# The forced KdV equation and passage through resonance

S.G. Glebov*, O.M. Kiselev†

**Abstract**

We construct a special asymptotic solution for the forced KdV equation. In the frame of the shallow water model this kind of the external driving force is related to the atmospheric disturbance. The perturbation slowly passes through a resonance and it leads to the solution exchange. The detailed asymptotic description of the process is presented.

# Contents



*Ufa State Petroleum Technical University, e-mail: glebskie@gmail.com
†Institute of Mathematics of USC RAS, e-mail: ok@ufanet.ru

# Introduction

Originally the KdV equation was derived as the fundamental equation governing the propagation of waves in the shallow water [1]. The same equation appeared in the theory of nonlinear ion acoustic waves in cold plasmas [2], [3]. Later it became clear that the KdV equation is a basic model for the description of internal waves in stratified fluids, propagating in waveguides which exist naturally in the ocean, or can be created in a laboratory [4],[5],[6]. The reduction of the KdV equation was justified ( see [7]).

In all applications mentioned above the KdV equation arises as an asymptotic limit. Taking into account additional physical factors it is easy to obtain some kinds of small perturbations for the KdV equation

$$u_t + \alpha u_x + 6uu_x + u_{xxx} = G. \tag{1}$$

The explicit asymptotic derivations for the forced KdV equation have been carried out for the water waves in a channel [8] - [11], internal waves in a shallow fluid [12] - [15], inertial waves in a narrow tube [16] etc.

In this article we study such kind derived forced KdV (fKdV) equation

$$U_t + 6UU_x + U_{xxx} = \varepsilon^2 f(\varepsilon x)\cos(\frac{S(\varepsilon^2 x, \varepsilon^2 t)}{\varepsilon^2}), \tag{2}$$

the right-hand side of (2) is the representation of the external force. This external force corresponds to the atmospheric disturbance in the frame of the shallow water model [17]. We suppose that the external force is small and has order $\varepsilon^2$, where $\varepsilon$ is a small parameter. The function $f(z)$ is smooth and rapidly vanishes as $|z| \to \infty$. The phase function of the perturbation $S(x,t)$ and all its derivatives are bounded.

Our goal is to investigate the influence of the perturbation on a small forced solution. This special solution corresponds to the forced oscillations and has order $\varepsilon^2$. A distinct feature of the considered problem is slowly varied frequency of the perturbation. There is a curve when the perturbation becomes resonant. It leads to the exchange of solution behaviour and change of the order of our special asymptotic solution. This change of the solution is typical and it is related to the slow passage through the resonance, see [18],[19],[20]. Here we present the detailed asymptotic description of the process.

The paper has a following structure. The second section contains the construction of the asymptotic solution that corresponds to forced oscillations.

The third section is devoted to the internal resonant reconstruction of the solution. And we construct the postresonant solution in the forth section. All mentioned above asymptotics are matched [21].

# 1 Forced Oscillations

## 1.1 Formal constructions

In this section we construct an asymptotic solution related to forced oscillations. The solution has order $\varepsilon^2$ and oscillates with the frequency of the perturbation.

It is more convenient for us to use a complex representation of the solution to (2). The solution has the WKB-form [22]

$$U(x,t,\varepsilon) = \sum_{n=2}^{\infty} \varepsilon^n \overset{n}{U}(x,t,\varepsilon) + complex\ conjugate, \tag{3}$$

where

$$\overset{n}{U}(x,t,\varepsilon) = \sum_{\psi\in\Omega_n} \overset{n}{u}_{\psi}(x_2,t_2,x_1)\exp\{i\psi(x_2,t_2)/\varepsilon^2\}. \tag{4}$$

The variables $x_m = \varepsilon^m x, t_m = \varepsilon^m t$ for $m = 1, 2,$ are slow. The set $\Omega_n$ of phase functions depends on the number of the correction term. For example, $\Omega_2 = \{\pm S\}$, $\Omega_3 = \{\pm S\}$, $\Omega_4 = \{\pm S, \pm 2S\}$ and the general formula for the phase set is

$$\Omega_n = \{\pm S, \pm 2S, \dots, \pm \left[\frac{n}{2}\right] S\}.$$

The validity of this formula can be obtained by direct calculations and taking into account the nonlinearity of (2).

Substitute expansion (3) into (2) and gather the terms of the same order with respect to small parameter $\varepsilon$ and exponents. It leads to equations for amplitudes $\overset{n}{u}_{\psi},\ \psi \in \Omega_n$. All these equations are algebraic.

The terms of order $\varepsilon^2$ give the equation for the leading-order term

$$\overset{2}{u}_S = \frac{-f}{[\partial_{t_2}S - (\partial_{x_2}S)^3]}. \tag{5}$$

The terms of order $\varepsilon^3$ give equation for $\overset{3}{u}_S$

$$\overset{3}{u}_S = \frac{-3(\partial_{x_2}S)^2\partial_{x_1}\overset{2}{u}_S}{[\partial_{t_2}S - (\partial_{x_2}S)^3]}. \tag{6}$$

Relations of order $\varepsilon^4$ give two equations for the amplitudes. On this step the solution contains the original phase $S$ related to the forced oscillations and the phase $2S$ generated by nonlinearity. Amplitudes are determined as follows

$$\begin{aligned}\overset{4}{u}_S &= \Big(\partial_{t_2}\overset{2}{u}_S + 3i\partial_{x_2}S\partial^2_{x1}\overset{2}{u}_S - 3(\partial_{x_2}S)^2\partial_{x_2}\overset{2}{u}_S \\ &- 3(\partial_{x_2}S)^2\partial_{x_1}\overset{3}{u}_S - 2\partial^2_{x_2}S\partial_{x_2}S\overset{2}{u}_S\Big) \times [\partial_{t_2}S - (\partial_{x_2}S)^3]^{-1}. \qquad (7)\end{aligned}$$

$$\overset{4}{u}_{2S} = \frac{-6\partial_{x_2}S\overset{2}{u}_S^2}{[\partial_{t_2}S - 4(\partial_{x_2}S)^3]}. \qquad (8)$$

The general formula for the $n$-th correction term $\overset{n}{u}_\psi$, $\psi \in \Omega_n$ has the form

$$\begin{aligned}\overset{n}{u}_\psi &= -\Big[-\partial_{t_2}\overset{n-2}{u} - 3i\partial_{x_2}S\partial^2_{x_1}\overset{n-2}{u} + 3(\partial_{x_2}S)^2\partial_{x_2}\overset{n-2}{u} + 3(\partial_{x_2})^2\partial_{x_1}\overset{n-1}{u} \\ &+ 2\partial^2_{x_2}S\partial_{x_2}S\overset{n-2}{u} + \partial^3_{x_1}\overset{n-3}{u} + \partial^3_{x_1x_1x_2}\overset{n-4}{u} + \partial^3_{x_1x_2x_2}\overset{n-5}{u} + \partial^3_{x_2x_2x_2}\overset{n-6}{u} \\ &- 3i\partial^2_{x_2}S\partial_{x_1}\overset{n-3}{u} - 3i\partial^2_{x_2}S\partial_{x_2}\overset{n-4}{u} - 6i\partial_{x_2}S\partial^2_{x_1x_2}\overset{n-3}{u} - 3i\partial_{x_2}S\partial^2_{x_2}\overset{n-4}{u} \\ &- i\partial^3_{x_2}S\overset{n-6}{u} + 6\sum_{\substack{k_1+k_2=n-1,\\ \psi_1+\psi_2=\psi}}\partial_{x_1}\overset{k_1}{u}_{\psi_1}\overset{k_2}{u}_{\psi_2} + 6\sum_{\substack{k_1+k_2=n-2,\\ \psi_1+\psi_2=\psi}}\partial_{x_2}\overset{k_1}{u}_{\psi_1}\overset{k_2}{u}_{\psi_2} \\ &+ 6i\sum_{\substack{k_1+k_2=n,\\ \psi_1+\psi_2=\psi}}\partial_{x_2}\psi_1\overset{k_1}{u}_{\psi_1}\overset{k_2}{u}_{\psi_2}\Big] \times \left[\partial_{t_2}\psi - (\partial_{x_2}\psi)^3\right]^{-1}. \qquad (9)\end{aligned}$$

It is easy to see that the power of the expression $[\partial_{t_2}\psi - (\partial_{x_2}\psi)^3]$ in the denominator of (9) increases along with the number $n$. Representation (3) of the solution for (2) becomes invalid when $[\partial_{t_2}\psi - (\partial_{x_2}\psi)^3]$ equals zero.

We consider the resonance with the external force. The equation

$$l[S] \equiv \partial_{t_2}S - (\partial_{x_2}S)^3 = 0$$

determines a curve $l$ on the plane of independent variables $(x_2, t_2)$. The curve of such a type is usually called resonant. We also suppose the expression $[\partial_{t_2}\psi - (\partial_{x_2}\psi)^3]$ for $\psi \neq S$ does not vanish on it.

## 1.2 The domain of validity for the forced oscillations solution (3)

In this subsection we describe the asymptotic behaviour of the coefficients of (3) as $[\partial_{t_2}S - (\partial_{x_2}S)^3] \to 0$ and determine the domain of validity for this asymptotic solution.

One can see that the solution (3) loses the asymptotic property in a neighborhood of the curve $l = 0$. The order of singularity of the coefficients for (3) as $[\partial_{t_2}S-(\partial_{x_2}S)^3] \to 0$ grows along with the number $n$ of the correction term.

The terms with the index $S$ have the maximal order of singularity as $l \to 0-$.

**Lemma 1.1.** *The terms* $\overset{n}{u}_S$ *have the asymptotic behaviour*

$$\overset{n}{u}_S = \sum_{k=-(n-1)}^{\infty} \overset{n}{u}{}_S^k l^{-k}, \quad \text{as} \quad l \to 0- . \tag{10}$$

*Proof.* The validity of this Lemma can be obtained by induction in the number $n$ of the correction term. The validity of this formula for small $n$ follows from formulas (5), (6), (7), (8).

The asymptotics of $\overset{n}{u}_S$ as $l \to 0-$ can be observed from relation (9). We suppose that the order of singularity for $\overset{n-1}{u}_S$ is $l^{-(n-2)}$ and then analyze the terms on right-hand side of (9). The increase of the order of singularities is caused by differentiation with respect to $x_2$ and $t_2$. The accordance between the order of differentiation with respect to $x_2$, $t_2$ and the number of correction terms implies the Lemma. □

The order of singularity as $l \to 0-$ for the amplitudes $\overset{n}{u}_\psi, \psi \neq S$ is smaller than that for $\overset{n}{u}_S$. This order can be calculated as in Lemma 1.1.

Asymptotic solution (3) is valid when

$$\varepsilon \max_{(x_2,t_2)} | \overset{n+1}{U} | = o\left( \max_{(x_2,t_2)} | \overset{n}{U} | \right), \quad \varepsilon \to 0.$$

Lemma 1.1 yields

$$-l \gg \varepsilon.$$

# 2 Inside the resonance

In this section we construct a formal asymptotic solution of equation (2) in a neighborhood of the resonance curve $l = 0$. We show the bifurcation of the solution under the resonant forcing.

## 2.1 Asymptotic solution. Derivation of equations

In the previous section we saw that no solution of the form (3) is available when $|l| \sim \varepsilon$. It leads to the change of the scale of independent variables. Here we use the scaled variable $\lambda = l/\varepsilon$.

We construct a solution of the form

$$U(x,t,\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n \overset{n}{W} (x_1, x_2, t_1, t_2, \varepsilon), \tag{11}$$

where

$$\overset{n}{W} = \sum_{\psi \in \Psi_n} \overset{n}{w}_\psi (x_1, x_2, t_1, t_2) \exp\{i\psi(x_2, t_2)/\varepsilon^2\}.$$

The set $\Psi_n$ depends on the number $n$ as follows $\Psi_n = \{\pm S, \dots, \pm nS\}$.

Substitute (11) into the original equation and gather the terms of the same order with respect to $\varepsilon$ and with the same exponents.

Terms of order $\varepsilon^2$ give us four equations.

$$\begin{aligned} &\left[\partial_{t_2} S - 4(\partial_{x_2} S)^3\right] \overset{2}{w}_{2S} = -3\partial_{x_2} S (\overset{1}{w}_S)^2, \\ &\partial_{t_1} \overset{1}{w}_S - 3(\partial_{x_2} S)^2 \partial_{x_1} \overset{1}{w}_S + i\lambda \overset{1}{w}_S = f. \end{aligned} \tag{12}$$

Two of them are equations for the complex conjugate functions.

It is easy to obtain the equations for higher-order correction terms of the asymptotic solution. The equations for amplitudes $\overset{n}{w}_S$ are differential.

$$\partial_{t_1} \overset{n}{w}_S - 3(\partial_{x_2} S)^2 \partial_{x_1} \overset{n}{w}_S + \lambda \overset{n}{w}_S = \overset{n}{f}_S . \tag{13}$$

Here

$$\begin{aligned} \overset{n}{f}_S \;=\; & -\partial_{t_2} \overset{n-1}{w}_S - 6 \sum_{\substack{k_1+k_2=n+1, \\ \psi_1+\psi_2=S}} \overset{k_1}{w}_{\psi_1} \left[\partial_{x_1} \overset{k_2-1}{w}_{\psi_2} + \partial_{x_2} \overset{k_2-2}{w}_{\psi_2} + i\partial_{x_2}\psi_2 \overset{k_2}{w}_{\psi_2} \right] \\ - \;& \partial^3_{x_1} \overset{n-2}{w}_S - 3\partial^3_{x_1x_1x_2} \overset{n-3}{w}_S - 3\partial^3_{x_1x_2x_2} \overset{n-4}{w}_S - 6i\partial_{x_2} S \partial^2_{x_1x_2} \overset{n-2}{w}_S \\ - \;& 3i\partial^2_{x_2} S \partial_{x_1} \overset{n-2}{w}_S - 3i\partial_{x_2} S \partial^2_{x_1} \overset{n-2}{w}_S + 3(\partial_{x_2} S)^2 \partial_{x_1} \overset{n}{w}_S - \partial^3_{x_2} \overset{n-5}{w}_S \\ - \;& 3i\partial^2_{x_2} S \partial_{x_2} \overset{n-3}{w}_S - 3i\partial_{x_2} S \partial^2_{x_2} \overset{n-3}{w}_S + 3(\partial_{x_2} S)^2 \partial_{x_2} \overset{n-1}{w}_S - i\partial^3_{x_2} S \overset{n-3}{w}_S \\ + \;& 3\partial_{x_2} S \partial^2_{x_2} S \overset{n-1}{w}_S . \end{aligned} \tag{14}$$

Functions $\overset{n}{w}_\psi$, $\psi \neq S$ are determined from algebraic equations

$$i\left[\partial_{t_2}\psi - (\partial_{x_2}\psi)^3\right] \overset{n}{w}_\psi = \overset{n}{f}_\psi$$

where $\overset{n}{f}_\psi$ has the same polynomial structure as that shown in (14).

## 2.2 Equations for coefficients of (11)

In this subsection we obtain solutions to the equations for coefficients of asymptotics.

First, we consider equation (13) for the coefficients of (11) that corresponds to the phase function $S$ of the perturbation. This equation can be solved by method of characteristics. The characteristics for (13) are determined by

$$\frac{dt_1}{d\sigma} = 1, \quad \frac{dx_1}{d\sigma} = -3(\partial_{x_2} S)^2. \tag{15}$$

and initial conditions

$$t_1|_{\sigma=0} = t_1^0, \quad x_1|_{\sigma=0} = x_1^0. \tag{16}$$

Here we use the notation $\xi$ and $\sigma$ for characteristic variables. We choose the point $(x_1^0, t_1^0)$ such that $\partial_{x_2} l|_{(x_1^0,t_1^0)} \neq 0$ as an origin. The variable $\sigma$ changes along the characteristics. We suppose that $\sigma = 0$ on the curve $\lambda = 0$. The variable $\xi$ measures the distance along the curve $\lambda = 0$ and the value $\xi = 0$ corresponds to the point $(x_1^0, t_1^0)$.

**Lemma 2.1.** *The Cauchy problem for characteristics has a solution when* $|\sigma| < c_1\varepsilon^{-1}, c_1 = const > 0$.

*Proof.* The Cauchy problem is equivalent to the following system

$$t_1 = t_1^0 + \sigma, \quad x_1 = x_1^0 - 3\int_0^\sigma (\partial_{x_2} S(\varepsilon x_1, \varepsilon t_1))^2 \, d\zeta. \tag{17}$$

After substitution $\tilde{t}_2 = (t_1 - t_1^0)\varepsilon, \tilde{x}_2 = (x_1 - x_1^0)\varepsilon$ we obtain

$$\tilde{t}_2 = \varepsilon\sigma, \quad \tilde{x}_2 = -3\int_0^{\varepsilon\sigma} \left(\partial_{x_2} S(\tilde{x}_2 + \varepsilon x_1^0, \varepsilon\zeta + \varepsilon t_1^0)\right)^2 d\zeta$$

The integrand is a smooth bounded function of $x_2, t_2$. There is a constant $c_1 > 0$ such that the integral operator is contraction when $\varepsilon|\sigma| < c_1$. □

**Lemma 2.2.** *In the domain* $|\sigma| \ll \varepsilon^{-1}$ *the asymptotics of solutions for Cauchy problem (15), (16), as* $\varepsilon \to 0$, *have the form*

$$x_1(\sigma,\xi,\varepsilon) - x_1^0(\xi) = -3\sigma\left(\partial_{x_2} S\right)^2 + 3\sum_1^N \varepsilon^n \sigma^{n+1} g_n(\varepsilon x_1, \varepsilon t_1) + O\left(\varepsilon^{N+1}\sigma^{N+2}\right), \tag{18}$$

$$t_1(\sigma,\xi,\varepsilon) - t_1^0(\xi) = \sigma, \tag{19}$$

*where*

$$g_n = -\frac{d^n}{d\sigma^n}\left(\partial_{x_2} S\right)^2|_{\sigma=0}$$

*Proof.* This lemma can be obtained by integration by parts in (17). □

The derivative along (15) has the form

$$\frac{d}{d\sigma} = \partial_{t_1} - 3(\partial_{x_2} S)^2 \partial_{x_1}$$

In order to obtain the asymptotics of the solution (11) across the resonance layer we should connect the transversal variable $\lambda$ and variable $\sigma$ along the characteristics.

**Lemma 2.3.** *Let $\sigma \ll \varepsilon^{-1}$, then*

$$\lambda = \varphi(\xi)\sigma + O(\varepsilon\sigma^2), \quad \varphi(\xi) = \frac{d\lambda}{d\sigma}|_{\sigma=0} \quad \sigma \to \infty.$$

*Proof.* For the variable $\lambda$ we obtain the representation

$$\lambda = \sum_{j=1}^{\infty} \lambda_j(x_1, t_1, \varepsilon)\sigma^j \varepsilon^{j-1},$$

where

$$\lambda_j(x_1, t_1, \varepsilon) = \frac{1}{j!}\frac{d^j}{d\sigma^j}\lambda(x_1, t_1, \varepsilon)|_{\sigma=0}$$

It yields

$$\lambda = \frac{d\lambda}{d\sigma}|_{\sigma=0}\sigma + O(\varepsilon\sigma^2 \frac{d^2\lambda}{d\sigma^2})$$

Let

$$|\frac{d^2\lambda}{d\sigma^2}| \geq \text{const}, \quad \xi \in R$$

The function $d\lambda/d\sigma$ is not equal to zero

$$\frac{d\lambda}{d\sigma} = \partial_{t_1}\lambda - 3(\partial_{x_2} S)^2 \partial_{x_1}\lambda \neq 0$$

It yields

$$\lambda = \varphi(\xi)\sigma + O(\varepsilon\sigma^2), \quad \varphi(\xi) = \frac{d\lambda}{d\sigma}|_{\sigma=0}$$

as desired. □

Along these characteristics the differential equation (13) for functions $\overset{n}{w}_S$ becomes ordinary differential equation

$$\frac{d \overset{n}{w}_S}{d\sigma} + i\lambda \overset{n}{w}_S = \overset{n}{f}_S \,. \tag{20}$$

A solution of this equation with asymptotics (10) has a form

$$\begin{aligned}\overset{n}{w}_S &= \exp\{-i\int_0^\sigma \lambda(x_1(\zeta),t_1(\zeta),\varepsilon)d\zeta\} \\ &\times \int_{-\infty}^{\sigma} \overset{n}{f}_S(x_1(\zeta),t_1(\zeta),\varepsilon)\exp\{i\int_0^\zeta \lambda(x_1(\mu),t_1(\mu),\varepsilon)d\mu\}d\zeta. \quad (21)\end{aligned}$$

The asymptotics of this solution when $\lambda\to-\infty$ can be obtained by integration by parts

$$\overset{n}{w}_S = \sum_{j=0}^{\infty}\left(\frac{\partial_{t_1}-3(\partial_{x_2}S)^2\partial_{x_1}}{i\lambda}\right)^j\left[\frac{\overset{n}{f}_S}{i\lambda}\right]. \quad (22)$$

It is easy to see that this internal representation is matched with the part (10) of the external solution that corresponds to forced oscillations.

## 2.3 Going out of the resonance. Asymptotics as $\lambda\to+\infty$

In this subsection we describe the going out of the resonance and obtain the asymptotic behaviour of internal resonant expansion (11) as $\lambda\to+\infty$.

Here we give two lemmas about the asymptotic behaviour of the coefficients of (11) and the phase function $\int_0^\sigma \lambda(x_1,t_1,\varepsilon)d\xi$ as $\lambda\to+\infty$. The first lemma allows us to determine the domain of validity for (11), and the second one shows the oscillations of the leading-order term after the passage through the resonant layer.

**Lemma 2.4.** *The asymptotic behaviour of* $\overset{n}{w}_S$ *when* $1\ll\lambda\ll\varepsilon^{-1}$ *has the form*

$$\overset{1}{w}_S = \exp\{-i\int_0^\sigma \lambda(x_1,t_1,\varepsilon)d\zeta\}\sum_{k=0}^{\infty}\overset{1}{w}{}_S^{(k)}\lambda^{-k}, \quad (23)$$

*and for* $n\ge 2$

$$\begin{aligned}\overset{n}{w}_S = \exp\{-i\int_0^\sigma \lambda(x_1,t_1,\varepsilon)d\zeta\}\sum_{j=0}^{2n-3}\lambda^j\sum_{k=0}^{j-1}\left(\ln^k|\lambda|\,\overset{n}{w}{}_S^{(j,k)}(\xi)\right) \\ +\sum_{j=0}^{\infty}\left(\frac{\partial_{t_1}-3(\partial_{x_2}S)^2\partial_{x_1}}{i\lambda}\right)^j\left[\frac{\overset{n}{f}_S}{i\lambda}\right]. \quad (24)\end{aligned}$$

*Proof.* The asymptotics of the coefficients $\overset{n}{w}_S$ are computed recurrently. First, we obtain the asymptotics for the leading-order term $\overset{1}{w}_S$

$$\overset{1}{w}_S = \exp\{-i\int_0^\sigma \lambda(x_1,t_1,\varepsilon)d\zeta\}\int_{-\infty}^{\sigma} \overset{1}{f}_S(x_1,t_1,\varepsilon)\exp\{i\int_0^\zeta \lambda(x_1,t_1,\varepsilon)d\mu\}d\zeta$$
$$= \exp\{-i\int_0^\sigma \lambda(x_1,t_1,\varepsilon)d\zeta\}\int_{-\infty}^{\infty} \overset{1}{f}_S(x_1,t_1,\varepsilon)\exp\{i\int_0^\zeta \lambda(x_1,t_1,\varepsilon)d\mu\}d\zeta$$
$$-\exp\{-i\int_0^\sigma \lambda(x_1,t_1,\varepsilon)d\zeta\}\int_{\sigma}^{\infty} \overset{1}{f}_S(x_1,t_1,\varepsilon)\exp\{i\int_0^\zeta \lambda(x_1,t_1,\varepsilon)d\mu\}d\zeta$$

By integration by parts we obtain formula (23), where

$$\overset{1}{w}{}_S^{(0)} = \int_{-\infty}^{\infty} f(x_1)\exp\{i\int_0^\zeta \lambda(x_1,t_1,\varepsilon)d\mu\}d\zeta. \qquad (25)$$

To calculate the asymptotics of the higher-order term $\overset{n}{w}_S$ we use the asymptotics with respect to $\sigma$ which were obtained on previous steps. In this case the representation (21) contains increasing terms with respect to $\sigma$. To obtain the asymptotics for $\overset{n}{w}_S$ we eliminate this growing part from the integral explicitly. The residual integral converges as $\sigma \to \infty$. It can be calculated in the same manner as it was shown for the leading-order term of the asymptotics.

To estimate the order of $\lambda$ as $\lambda \to +\infty$ it is necessary to note that the highest order integrand consists in nonlinear term related to $UU_x$. It is easy to trace the procedure of increasing of singularity for integrand. The first integration, when $n = 1$, yields

$$\overset{1}{w}_S \sim \text{const}, \quad \lambda \to +\infty.$$

The next integration gives

$$\overset{2}{w}_S \sim \lambda, \quad \lambda \to +\infty.$$

After substitution and $n-2$ integrations of the eliminated part of asymptotics we obtain representation (24). □

## 2.4 The domain of validity for the internal solution

Lemma 2.4 allows us to determine the domain of validity for (11). Representation (11) is asymptotic when

$$\varepsilon \max_{x_2,t_2,x_1,t_1} |\overset{n+1}{W}| = o\left(\max_{x_2,t_2,x_1,t_1} |\overset{n}{W}|\right), \quad \varepsilon \to 0. \qquad (26)$$

Lemma 2.4 and condition (26) give $\lambda \ll \varepsilon^{-1/2}$. In terms of the external variable $l$ we obtain $l \ll \varepsilon^{1/2}$.

## 2.5 Asymptotics of the phase function as $\lambda \to +\infty$

To determine the oscillations after the passage through the resonance we should investigate the asymptotic behaviour of the phase function as $\lambda \to \infty$.

**Lemma 2.5.** *For $\lambda \to \infty$, we get*

$$\int_0^\sigma \lambda d\zeta = \frac{S}{\varepsilon^2} + \frac{1}{\varepsilon}\Big[\partial_{x_2} S(x_1 - x_1^0) - (\partial_{x_2} S)^3 (t_1 - t_1^0)\Big] + O\left(\varepsilon\lambda^3\right). \tag{27}$$

*Proof.* To obtain (27), we use the asymptotics from Lemma 2.3

$$\begin{aligned} \int_0^\sigma \lambda d\zeta &= \int_0^\sigma \Big[ \left(\partial_{t_2} l - 3(\partial_{x_2} S)^2 \partial_{x_2} l\right)\zeta + O\left(\varepsilon\zeta^2\right) \Big] d\zeta \\ &= \left(\partial_{t_2} l - 3(\partial_{x_2} S)^2 \partial_{x_2} l\right) \frac{\sigma^2}{2} + O\left(\varepsilon\sigma^3\right). \end{aligned}$$

The asymptotics of the phase function $S(x_2, t_2)/\varepsilon^2$ in a neighborhood of the curve $l = 0$ is represented by a partial sum of the Taylor series

$$\begin{aligned} \frac{S}{\varepsilon^2} &+ \frac{1}{\varepsilon}\Big(\partial_{x_2} S(x_1 - x_1^0) + \partial_{t_2} S(t_1 - t_1^0)\Big) \\ &+ \frac{1}{2}\Big(\partial^2_{x_2} S(x_1 - x_1^0)^2 + 2\partial^2_{x_2 t_2} S(x_1 - x_1^0)(t_1 - t_1^0) + \partial^2_{t_2} S(t_1 - t_1^0)^2\Big) \\ &+ O\Big(\varepsilon(|x_1 - x_1^0| + |t_1 - t_1^0|)^3\Big). \end{aligned}$$

Substituting the asymptotic representation of $|x_1 - x_1^0|$ and $|t_1 - t_1^0|$ in $\varepsilon$ from Lemma 2.2 and combining it with the result of Lemma 2.3 completes the proof. □

The asymptotic representation of (11) as $\lambda \to +\infty$ contains diverse modes. The leading-order term contains the oscillations with an additional phase. This result follows from Lemma 2.4. We denote this new phase by $\varphi(x_2, t_2)/\varepsilon^2$. The asymptotics of this phase function is obtained in Lemma 2.5. The additional mode and nonlinearity of the original equation lead to complicated structure of the phase set for higher-order terms of the solution.

# 3 After the passage

In this section we construct the asymptotics of the solution after the passage through the resonance. The constructed solution has order $\varepsilon$ and oscillates. The equations for the coefficients are derived by multiscale method. Initial conditions for these equations are obtained by matching method.

## 3.1 Formal constructions. Equations for coefficients

The domain of validity of the internal solution shows that it is necessary to introduce the new external variable $l_3 = \varepsilon^{-1/2} l$. We construct a solution of the form

$$U(x,t,\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^{\frac{1+n}{2}} \overset{n}{V}, \tag{28}$$

where

$$\overset{n}{V} = \sum_{k=0}^{(n-1)/2} \ln^k(\varepsilon) \left[ \sum_{\pm\varphi} \exp\{\pm i\varphi/\varepsilon^2\} \overset{n,k}{\Psi}_{\pm\varphi} + \sum_{\chi} \exp\{i\chi/\varepsilon^2\} \overset{n,k}{\Psi}_{\chi} \right].$$

The amplitudes $\overset{n,k}{\Psi}_{\chi} = \overset{n,k}{\Psi}_{\chi}(t_2, x_3, t_3)$ depend on the new scaled variables $x_3 = \varepsilon^{3/2} x$ and $t_3 = \varepsilon^{3/2} t$.

Substitute (28) into (2) and gather the terms of the same order with respect to $\varepsilon$.

Terms of order $\varepsilon$ give us the equation for $\varphi$

$$\partial_{t_2}\varphi - (\partial_{t_2}\varphi)^3 = 0 \tag{29}$$

and

$$\varphi|_{l=0} = S|_{l=0}. \tag{30}$$

Terms of order $\varepsilon^{3/2}$ lead to equation (29) also.

Terms of order $\varepsilon^2$ allow one to obtain the relations for $\overset{3,0}{\Psi}_{\chi}$, where $\chi = S, 2\varphi$. The equation for $\overset{3,0}{\Psi}_S$ coincides with (5) of Section 1. The coefficient $\overset{3,0}{\Psi}_{2\varphi}$ is determined by

$$(\partial_{x_2}\varphi)^2 \overset{3,0}{\Psi}_{2\varphi} = \left( \overset{1,0}{\Psi}_{\varphi} \right)^2. \tag{31}$$

Note that equation (31) is not yet solved. The coefficient $\overset{1,0}{\Psi}_{\varphi}$ is not determined.

The terms of order $\varepsilon^{5/2}$ give a homogeneous transport equation for $\overset{1,0}{\Psi}_{\pm\varphi}$

$$\partial_{t_3} \overset{1,0}{\Psi}_{\pm\varphi} - 3(\partial_{x_2}\varphi)^2 \partial_{x_3} \overset{1,0}{\Psi}_{\pm\varphi} = 0. \tag{32}$$

This equation allows one to determine the dependence of the leading-order term with respect to the variable $\zeta$. Equation (32) along the characteristics

$$\frac{dt_3}{d\zeta} = 1, \quad \frac{dx_3}{d\zeta} = -3(\partial_{x_2}\varphi)^2 \tag{33}$$

looks like

$$\frac{d \overset{1,0}{\Psi}_{\pm\varphi}}{d\zeta} = 0. \tag{34}$$

Equation (34) shows that $\overset{1,0}{\Psi}_{\pm\varphi}$ does not depend on $\zeta$. Here we denote the variable along the characteristic by $\zeta$ and the transversal variable by $\xi$. The variable $\xi$ is defined by

$$\frac{dt_3}{d\xi} = 3(\partial_{x_2}\varphi)^2, \quad \frac{dx_3}{d\xi} = 1 \tag{35}$$

Terms of order $\varepsilon^3$ give us a inhomogeneous equation for $\overset{2,0}{\Psi}_{\varphi}$,

$$\partial_{t_3} \overset{2,0}{\Psi}_{\varphi} - 3(\partial_{x_2}\varphi)^2 \partial_{x_3} \overset{2,0}{\Psi}_{\varphi} = -\partial_{t_2} \overset{1,0}{\Psi}_{\varphi} + 3\partial_{x_2}\varphi\partial^2_{x_2}\varphi \overset{1,0}{\Psi}_{\varphi} + i\partial_{x_2}\varphi \overset{1,0^*}{\Psi}_{\varphi} \overset{3,0}{\Psi}_{2\varphi}. \tag{36}$$

As we showed above, the leading-order term $\overset{1,0}{\Psi}_{\varphi}$ does not depend on the variable $\zeta$. To construct a bounded solution of (36) we require that the right-hand side of this equation be equal to zero. Using formula (31) we obtain

$$\partial_{x_2}\varphi\partial_{t_2} \overset{1,0}{\Psi}_{\varphi} - 3(\partial_{x_2}\varphi)^2 \partial^2_{x_2}\varphi \overset{1,0}{\Psi}_{\varphi} + i| \overset{1,0}{\Psi}_{\varphi} |^2 \overset{1,0}{\Psi}_{\varphi} = 0. \tag{37}$$

It allows one to determine the leading-order term of the asymptotics. The Cauchy condition for (37) is obtained by a matching procedure. The initial datum (25) is the leading-order term in asymptotic expansion (23),

$$\overset{1,0}{\Psi}_{\varphi} |_{l=0} = \int_{-\infty}^{\infty} d\zeta f(x_1) \exp\{i \int_0^{\zeta} d\mu\lambda(x_1, t_1, \varepsilon)\}. \tag{38}$$

The higher-order correction terms are determined by the same manner. The amplitude $\overset{n,k}{\Psi}_{\varphi}$ satisfies to ordinary differential equations and $\overset{n,k}{\Psi}_{\chi}$ satisfies to algebraic equations.

The differential equations for $\overset{n,k}{\Psi}_{\varphi}$ have the form

$$\partial_{t_2} \overset{n,k}{\Psi}_{\varphi} - 3\partial_{x_2}\varphi\partial^2_{x_2}\varphi \overset{n,k}{\Psi}_{\varphi} = \overset{n,k}{\mathcal{F}}_{\varphi}, \tag{39}$$

where

$$\overset{n,k}{\mathcal{F}}_{\varphi} = -\partial^3_{x_2}\varphi \overset{n-4,k}{\Psi}_{\varphi} - 3i\partial^2_{x_2}\varphi\partial_{x_3} \overset{n-3,k}{\Psi}_{\varphi} - 3i\partial_{x_2}\varphi\partial^2_{x_3} \overset{n-2,k}{\Psi}_{\varphi} - \partial^3_{x_3} \overset{n-5,k}{\Psi}_{\varphi}$$
$$-6 \sum_{\substack{n_1+n_2=n,\\ k_1+k_2=k,\\ \alpha_1+\alpha_2=\varphi}} i\partial_{x_2}\alpha_1 \overset{n_1,k_1}{\Psi}_{\alpha_1} \overset{n_2,k_2}{\Psi}_{\alpha_2} - 6 \sum_{\substack{n_1+n_2=n-1,\\ k_1+k_2=k,\\ \alpha_1+\alpha_2=\varphi}} \partial_{x_3} \overset{n_1,k_1}{\Psi}_{\alpha_1} \overset{n_2,k_2}{\Psi}_{\alpha_2}. \tag{40}$$

Algebraic equations look like the equation from Section 1,

$$i\left[\partial_{t_2}\chi - (\partial_{x_2}\chi)^3\right] \overset{n,k}{\Psi}_{\chi} = \overset{n,k}{\mathcal{F}}_{\chi}, \tag{41}$$

where the right-hand side $\overset{n,k}{\mathcal{F}}_{\chi}$ depends on previous terms of the asymptotic expansion. The structure of $\overset{n,k}{\mathcal{F}}_{\chi}$ is similar to (40) with index $\varphi$ replaced by $\chi$. When $\chi = S$, the corresponding functions $\overset{n,k}{\Psi}_{\chi}$ have singularities on the resonant curve $l = 0$. The analysis of these singularities is carried out in much the same way as in Section 1.

### 3.2 The domain of validity for postresonant expansion (28)

As is mentioned above the amplitudes of (28) have singularities on the resonant curve $l = 0$. These singularities appear due to the structure of the right-hand side of equations (39) and (41). Here we determine the asymptotic behaviour of the coefficient of (28) and domain of validity of this representation of the solution.

First of all we calculate the order of singularity for amplitudes $\overset{n,k}{\Psi}_{\chi}$ with $\chi = \pm S$. It is easy to see that the first appearance of this forced amplitudes take place when $n = 3, k = 0$. They are determined from algebraic equations

$$l[S] \overset{3,0}{\Psi}_{S} = f.$$

It yields

$$\overset{3,0}{\Psi}_{S} = O(l^{-1}), \quad l \to 0+.$$

The direct calculations give

$$\overset{n,0}{\Psi}_S = O(l^{-[(n+1)/4]}), \quad l \to 0+. \tag{42}$$

It should be noted that the singularities of $\overset{n,0}{\Psi}_S$ have the strongest order and these terms determine the domain of validity for (28).

The asymptotic behaviour for $\overset{n,0}{\Psi}_\chi,\ \chi \neq S$ and $\overset{n,k}{\Psi}_S,\ k > 0$ obtained either from differential equations (39) or algebraic equations (41) shows that their order of singularity is weaker.

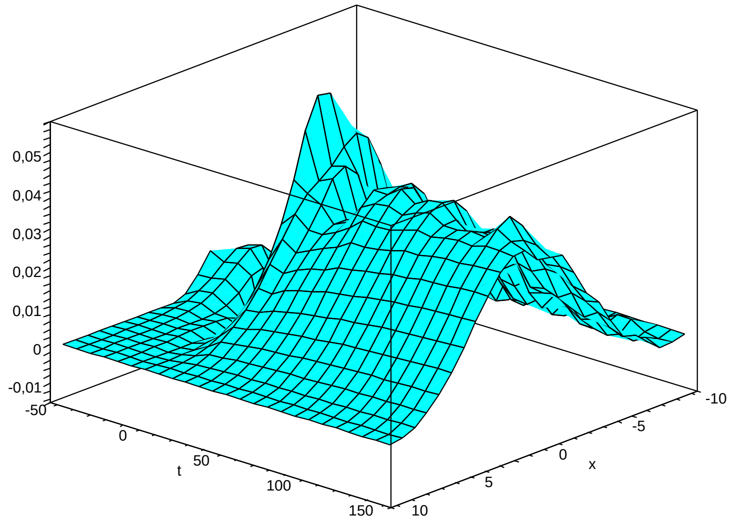
It gives the domain of validity for (28)

$$l \gg \varepsilon.$$

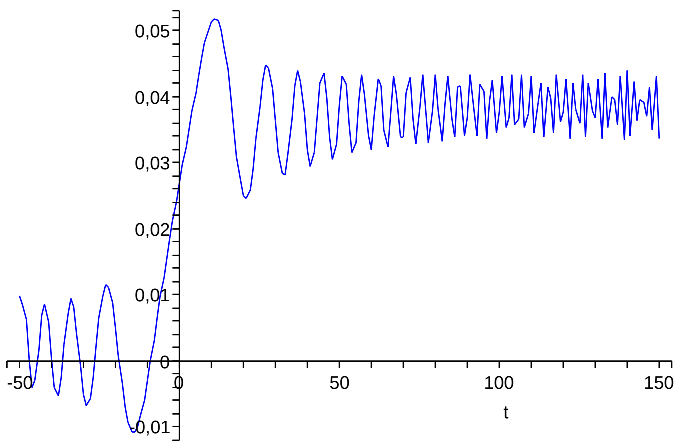
# 4 Numerical simulations

In this section we present a result of the numerical simulations for (2) and explain the behaviour of the solution. We take $S(x_2, t_2) = t_2^2$, $f(y) = \exp\{-y^2\}$ for simplicity. In this case the resonance takes place in a neighborhood of the line $t = 0$. Calculations were made under $\varepsilon = 0.1$. The initial datum is given at $t_0 = -50$. Here we present the 3D picture of the solution and his profile for $x = 0$.

We see that the solution has order $\varepsilon^2$ and oscillates with the frequency of perturbation in the preresonant domain. The solution grows up to the value $O(\varepsilon)$ in a neighborhood of the curve $t = 0$. And after the passage through the resonance domain it oscillates, too. These oscillations are realized with the frequency related to the new generated phase $\varphi$.




In this case the leading-order term $\overset{1,0}{\Psi}_{\varphi}$ is constant and oscillations take place around the constant value defined by formula (38).




**Acknowledgements** . We are grateful to N. Tarkhanov for helpful comments and discussions. This article was written during the stay of the first author at the Institute of Mathematics, University of Potsdam. He gratefully acknowledges the financial support of the Deuthcher Akademischer Austauschdienst. The work was also supported by grants RFBR 03-01-00716, Leading Scientific Schools 1446.2003.1 and INTAS 03-51-4286.

# References


[1] Korteweg D.J. and G. de Vries. On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves, Philos. Mag., 1895, V. 39,(5), p.422-443.

[2] Zabusky N.J. and M.D. Kruskal, Interaction of "solitons" in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. 1965, 15, p.240-243.

[3] Washimi H., T. Taniuti, Propagation of ion-acoustic solitary waves of small amplitudes, Phys. Rev. Lett., 1966, 17, p.996.

[4] S.A. Maslowe and L.G. Redekopp, Long nonlinear waves in stratified shear flows, J. Fluid Mech., 1980, 101, p.321-348.

[5] K.-K. Tung, D.R.S. Ko and J.J. Chang, Stud. Appl. Math., 1981, 65, p.189.

[6] R. Grimshaw, in Advances in Coastal and Ocean Engineering, ed. P.L.-F. Liu, World Scientific Publishing Company, Singapore, 1997, 3, 1.

[7] Kalyakin L.A., Long-wave asymptotics. Integrable equations as asymptotic limit of nonlinear systems, Uspehi matem. nauk, 1989. V. 44, Is. 1.

[8] Akylas T.R., On the excitation of long nonlinear water waves by moving pressure distribution, J.Fluid Mech., 1984, 141, p.455-466.

[9] Cole S.L., Transient waves produced by flow past a bump, Wave Motion, 1985, 7, p.579-587.

[10] Lee S.J., Yates G.T., and Wu T.Y., Experiments and analyses of upstream-advancing solitary waves generated by moving disturbances. J.Fluid Mech., 1989, 199, p.569-593.

[11] Wu T.Y., Generation of upstream advancing solitons by moving disturbances. J.Fluid Mech., 1987, 184, p.75-99.

[12] Grimshaw R. and Smyth N., Resonant flow of a stratified fluid over topography. J.Fluid Mech., 1986, 169, p.429-464.

[13] Melville W.K., Helfrich K.R., Transcritical two-layer flow over topography. J.Fluid Mech., 1987, 178, p.31-52.

[14] Clarke, S.R. and Grimshaw, R.H.J., Resonantly generated internal wave in a contraction. J. Fluid Mech., 1994, 274, p.139-161.

[15] Clarke, S.R. and Grimshaw, R.H.J., Weakly-nonlinear internal wave fronts trapped in contractions. J. Fluid Mech., 2000, 415, p.323-345.

[16] Grimshaw R., Resonant flow of a rotating fluid past an obstacle: the general case. Stud. Appl.Maths., 1990, 83, p.249-269.

[17] Pelinovsky E., Talipova T., Kurkin A., Kharif C. Nonlinear mechanism of tsunami wave generation by atmospheric disturbance, Natural Hazrd and Earth System Science, 2001, v.1, p.243-250.

[18] S.G.Glebov, O.M. Kiselev, V.A.Lazarev. Birth of solitons during passage through local resonance. Proceedings of the Steklov Institute of Mathematics. Suppl., 2003, issue 1, S84-S90.

[19] S.G.Glebov, O.M. Kiselev, V.A. Lazarev. Slow passage through resonance for a weakly nonlinear dispersive wave. SIAM Journal of Applied Mathematics, 2005, Vol. 65, N.6, p.2158-2177.

[20] S.G. Glebov, O.M. Kiselev. The stimulated scattering of solitons on a resonance. Journal of Nonlinear Mathematical Physics, 2005, v.12, n3, p.330-341.

[21] A. M. Il'in Matching of Asymptotic Expansions of Solutions of Boundary Value Problem, AMS, 1992.

[22] A. Jeffrey and T. Kawahara, Asymptotic Methods in Nonlinear Wave Theory, Pitman, Boston, 1982.