Regularization of the Cauchy Problem for the System of Elasticity Theory in R^m

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Abstract

In this paper we consider the regularization of the Cauchy problem for a system of second order differential equations with constant coefficients.

Key words: the Cauchy problem, Lame system, elliptic system, ill-posed problem, Carleman matrix, regularization, Laplace equation.

Introduction

As is well known, the Cauchy problem for elliptic equations is ill-posed, the solution of the problem is unique, but unstable (Hadamard's example). For ill-posed problems, one does not prove existence theorems, the existence is assumed a priori. Moreover, the solution is assumed to belong to a given subset of a function space, usually a compact one [4]. The uniqueness of the solution follows from the general Holmgren theorem [9].

Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ be points in R^m, D_ρ be a bounded simply connected domain in R^m whose boundary consists of a cone surface

$$\Sigma: \quad \alpha_1 = \tau y_m, \quad \alpha_1^2 = y_1^2 + \ldots + y_{m-1}^2, \quad \tau = tg\frac{\pi}{2\rho}, \quad y_m > 0, \quad \rho > 1$$

and a smooth surface S, lying in the cone.

In the domain D_ρ, consider the system of elasticity theory

$$\mu \Delta U(x) + (\lambda + \mu)grad \ \ divU(x) = 0;$$

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here $U = (U_1, \ldots, U_m)$ is the displacement vector, Δ is the Laplace operator, λ and μ are the Lame constants. For brevity, it is convenient to use matrix-valued notation. Let us introduce the matrix differential operator

$$A(\partial_x) = \|A_{ij}(\partial_x)\|_{m \times m},$$

where

$$A_{ij}(\partial_x) = \delta_{ij} \mu \Delta + (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then the elliptic system can be written in matrix form

$$A(\partial_x)U(x) = 0. \tag{1}$$

Statement of the problem. Assume the Cauchy data of a solution U are given on S,

$$U(y) = f(y), \quad y \in S,$$

$$T(\partial_y, n(y))U(y) = g(y), \quad y \in S, \tag{2}$$

where $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_m)$ are prescribed continuous vector functions on S, $T(\partial_y, n(y))$ is the strain operator, i.e.,

$$T(\partial_y, n(y)) = \|T_{ij}(\partial_y, n(y))\|_{m \times m} = \left\| \lambda n_i \frac{\partial}{\partial y_j} + \mu n_j \frac{\partial}{\partial y_i} + \mu \delta_{ij} \frac{\partial}{\partial n} \right\|_{m \times m}.$$

δ_{ij} is the Kronecker delta, and $n(y) = (n_1(y), \ldots, n_m(y))$ is the unit normal vector to the surface S at a point y.

It is required to determine the function $U(y)$ in D, i.e., find an analytic continuation of the solution of the system of equations in a domain from the values of f and g on a smooth part S of the boundary.

On establishing uniqueness in theoretical studies of ill - posed problems, one comes across important questions concerning the derivation of estimates of conditional stability and the construction of regularizing operators.

Suppose that, instead of $f(y)$ and $g(y)$, we are given their approximations $f_\delta(y)$ and $g_\delta(y)$ with accuracy δ, $0 < \delta < 1$ (in the metric of C) which do not necessarily belong to the class of solutions. In this paper, we construct a family of functions $U(x, f_\delta, g_\delta) = U_{\sigma\delta}(x)$ depending on the parameter σ and prove that under certain conditions and a special choice of the parameter $\sigma(\delta)$ the family $U_{\sigma\delta}(x)$ converges in the usual sense to the solution $U(x)$ of problem (1),(2), as $\delta \to 0$

Following A.N.Tikhonov, $U_{\sigma\delta}(x)$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of problem [13].

Using results of [4],[14] concerning the Cauchy problem for the Laplace equation, we succeeded in constructing a Carleman matrix in explicit form and on in constructing a

regularized solution of the Cauchy problem for the system (1). Since we refer to explicit formulas, it follows that the construction of the Carleman matrix in terms of elementary and special functions is of considerable interest. For $m = 2, 3$ the problem under consideration coincides with the Cauchy problem for the system of elasticity theory describing statics of an isotropic elastic medium. In these cases problem (1),(2) was studied for special classes of domains in [5], [6], [7], [8].

Further, the Cauchy problem for systems describing steady-state elastic vibrations, for systems of thermoelasticity, and for systems of Navier-Stokes was studied in [15], [2], [10], [3].

Earlier, is was proved in [12], [11] that the Carleman matrix exists in any Cauchy problem for solutions of elliptic systems whenever the Cauchy data are given on a boundary set of positive measure.

1.Construction of the matrix of fundamental solution for the system of elasticity of a special form

Definition 1. $\Gamma(y, x) = ||\Gamma_{ij}(y, x)||_{m \times m}$, is called the matrix of fundamental solutions of system (1), where

$$\Gamma_{ij}(y, x) = \frac{1}{2\mu(\lambda + 2\mu)}((\lambda + 3\mu)\delta_{ij}q(y, x) - (\lambda + \mu)(y_j - x_j)\frac{\partial}{\partial x_i}q(y, x)), \quad i, j = 2, ..., m,$$

$$q(y, x) = \begin{cases} \frac{1}{(2-m)\omega_m} \cdot \frac{1}{|y-x|^{m-2}}, & m > 2 \\ \frac{1}{2\pi}ln|y - x|, & m = 2, \end{cases}$$

and ω_m is the area of unit sphere in R^m.

The matrix $\Gamma(y, x)$ is symmetric and its columns and rows satisfy equation (1) at an arbitrary point $x \in R^m$, except $y = x$. Thus, we have

$$A(\partial_x)\Gamma(y, x) = 0, \quad y \neq x.$$

Developing Lavrent'ev's idea concerning the notion of Carleman function of the Cauchy problem for the Laplace equation [4], we introduce the following notion.

Definition 2. By a Carleman matrix of problem (1),(2) we mean an $(m \times m)$ matrix $\Pi(y, x, \sigma)$ satisfying the following two conditions:

$$1) \ \Pi(y, x, \sigma) = \Gamma(y, x) + G(y, x, \sigma),$$

where σ is a positive numerical parameter and, with respect to the variable y, the matrix $G(y, x, \sigma)$ satisfies system (1) everywhere in the domain D.

2) The relation holds

$$\int_{\partial D \backslash S} (|\Pi(y, x, \sigma)| + |T(\partial_y, n)\Pi(y, x, \sigma)|)ds_y \leq \varepsilon(\sigma),$$

where $\varepsilon(\sigma) \to 0$, as $\sigma \to \infty$, uniformly in x on compact subsets of D ; here and elsewhere $|\Pi|$ denotes the Euclidean norm of the matrix $\Pi = ||\Pi_{ij}||$, i.e., $|\Pi| = (\sum\limits_{i,j=1}^{m} \Pi_{ij}^2)^{\frac{1}{2}}$. In particular $|U| = (\sum\limits_{i=1}^{m} U_i^2)^{\frac{1}{2}}$ for a vector $U = (U_1, ..., U_m)$.

Definition 3. A vector function $U(y) = (U_1(y), ..., U_m(y))$ is said to be regular in D, if it is continuous together with its partial derivatives of second order in D and those of first order in $\overline{D} = D \bigcup \partial D$.

In the theory of partial differential equations, an important role is played by representations of solutions of these equations as functions of potential type. As an example of such representations, we show the formula of Somilian-Bettis [11] below.

Theorem 1. Any regular solution $U(x)$ of equation (1) in the domain D is represented by the formula

$$U(x) = \int\limits_{\partial D} (\Gamma(y, x)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Gamma(y, x)\}^* U(y)) ds_y, \quad x \in D. \qquad (3)$$

Here A^* is conjugate to A.

Suppose that a Carleman matrix $\Pi(y, x, \sigma)$ of the problem (1),(2) exists. Then for the regular functions $v(y)$ and $u(y)$ the following holds

$$\int\limits_{\partial D_\rho} [v(y)\{A(\partial_y)U(y)\} - \{A(\partial_y)v(y)\}^* U(y)] dy =$$

$$= \int\limits_{\partial D_\rho} [v(y)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)v(y)\}U(y)] ds_y.$$

Substituting $v(y) = G(y, x, \sigma)$ and $u(y) = U(y)$, a regular solution system (1), into the above equality, we get

$$\int\limits_{\partial D_\rho} [G(y, x, \sigma)\{A(\partial_y)U(y)\} - \{A(\partial_y)G(y, x, \sigma)\}^* U(y)] dy = 0. \qquad (4)$$

Adding (3) and (4) gives the following theorem.

Theorem 2. Any regular solution $U(x)$ of equation (1) in the domain D_ρ is represented by the formula

$$U(x) = \int\limits_{\partial D_\rho} (\Pi(y, x, \sigma)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \sigma)\}^* U(y)) ds_y, \quad x \in D_\rho, \qquad (5)$$

where $\Pi(y, x, \sigma)$ is a Carleman matrix.

Suppose that $K(\omega)$, $\omega = u + iv$ (u, v are real), is an entire function taking real values on the real axis and satisfying the conditions

$$K(u) \neq 0, \ \sup\limits_{v \geq 1} |v^p K^{(p)}(\omega)| = M(p, u) < \infty, \ p = 0, ..., m, \ u \in R^1.$$

Let

$$s = \alpha^2 = (y_1 - x_1)^2 + \ldots + (y_{m-1} - x_{m-1})^2.$$

For $\alpha > 0$, we define the function $\Phi(y, x)$ by the following relations:

if $m = 2$, then

$$-2\pi K(x_2)\Phi(y, x) = \int_0^\infty Im[\frac{K(i\sqrt{u^2 + \alpha^2} + y_2)}{i\sqrt{u^2 + \alpha^2} + y_2 - x_2}]\frac{udu}{\sqrt{u^2 + \alpha^2}}, \tag{6}$$

if $m = 2n + 1, \ n \geq 1$, then

$$C_m K(x_m)\Phi(y, x) = \frac{\partial^{n-1}}{\partial s^{n-1}} \int_0^\infty Im[\frac{K(i\sqrt{u^2 + \alpha^2} + y_m)}{i\sqrt{u^2 + \alpha^2} + y_m - x_m}]\frac{du}{\sqrt{u^2 + \alpha^2}}; \tag{7}$$

where $C_m = (-1)^{n-1} \cdot 2^{-n}(m-2)\pi\omega_m(2n-1)!$,

if $m = 2n, \ n \geq 2$, then

$$C_m K(x_m)\Phi(y, x) = \frac{\partial^{n-2}}{\partial s^{n-2}}Im\frac{K(\alpha i + y_m)}{\alpha(\alpha + y_m - x_m)}, \tag{8}$$

where $C_m = (-1)^{n-1}(n-1)!(m-2)\omega_m$.

Lemma 1. The function $\Phi(y, x)$ can be expressed as

$$\Phi(y, x) = \frac{1}{2\pi}ln\frac{1}{r} + g_2(y, x), \ \ m = 2, \ \ r = |y - x|,$$

$$\Phi(y, x) = \frac{r^{2-m}}{\omega_m(m-2)} + g_m(y, x), \ \ m \geq 3, \ \ r = |y - x|,$$

where $g_m(y, x), \ m \geq 2$ is a function defined for all values of y, x and harmonic in the variable y in all of R^m.

With the help of function $\Phi(y, x)$ we construct a matrix:

$$\Pi(y, x) = ||\Pi_{ij}(y, x)||_{m \times m} = \left\|\frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)}\delta_{ij}\Phi(y, x) - \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)}(y_j - x_j)\frac{\partial}{\partial y_i}\Phi(y, x)\right\|_{m \times m},$$

$$i, j = 1, 2, ..., m. \tag{9}$$

2. The solution of problems (1), (2) in domain D_ρ

I. Let $x_0 = (0, \ldots, 0, x_m) \in D_\rho$. We adopt the notation

$$\beta = \tau y_m - \alpha_0, \ \ \gamma = \tau x_m - \alpha_0, \ \ \alpha_0^2 = x_1^2 + \ldots + x_{m-1}^2, \ \ r = |x - y|,$$

$$s = \alpha^2 = (y_1 - x_1)^2 + \ldots + (y_{m-1} - x_{m-1})^2, \ \ w = i\tau\sqrt{u^2 + \alpha^2} + \beta, \ \ w_0 = i\tau\alpha + \beta.$$

We now construct Carleman's matrix for the problem (1), (2) for the domain D_ρ. The Carleman matrix is explicitly expressed by Mittag-Löffler's a entire function. It is defined by series [1]

$$E_\rho(w) = \sum_{n=0}^\infty \frac{w^n}{\Gamma\left(1 + \frac{n}{\rho}\right)}, \ \ \rho > 0, \ \ E_1(w) = \exp w,$$

where $\Gamma(\cdot)$ is the Euler function.

Denote by $\gamma = \gamma(1,\theta)$, $0 < \theta < \frac{\pi}{\rho}$, $\rho > 1$ the contour in the complex plane w, running in the direction nondecreasing $arg w$ and consisting of the following part's.

1)ray $\quad arg w = -\theta, \quad |w| \geq 1$,

2)arc $\quad -\theta \leq arg w \leq \theta \quad$ of the circle $\quad |w| = 1$,

3)ray $\quad arg w = \theta, \quad |w| \geq 1$.

The contour γ divides complex plane into two parts: D^- and D^+ lying on the left and on the right from γ, respectively. Suppose that $\frac{\pi}{2\rho} < \theta < \frac{\pi}{\rho}$, $\rho > 1$. Then the formula holds

$$E_\rho(w) = \exp w^\rho + \Psi_\rho(w), \quad w \in D^+$$

$$E_\rho(w) = \Psi_\rho(w), \quad E'_\rho(w) = \Psi'_\rho(w), \quad w \in D^-, \tag{10}$$

where

$$\Psi_\rho(w) = \frac{\rho}{2\pi i} \int\limits_\gamma \frac{\exp \zeta^\rho}{\zeta - w} d\zeta, \qquad \Psi'_\rho(w) = \frac{\rho}{2\pi i} \int\limits_\gamma \frac{\exp \zeta^\rho}{(\zeta - w)^2} d\zeta. \tag{11}$$

$$Re\Psi_\rho(w) = \frac{\Psi_\rho(w) + \Psi_\rho(\overline{w})}{2} = \frac{\rho}{2\pi i} \int\limits_\gamma \frac{\exp \zeta^\rho(\zeta - Rew)}{(\zeta - w)(\zeta - \overline{w})} d\zeta,$$

$$Im\Psi_\rho(w) = \frac{\Psi_\rho(w) - \Psi_\rho(\overline{w})}{2i} = \frac{\rho Im w}{2\pi i} \int\limits_\gamma \frac{\exp \zeta^\rho}{(\zeta - w)(\zeta - \overline{w})} d\zeta, \tag{12}$$

$$\frac{Im\Psi'_\rho(w)}{Imw} = \frac{\rho}{2\pi i} \int\limits_\gamma \frac{2\exp \zeta^\rho(\zeta - Rew)}{(\zeta - w)^2(\zeta - \overline{w})^2} d\zeta.$$

In what follows, we take $\theta = \frac{\pi}{2\rho} + \frac{\varepsilon_2}{2}$, $\rho > 1$, $\varepsilon_2 > 0$. It is clear that if $\frac{\pi}{2\rho} + \varepsilon_2 \leq |arg w| \leq \pi$, then $w \in D^-$ and $E_\rho(w) = \Psi_\rho(w)$.

Set

$$E_{k,q}(w) = \frac{\rho}{2\pi i} \int\limits_\gamma \frac{\zeta^q \exp \zeta^\rho}{(\zeta - w)^k(\zeta - \overline{w})^k} d\zeta, \quad k = 1, 2, \ldots, \quad q = 0, 1, 2, \ldots.$$

If $\frac{\pi}{2\rho} + \frac{\varepsilon_2}{2} \leq |arg w| \leq \pi$, then the inequalities are valid

$$|E_\rho(w)| \leq \frac{M_1}{1 + |w|}, \quad |E'_\rho(w)| \leq \frac{M_2}{1 + |w|^2},$$

$$|E_{k,q}(w)| \leq \frac{M_3}{1 + |w|^{2k}}, \quad k = 1, 2, \ldots, \tag{13}$$

where M_1, M_2, M_3 are constants.

Suppose that in formula (10) $\theta = \frac{\pi}{2\rho} + \frac{\varepsilon_2}{2} < \frac{\pi}{\rho}$, $\rho > 1$. Then $E_\rho(w) = \Psi_\rho(w), \cos \rho\theta < 0$ and

$$\int\limits_\gamma |\zeta|^q \exp(\cos \rho\theta |\zeta|^q) |d\zeta| < \infty, \quad q = 0, 1, 2, \ldots. \tag{14}$$

In this case for sufficiently large $|w|$ $(w \in D^+, \overline{w} \in D^-)$, we have

$$\min_{\zeta \in \gamma} |\zeta - w| = |w| \sin \frac{\varepsilon_2}{2}, \quad \min_{\zeta \in \gamma} |\zeta - \overline{w}| = |w| \sin \frac{\varepsilon_2}{2}. \tag{15}$$

Now from (10) and

$$\frac{1}{\zeta - w} = -\frac{1}{w} + \frac{\zeta}{w(\zeta - \overline{w})},$$

$$\frac{1}{\zeta - \overline{w}} = -\frac{1}{\overline{w}} + \frac{\zeta}{\overline{w}(\zeta - \overline{w})}, \tag{16}$$

for large $|w|$ we obtain

$$\left| E_\rho(w) - \Gamma^{-1}\left(1 - \frac{1}{\rho}\right)\frac{1}{w} \right| \leq \frac{\rho}{2\pi \sin \frac{\varepsilon_2}{2}} \frac{1}{|w|^2}.$$

$$\int_\gamma |\zeta| \exp\left[\cos \rho\theta|\zeta|^\rho\right] |d\zeta| \leq \frac{const}{|w|^2},$$

$$\Gamma^{-1}\left(1 - \frac{1}{\rho}\right) = \frac{\rho}{2\pi i} \int_\gamma \exp\left(\zeta^\rho\right) d\zeta.$$

From this it follows that

$$|E_\rho(w)| \leq \frac{M_1}{1 + |w|}.$$

From (11),(15) and

$$\frac{1}{(\zeta - w)^2} = \frac{1}{w^2} - \frac{2\zeta}{w^2(\zeta - w)} + \frac{\zeta^2}{w^2(\zeta - w)^2}$$

for large $|w|$, we obtain

$$\left| E_\rho'(w) - \Gamma^{-1}\left(1 - \frac{1}{\rho}\right)\frac{1}{w^2} \right| \leq \frac{const}{|w|^3}$$

or

$$|E_\rho'(w)| = \frac{M_2}{1 + |w|^2}.$$

For $k = 1, 2, \ldots$ we have from (16)

$$\frac{1}{(\zeta - w)^k(\zeta - \overline{w})^k} = \left[\frac{(-1)^k}{w^k} + \ldots + \frac{\zeta^k}{w^k(\zeta - w)^k}\right]\left[\frac{(-1)^k}{\overline{w}^k} + \ldots + \frac{\zeta^k}{\overline{w}^k(\zeta - \overline{w})^k}\right] =$$

$$= \frac{1}{|w|^{2k}} - \frac{k}{|w|^{2k+1}|\zeta - w|} + \ldots.$$

From this for large $|w|$, (14) and (15) we get

$$\left| E_{k,q}(w) - \Gamma^{-1}\left(1 - \frac{1}{\rho}\right)\frac{1}{|w|^{2k}} \right| \leq \frac{const}{|w|^{2k+1}}$$

or

$$|E_{k,q}'(w)| = \frac{M_3}{1 + |w|^{2k}}, \quad k = 1, 2, \ldots.$$

Therefore, since

$$(\zeta - w)(\zeta - \overline{w}) = \zeta^2 - 2\zeta(y_m - x_m) + u^2 + \alpha^2 + (y_m - x_m)^2, \quad \alpha^2 = s,$$

then

$$\frac{\partial^{n-1}}{\partial s^{n-1}} \frac{1}{(\zeta - w)(\zeta - \overline{w})} = \frac{(-1)^{n-1}(n-1)!}{(\zeta - w)^n(\zeta - \overline{w})^n}.$$

Now from (11) we obtain

$$\frac{d^{n-1}}{ds^{n-1}} Re E_\rho(w) = \frac{(-1)^{n-1}(n-1)!\rho}{2\pi i} \int_\gamma \frac{(\zeta - (y_m - x_m))\exp \zeta^\rho}{(\zeta - w)^n(\zeta - \overline{w})^n} d\zeta,$$

$$\frac{d^{n-1}}{ds^{n-1}} \frac{Im E_\rho(w)}{\sqrt{u^2 + \alpha^2}} = \frac{(-1)^{n-1}(n-1)!\rho}{\pi i} \int_\gamma \frac{\exp \zeta^\rho}{(\zeta - w)^n(\zeta - \overline{w})^n} d\zeta,$$

Then from (12) we have

$$\left| \frac{d^{n-1}}{ds^{n-1}} Re E_\rho(w) \right| \le \frac{const \cdot r}{1 + |w|^2}$$

$$\left| \frac{d^{n-1}}{ds^{n-1}} \frac{Im E_\rho(w)}{\sqrt{u^2 + \alpha^2}} \right| \le \frac{const \cdot r}{1 + |w|^2}.$$

Now for $\sigma > 0$, we set in formulas (6)-(9)

$$K(w) = E_\rho(\sigma^{\frac{1}{\rho}} w), \quad K(x_m) = E_\rho(\sigma^{\frac{1}{\rho}} \gamma). \tag{17}$$

Then, for $\rho > 1$ we obtain

$$\Phi(y, x) = \Phi_\sigma(y, x) = \frac{\varphi_\sigma(y, x)}{c_m E_\rho(\sigma^{\frac{1}{\rho}} \gamma)}, \quad y \ne x,$$

where $\varphi_\sigma(y, x)$ is defined as follows:

if $m = 2$, then

$$\varphi_\sigma(y, x) = \int_0^\infty Im \frac{E_\rho(\sigma^{\frac{1}{\rho}} w)}{i\sqrt{u^2 + \alpha^2} + y_2 - x_2} \frac{u du}{\sqrt{u^2 + \alpha^2}};$$

if $m = 2n + 1$, $n \ge 1$, then

$$\varphi_\sigma(y, x) = \frac{d^{n-1}}{ds^{n-1}} \int_0^\infty Im \frac{E_\rho(\sigma^{\frac{1}{\rho}} w)}{i\sqrt{u^2 + \alpha^2} + y_m - x_m} \frac{u du}{\sqrt{u^2 + \alpha^2}}, y \ne x;$$

if $m = 2n$, $n \ge 2$, then

$$\varphi_\sigma(y, x) = \frac{d^{n-2}}{ds^{n-2}} Im \frac{E_\rho\left(\sigma^{\frac{1}{\rho}} w\right)}{\alpha(i\alpha + y_m - x_m)}, y \ne x.$$

We now define the matrix $\Pi(y, x, \sigma)$ by formula (9) for $\Phi(y, x) = \Phi_\sigma(y, x)$.

In the work [14] there is proved

Lemma 2. The function $\Phi_\sigma(y, x)$ can be expressed as

$$\Phi_\sigma(y, x) = \frac{1}{2\pi} ln \frac{1}{r} + g_2(y, x, \sigma), \quad m = 2, \quad r = |y - x|,$$

$$\Phi_\sigma(y, x) = \frac{r^{2-m}}{\omega_m(m - 2)} + g_m(y, x, \sigma), \quad m \geq 3, \quad r = |y - x|,$$

where $g_m(y, x, \sigma)$, $m \geq 2$ is a function defined for all y, x and harmonic in the variable y in all of R^m.

Using Lemma 2, we obtain.

Theorem 3. The matrix $\Pi(y, x, \sigma)$ given by (7)-(9) is a Carleman matrix for problem (1), (2).

We first consider some properties of function $\Phi_\sigma(y, x)$

I. Let $m = 2n + 1$, $n \geq 1$, $x \in D_\rho$, $y \neq x, \sigma \geq \sigma_0 > 0$, then

1) for $\beta \leq \alpha$ the following inequality holds:

$$|\Phi_\sigma(y, x)| \leq C_1(\rho) \frac{\sigma^{m-2}}{r^{m-2}} \exp(-\sigma \gamma^\rho),$$

$$\left| \frac{\partial \Phi_\sigma}{\partial n}(y, x) \right| \leq C_2(\rho) \frac{\sigma^m}{r^{m-1}} \exp(-\sigma \gamma^\rho), \quad y \in \partial D_\rho,$$

$$\left| \frac{\partial}{\partial x_i} \frac{\partial \Phi_\sigma}{\partial n}(y, x) \right| \leq C_3(\rho) \frac{\sigma^{m+2}}{r^m} \exp(-\sigma \gamma^\rho), \quad i = 1, ..., m, \tag{18}$$

2) for $\beta > \alpha$ the following inequalities hold:

$$|\Phi_\sigma(y, x)| \leq C_4(\rho) \frac{\sigma^{m-2}}{r^{m-2}} \exp(-\sigma \gamma^\rho + \sigma Re\omega_0^\rho),$$

$$\left| \frac{\partial \Phi_\sigma}{\partial n}(y, x) \right| \leq C_5(\rho) \frac{\sigma^m}{r^{m-1}} \exp(-\sigma \gamma^\rho + \sigma Re\omega_0^\rho), \quad y \in \partial D_\rho,$$

$$\left| \frac{\partial}{\partial x_i} \frac{\partial \Phi_\sigma}{\partial n}(y, x) \right| \leq C_6(\rho) \frac{\sigma^{m+2}}{r^m} \exp(-\sigma \gamma^\rho + \sigma Re\omega_0^\rho), \quad i = 1, ..., m. \tag{19}$$

II. Let $m = 2n$, $n \geq 2$, $x \in D_\rho$, $x \neq y$, $\sigma \geq \sigma_0 > 0$, then

1) for $\beta \leq \alpha$ the following inequalities hold:

$$|\Phi_\sigma(y, x)| \leq \widetilde{C}_1(\rho) \frac{\sigma^{m-3}}{r^{m-2}} \exp(-\sigma \gamma^\rho),$$

$$\left| \frac{\partial \Phi_\sigma}{\partial n}(y, x) \right| \leq \widetilde{C}_2(\rho) \frac{\sigma^m}{r^{m-1}} \exp(-\sigma \gamma^\rho), \quad y \in \partial D_\rho,$$

$$\left| \frac{\partial}{\partial x_i} \frac{\partial \Phi_\sigma}{\partial n}(y, x) \right| \leq \widetilde{C}_3(\rho) \frac{\sigma^{m+2}}{r^m} \exp(-\sigma \gamma^\rho), \quad y \in \partial D_\rho, \quad i = 1, ..., m, \tag{20}$$

2) for $\beta > \alpha$ the following inequalities hold:

$$|\Phi_\sigma(y, x)| \leq \widetilde{C}_4(\rho) \frac{\sigma^{m-2}}{r^{m-2}} \exp(-\sigma \gamma^\rho + \sigma Re\omega_0^\rho),$$

$$\left|\frac{\partial \Phi_\sigma}{\partial n}(y,x)\right| \leq \widetilde{C}_5(\rho)\frac{\sigma^m}{r^{m-1}}\exp(-\sigma\gamma^\rho + \sigma Re\omega_0^\rho), \ \ y \in \partial D_\rho,$$

$$\left|\frac{\partial}{\partial x_i}\frac{\partial \Phi_\sigma}{\partial n}(y,x)\right| \leq \widetilde{C}_6(\rho)\frac{\sigma^{m+2}}{r^m}\exp(-\sigma\gamma^\rho + \sigma Re\omega_0^\rho), \ \ y \in \partial D_\rho, \ \ i=1,...,m. \qquad (21)$$

III. Let $m=2, \ x \in D_\rho, \ x \neq y, \ \sigma \geq \sigma_0 > 0$, then

1) if $\beta \leq \alpha$, then

$$|\Phi_\sigma(y,x)| \leq C_7(\rho)E^{-1}(\sigma^{\frac{1}{\rho}}\gamma)ln\frac{1+r^2}{r^2},$$

$$\left|\frac{\partial \Phi_\sigma}{\partial y_i}(y,x)\right| \leq C_8(\rho)\frac{E_\rho^{-1}(\sigma^{\frac{1}{\rho}}\gamma)}{r}, \qquad (22)$$

2) if $\beta > \alpha$, then

$$|\Phi_\sigma(y,x)| \leq \widetilde{C}_7(\rho)E^{-1}(\sigma^{\frac{1}{\rho}}\gamma)(ln\frac{1+r^2}{r^2})\exp(\sigma Re\omega_0^\rho),$$

$$\left|\frac{\partial \Phi_\sigma}{\partial y_i}(y,x)\right| \leq \widetilde{C}_8(\rho)E_\rho^{-1}(\sigma^{\frac{1}{\rho}}\gamma)\frac{1}{2}\exp(\sigma Re\omega_0^\rho). \qquad (23)$$

Here all coefficients $C_i(\rho)$ and $\widetilde{C}_i(\rho), \quad i=1,\dots,8$, depend on ρ.

Proof Theorem 3. From the definition of $\Pi(y,x,\sigma)$ and Lemma 1, we have

$$\Pi(y,x,\sigma) = \Gamma(y,x) + G(y,x,\sigma),$$

where

$$G(y,x,\sigma) = ||G_{kj}(y,x,\sigma)||_{m\times m} =$$

$$= \left\| \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)}\delta_{kj}g_m(y,x,\sigma) - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}(y_j - x_j)\frac{\partial}{\partial y_i}g_m(y,x,\sigma)\right\|_{m\times m}.$$

Prove that $A(\partial_y)G(y,x,\sigma) = 0$. Indeed, since $\Delta_y g_m(y,x,\sigma) = 0, \ \ \Delta_y = \sum\limits_{k=1}^{m}\frac{\partial^2}{\partial y_k^2}$ and for the jth column $G^j(y,x,\sigma)$:

$$divG^j(y,x,\sigma) = \frac{1}{2\mu(\lambda+2\mu)} \cdot \frac{\partial}{\partial y_j}g_m(y,x,\sigma),$$

then for the kth components of $A(\partial_y)G^j(y,x,\sigma)$ we obtain

$$\sum_{i=1}^{m}A(\partial_y)_{ki}G_{ij}(y,x,\sigma) = \mu\Delta_y[\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\cdot\delta_{kj}g_m(y,x,\sigma) - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}(y_j - x_j)\frac{\partial}{\partial y_k}g_m(y,x,\sigma)] +$$

$$+(\lambda+\mu)\frac{\partial}{\partial y_k}divG^j(y,x,\sigma) = -\frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\frac{\partial^2}{\partial y_j^2}g_m(y,x,\sigma) + \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\frac{\partial^2}{\partial y_j^2}g_m(y,x,\sigma) = 0$$

Therefore, each column of the matrix $G(y,x,\sigma)$ satisfies to system (1) in the variable y everywhere on R^m.

The second condition of Carleman's matrix follows from inequalities (18)-(23). The theorem proved.

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For fixed $x \in D_\rho$ we denote by S^* the part of S, where $\beta \geq \alpha$. It $x = x_0 = (0, \ldots, 0, x_m) \in D_\rho$, then $S = S^*$. In the point $(0, \ldots, 0) \in D_\rho$, suppose that

$$\frac{\partial U}{\partial n}(0) = \frac{\partial U}{\partial y_m}(0), \quad \frac{\partial \Phi_\sigma(0, x)}{\partial n} = \frac{\partial \Phi_\sigma(0, x)}{\partial y_m}.$$

Let

$$U_\sigma(y) = \int\limits_{S^*} [\Pi(y, x, \sigma)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \sigma)\}^* U(y)] ds_y, \quad x \in D_\rho. \tag{24}$$

Theorem 4. Let $U(x)$ be a regular solution of system (1) in D_ρ, such that

$$|U(y)| + |T(\partial_y, n)U(y)| \leq M, \quad y \in \Sigma. \tag{25}$$

Then,

1) if $m = 2n + 1$, $n \geq 1$, and for the $x \in D_\rho$, $\sigma \geq \sigma_0 > 0$, the following estimate is valid:

$$|U(x) - U_\sigma(x)| \leq MC_1(x)\sigma^{m+1} \exp(-\sigma\gamma^\rho).$$

2) In case $m = 2n$, $n \geq 1$, $x \in D_\rho$, $\sigma \geq \sigma_0 > 0$, the following estimate is valid

$$|U(x) - U_\sigma(x)| \leq MC_2(x)\sigma^m \exp(-\sigma\gamma^\rho),$$

where

$$C_k(x) = C_k(\rho) \int\limits_{\partial D_\rho} \frac{ds_y}{r^m}, \quad k = 1, 2,$$

$C_k(\rho)$ is a constant depending on ρ.

Proof. From formula (5)

$$U(x) = \int\limits_{S^*} [\Pi(y, x, \sigma)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \sigma)\}^* U(y)] ds_y +$$

$$+ \int\limits_{\partial D_\rho \backslash S^*} [\Pi(y, x, \sigma)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \sigma)\}^* U(y)] ds_y, \quad x \in D_\rho,$$

therefore, (24) implies

$$|U(x) - U_\sigma(x)| \leq \int\limits_{\partial D_\rho \backslash S^*} [\Pi(y, x, \sigma)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \sigma)\}^* U(y)] ds_y \leq$$

$$\leq \int\limits_{\partial D_\rho \backslash S^*} [|\Pi(y, x, \sigma)| + |T(\partial_y, n)\Pi(y, x, \sigma)|] [|T(\partial_y, n)\Pi(y, x, \sigma)| + |U(y)|] ds_y.$$

Therefore for $\beta \leq \alpha$ we obtain from inequalities (18)-(23), and condition (25) for $m = 2n + 1$, $n \geq 1$

$$|U(x) - U_\sigma(x)| \leq MC_1(\rho)\sigma^{m+1} \exp(-\sigma\gamma^\rho) \int\limits_{\partial D_\rho} \frac{ds_y}{r^m},$$

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and for $m = 2n$, $n \geq 1$ we obtain

$$|U(x) - U_\sigma(x)| \leq MC_2(\rho)\sigma^m \exp(-\sigma\gamma^\rho) \int\limits_{\partial D_\rho} \frac{ds_y}{r^m}.$$

The theorem proved.

Now we write out a result that allows us to calculate $U(x)$ approximately if, instead of $U(y)$ and $T(\partial_y, n)U(y)$ their continuous approximations $f_\delta(y)$ and $g_\delta(y)$ are given on the surface S:

$$\max_S |U(y) - f_\delta(y)| + \max_S |T(\partial_y, n)U(y) - g_\delta(y)| \leq \delta, \ \ 0 < \delta < 1. \tag{26}$$

We define a function $U_{\sigma\delta}(x)$ by setting

$$U_{\sigma\delta}(x) = \int\limits_{S^*} [\Pi(y, x, \sigma)g_\delta(y) - \{T(\partial_y, n)\Pi(y, x, \sigma)\}^* f_\delta(y)]ds_y, \ \ x \in D_\rho, \tag{27}$$

where $\sigma = \frac{1}{R^\rho}\ln\frac{M}{\delta}$, $R^\rho = \max\limits_{y \in S} Re\omega_0^\rho$.

Then the following theorem holds.

Theorem 5. Let $U(x)$ be a regular solution of system (1) in D_ρ such that

$$|U(y)| + |T(\partial_y, n)U(y)| \leq M, \ \ y \in \partial D_\rho.$$

Then,

1) if $m = 2n + 1$, $n \geq 1$, the following estimate is valid:

$$|U(x) - U_{\sigma\delta}(x)| \leq C_1(x)\delta^{(\frac{\gamma}{R})^\rho}\left(\ln\frac{M}{\delta}\right)^{m+1},$$

2) if $m = 2n$, $n \geq 1$, the following estimate is valid:

$$|U(x) - U_{\sigma\delta}(x)| \leq C_2(x)\delta^{(\frac{\gamma}{R})^\rho}\left(\ln\frac{M}{\delta}\right)^{m},$$

where

$$C_k(x) = C_k(\rho) \int\limits_{\partial D_\rho} \frac{ds_y}{r^m}, \ \ k = 1, 2.$$

Proof. From formula (5) and (27) we have

$$U(x) - U_{\sigma\delta}(x) = \int\limits_{\partial D_\rho \backslash S^*} [\Pi(y, x, \sigma)\{T(\partial_y, n)U(y)\} - \{T(\partial_y, n)\Pi(y, x, \sigma)\}^* U(y)]ds_y +$$

$$+ \int\limits_{S^*} [\Pi(y, x, \sigma)\{T(\partial_y, n)U(y) - g_\delta(y)\} + \{T(\partial_y, n)\Pi(y, x, \sigma)\}^*(U(y) - f_\delta(y))]ds_y = I_1 + I_2.$$

By Theorem 4 for $m = 2n + 1$, $n \geq 1$,

$$|I_1| = MC_1(\rho)\sigma^{m+1} \exp(-\sigma\gamma^\rho) \int\limits_{\partial D_\rho} \frac{ds_y}{r^m},$$

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and for the $m = 2n,\ \ n \geq 1$

$$|I_1| = MC_2(\rho)\sigma^m \exp(-\sigma\gamma^\rho) \int\limits_{\partial D_\rho} \frac{ds_y}{r^m}.$$

Now consider $|I_2|$:

$$|I_2| = \int\limits_{S^*} \left(|\Pi(y,x,\sigma)| + |T(\partial_y, n)\Pi(y,x,\sigma)|\right)\left(|T(\partial_y, n)U(y) - g_\delta(y)| + |U(y) - f_\delta(y)|\right)ds_y.$$

By Lemma 2 and condition (28) we obtain for $m = 2n + 1,\ \ n \geq 1$

$$|I_2| = \widetilde{C}_1(\rho)\sigma^{m+1}\delta \exp(-\sigma\gamma^\rho + \sigma Rew_0^\rho) \int\limits_{\partial D_\rho} \frac{ds_y}{r^m}$$

and for $m = 2n,\ \ n \geq 1$,

$$|I_2| = \widetilde{C}_2(\rho)\sigma^m \delta \exp(-\sigma\gamma^\rho + \sigma Rew_0^\rho) \int\limits_{\partial D_\rho} \frac{ds_y}{r^m}.$$

Therefore, from

$$\sigma = \frac{1}{R^\rho} ln\frac{M}{\delta}, \ \ R^\rho = \max_{y \in S} Re\omega_0^\rho.$$

we obtain the desired result.

Corollary 1. The limit relation

$$\lim_{\sigma \to \infty} U_\sigma(x) = U(x), \ \ \lim_{\delta \to 0} U_{\sigma\delta}(x) = U(x)$$

hold uniformly on any compact set from D_ρ.

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