The Cauchy problem for the Lame system in infinite domains in R^m

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Abstract

We consider the problem of analytic continuation of the solution of the multidimensional Lame system in infinite domains through known values of the solution and the corresponding strain tensor on a part of the boundary, i.e,the Cauchy problem.

Key words: the Cauchy problem, system Lame, elliptic system, ill-posed problem, Carleman matrix, regularization, Laplace equation.

Introduction

We consider the problem of analytic continuation of the solution of the multidimensional Lame system in infinite domains through known values of the solution and the corresponding strain tensor on a part of the boundary, i.e,the Cauchy problem.

Let R^m, $(m \geq 2)$, be $m-$ dimensional real Euclidean space, D a domain in R^m with piecewise smooth boundary ∂D , S a part of ∂D with smooth boundary.

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Suppose that $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$ are points in R^m and $U(x)$ satisfies in domain D the following uniform system of Lame equations:

$$LU(x) = 0, \tag{1}$$

were $L = \mu\Delta + (\lambda + \mu)\ grad\ div$, Δ is the Laplace operator, λ, μ are the Lame constants, and $\mu \neq 0$, $\lambda \neq -2\mu$.

Set

$$\begin{aligned} U(y) &= f(y), \\ T(\partial_y, n)U(y) &= g(y), \ \ y \in S \end{aligned} \tag{2}$$

here $f(y) = (f_1(y), ..., f_m(y))$ and $g(y) = (g_1(y), ..., g_m(y))$ are given continuous vector functions on S, $T(\partial_y, n)$ is the stress operator,i.e.,

$$T(\partial_y, n) = \mu\delta_{ij}\frac{\partial}{\partial n} + \lambda n_i(y)\frac{\partial}{\partial y_j} + \mu n_j(y)\frac{\partial}{\partial y_i}, \ \ i, j = 1, ..., m,$$

where δ_{ij} is the Kronecker delta, and $n = (n_1, ..., n_m)$ is the unit normal vector to the surface S.

Given f and g determine the function $U(y)$ in D.

The Lame system is elliptic. The Cauchy problem for elliptic equations is unstable with respect to small variations of the data, i.e., it is an ill-posed problem. For ill-posed problems, one does not prove the existence theorem; the existence is assumed a priori. Moreover, the solution is assumed to belong to some given subset of the function space, usually a compact one (the Tikhonov well-posedness class) [5], [13]. The uniqueness of the solution of problem (1), (2) follows from the general Holmgren theorem.

In this paper, we construct an approximate solution of problem (1), (2) by using Carleman's function.

Extending Lavrent'ev's idea, Yarmukhamedov constructed the Carleman function for the Cauchy problem for the Laplace and Helmholtz equations [14], [15].

It was proved earlier [1], [12] that for any Cauchy problem for solutions of elliptic systems, the Carleman matrix exists if the Cauchy data are given on the boundary set of positive measure. Since we deal here with explicit formulas, the construction of the Carleman matrix in elementary and special functions is of considerable interest.

Starting from of the Carleman function for the Cauchy problem for the Laplace equation, was constructed Carleman matrix for the Cauchy problem for the system of elasticity theory for $m = 2, 3$ in works [3], [6], [7], [8], [9].

Since f and g can be given approximately, the solution of problem found only approximately.

Suppose that instead of $f(y)$ and $g(y)$ we know their approximations $f_\delta(y)$ and $g_\delta(y)$ with accuracy δ, $0 < \delta < 1$ (in the uniform metric); these approximations do not necessarily belong to the class of existence of the solution. Let us construct a family of functions $U(x, f_\delta, g_\delta) = U_{\sigma\delta}(x)$ that depend on a parameter σ, and let us prove that under some conditions and for a special choice of the parameter $\sigma(\delta)$, the family $U_{\sigma\delta}(x)$ converges in the usual sense to the solution $U(x)$ of problem (1), (2) as $\delta \to 0$. Following Tikhonov [13] we call $U_{\sigma\delta}(x)$ a regularized solution of the problem. The regularized solution defines a stable method for approximately solving problem (1), (2).

In the work [11] the problem is solved by the polynomial approximation method and yields a solvability criterion of the problem, too.

1 Construction of the matrix of fundamental solution for the system of elasticity of a special form

Definition 1.1. *The matrix* $\Gamma(y,x) = ||\Gamma_{ij}(y,x)||_{m\times m}$, *is called the matrix of fundamental solutions of system (1), where*

$$\Gamma_{ij}(y,x) = \frac{1}{2\mu(\lambda+2\mu)}((\lambda+3\mu)\delta_{ij}q(y,x) - (\lambda+\mu)(y_j - x_j)\frac{\partial}{\partial x_i}q(y,x)),$$

$i, j = 1, 2, ..., m,$

$$q(y,x) = \begin{cases} \frac{1}{(2-m)\omega_m}\cdot\frac{1}{|y-x|^{m-2}}, & m > 2, \\ \frac{1}{2\pi}\, ln|y-x|, & m = 2, \end{cases}$$

and ω_m- *is the area of unit sphere in* R^m.

The matrix $\Gamma(y,x)$ is symmetric and each of its columns and rows satisfy equation (1) at an arbitrary point $x \in R^m$, except $y = x$. Thus, we have

$$L\Gamma(y,x) = 0, \quad y \neq x.$$

Developing Lavrent‘ev's idea concerning the notion of Carleman function of the Cauchy problem for the Laplace equation [5], we introduce the following notion.

Definition 1.2. *By the Carleman matrix of problem (1),(2) we mean an* $(m \times m)$ *matrix* $\Pi(y,x,\sigma)$ *satisfying the following two conditions:*

$$1)\ \ \Pi(y,x,\sigma)=\Gamma(y,x)+G(y,x,\sigma),$$

where σ *is a positive numerical parameter and, with respect to the variable* y*, the matrix* $G(y,x,\sigma)$ *satisfies system (1)every where in the domain* D*.*

2) The following relation holds:

$$\int\limits_{\partial D\setminus S}(|\Pi(y,x,\sigma)|+|T(\partial_y,n)\Pi(y,x,\sigma)|)ds_y\le\varepsilon(\sigma),$$

where $\varepsilon(\sigma)\to 0,$ *as* $\sigma\to\infty$ *uniformly* x *on compact subsets of* D *; here and elsewhere* $|\Pi|$ *denotes the Euclidean norm of the matrix* $\Pi=||\Pi_{ij}||$*, i.e.,* $|\Pi|=(\sum\limits_{i,j=1}^{m}\Pi_{ij}^2)^{\frac{1}{2}}$*, in particular* $|U|=(\sum\limits_{i=1}^{m}U_i^2)^{\frac{1}{2}}$ *for the vector* $U=(U_1,...,U_m)$*.*

Definition 1.3. *A vector function* $U(y)=(U_1(y),...,U_m(y))$ *is said to be regular in* D*, if it is continuous together with its partial derivatives of second order in* D *and those of first order in* $\overline{D}=D\bigcup\partial D$*.*

In the theory of partial differential equations, an important role is played by representations of solutions of these equations as functions of potential type. As an example of such representations, we present Somilian-Betti formula [4] below.

Theorem 1.4. *Any regular solution* $U(x)$ *of equation (1) in the domain* D *is specified by the formula*

$$U(x)=\int\limits_{\partial D}(\Gamma(y,x)\{T(\partial_y,n)U(y)\}-U(y)\{T(\partial_y,n)\Gamma(y,x)\})ds_y,\ \ x\in D.$$

Since the Carleman matrix differs from the matrix of fundamental solutions by a solution of the transposed system, it follows that Somilian-Betti formula remains valid if the fundamental solution is replaced by the Carleman matrix. Thus, we obtain the following assertion.

Theorem 1.5. *Any regular solution* $U(x)$ *of equation (1) in the domain* D *is specified by the formula*

$$U(x)=\int\limits_{\partial D}(\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\}-U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\})ds_y,\ \ x\in D, \tag{3}$$

where $\Pi(y,x,\sigma)$ *the Carleman matrix.*

Suppose that $K(\omega)$, $\omega = u + iv$ (u, v are real), is an entire function taking real values the real axis and satisfying the conditions

$$K(u) \neq 0, \ \sup_{v \geq 1} |v^p K^{(p)}(\omega)| = M(p,u) < \infty, \ \ p = 0, ..., m, \ \ u \in R^1.$$

Let

$$s = \alpha^2 = (y_1 - x_1)^2 + ... + (y_{m-1} - x_{m-1})^2.$$

For $\alpha > 0$, we define the function $\phi(y, x)$ by the following relations: if $m = 2$, then

$$-2\pi K(x_2)\phi(y,x) = \int_0^\infty Im[\frac{K(i\sqrt{u^2+\alpha^2}+y_2)}{i\sqrt{u^2+\alpha^2}+y_2-x_2}]\frac{udu}{\sqrt{u^2+\alpha^2}},$$

if $m = 2n + 1$, $n \geq 1$, then

$$C_m K(x_m)\phi(y,x) = \frac{\partial^{n-1}}{\partial s^{n-1}} \int_0^\infty Im[\frac{K(i\sqrt{u^2+\alpha^2}+y_m)}{i\sqrt{u^2+\alpha^2}+y_m-x_m}]\frac{du}{\sqrt{u^2+\alpha^2}}, \quad (4)$$

where $C_m = (-1)^{n-1} \cdot 2^{-n}(m-2)\pi\omega_m(2n-1)!$, if $m = 2n$, $n \geq 2$, then

$$C_m K(x_m)\phi(y,x) = \frac{\partial^{n-2}}{\partial s^{n-2}} Im \frac{K(\alpha i + y_m)}{\alpha(\alpha + y_m - x_m)}, \quad (5)$$

where $C_m = (-1)^{n-1}(n-1)!(m-2)\omega_m$.

With the help of function $\phi(y, x)$ we construct a matrix

$$\begin{aligned}\Pi(y,x) &= ||\Pi_{ij}(y,x)||_{m\times m} \\ &= \left|\left| \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\delta_{ij}\phi(y,x) - \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}(y_j-x_j)\frac{\partial}{\partial y_i}\phi(y,x)\right|\right|_{m\times m}, \\ & i,j = 1,2,...,m. \end{aligned} \quad (6)$$

Now suppose D is an infinite domain in R^m, $m \geq 2$, lying in the shall: $0 < y_m < h$, $h = \frac{\pi}{\rho}$, $\rho > 0$. Moreover, its boundary consists of a smooth surface S lying in the half-space $y_m > 0$ and of the hyperplane $\partial D \backslash S : y_m = 0$.

Suppose that for the some $b_0 > 0$ the boundary area satisfies the conditions

$$\int_{\partial D} \exp\{-b_0 \exp \rho_0 |y'|\} ds_y < \infty, \ \ 0 < \rho_0 < \rho, \ \ y' = (y_1, ..., y_{m-1}) \in R^{m-1}. \quad (7)$$

For $\sigma \geq 0$ in 4-6, put

$$\begin{aligned} K(\omega) &= \exp(\sigma\omega - bchi\rho_1(\omega - \frac{h}{2}) - b_1 chi\rho_0(\omega - \frac{h}{2})), \\ K(x_m) &= \exp(\sigma x_m + b\cos\rho_1(x_m - \frac{h}{2}) - b_1\cos\rho_0(x_m - \frac{h}{2})), \end{aligned}$$

where

$$\omega = y_m + iv, \ \ \rho_0, \ \ \rho_1 \in (0, \rho), \ \ 0 < x_m < h, \ \ b > 0, \ \ b_1 > b_0(\cos\rho_0\frac{h}{2})^{-1}.$$

Take $\phi(y, x, \sigma) = \phi(y, x)$ and $\Pi(y, x, \sigma) = \Pi(y, x)$. In [14], the following assertion was proved.

Lemma 1.6. *The function $\phi(y, x, \sigma)$ can be expressed as*

$$\begin{aligned} \phi(y, x, \sigma) &= \frac{1}{2\pi} ln\frac{1}{r} + g_2(y, x, \sigma), \quad m = 2, \quad r = |y - x|, \\ \phi(y, x, \sigma) &= \frac{r^{2-m}}{\omega_m(m-2)} + g_m(y, x, \sigma), \quad m \geq 3, \quad r = |y - x|, \end{aligned}$$

where $g_m(y, x, \sigma)$, $m \geq 2$ is a function defined for all values y, x and harmonic in the variable y in all of R^m.

Using in Lemma 1.6, we obtain.

Lemma 1.7. *The matrix $\Pi(y, x, \sigma)$ is the Carleman matrix for problem* (1), (2).

Proof. By (4 - (6) and Lemma 1.6, we have

$$\Pi(y, x, \sigma) = \Gamma(y, x) + G(y, x, \sigma),$$

where

$$\begin{aligned} G(y, x, \sigma) &= ||G_{kj}(y, x, \sigma)||_{m\times m} \\ &= \left|\left| \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)}\delta_{kj} g_m(y, x, \sigma) - \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)}(y_j - x_j)\frac{\partial}{\partial y_i} g_m(y, x, \sigma) \right|\right|_{m\times m}. \end{aligned}$$

□

Prove that $L_yG(y,x,\sigma)=0$. Indeed, since $\Delta_y g_m(y,x,\sigma)=0$, $\Delta_y=\sum\limits_{k=1}^{m}\frac{\partial^2}{\partial y_k^2}$ and for the $j-$ column $G^j(y,x,\sigma)$,

$$divG^j(y,x,\sigma)=\frac{1}{2\mu(\lambda+2\mu)}\cdot\frac{\partial}{\partial y_j}g_m(y,x,\sigma),$$

then for the k-th component of the vector $L_yG^j(y,x,\sigma)$ we obtain

$$\begin{aligned}
&\sum_{i=1}^{m}(L_y)_{ki}G_{ij}(y,x,\sigma)\\
&= \mu\Delta_y\left[\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\cdot\delta_{ij}g_m(y,x,\sigma)-\frac{\lambda+\mu}{2\mu(\lambda+2\mu)}(y_j-x_j)\frac{\partial}{\partial y_k}g_m(y,x,\sigma)\right]\\
&\quad+(\lambda+\mu)\frac{\partial}{\partial y_k}divG^j(y,x,\sigma)\\
&= -\frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\frac{\partial^2}{\partial y_j^2}g_m(y,x,\sigma)+\frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\frac{\partial^2}{\partial y_j^2}g_m(y,x,\sigma)=0
\end{aligned}$$

Hence each column of the matrix $G(y,x,\sigma)$ satisfies Lame system in the variable y every where in R^m.

We now check the second property of the Carleman matrix. We estimate

$$\int\limits_{\partial D\backslash S}[|\Pi(y,x,\sigma)|+|T(\partial_y,n)\Pi(y,x,\sigma)|]ds_y.$$

To estimate the integral, we have to estimate

$$\frac{\partial^q\phi(y,x,\sigma)}{\partial y_1^{q_1}...\partial y_m^{q_m}},\quad q=q_1+...+q_m,\quad q=0,1,2.$$

From the construction of $\phi(y,x,\sigma)$ for

$$K(\omega)=\exp(\sigma\omega-bchi\rho_1(\omega-\frac{h}{2})-b_1chi\rho_0(\omega-\frac{h}{2})),$$

we obtain

$$\begin{aligned}
&Im\frac{K(i\theta+y_m)}{i\theta+y_m-x_m}\\
&= \frac{1}{\theta^2+(y_m-x_m)^2}(-\theta\cos\sigma+(y_m-x_m)\sin\sigma\theta \qquad (8)\\
&\quad\times\exp\left(\sigma y_m-b\cos\rho_1(y_m-\frac{h}{2})ch\theta\rho_1-b_1\cos\rho_0(y_m-\frac{h}{2})ch\theta\rho_0\right),
\end{aligned}$$

where $\theta = \sqrt{u^2+\alpha^2}$, by $m = 2n+1,\ \ n \geq 1$ and $\theta = \alpha$ for $m = 2n,\ \ n \geq 2$.
Then for $m = 2n+1$ and $s = \alpha^2$

$$\begin{aligned} &C_m\phi(y,x,\sigma) \\ &\quad = \frac{\exp\sigma y_m}{K(x_m)}\frac{\partial^{n-1}}{\partial s^{n-1}}\int\limits_0^\infty \frac{\exp(-b\cos\rho_1(y_m-\frac{h}{2})ch\rho_1\sqrt{u^2+s})}{u^2+s+(y_m-x_m)^2} \\ &\quad\times((y_m-x_m)\sin\sigma\sqrt{u^2+s}-\sqrt{u^2+\alpha^2}\cos\sigma\sqrt{u^2+s}) \\ &\quad\times\exp(-b_1\cos\rho_0(y_m-\frac{h}{2})ch\rho_0\sqrt{u^2+s})\frac{du}{\sqrt{u^2+s}}. \end{aligned}$$

Since the function $ch\rho\sqrt{u^2+s}$ growths by $u \geq 0$, and

$$\frac{-\pi}{2} < \frac{-\rho_1 h}{2} < \rho_1(y_m-\frac{h}{2}) < \frac{\rho_1 h}{2} < \frac{\pi}{2},\ \ \rho_1<\rho,\ \ \frac{-\pi}{2} < \rho_0(y_m-\frac{h}{2}) < \frac{\pi}{2},$$

then function

$$\frac{1}{\sqrt{u^2+s}}\exp(-b_1\cos\rho_1(y_m-\frac{h}{2})ch\rho_1\sqrt{u^2+s}-b_1\cos\rho_0(y_m-\frac{h}{2})ch\rho_0\sqrt{u^2+s}),$$

is decreasing for $u \geq 0$. By the mean value theorem for improper integrals [2], for some $\xi \geq 0$, we have

$$\begin{aligned} C_m\phi(y,x,\sigma) &= \frac{\exp\sigma y_m}{K(x_m)}\frac{\partial^{n-1}}{\partial s^{n-1}} \\ &\times\exp\left(-b\cos\rho_1(y_m-\frac{h}{2})ch\rho_1\sqrt{s}-b_1\cos\rho_0(y_m-\frac{h}{2})ch\rho_1\sqrt{s}\right) \\ &\times\frac{1}{\sqrt{s}}\int\limits_0^\xi \frac{\left((y_m-x_m)\sin\sigma\sqrt{u^2+s}-\sqrt{u^2+\alpha^2}\cos\sigma\sqrt{u^2+s}\right)du}{(u^2+s+(y_m-x_m)^2)}. \end{aligned} \tag{9}$$

Hence

$$\begin{aligned} |\phi(y,x,\sigma)| &\leq C_m^{-1}C_1(\rho)\sigma^{n-1}s^{\frac{1-2n}{2}} \\ &\times\exp\left(-b\cos\rho_1(y_m-\frac{h}{2})ch\rho_1\sqrt{s}-b_1\cos\rho_0(y_m-\frac{h}{2})ch\rho_1\sqrt{s}\right) \\ &\times\exp\left(\sigma(y_m-x_m)+b\cos\rho_1(x_m-\frac{h}{2})+b_1\cos\rho_0(x_m-\frac{h}{2})\right), \end{aligned}$$

$$\left|\frac{\partial^q \phi(y,x,\sigma)}{\partial y_1^{q_1}...\partial y_m^{q_m}}\right| \leq C_m^{-1} C_1(\rho)\sigma^{n-1+k} s^{\frac{1-2n-2k}{2}}$$
$$\times \left(-b\cos\rho_1(y_m-\frac{h}{2})ch\rho_1\sqrt{s} - b_1\cos\rho_0(y_m-\frac{h}{2})ch\rho_1\sqrt{s}\right)$$
$$\times \exp\left(\sigma(y_m-x_m) + b\cos\rho_1(y_m-\frac{h}{2}) + b_1\cos\rho_0(y_m-\frac{h}{2})\right),$$

where $q = q_1 + \cdots + q_m$, $q = 1, 2$. Then

$$|\Pi(y,x,\sigma)| + |T(\partial_y, n)\Pi(y,x,\sigma)| \leq C(\rho)\cdot s^{\frac{1-2n}{2}}\sigma^{n+1}$$
$$\times \exp(-b\cos\rho_1(y_m-\frac{h}{2})ch\rho_1\sqrt{s} - b_1\cos\rho_0(y_m-\frac{h}{2})ch\rho_1\sqrt{s})$$
$$\times \exp\left(\sigma(y_m-x_m) + b\cos\rho_1(x_m-\frac{h}{2}) + b_1\cos\rho_0(x_m-\frac{h}{2})\right).$$

From this inequality by condition (7), we obtain

$$\int\limits_{\partial D\backslash S} [|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]ds_y$$
$$\leq \int\limits_{\partial D\backslash S} [|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]ds_y \tag{10}$$
$$\leq C(\rho)\sigma^{n+1}\cdot\exp(-\sigma x - m),\ \ x_m > 0.$$

For $m = 2n$ estimate (10) easily follows from (8. The lemma 2 proved.

2 The formula of Somilian-Betti in infinite domains and its applications

I. Let D be a domain in R^m, $m \geq 2$, with piecewise smooth boundary ∂D. We denote by $A(D)$ the space for all solutions of system (1) in D (if D infinite domain, then the regularity is required in finite points of ∂D).

If D is a bounded domain and $U(x) \in A(D)$, then the following formula

(3) holds:

$$\int\limits_{\partial D} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y$$
$$= \begin{cases} U(x), & x \in D, \\ 0, & x \notin D. \end{cases}$$

Let $U(x) \in A(D)$, where D is an infinite domain. Write

$$D_R = D \cap \{x \in R^m : \ |x| < R\}, \quad D_R^\infty = D \backslash D_R, \ \ R > 0.$$

If for each fixed $x \in D$ we have

$$\lim_{R\to\infty} \int\limits_{\partial D_R^\infty} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y = 0,$$

then

$$\int\limits_{\partial D} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y$$
$$= \begin{cases} U(x), & x \in D \\ 0, & x \notin \overline{D}. \end{cases} \qquad (11)$$

Indeed, by fixed $x \in D$ $(|x| < R)$ we have

$$\int\limits_{\partial D} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y$$
$$= \int\limits_{\partial D_R} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y$$
$$+ \int\limits_{\partial D_R^\infty} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y$$
$$= U(x)$$
$$+ \int\limits_{\partial D_R^\infty} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y.$$

Letting $R \to \infty$, we obtain (11).

II. Suppose that an infinite domain D lies in a shall: $0 < y_m < h$, $h = \frac{\pi}{\rho}$, $\rho > 0$, moreover, the boundary area satisfies condition (7).

Let $U(x) \in A(D)$ satisfy a growth condition

$$\begin{aligned} &|U(y)| + |T(\partial_y, n)U(y)| \leq C \exp(\exp \rho_2 |y'|), \\ &\rho_2 < \rho, y = (y', y_m) \in D. \end{aligned} \tag{12}$$

In formula (4), (6) we take $K(\omega) = \exp(-bchi\rho_1(\omega - \frac{h}{2}) - b_1 chi\rho_0(\omega - \frac{h}{2}))$, where

$$\omega = i\sqrt{u^2 + \alpha^2} + y_m,$$

$0 < \rho_2 < \rho_1 < \rho$, $0 < x_m < h$, $b > 0$, $b_1 > b_0(\cos \rho_0 \frac{h}{2})^{-1}$, $\alpha = |y' - x'|$. In order to obtain the formula (11) for the domain D, we prove that

$$\lim_{R\to\infty} \int\limits_{\partial D_R^\infty} [\Pi(y, x, \sigma)\{T(\partial_y, n)U(y)\} - U(y)\{T(\partial_y, n)\Pi(y, x, \sigma)\}]ds_y = 0. \tag{13}$$

Indeed, by (6) and (12) for fixed $x \in D$ we get

$$\begin{aligned} &\left| \int\limits_{\partial D_R^\infty} [\Pi(y, x, \sigma)\{T(\partial_y, n)U(y)\} - U(y)\{T(\partial_y, n)\Pi(y, x, \sigma)\}]ds_y \right| \\ &\leq \int\limits_{\partial D_R^\infty} (|\Pi(y, x, \sigma)| + |T(\partial_y, n)\Pi(y, x, \sigma)|) \\ &\times \quad (|U(y)| + |T(\partial_y, n)U(y)|)\, ds_y \leq \int\limits_{\partial D_R^\infty} \widetilde{C} s^{\frac{1-2n}{2}} \sigma^2 \\ &\times \quad \exp\left(-b \cos \rho_1 (y_m - \frac{h}{2}) ch\rho_1 \sqrt{s} - b_1 \cos \rho_0 (y_m - \frac{h}{2}) ch\rho_0 \sqrt{s}\right) \\ &\times \quad \exp\left(\sigma(y_m - x_m) - b \cos \rho_1 (x_m - \frac{h}{2}) - b_1 \cos \rho_0 (x_m - \frac{h}{2})\right) \\ &\times \quad \exp(\exp \rho_2 |y'|) ds_y \leq \widetilde{C} \exp(-\sigma x_m) \sigma^2 \end{aligned}$$

$$\times \exp\left(\sigma h - b\cos\rho_1(x_m - \frac{h}{2}) - b_1\cos\rho_0(x_m - \frac{h}{2})\right)$$

$$\int\limits_{\partial D_R^\infty} \frac{1}{|y'|^{2n-1}} \cdot \exp\left(-b\cos\rho(y_m - \frac{h}{2})ch\rho_1|y'|\right.$$

$$\left.-b_1\cos\rho_0(y_m - \frac{h}{2})ch\rho_0|y'| + \exp\rho_2|y'|\right)ds_y.$$

From this inequality, (13) easily follows.

III. Suppose that D is aninfinite simply connected domain in R^m, with smooth boundary ∂D, lying in half-spaces $y_m > 0$, defined by the equation $y_m = f(y'), y' \in R^{m-1}$ and satisfying the conditions $|grad\ \ f(y')| \leq C < \infty$.

Let

$$K(\omega) = (\omega + x_m)^{-k}\exp(-\varepsilon(\omega + 1)^{\rho_1}),$$

$$\omega = i\sqrt{u^2 + \alpha^2} + y_m,\ \ \varepsilon > 0,\ \ y_m > 0,\ \ x_n > 0,\ \ \rho_1 \in (0,1),\ \ k \in N.$$

Denote by $B_\rho(D) = \{U(y) : U(y) \in A(D),\ \ |U(y)| + |T(\partial_y, n)U(y)| \leq \exp|y|^\rho,\ \ \rho > 0\}$. Let $\rho < \rho_1 < 1$, then for $U(x) \in B_\rho(D)$ condition (13) holds and hence formula (11) is valid. Indeed from (4), (5), (6) for $K(\omega) = (\omega + x_m)^{-k}\exp(-\varepsilon(\omega + 1)^{\rho_1})$ we find $Im\frac{K(i\theta + y_m)}{i\theta + y_m - x_m}$, where $\theta = \sqrt{u^2 + \alpha^2}$ for $m = 2n + 1$, $n \geq 1$ and $\theta = \alpha$ for $m = 2n$, $n \geq 2$.

$$\begin{aligned}
&Im\frac{K(i\theta + y_m)}{i\theta + y_m - x_m} \\
&= Im\frac{(i\theta + y_m + x_m)^{-k}\exp(-\varepsilon(i\theta + y - m + 1)^{\rho_1})}{i\theta + y_m - x_m} \\
&= \left(\theta^2 + (y_m + x_m)^2\right)^{\frac{-k}{2}} \cdot \exp\left(-\varepsilon\left(\sqrt{\theta^2 + (y_m + 1)^2}\right)^{\rho_1}\cos\rho_1\varphi\right) \\
&\quad \times[\theta\cos(k\psi + \varepsilon\left(\sqrt{\theta^2 + (y_m + 1)^2}\right)^{\rho_1}\sin\rho_1\varphi) \\
&\quad -(y_m - x_m)\sin(k\psi + \varepsilon\left(\sqrt{\theta^2 + (y_m + 1)^2}\right)^{\rho_1}\sin\rho_1\varphi)], \qquad (14)
\end{aligned}$$

where

$$\varphi = \arctan\frac{\theta}{y_m + 1},\quad \psi = \arctan\frac{\theta}{y_m + x_m}.$$

Now for $m = 2n+1$, $\theta = \sqrt{u^2+\alpha^2}$, $s = \alpha^2$, we define

$$C_m\phi(y,x)$$
$$= \frac{1}{K(x_m)} \frac{\partial^{n-1}}{\partial s^{n-1}} \int\limits_0^\infty \frac{\exp\left(-\varepsilon\left(\sqrt{u^2+s+(y_m+1)^2}\right)^{\rho_1} \cos\rho_1\varphi\right)}{[s+u^2+(y_m+1)^2]^{\frac{k}{2}}}$$
$$\times(\sqrt{u^2+s}\cos\left(k\psi+\varepsilon(u^2+s+(y_m+1)^2)^{\rho_1}\sin\rho_1\varphi\right)$$
$$-(y_m-x_m)\cdot\sin\left(k\psi+\varepsilon(u^2+s+(y_m+1)^2)^{\rho_1}\sin\rho_1\varphi\right))\frac{du}{\sqrt{u^2+s}}.$$

Since $-\pi < \arg\omega < \pi$, then $\frac{-\pi}{2} < \varphi < \frac{\pi}{2}$ and $\frac{-\pi}{2} < \frac{-\pi\rho_1}{2} < \rho_1\varphi < \frac{\pi}{2}$ therefore $\cos\rho_1\varphi \geq \delta_0 > 0$.

Hence, the integral on the right hand side converges uniformly. Moreover, since function $\exp\left(-\varepsilon(u^2+\alpha^2(y_m+1)^2)^{\frac{\rho_1}{2}}\right)$ is -decreasing for $u \geq 0$, and the integral converges by the Dirichlet criterion, then the mean value theorem is appeicable [2] , i.e., $\exists\xi \geq 0$, that

$$C_m\phi(y,x)$$
$$= \frac{1}{K(x_m)} \frac{\partial^{n-1}}{\partial s^{n-1}} \exp\left(-\varepsilon\left(\xi^2+s+(y_m+1)^2\right)^{\frac{\rho_1}{2}}\cos\rho_1\varphi\right)$$
$$\times\int\limits_0^\xi \frac{1}{(s+u^2+(y_m+1)^2)^{\frac{k}{2}}}[\cos\left(k\psi+\varepsilon(u^2+s+(y_m+1)^2)^{\frac{\rho_1}{2}}\sin\rho_1\varphi\right)$$
$$-\frac{y_m-x_m}{\sqrt{u^2+s}}\sin\left(k\psi+\varepsilon(u^2+s+(y_m+1)^2)^{\frac{\rho_1}{2}}\sin\rho_1\varphi\right)]du.$$

Then

$$\left|\frac{\partial^q\phi(y,x)}{\partial y_1^{q_1}...\partial y_m^{q_m}}\right| \leq C\varepsilon^n\frac{1}{K(x_m)}\exp\left(-\varepsilon\left(\xi^2+s+(y_m+1)^2\right)^{\frac{\rho_1}{2}}\cos\rho_1\varphi\right) \quad (15)$$

where $C = const$, $q = 0,1,2,\ \ q_1+...+q_m = q$.

Then by fixed $x \in D$

$$\left|\int\limits_{\partial D_R^\infty} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y\right|$$

$$\begin{aligned}
&\leq \int\limits_{\partial D_R^\infty} [|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|] \cdot [|U(y)| + |T(\partial_y,n)U(y)|]\, ds_y \\
&\leq \frac{C\varepsilon^{n+2}}{K(x_m)} \int\limits_{\partial D_R^\infty} \sum_{q=1}^{n} \left(\xi^2 + s + (y_m+1)^2\right)^{\frac{\rho_1}{2}-q} \\
&\quad \times \exp\left(-\varepsilon\left(\xi^2 + s + (y_m+1)^2\right)^{\frac{\rho_1}{2}} \cos\rho_1\varphi + |y|^\rho\right) ds_y \\
&= \frac{C\varepsilon^{n+2}\exp(-\varepsilon(x_m+1)^{\rho_1})}{(2x_m)^k} \int\limits_{\partial D_R^\infty} \sum_{q=1}^{n} \left(\xi^2 + s + (y_m+1)^2\right)^{\frac{\rho_1}{2}-q} \\
&\quad \times \exp\left(|y|^\rho - \varepsilon\left(\xi^2 + s + (y_m+1)^2\right)^{\frac{\rho_1}{2}} \cos\rho_1\varphi\right) ds_y \\
&\leq \frac{C\varepsilon^{n+2}\exp(-\varepsilon(x_m+1)^{\rho_1})}{(2x_m)^k} \cdot \int\limits_{\partial D_R^\infty} \sum_{q=1}^{n} \left(\xi^2 + \alpha^2 + (y_m+1)^2\right)^{\frac{\rho_1}{2}-q} \\
&\quad \times \exp\left(|y|^\rho - \varepsilon\left(\alpha^2 + (y_m+1)^2\right)^{\frac{\rho_1}{2}} \cos\rho_1\varphi\right) ds_y.
\end{aligned}$$

Since $|y|^\rho \sim \left(\sum\limits_{j=1}^{m=1}(y_j - x_j)^2 + (y_m+1)^2\right)^{\frac{\rho}{2}}$, $y \to \infty$, then for $0 < \rho < \rho_1 < 1$ and for some values $\widetilde{c}$ and $\varepsilon_0 > 0$. we have

$$\exp\left(|y|^\rho - \varepsilon\left(\alpha^2 + (y_m+1)^2\right)^{\frac{\rho_1}{2}} \cos\rho_1\varphi\right) \leq \widetilde{c}\exp\left(-\varepsilon_0|y|^\rho\right) \tag{16}$$

Hence, (13) is valid.

We now prove condition (13) and, hence formula (11) for $m = 2n$. By formula 5 and (14) we obtain

$$\begin{aligned}
\phi(y,x) &= [C_m K(x_m)]^{-1} \frac{\partial^{n-2}}{\partial s^{n-2}} Im \frac{K(\alpha i + y_m)}{\alpha(i\alpha + y_m - x_m)} \\
&= [C_m K(x_m)]^{-1} \frac{\partial^{n-2}}{\partial s^{n-2}} \left(s + (y_m + x_m)^2\right)^{\frac{-k}{2}} \\
&\quad \times \exp\left(-\varepsilon\left(s + (y_m+1)^2\right)^{\frac{\rho_1}{2}} \cos\rho_1\varphi\right) \\
&\quad \times [\cos\left(k\psi + \varepsilon(s + (y_m+1)^2)^{\frac{\rho_1}{2}} \sin\rho_1\varphi\right) \\
&\quad - \frac{(y_m - x_m)}{\sqrt{s}} \sin\left(k\psi + \varepsilon(s + (y_m+1)^2)^{\frac{\rho_1}{2}} \sin\rho_1\varphi\right)].
\end{aligned}$$

Since $\cos\rho_1\varphi > 0$, $\rho_1 < 1$ and k is a positive integer, then

$$\left|\frac{\partial^q \phi(y,x)}{\partial y_1^{q_1}...\partial y_m^{q_m}}\right| \le C[K(x_m)]^{-1}\exp\left(-\varepsilon\left(s+(y_m+1)^2\right)^{\frac{\rho_1}{2}}\cos\rho_1\varphi\right),$$

$q = 0,1,2$, $q = q_1 + ... + q_m$.

Hence, by (16),

$$\left|\int\limits_{\partial D_R^\infty}[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y\right|$$

$$\le \int\limits_{\partial D_R^\infty}[|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]\cdot[|U(y)| + |T(\partial_y,n)U(y)|]\,ds_y$$

$$\le C[K(x_m)]^{-1}\cdot\int\limits_{\partial D_R^\infty}\exp\left(|y|^\rho - \varepsilon\left(\alpha^2+(y_m+1)^2\right)^{\frac{\rho_1}{2}}\cos\rho_1\varphi\right)ds_y$$

$$\le C[K(x_m)]^{-1}\int\limits_{\partial D_R^\infty}\exp\left(-\varepsilon_0|y|^\rho\right)ds_y.$$

From this inequality we obtain (13) and (11).

Suppose that D is a domain in half-space $y_m > 0$, with boundary stretched to infinity and given by the equation $y_m = f(y^{'})$, where $|grad\ \ f(y')| \le c < \infty$, $y' \in R^{m-1}$ and $0 \le f \le y_0 < \infty$.

Theorem 2.1. *Let $U(x) \in B_\rho(D)$ satisfy the boundary condition*

$$\int\limits_{\partial D}\frac{|U(y)|}{1+|y|^m}ds_y < \infty, \quad \int\limits_{\partial D}\frac{|T(\partial_y,n)U(y)|}{1+|y|^{m-1}}ds_y < \infty. \tag{17}$$

If $\rho < 1$, then the formula (11) holds. Moreover

$$\phi(y,x) = \frac{1}{2\pi}ln\frac{(y_1-x_1)^2+(y_2+x_2)^2}{(y_1-x_1)^2+(y_2-x_2)^2}, \ \ m=2,$$

$$\phi(y,x) = \frac{1}{(m-2)\omega_m}\left[\frac{1}{r^{m-2}}+\frac{1}{r_1^{m-2}}\right] \ \ m \ge 3,$$

$$r^2 = |y'-x'|^2 + (y_m - x_m)^2, \ \ r_1^2 = |y'-x'|^2 + (y_m + x_m)^2$$

and $\Pi(y,x)$ is determined by formula (6).

Proof. From (15) and 6 for $k = 1$ we obtain an asymptotic estimate

$$|\Pi(y,x)| = 0\left(|y|^{-m}\right), \;\; y \to \infty, \;\; y \in \partial D$$

$$|T(\partial_y, n)\Pi(y,x)| = 0\left(|y|^{-m+1}\right), \;\; y \to \infty, \;\; y \in \partial D \tag{18}$$

Then from condition (17) it follows that the integral on the left-hand side of (13) uniformly converges relative to parameter $\varepsilon \geq 0$. Assume $\varepsilon = 0$ in formula for $K(\omega)$. Then for $m = 2$ we have

$$\begin{aligned}
&\phi(y,x)\\
= &-\frac{x_2}{\pi}\int_0^\infty Im[\frac{1}{\left(i\sqrt{u^2+\alpha^2}+y_2+x_2\right)\left(i\sqrt{u^2+\alpha^2}+y_2-x_2\right)}]\frac{udu}{\sqrt{u^2+\alpha^2}}\\
= &-\frac{x_2}{\pi}\int_0^\infty Im[\frac{1}{(it+y_2+x_2)}\cdot\frac{1}{(it+y_2-x_2)}]dt\\
= &-\frac{x_2y_2}{\pi}\int_\alpha^\infty Im[\frac{1}{(t^2+(y_2+x_2)^2)}\cdot\frac{tdt}{(t^2+(y_2-x_2)^2)}\\
= &\frac{1}{4\pi}ln\left[(y_1-x_1)^2+(y_2+x_2)^2\right]-\frac{1}{2\pi}lnr\\
= &\frac{1}{2\pi}ln\frac{(y_1-x_1)^2+(y_2+x_2)^2}{(y_1-x_1)^2+(y_2-x_2)^2}.
\end{aligned}$$

If $m = 2n+1, \;\; n \geq 1$, then

$$\begin{aligned}
\phi(y,x) &= \frac{2x_m}{C_m}\frac{d^{n-1}}{ds^{n-1}}\int_0^\infty Im\frac{i\sqrt{u^2+s}+y_m+x_m}{i\sqrt{u^2+s}+y_m-x_m}\frac{du}{\sqrt{u^2+s}}\\
&= \frac{1}{C_m}\frac{d^{n-1}}{ds^{n-1}}\int_0^\infty\left[-\frac{1}{u^2+r^2}+\frac{1}{s}\right]du\\
&= \frac{\pi}{2}\frac{1}{C_m}\frac{d^{n-1}}{ds^{n-1}}\left(\frac{1}{r}-\frac{1}{r_1}\right)\\
&= \frac{1}{(m-r)\omega_n}\left(\frac{1}{r^{m-2}}-\frac{1}{r_1^{m-2}}\right),\\
&\quad r = |y-x|, \;\; r_1^2 = s+(y_m+x_m)^2.
\end{aligned}$$

If $m = 2n,\ \ n \geq 2$, then

$$\begin{aligned}
\phi(y,x) &= \frac{2x_m}{C_m}\frac{d^{n-2}}{ds^{n-2}}\int\limits_0^\infty Im\frac{(i\alpha+y_m+x_m)^{-1}}{\alpha(i\alpha+y_m-x_m)} \\
&= \frac{1}{C_m}\frac{d^{n-2}}{ds^{n-2}}\int\limits_0^\infty \left[-\frac{1}{r}+\frac{1}{r_1}\right]\frac{1}{C_m}(-1)\cdot(-2)...\cdot(-n+1) \\
&\quad \times\left(-\frac{1}{r^{2(n-1)}}+\frac{1}{r_1^{2(n-1)}}\right) \\
&= \frac{1}{(m-r)\omega_n}\left(\frac{1}{r^{m-2}}-\frac{1}{r_1^{m-2}}\right).
\end{aligned}$$

The theorem is proved. □

Theorem 2.2. *Let $U(x) \in B_\rho(D)$, where D is half-space $y_m > 0$, and*

$$\int\limits_{y_m=0}\frac{|U(y)|}{1+|y|^m}ds_y < \infty,\quad \int\limits_{y_m=0}\frac{|T(\partial_y,n)||U(y)|}{1+|y|^{m-1}}ds_y < \infty. \tag{19}$$

If $\rho < 1$, then for $x_m > 0$ the formula holds

$$U(x) = \int\limits_{y_m=0}[\Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\}]ds_y,\ \ x \in D.$$

Proof. We set

$$K(\omega) = (\omega + x_m)^{-k}\exp(-\varepsilon(\omega+1)^\rho),$$

$$\omega = i\sqrt{u^2+\alpha^2}+y_m,\ \ \varepsilon > 0,\ \ y_m \geq 0,\ \ 0 < \rho < 1,\ \ k \in N.$$

We construct the function $\phi(y,x)$ and matrix $\Pi(y,x)$ by formula 4-6. Let $D_R = \{x \in D : |x| < R\}$, $\partial D_R = \{x \in D : |x| = R\}$, $D_R^\infty = D\backslash D_R$. Then by $x \in D$

$$\begin{aligned}
&\int\limits_{\partial D}[\Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\}]ds_y \\
&= \int\limits_{\partial D_R}[\Pi(y,x)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x)\}]ds_y
\end{aligned}$$

$$+\int\limits_{\partial D_R^\infty}[\Pi(y,x)\{T(\partial_y,n)U(y)\}-U(y)\{T(\partial_y,n)\Pi(y,x)\}]ds_y$$

$$= \quad U(x)+\int\limits_{\partial D_R^\infty}[\Pi(y,x)\{T(\partial_y,n)U(y)\}-U(y)\{T(\partial_y,n)\Pi(y,x)\}]ds_y.$$

But the last integral converges uniformly to zero when $R\to\infty$, by condition (18) and (19). The theorem is proved. □

3 Regularization of the solution of the Cauchy problem for the multidimensional Lame system in infinite domains

Suppose that $D\subset R^m$ lies in a shall $0<y_m<h$, $h=\frac{\pi}{\rho}$, $\rho>0$, and its boundary consists of hyperplane $y_m=0$ and of some smooth surface S, with given equation $y_m=f(y')$, $y'\in R^{m-1}$, where $0<f(y')\le h$ and $|grad\ f(y')|\le c\le\infty$.

Suppose that $U(x)\in A(D)$ and

$$|U(y)|+|T(\partial y,n)U(y)|\le M,\ \ y\in\partial D. \tag{20}$$

Then formula (11), is valid, where

$$K(\omega)=(\omega-x_m+2h)^{-1}\exp\sigma\omega,\ \ \omega=i\sqrt{u^2+\alpha^2}+y_m,$$

$$K(x_m)=(2h)^{-1}\exp\sigma x_m,\ \ 0<x_m<h.$$

We denote

$$U_\sigma(x)=\int\limits_S[\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\}-U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y,\ \ x\in D.$$

Theorem 3.1. *Let $U(x)\in A(D)$ satisfy the boundary condition* (20). *Then*

$$|U(x)-U_\sigma(x)|\le MC(\rho,x)\sigma^{n+1}\exp(-\sigma x_m),x\in D, \tag{21}$$

where $C(\rho,x)=C(\rho)\int\limits_{y_m=0}\frac{ds_y}{r^{m+1}}$.

In the sequel we denote by $C(\rho)$ any constant depending on p. It may different in diverse applications.

Proof. From formula (11), we have

$$\begin{aligned}
&|U(x) - U_\sigma(x)| \\
&\leq \int\limits_{y_m=0} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y)\} - U(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]\, ds_y \\
&\leq \int\limits_{y_m=0} [\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]\, [|U(y)| + |T(\partial_y,n)U(y)|]\, ds_y \\
&\leq M \int\limits_{y_m=0} [\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]\, ds_y.
\end{aligned}$$

We estimate

$$I = \int\limits_{y_m=0} [\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]\, ds_y.$$

From the formula (4), (5) and (6):

$$\begin{aligned}
Im\frac{K(i\theta + y_m)}{i\theta + y_m - x_m} &= \frac{\exp \sigma y_m}{(u^2+r^2)(r^2+4h(y_m-x_m)+4h^2+u^2)} \\
&\times \left[((y_m - x_m)(y_m - x_m + 2h) - \theta^2)\right. \\
&\times \left.\sin\sigma\theta - 2\theta(y_m - x_m + h)\cos\sigma\theta\right].
\end{aligned}$$

where $\theta = \sqrt{u^2+\alpha^2}$ for $m = 2n+1$, $n \geq 1$ and $\theta = \alpha$ for $m = 2n$, $n \geq 2$. □

Let for $m = 2n+1$, $n \geq 1$,

$$\begin{aligned}
&C_m\phi(y,x,\sigma) = 2h \exp\sigma(y_m - x_m)\frac{d^{n-1}}{ds^{n-1}} \\
&\times \int\limits_0^\infty \frac{1}{(u^2+s+(y_m-x_m))^2(s+(y_m-x_m)^2+u^2+4h(y_m-x_m)+4h^2)} \\
&\times \Big[((y_m - x_m)(y_m - x_m + 2h) - s - u^2) \\
&\times \sin\sigma\sqrt{u^2+s} - 2\sqrt{u^2+s}(y_m - x_m + h)\cos\sigma\sqrt{u^2+s}\Big] \cdot \frac{du}{\sqrt{u^2+s}}.
\end{aligned}$$

Here the mean value theorem is applicable, is since $\frac{1}{s+(y_m-x_m)^2+u^2}$ decreasing and bounded, therefore $\exists\xi \geq 0$ such that

$$C_m\phi(y,x,\sigma)$$
$$2h\exp(\sigma y_m - \sigma x_m)\cdot\frac{d^{n-1}}{ds^{n-1}}\cdot\frac{1}{s+(y_m-x_m)^2+\xi^2}$$
$$\times\int\limits_0^{\xi}\left[\frac{(y_m-x_m)(y_m-x_m+2h)-s-u^2}{u^2+s+(y_m-x_m+2h)^2}\frac{\sin\sigma\sqrt{u^2+s}}{\sqrt{u^2+s}}\right.$$
$$\left.-\frac{y_m-x_m+h}{u^2+s+(y_m-x_m)^2}\cos\sigma\sqrt{u^2+s}\right]du$$
$$=\frac{(-1)^{n-1}(n-1)!2h(\exp(\sigma y_m-\sigma x_m))}{(s+(y_m-x_m)^2+\xi^2)^n}\cdot\frac{d^{n-1}}{ds^{n-1}}$$
$$\times\int\limits_0^{\xi}(\frac{(y_m-x_m)(y_m-x_m+2h)-s-u^2}{u^2+s+(y_m-x_m+2h)^2}$$
$$\times\frac{\sin\sigma\sqrt{u^2+s}}{\sqrt{u^2+s}}-\frac{y_m-x_m+h}{u^2+s+(y_m-x_m)^2}\cos\sigma\sqrt{u^2+s})du$$
$$\leq\frac{C\sigma^{n-1}\exp(\sigma y_m-\sigma x_m)}{r^{m-1}}$$

Hence, we have

$$\left|\frac{\partial^q\phi(y,x,\sigma)}{\partial y_1^{q_1}...\partial y_m^{q_m}}\right| \leq \frac{C\sigma^{n+1}\exp(\sigma y_m-\sigma x_m)}{r^{m+1}}, \quad q=q_1+...+q_m, \;\; q=0,1,2.$$

and

$$|\Pi(y,x,\sigma)|+|T(\partial_y,n)\Pi(y,x,\sigma)|\leq\frac{C\sigma^{n+1}\exp(\sigma y_m-\sigma x_m)}{r^{m+1}}. \tag{22}$$

Then $I \leq MC(\rho,x)\sigma^{n+1}\exp(-\sigma x_m)$.

For $m=2n$ the Theorem is proved similarly.

Let us formulate a result that allows us to approximately solve the problem, if $U(y)$ and $T(\partial_y,n)U(y)$ on S are replaced by their continuous approximations $f_\delta(y)$ and $g_\delta(y)$, respectively, i.e.,

$$\max_S|U(y)-f_\delta(y)|+\max_S|T(\partial_y,n)U(y)-g_\delta(y)|<\delta,\;\; 0<\delta<1.$$

We define the functions $U_{\sigma\delta}(x)$ by

$$U_{\sigma\delta}(x) = \int_S [\Pi(y,x,\sigma)g_\delta(y) - f_\delta(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]\,ds_y, \;\; x \in D.$$

The following theorem is valid.

Theorem 3.2. *Let $U(x) \in A(D)$ satisfy the boundary condition* (20). *Then we have the estimate*

$$|U(x) - U_{\sigma\delta}(x)| \le M^{1-\frac{x_m}{n}} C(\rho,x)\delta^{\frac{x_m}{n}} \left[ln\left(\frac{M}{\delta}\right)\right]^{n+1}, \quad x \in D,$$

where $\sigma = h^{-1} ln \frac{M}{\delta}$, $C(\rho,x) = C(\rho) \int\limits_{\partial D} \frac{ds_y}{r^{m+1}}$.

Proof. From formula (11), (21), (22) and by assumption we obtains

$$\begin{aligned}
&|U(x) - U_{\sigma\delta}(x)| \\
&\quad \le \Big| \int_{y_m=0} [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y) - g_\delta(y)\} \\
&\quad + (U(y) - f_\delta(y))\{T(\partial_y,n)\Pi(y,x,\sigma)\}]\,ds_y \Big| \\
&\quad + \Big| \int_S [\Pi(y,x,\sigma)\{T(\partial_y,n)U(y) - g_\delta(y)\} \\
&\quad + (U(y) - f_\delta(y))\{T(\partial_y,n)\Pi(y,x,\sigma)\}]\,ds_y \Big| \\
&\quad \le MC(\rho,x)\sigma^{n+1}\exp(-\sigma x_m) \\
&\quad + \delta \int_S [|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]\,ds_y \\
&\quad \le MC(\rho,x)\sigma^{n+1}\exp(-\sigma x_m) \\
&\quad + \delta \int_S \frac{C_2(\rho)\sigma^{n+1}\exp(\sigma y_m - \sigma x_m)}{r^{m+1}} ds_y \\
&\quad \le C(\rho,x)\sigma^{n+1}(M\exp(-\sigma x_m) + \delta\exp(h\sigma - \sigma x_m)) \\
&\quad = C(\rho,x)\sigma^{n+1}\exp(-\sigma x_m)(M + \delta \exp h\sigma).
\end{aligned}$$

Now, for the choice of the parameter $M = \delta \exp h\sigma$, we derive

$$|U(x) - U_{\sigma\delta}(x)| \le C(\rho, x) M^{1-\frac{x_m}{n}} h^{-(n+1)} \left(ln \frac{M}{\delta} \right)^{n+1} \delta^{\frac{x_m}{n}}.$$

The theorem is proved. □

Corollary 3.3. *The limit relations*

$$\lim_{\sigma \to \infty} U_\sigma(x) = U(x), \;\; \lim_{\delta \to 0} U_{\sigma\delta}(x) = U(x)$$

hold uniformly on each compact subset of D.

By slightly modifying the techniques, it is possible to generalize obtained results. Concerning the surface S let‘s assume that it is smooth and its area satisfies the growth condition

$$\int_S \exp\{-b_0 ch\rho_0 |y'|\} ds_y < \infty, \;\; 0 < \rho_0 < \rho. \tag{23}$$

Let $U(x) \in A(D)$ satisfy a growth condition

$$|U(y)| + |T(\partial_y, n)U(y)| \le C \exp\left(a \cos \rho_1 (y_m - \frac{h}{2}) \cdot \exp \rho_1 |y'| \right), \;\; y \in \partial D. \tag{24}$$

Suppose that given $f_\delta(y)$, $g_\delta(y)$ in S, satisfies following inequality:

$$|f_\delta(y)| + |g_\delta(y)| \le \exp\left(a \cos \rho_1 (y_m - \frac{h}{2}) \cdot \exp \rho_1 |y'| \right), \;\; y \in S, \tag{25}$$

$$\begin{aligned} &|U(y) - f_\delta(y)| + |T(\partial y, n)U(y) - g_\delta(y)| \\ &\quad \le \delta \exp\left(a \cos \rho_1 (y_m - \frac{h}{2}) \cdot \exp \rho_1 |y'| \right), \;\; y \in S, \end{aligned}$$

where $0 < \delta < 1$, $0 < \rho_1 < \rho$, $a \ge 0$. Put

$$K(\omega) = \exp\left(\sigma\omega - bchi\rho_1 \left(\omega - \frac{h}{2} \right) - b_1 chi\rho_0 \left(\omega - \frac{h}{2} \right) \right),$$

where $\omega = i\sqrt{u^2 + \alpha^2} + y_m$, $0 < x_m < h$, $b \ge 0$, $b_1 \ge b_0 (\cos \rho_0 \frac{h}{2})^{-1} + \varepsilon$, $\varepsilon > 0$, $\alpha = |y' - x'|$, $b = 2a \exp \rho_1 |x'|$, $\sigma = h^{-1} ln\delta^{-1}$.

Under these conditions, we define

$$U_{\sigma\delta}(x) = \int\limits_S [\Pi(y,x,\sigma)g_\delta(y) - f_\delta(y)\{T(\partial_y,n)\Pi(y,x,\sigma)\}]ds_y, \ \ x \in D.$$

Theorem 3.4. *Let $U(x) \in A(D)$ satisfy conditions* (24),(25) *and surface S satisfy condition* (23)*, then*

$$|U(x) - U_{\sigma\delta}(x)| \\ \leq C(x)\delta^{\frac{x_m}{h}}\left(ln\frac{1}{\delta}\right)\cdot\exp\left(2a\cos\rho_1(x_m - \frac{h}{2})\cdot\exp\rho_1|x'|\right), x \in D, \tag{26}$$

where

$$C(x) = C(\rho)\int\limits_{\partial D}\frac{1}{r^m}\exp(-b_0 ch\rho_0\alpha) \\ \times \exp\left[\frac{b}{2}\cos\rho_1\frac{h}{2}\left(\exp\rho_1\left(|y'| - |x'|\right) - \exp\rho_1|y' - x'|\right)\right]ds_y.$$

Proof. By the assumptions of the theorem

$$\begin{aligned} &|U(x) - U_{\sigma\delta}(x)| \\ &\leq \int\limits_S [|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|] \\ &\times[U(y) - f_\delta(y)| + |T(\partial_y,n)U(y) - g_\delta(y)|]ds_y \\ &+ \int\limits_{y_m=0} [|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|] \\ &\times[|U(y)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]ds_y \qquad (27) \\ &\leq \delta\int\limits_S \exp\left(a\cos\rho_1(y_m - \frac{h}{2})\cdot\exp\rho_1|y'|\right) \\ &\times[|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]ds_y \\ &+C\int\limits_{y_m=0} \exp\left(a\cos\rho_1\frac{h}{2})\cdot\exp\rho_1|y'|\right) \\ &\times[|\Pi(y,x,\sigma)| + |T(\partial_y,n)\Pi(y,x,\sigma)|]ds_y. \end{aligned}$$

From (9) we obtain

$$\begin{aligned}
&|\Pi(y,x,\sigma)| + |T(\partial_y, n)\Pi(y,x,\sigma)| \\
&\leq \frac{C(\rho)}{r^m}\sigma^{n+1}\exp(\sigma(y_m - x_m)) \\
&\times \exp\left(\frac{-b}{2}\cos\rho_1(y_m - \frac{h}{2})\exp\rho_1\alpha - b_1\cos\rho_0(y_m - \frac{h}{2})ch\rho_0\alpha\right) \\
&\times \exp\left(b\cos\rho_1(x_m - \frac{h}{2})\right).
\end{aligned}$$

Then from (27)

$$\begin{aligned}
&|U(x) - U_{\sigma\delta}(x)| \\
&\leq C\sigma^{m+1}\delta\int_S \exp\left(a\cos\rho_1(y_m - \frac{h}{2})\cdot\exp\rho_1|y'|\right) \\
&\times \exp\left(b\cos\rho_1(x_m - \frac{h}{2})\right)\cdot\frac{1}{r^m}\exp\sigma(y_m - x_m) \\
&\times \exp\left(-\frac{b}{2}\cos\rho_1(y_m - \frac{h}{2})\exp\rho_1\alpha - b_0 ch\rho_0\alpha\right)ds_y \\
&+C\sigma^{n+1}\int_{y_m=0} \exp\left[a\cos\rho_1\frac{h}{2}\exp\rho_1|y'|\right] \\
&\times \exp\left(-\sigma x_m + b\cos\rho_1(x_m - \frac{h}{2})\right) \\
&\times \exp\left(-\frac{b}{2}\cos\rho_1\frac{h}{2}\exp\rho_1\alpha\right)\cdot\exp(-b_0 ch\rho_0\alpha)\frac{1}{r^m}ds_y \\
&\leq C\sigma^{n+1}\delta\exp\sigma h\exp(-\sigma x_m) \\
&\times \int_S \exp\left[\frac{b}{2}\cos\rho_1(y_m - \frac{h}{2})\exp\rho(|y'| - |x'|)\right. \\
&\left. - \frac{b}{2}\cos\rho_1(y_m - \frac{h}{2})\exp\rho_1\alpha\right] \\
&\times \exp\left(-b\cos\rho_0(y_m - \frac{h}{2})ch\rho_0\alpha\right) \\
&\times \exp\left[b\cos\rho_1(x_m - \frac{h}{2})\right]\frac{1}{r^m}ds_y + C\sigma^{n+1}\exp(-\sigma x_m)
\end{aligned}$$

$$\int\limits_{y_m=0} \exp\left[\frac{b}{2}\cos\frac{\rho_1 h}{2}) \exp\rho_1(|y'|-|x'|) - \frac{b}{2}\cos\frac{\rho_1 h}{2}) \exp\rho_1|y'-x'|\right]$$
$$\times \exp\left[b\cos\rho_1(x_m - \frac{h}{2}) - b_0 ch\rho_0\alpha\right]\frac{1}{r^m}ds_y$$
$$\le \sigma^{n+1}(\delta\exp\sigma h + 1)\exp(-\sigma x_m)\exp\left(b\cos\rho_1(x_m - \frac{h}{2})\right)$$
$$\times \int\limits_{\partial D} \exp\left[\frac{b}{2}\cos\frac{\rho_1 h}{2}\,(\exp\rho_1(|y'|-|x'|) - \exp\rho_1|y'-x'|)\right.$$
$$\left.\times \exp[-b_0 ch\rho_0\alpha]\right]\frac{1}{r^m}ds_y.$$

If we now take $\sigma = h^{-1}ln\frac{1}{\delta}$ and $b = 2a\exp\rho_1|x'|$, then we obtain (26). The theorem is proved. □

Corollary 3.5. *The limit relation*

$$\lim_{\delta\to 0} U_{\sigma\delta}(x) = U(x)$$

holds uniformly on each compact subset of D.

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