left\|f(0)\right\|_{L^2}))\end{aligned}$$

$$\left\|v_2\right\|_{2,2}\le e^{c_1t_0}(\left\|\varphi\right\|_{H^2}+\left\|\psi\right\|_{H^1}+ct_0(M+\left\|f(0)\right\|_{L^2})).$$

Thus for $t_0>0$ small enough, G is contract.

From process above, we have proved the existence of solution for the problem (9). Since G is contract, then the solution is unique.

5 Global existence of Solutions for $n = 1$

In this section, we establish the global existence and uniqueness of the solution $(u, v) \in X_\infty \times Y_\infty$ of (9) in the case of $n = 1$ and $g(u, v) = -\gamma v + f(u)$. Here we suppose that

$$f(x) \in C_0^2(\mathbf{R}), \quad \sigma = \frac{5}{4}. \tag{37}$$

Observe that, for $n = 1$, $\sigma = \frac{5}{4}$ can simultaneously satisfy the condition (10) and (11). So from the result of Theorem 4.1, the problem (9) has a unique local solution $(u, v) \in X_{t_0} \times Y_{t_0}$ for some $t_0 > 0$ small enough.

Actually we can obtain following more strong result:

Theorem 5.1. *If $n = 1$, $g(u, v) = -\gamma v + f(u)$ and σ and f satisfy the condition (37), then for each initial data $u_0 \in H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$ and $u_0 \geq 0$, $\varphi \in H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\}$ and $\psi \in H^1(\Omega)$, the problem (9) has a unique global solution $(u, v) \in X_\infty \times Y_\infty$.*

If $u_0 \geq 0$, then from the first equation of (9), we can deduce that the local solution (u, v) satisfies

$$\|u(t, \cdot)\|_{L^1} = \|u_0\|_{L^1} \tag{38}$$

Next, we have

Lemma 5.2. *Let $s \leq 2$, the local solution $(u, v) \in X_{t_0} \times Y_{t_0}$ of (9), for $g(u, v) = -\gamma v + f(u)$, satisfies*

$$\|v(t, \cdot)\|_{H^s} \leq e^{ct_0}(c_0 + \int_0^{t_0} \|f(u(\tau, \cdot))\|_{H^{s-1}} d\tau), \quad 0 \leq t \leq t_0, \tag{39}$$

where $c_0 = \|\varphi\|_{H^2} + \|\psi\|_{H^1}$ and c is independent of t_0.

Proof: For $U = (v, w)$ and $F(U) = (0, (1 - \gamma)v + f(u))$, in terms of (21), we know that

$$U = T(t)U_0 + \int_0^t T(t - \tau)F(U(\tau))d\tau$$

where $w = v_t$ and (u, v) is the solution of (9).

By using (22), we know that

$$\begin{aligned}
&\|U(t)\|_{H^1 \times L^2} \leq \|T(t)U_0\|_{H^1 \times L^2} + \textstyle\int_0^t \|T(t - \tau)F(U(\tau))\|_{H^1 \times L^2} d\tau \\
&\leq \|U_0\|_{H^1 \times L^2} + \textstyle\int_0^t \|F(U(\tau))\|_{H^1 \times L^2} d\tau \\
&= \|\varphi\|_{H^1} + \|\psi\|_{L^2} + \textstyle\int_0^t \|(1 - \gamma)v + f(u)\|_{L^2} d\tau \\
&\leq \|\varphi\|_{H^1} + \|\psi\|_{L^2} + c\textstyle\int_0^t \|v\|_{L^2} d\tau + \int_0^t \|f(u)\|_{L^2} d\tau \\
&\leq \|\varphi\|_{H^1} + \|\psi\|_{L^2} + c\textstyle\int_0^t \|U(\tau)\|_{H^1 \times L^2} d\tau + \int_0^{t_0} \|f(u)\|_{L^2} d\tau, \quad 0 \leq t \leq t_0.
\end{aligned} \tag{40}$$

So the Gronwall's inequality indicates

$$\begin{aligned}
&\|U(t)\|_{H^1 \times L^2} \leq e^{ct}(\|\varphi\|_{H^1} + \|\psi\|_{L^2} + \textstyle\int_0^{t_0} \|f(u)\|_{L^2} d\tau) \\
&\leq e^{ct_0}(\|\varphi\|_{H^2} + \|\psi\|_{H^1} + \textstyle\int_0^{t_0} \|f(u)\|_{L^2} d\tau), \quad 0 \leq t \leq t_0.
\end{aligned} \tag{41}$$

Since $H^s \times H^{s-1} \subset H^1 \times L^2$ for $s > 1$, we denote $T(t) \mid_{H^s \times H^{s-1}}$ as the restriction of $T(t)$ on $H^s \times H^{s-1}$, the norm of $T(t) \mid_{H^s \times H^{s-1}}$ satisfies also the estimate (22). Thus, by similar process of (40) and (41), we can deduce that

$$\|U(t)\|_{H^s \times H^{s-1}} \leq e^{ct_0}(\|\varphi\|_{H^2} + \|\psi\|_{H^1} + \int_0^{t_0} \|f(u)\|_{H^{s-1}} d\tau), \quad 0 \leq t \leq t_0. \tag{42}$$

If $s < 1$, then $H^1 \times L^2 \subset H^s \times H^{s-1}$, we use Hahn-Banach theorem to get that the operator $T(t)$ can be continuously extended on $H^s \times H^{s-1}$ and the norm of $T(t)$ is invariable. Thus for $s < 1$, we have also that

$$\|U(t)\|_{H^s \times H^{s-1}} \leq e^{ct_0}(\|\varphi\|_{H^2} + \|\psi\|_{H^1} + \int_0^{t_0} \|f(u)\|_{H^{s-1}} d\tau), \quad 0 \leq t \leq t_0. \tag{43}$$

Lemma 5.2 can be deduced directly by (41), (42) and (43).

Proof of theorem 5.1:

For the unique local solution $(u, v) \in X_{t_0} \times Y_{t_0}$ of (9), if we take s=1/2 in (39), then

$$\|v(t,\cdot)\|^2_{H^{\frac{1}{2}}} \leq ce^{t_0}(c_0 + \int_0^{t_0} \|f(u(\tau,\cdot))\|^2_{H^{-\frac{1}{2}}} d\tau), \quad 0 \leq t \leq t_0. \tag{44}$$

Since $n = 1$, then from Sobolev imbedding theorems, we can deduce that $W^{0,1}(\Omega) \hookrightarrow H^{-\frac{1}{2}}(\Omega)$. Hence we have

$$\begin{aligned} &\|v(t,\cdot)\|^2_{H^{\frac{1}{2}}} \leq ce^{t_0}(c_0 + \textstyle\int_0^{t_0} \|f(u(\tau,\cdot))\|^2_{H^{-\frac{1}{2}}} d\tau) \\ &\leq ce^{t_0}(c_0 + \textstyle\int_0^{t_0} \|f(u(\tau,\cdot))\|^2_{L^1} d\tau) \\ &\leq ce^{t_0}(c_0 + \textstyle\int_0^{t_0} (M_1 \|u\|_{L^1} + \|f(0)\|_{L^1})^2 d\tau) \\ &= ce^{t_0}(c_0 + t_0(M_1 \|u_0\|_{L^1} + \|f(0)\|_{L^1})^2), \quad 0 \leq t \leq t_0, \end{aligned} \tag{45}$$

where $M_1 = \|f\|_{C^2}$.

On the other hand, for each $s \leq \sigma$ and $0 \leq \sigma_0 < 2$, we have that

$$\begin{aligned} &\|u(t,\cdot)\|_{H^s} \leq c\|u_0\|_{H^\sigma} + ct_0^{1-\frac{\sigma_0}{2}} \|\nabla(u\nabla v)\|_{H^{s-\sigma_0}} \\ &\leq c\|u_0\|_{H^\sigma} + ct_0^{1-\frac{\sigma_0}{2}} \|u\nabla v\|_{H^{s-\sigma_0+1}}, \quad 0 \leq t \leq t_0, \end{aligned} \tag{46}$$

Especially for $s = -\frac{1}{2} + \frac{1}{4}$ and $\sigma_0 = 2 - \frac{1}{8}$, we have

$$\|u(t,\cdot)\|_{H^{-\frac{1}{2}+\frac{1}{4}}} \leq c\|u_0\|_{H^\sigma} + ct_0^{\frac{1}{16}} \|u\nabla v\|_{H^{-1-\frac{1}{8}}}, \quad 0 \leq t \leq t_0. \tag{47}$$

By Sobolev imbedding theorems and (45),

$$\begin{aligned} &\|u\nabla v\|_{H^{-1-\frac{1}{8}}} \leq c\|u\|_{H^{-1-\frac{1}{8}}} \cdot \|\nabla v\|_{W^{-1-\frac{1}{8},\infty}} \\ &\leq c\|u\|_{H^{-1}} \cdot \|\nabla v\|_{H^{-\frac{1}{2}}} \\ &\leq c\|u\|_{L^1} \cdot \|v\|_{H^{\frac{1}{2}}} \\ &\leq c\|u_0\|_{L^1} \cdot e^{\frac{1}{2}t_0}(c_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}(M_1\|u_0\|_{L^1} + \|f(0)\|_{L^1})), \quad 0 \leq t \leq t_0. \end{aligned} \tag{48}$$

Thus

$$\begin{aligned} &\|u(t,\cdot)\|_{H^{-\frac{1}{4}}} \leq c\|u_0\|_{H^\sigma} + ct_0^{\frac{1}{16}} \|u\nabla v\|_{H^{-1-\frac{1}{8}}} \\ &\leq c\|u_0\|_{H^\sigma} + ct_0^{\frac{1}{16}} \|u_0\|_{L^1} \cdot e^{\frac{1}{2}t_0}(c_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}(M_1\|u_0\|_{L^1} + \|f(0)\|_{L^1})), \quad 0 \leq t \leq t_0. \end{aligned} \tag{49}$$

Take $s=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ in (39), then (39) and (49) give

$$\begin{aligned}&\|v(t,\cdot)\|^2_{H^{\frac{3}{4}}}\le ce^{t_0}(c_0+\textstyle\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{H^{\frac{3}{4}-1}}d\tau)\\&\le ce^{t_0}(c_0+t_0(M_1\sup_{0\le\tau\le t_0}\|u(\tau,\cdot)\|_{H^{-\frac{1}{4}}}+\|f(0)\|_{H^{-\frac{1}{4}}})^2)\\&\le ce^{t_0}(c_0+t_0(M_1(c\|u_0\|_{H^\sigma}+ct_0^{\frac{1}{16}}\|u_0\|_{L^1}\cdot e^{\frac{1}{2}t_0}(c_0^{\frac{1}{2}}\\&+t_0^{\frac{1}{2}}(M_1\|u_0\|_{L^1}+\|f(0)\|_{L^1}))+\|f(0)\|_{H^{-\frac{1}{4}}}))^2),\quad 0\le t\le t_0.\end{aligned}\tag{50}$$

Take $s=-\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=0$ and $\sigma_0=2-\frac{1}{8}$ in (46) again, we obtain that

$$\begin{aligned}&\|u(t,\cdot)\|_{L^2}\le c\|u_0\|_{H^\sigma}+ct_0^{1-\frac{\sigma_0}{2}}\|\nabla(u\nabla v)\|_{H^{-\sigma_0}}\\&\le c\|u_0\|_{H^\sigma}+ct_0^{\frac{1}{16}}\|u\nabla v\|_{H^{-\sigma_0+1}}\\&\le c\|u_0\|_{H^\sigma}+ct_0^{\frac{1}{16}}\|u\nabla v\|_{H^{-1+\frac{1}{8}}},\quad 0\le t\le t_0.\end{aligned}\tag{51}$$

Since we know that

$$\begin{aligned}&\|u\nabla v\|_{H^{-1+\frac{1}{8}}}\le c\|u\|_{H^{-1+\frac{1}{8}}}\cdot\|\nabla v\|_{W^{-1+\frac{1}{8},\infty}}\\&\le c\|u\|_{H^{-\frac{1}{4}}}\cdot\|\nabla v\|_{H^{-\frac{1}{2}+\frac{1}{4}}}\\&\le c\|u\|_{H^{-\frac{1}{4}}}\cdot\|v\|_{H^{\frac{3}{4}}},\quad 0\le t\le t_0.\end{aligned}\tag{52}$$

We can get that

$$\begin{aligned}&\|u(t,\cdot)\|_{L^2}\le c\|u_0\|_{H^\sigma}+ct_0^{1-\frac{\sigma_0}{2}}\|\nabla(u\nabla v)\|_{H^{-\sigma_0}}\\&\le c\|u_0\|_{H^\sigma}+ct_0^{\frac{1}{16}}\|u\nabla v\|_{H^{-1+\frac{1}{8}}}\\&\le c\|u_0\|_{H^\sigma}+ct_0^{\frac{1}{16}}\cdot\|u\|_{H^{-\frac{1}{4}}}\cdot\|v\|_{H^{\frac{3}{4}}},\quad 0\le t\le t_0.\end{aligned}\tag{53}$$

From (49) and (50), we have obtained that $\|u(t,\cdot)\|_{L^2}$ grows by a bounded manner in time.

Again we take $s=\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$ in (39), then (39) and (53) imply that $\|v(t,\cdot)\|_{H^1}$ grows also by a bounded manner in time.

Taking $s=-\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{1}{4}$ and $\sigma_0=2-\frac{1}{8}$ in (46) once more, since $\|v(t,\cdot)\|_{H^1}$ grows by a bounded manner in time, similar to which we have done in (51), (52) and (53), we can deduce that $\|u(t,\cdot)\|_{H^{\frac{1}{4}}}$ grows by a bounded manner in time.

Let us repeat processes above four times, we can prove that $\|u(t,\cdot)\|_{H^{\frac{5}{4}}}$ and $\|v(t,\cdot)\|_{H^2}$ grow by a bounded manner in time, as required.

References

[1] E. Keller, Mathematical aspects of bacterial chemotaxis, Antibiotics and Chemotherapy, 19, 79-93, (1974).

[2] E.F. Keller and L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol. 26, 399-415, (1970).

[3] E.F. Keller and L.A. Segel, Travelling bands of chemotactic bacteria: A theoretical analysis, J. Theor. Biol., 30, 235-248, (1971).

[4] M.E. Taylor, Partial Differential Equations III, Springer, New York, (1996).

[5] B. Zeidler, Nonlinear Functional Analysis and its Applications III, Springer, New York, (1985).
[6] Alain Haraux, Nonlinear Evolution Equations Global Behavior of Solutions, Springer-verlag