Uniform Compact Attractors for a Nonlinear Non-autonomous Equation of Viscoelasticity

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Abstract

In this paper we establish the regularity, exponential stability of global (weak) solutions and existence of uniform compact attractors of semiprocesses, which are generated by the global solutions, of a two-parameter family of operators for the nonlinear 1-d non-autonomous viscoelasticity. We employ the properties of the analytic semigroup to show the compactness for the semiprocess generated by the global solutions.

Key words: exponential stability, semiprocess, absorbing set, C_0-semigroup, uniform compact attractor.

1 Introduction

In this paper we prove the regularity, exponential stability of global solutions and existence of uniform compact attractors of semiprocesses, generated by the global solutions, of a two-parameter family of operators for the following nonlinear 1-d non-autonomous system of viscoelasticity (see, e.g., [5-10, 19-22, 24, 35-37])

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$$y_{tt} = \sigma(y_x)y_{xx} + y_{xtx} + f(x,t),\ (x,t) \in (0,1) \times (\tau, +\infty), \tag{1}$$
$$y(0,t) = y(1,t) = 0, \quad t \geq \tau, \tag{2}$$
$$y(x,\tau) = y_0^\tau(x), y_t(x,\tau) = y_1^\tau(x), \quad x \in (0,1), \tau \in \mathbb{R}^+ = [0, +\infty), \tag{3}$$

where $y = y^\tau(x,t)(\tau \in \mathbb{R}^+)$ is an unknown function, σ is a real function defined on $\mathbb{R}$, $f = f(x,t)$ is an external forcing term.
Since the 1960s, the global well-posedness, asymptotic behaviour of solutions and the investigation of the associated infinite-dimensional dynamical system became the most essential aspects in the field of nonlinear evolution equations. For instance, the global well-posedness and large-time behaviour of solutions to the problem (1)-(3) have attracted many mathematicians (see., e.g., [5-10,19-22,24,35-37] and the references cited therein), and the development has seen great progress since then. Let us recall some of these achievements. When $\sigma(s)$ is smooth enough, $\sigma(s) > 0$, for all $s \in \mathbb{R}, f \equiv 0$ (autonomous case), Greenberg et al [8, 10] established existence, uniqueness and stability of smooth solutions and Greenberg et al [9], and Nishihara [22] proved the exponential decay of smooth solutions. Yamada [36] weakened the regularity of initial data to obtain the global existence and exponential stability of smooth solutions to the non-autonomous problem (1)-(3) (i.e., $f \not\equiv 0$). MacCamy [21] also obtained existence, uniqueness and stability of smooth solutions to the more general equation $u_{tt} = \frac{\partial}{\partial x}(\sigma(u_x)u_x + \lambda(u_x)u_{xt})$. When $\sigma = \sigma(s)$ changes sign in $s \in \mathbb{R}$, which corresponds to the model of phase transition, Andrew and Ball [1,2], and Pego [24] established global existence, uniqueness and/or asymptotic behaviour of (weak) solutions for the equation (1) with some boundary conditions. Kuttler and Hicks [19] proved the global existence and uniqueness of weak solutions for the more general equation $u_{tt} = (\sigma(u_x))_x + (\alpha(u_x)u_{xt})_x + f$ with some different boundary conditions from (2). Liu et al [20] proved the global existence of solutions to the initial boundary value problem or the periodic boundary problem or initial value problem of (1). Tsutsumi [35] obtained the global existence of solutions to the initial boundary value problem of (1). Yang and Song [37] established blowup results for the initial boundary value problem of (1) with four types of boundary conditons and special versions of $\sigma(s)$.
As far as the associated infinite-dimensional dynamics is concerned, we refer to the works [3, 4, 7, 13, 17, 18, 25, 26, 28-34, 38, 39] and the references therein for related models. Generally speaking, for a given nonlinear evolution equation, once a global solution for all time $t > 0$ has been established, a natural and interesting question is to ask the asymptotic behaviour of the global solution as t tends to infinity. The study of the asymptotic behaviour of global solutions to nonlinear evolution equations as time goes to infinity can be divided into two categories. The first one is to investigate the asymptotic behaviour of a solution for any *given* initial datum. The second one is to investigate the asymptotic behaviour of all global solutions when the initial data vary *in a bounded set*. The second category corresponds to the infinite-dimensional dynamics for nonlinear evolution equations. In this paper, we shall study these

two categories of asymptotic behaviour of global solutions to the problem (1)-(3).
Let us now compare our results with those of other authors. Hoff and Ziane [17, 18] proved the existence of a compact (global) attractor for the one-dimensional *isentropic* compressible viscous flow in a finite interval. Concerning a one-dimensional and a multi-dimensional spherically symmetric heat-conductive viscous *non-isentropic* ideal gas, Zheng and Qin [38, 39] proved the existence of maximal (universal)*weak* attractors. Qin and Rivera [28, 29] established the existence of universal *weak* attractors for a one-dimensional heat-conductive real gas and for a compressible flow between two horizontal parallel plates in $\mathbb{R}^3$. Furthermore, Qin et al [25, 26, 30] recently established the existence of a maximal *weak* attractor in H^4, which corresponds to the orbit governed by the classical solution, for a one-dimensional heat-conductive real gas and a multi-dimensional spherically symmetric heat-conductive viscous *non-isentropic* ideal gas and for the one-dimensional thermoviscoelasticity. Note that the attractors established in [25, 26, 28-30, 38, 39] are all in the weak sense, that is, the orbits associated with the global solution are compact in the weak topology in H^i $(i = 1, 2, 4)$, where the abstract framework in [11] was used and the sequence of closed subsequences was established to overcome the lack of compactness (in the strong topology) of the orbit generated by the global solution. In this paper, we employ the properties of an analytic semigroup and delicate new estimates to establish the existence of uniform compact attractors in the strong sense (i.e., in the strong topology of $H^i\,(i = 1, 2, 4)$) for the semiprocesses of a two-parameter family of operators. This is the first new result of this paper. Note that the problem (1)-(3) is a non-autonomous problem with the non-autonomous term f, so the situation here is quite different from those encountered in [25, 26, 28-30, 38, 39] where, caused by the lack of the non-autonomous term f in (1), it is not necessary to consider the influence of this non-autonomous term to the existence of global solutions and weak attractors, while in this paper, because of the non-autonomous term f in (1), we have to investigate in detail the influence of f to the existence of global solutions and uniform (with respect to f) compact attractors in $H^i\,(i = 1, 2, 4)$. This is the second new result of this paper. Moreover, it is well known that continuous dependence of solutions on initial data is very important, especially when we study infinite-dimensional dynamics (which is equivalent to the fact that the associated semigroup or semiprocess of a two-parameter family of operators is continuous with respect to initial data or the semigroup or semiprocess of a two-parameter family of operators as an operator is continuous for any but fixed time t and parameter τ), that is, the semiprocess is $(H^i_+ \times \Sigma, H^i_+)$-continuous $(i = 1, 2, 4)$ (see the definitions of H^i_+ and Σ below). For example, we refer to the works by Hoff and Serre [14], Hoff [16] and Hoff and Zarnowski [15] and the references therein. In this paper, we establish such a $(H^i_+ \times \Sigma, H^i_+)$-continuity of the semiprocess. This is the third new result of the present paper. The differences between this and the above mentioned papers [25, 26, 28-30, 38, 39] essentially lies in

the following points: semiprocess via semigroup; non-autonomous system via autonomous system; uniform (strong) compact attractor via weak compact attractor and $(H_+^i \times \Sigma, H_+^i)-$continuity of semiprocess via $(H_+^i, H_+^i)-$ continuity of semigroup (i.e., continuous dependence of solutions in H_+^i on initial data, $i = 1, 2, 4$).
The main objectives of this paper can be summarized as follows:
(1) establish existence and exponential stability of global solutions in H_+^i $(i = 1, 2, 4)$ and the existence of the semiprocess of a two-parameter family of operators generated by the global solutions in H_+^i $(i = 1, 2, 4)$ for any fixed external forcing term $f(x, t)$;
(2) prove existence of uniform (with respect to $f \in \Sigma$) compact attractors for semiprocesses, generated by the global solutions in H_+^i $(i = 1, 2, 4)$, of a two-parameter family of operators for any external forcing term $f(x, t)$ varying in Σ which is a symbol set or a space specified for the different cases later on.
Here we observe that the global (regular) solution in H_+^i $(i = 1, 2)$ is not a classical solution. In order to investigate the infinite-dimensional dynamics determined by the global classical solutions, in this paper we first establish the existence of global solutions, the semiprocess of two-parameter family of operators and uniform compact attractor in H_+^4; then by the standard Sobolev embedding theorem, it turns out that the global solution in H_+^4 is in fact a classical solution, and the uniform compact attractor in H_+^4 corresponds indeed to the semiprocess governed by the classical solutions.
Now we define

$$H_+^i = \{(u, v) \in H^i[0, 1] \times H^i[0, 1] : v|_{x=0} = v|_{x=1} = 0, \int_0^1 u dx = 0\}, i = 1, 2, 4$$

which become three metric spaces when equipped with the metrics induced from the usual norms. Here H^1, H^2, H^4 are the usual Sobolev spaces.
We first transform problem (1)-(3) into the following system

$$u_t = v_x, (x, t) \in (0, 1) \times (\tau, +\infty), \tag{4}$$
$$v_t = \sigma(u)u_x + v_{xx} + f(x, t), (x, t) \in (0, 1) \times (\tau, +\infty), \tag{5}$$
$$v(0, t) = v(1, t) = 0, t \geq \tau, \tag{6}$$
$$u(\tau, x) = y_x^\tau \equiv u_0^\tau, v(\tau, x) = y_1^\tau(x) \equiv v_0^\tau(x), x \in (0, 1), \tau \in \mathbb{R}^+ \tag{7}$$

where

$$u = u^\tau(x, t) = y_x^\tau(x, t),\ v = v^\tau(x, t) = y_t^\tau(x, t).$$

Here we can regard the external forcing term f as a symbol for this non-autonomous system (4)-(7) with a parameter $\tau \in \mathbb{R}^+$.
The notation in this paper will be as follows:
$H^i = W^{i,2}, H_0^i = W_0^i (i = 1, 2, 4, 6)$ denote the usual (Sobolev) spaces on $(0, 1)$.
In addition $\| \cdot \|_B$ denotes the norm in the space B; we also set $\| \cdot \| = \| \cdot \|_{L^2}$.
We denote by $W^{k,\bar{p}}(I, B), k \in \mathbb{N}_0, 1 \leq \bar{p} \leq \infty$ the space of functions whose

derivatives up to the order k are $L^{\bar{p}}$ integrable from $I \subseteq \mathbb{R}$ into a Banach space B. Subscripts t and x denote the (partial) derivatives in t and x, respectively. Here we only state results for the problem (4)-(7), while the analogous conclusions for the problem (1)-(3) can be easily drawn and will not be presented in this paper.
Our main results are as follows:

Theorem 1 *Let*

$$\sigma(s) \in C^1(\mathbb{R}), \sigma(s) > 0, s \in \mathbb{R}, \tag{8}$$

and set

$$E_1 = L^2(\mathbb{R}^+, L^2(0,1)) \cap L^1(\mathbb{R}^+, L^2(0,1)).$$

Then for any fixed $f \in E_1$ and for any $(u_0^\tau, v_0^\tau) \in H_+^1, \tau \in \mathbb{R}^+$, problem (4)-(7) admits a unique global (regular) weak solution $(u^\tau(t), v^\tau(t)) \in H_+^1$, which generates a unique semiprocess $\left\{U_f^{(1)}(t,\tau)\right\}$ on H_+^1 of a two-parameter family of operators such that for any $t \geq \tau \geq 0$,

$$U_f^{(1)}(t,\tau)(u_0^\tau, v_0^\tau) = (u^\tau(t), v^\tau(t)) \in H_+^1, \tag{9}$$

$$\|(u^\tau(t), v^\tau(t))\|_{H^1}^2 + \int_\tau^t (\|u^\tau\|_{H^1}^2 + \|v^\tau\|_{H^2}^2 + \|v_t^\tau\|^2)(s)ds \leq C_1(\tau); \tag{10}$$

here and in the sequel $C_1(\tau) = C_1(\|(u_0^\tau, v_0^\tau)\|_{H_+^1}, \|f\|_{E_1}) > 0$ is a generic constant. If we further assume that

$$\|f(t)\| \leq \hat{C}_1 e^{-\gamma_1 t}, \ \ t \geq \tau \geq 0 \tag{11}$$

for some positive constants γ_1 and $\hat{C}_1$, then there exists a constant $\beta_1 = \beta_1(C_1(\tau)) > 0$ such that for any fixed $\beta \in (0, \beta_1]$, and any $t \geq \tau \geq 0$ we have

$$e^{\beta t}\|(u^\tau(t), v^\tau(t))\|_{H^1}^2 + \int_\tau^t e^{\beta s}(\|u^\tau\|_{H^1}^2 + \|v^\tau\|_{H^2}^2 + \|v_t^\tau\|^2)(s)ds \leq C_1(\tau)e^{\beta\tau}. \tag{12}$$

The semiprocess $\left\{U_f^{(1)}(t,\tau)\right\}$ $(f \in E_1, t \geq \tau \geq 0)$ is $\left(H_+^1 \times E_1, H_+^1\right)$-continuous in the sense that for all fixed t and $\tau, t \geq \tau, \tau \in \mathbb{R}^+$, the mapping $\left((u,v), f\right) \to U_f^{(1)}(t,\tau)(u,v)$ is continuous from $H_+^1 \times E_1$ to H_+^1. Moreover, every fixed *semiprocess $\left\{U_f^{(1)}(t,\tau)\right\}$ (i.e., $f \in E_1$ fixed) possesses a (non-uniform) compact attractor $\mathcal{A}_{\{f\}}^{(1)}$ in H_+^1.*

The following result concerns the uniform compact attractor in H_+^1. Observe the following important fact: The properly defined (uniform) attractor $\mathcal{A}$ of problem (4)-(7) with the symbol f_0 must be simultaneously the attractor of each problem (4)-(7) with the symbol $f(t) \in H_+(f_0)$, which is called the hull of f_0 and defined as

$$\Sigma = H_+(f_0) = \left[f_0(t+h)|h \in \mathbb{R}^+\right]_E,$$

where $\left[\,\cdot\,\right]_E$ denotes the closure in some Banach space E.

Theorem 2 *Assume that* (8) *holds, and let*

$$\Sigma_1 = H_+(f_1) = \left[f_1(t+h)|h \in \mathbb{R}^+\right]_{E_1},$$

where $f_1 \in E_1$ is an arbitrary but fixed function. Then for any $f \in \Sigma_1$ and for any $(u_0^\tau, v_0^\tau) \in H_+^1, \tau \in \mathbb{R}^+$, problem (4)-(7) *admits a unique global* (*regular*) *weak solution $(u^\tau(t), v^\tau(t)) \in H_+^1$, which generates a unique semiprocess $\left\{U^{(1)}_{f|\Sigma_1}(t,\tau)\right\}$ on H_+^1 of a two-parameter family of operators such that* (9)-(10) *hold, where $\left\{U^{(1)}_f(t,\tau)\right\}$ should be replaced by $\left\{U^{(1)}_{f|\Sigma_1}(t,\tau)\right\}$; if* (11) *holds, then estimate* (12) *still holds where $C_1(\tau)$ should be replaced by $C_1^*(\tau) = C_1(\|(u_0^\tau, v_0^\tau)\|_{H^1}, \|f_1\|_{E_1})$. Moreover, the semiprocess $\left\{U^{(1)}_{f|\Sigma_1}(t,\tau)\right\}(f \in \Sigma_1, t \geq \tau \geq 0)$ possesses a uniform* (*with respect to $f \in \Sigma_1$*) *compact attractor $\mathcal{A}_{\Sigma_1}$ satisfying*

$$\bigcup_{f\in\Sigma_1} \mathcal{A}^{(1)}_{\{f\}} \subseteq \mathcal{A}_{\Sigma_1}.$$

Remark 3 *When $\tau = 0$, the solution $(u^0(t), v^0(t)) \equiv (u(t), v(t))$ is the global solution in H_+^1 to problem* (4)-(7). *Moreover, if $f \equiv$ constant, which corresponds to the autonomous system, then the semiprocess $\left\{U^{(1)}_f(t,0)\right\}$ reduces to a C_0-semigroup on H_+^1.*

The following theorem concerns the case where the symbol space takes the form $\hat{\Sigma}_1 = H_+(\hat{f}_1)$ where $\hat{f}_1 \in E_1 \subseteq \hat{E}_1 \equiv L^2_{\text{loc},w}(\mathbb{R}^+, L^2(0,1))$ is a translation compact function in $\hat{E}_1$ in the weak topology, which means that $H_+(\hat{f}_1)$ is compact in $\hat{E}_1$. To this end, we consider the Banach space $L^p(\mathbb{R}^+, E)$ $(1 \leq p < +\infty)$ of functions $g(s), s \in \mathbb{R}^+$ with values in a Banach space E that are locally integrable to the power p in the Bochner sense. In particular, for any time interval $[t_1, t_2] \subseteq \mathbb{R}^+, \int_{t_1}^{t_2} \|g(s)\|_E^p ds < +\infty$. We denote by $L^p_{\text{loc},w}(\mathbb{R}^+, E)$ $(1 < p < +\infty)$ the space $L^p_{\text{loc}}(\mathbb{R}^+, E)$ $(1 < p < +\infty)$ endowed with the local weak convergence topology of; here E is a reflexive separable Banach space. Generally, we still denote by $W^{m,p}_{\text{loc},w}(\mathbb{R}^+, E)$ ($m \geq 0$ an integer,$1 < p < +\infty$) the space $W^{m,p}_{\text{loc}}(\mathbb{R}^+, E)$ endowed with the local weak convergence topology, here the space $W^{m,p}_{\text{loc}}(\mathbb{R}^+, E)$ consists of all elements $g(s)$ whose all time derivatives $\partial_s^i g(s)$ $(0 \leq i \leq m)$ up to order m belong to $L^p_{\text{loc}}(\mathbb{R}^+, E)$. Thus

$W^{0,p}_{\text{loc},w}(\mathbb{R}^+, E) \equiv L^p_{\text{loc},w}(\mathbb{R}^+, E), W^{m,2}_{\text{loc},w}(\mathbb{R}^+, E) \equiv H^m_{\text{loc},w}(\mathbb{R}^+, E)$. Similarly, we introduce a Banach space $W^{m,p}_b(\mathbb{R}^+, E)$ of functions $g(s) \in L^p_{\text{loc}}(\mathbb{R}^+, E)$ characterised by the conditions $\partial^i_s g(s) \in L^p_{\text{loc}}(\mathbb{R}^+, E)(0 \le i \le m)$, and

$$\sum_{i=0}^{m} \sup_{t\in\mathbb{R}^+} \int_t^{t+1} \|\partial^i_s g(s)\|^p_E ds < +\infty$$

and equipped the norm

$$\|g\|_{W^{m,p}_b(\mathbb{R}^+,E)} = \Big\{ \sum_{i=0}^{m} \sup_{t\in\mathbb{R}^+} \int_t^{t+1} \|\partial^i_s g(s)\|^p_E ds \Big\}^{1/p}.$$

In particular, we have,

$$W^{0,p}_b(\mathbb{R}^+, E) \equiv L^p_b(\mathbb{R}^+, E), \quad W^{m,2}_b(\mathbb{R}^+, E) \equiv H^m_b(\mathbb{R}^+, E).$$

Theorem 4 *Assume that* (8) *holds, and let*

$$\hat{\Sigma}_1 = H_+(\hat{f}_1) = \Big[\hat{f}_1(t+h)|h \in \mathbb{R}^+\Big]_{\hat{E}_1}$$

where $\hat{f}_1 \in E_1$ is an arbitrary but fixed function. Then for any $f \in \hat{\Sigma}_1$ and for any $(u^\tau_0, v^\tau_0) \in H^1_+, \tau \in \mathbb{R}^+$, problem (4)-(7) *admits a unique global (regular) weak solution $(u^\tau(t), v^\tau(t)) \in H^1_+$, which generates a unique semiprocess $\Big\{U^{(1)}_{f|\hat{\Sigma}_1}(t,\tau)\Big\}$ on H^1_+ of a two-parameter family of operators such that* (9)-(10) *hold where $\Big\{U^{(1)}_f(t,\tau)\Big\}$ should be replaced by $\Big\{U^{(1)}_{f|\hat{\Sigma}_1}(t,\tau)\Big\}$; if* (11) *holds, then estimate* (12) *still holds where $C_1(\tau)$ should be replaced by $\hat{C}^*_1(\tau) = \hat{C}^*_1(\|(u^\tau_0, v^\tau_0)\|_{H^1}, \|\hat{f}_1\|_{E_1})$. Moreover, the semiprocess $\Big\{U^{(1)}_{f|\hat{\Sigma}_1}(t,\tau)\Big\}(f \in \hat{\Sigma}_1, t \ge \tau \ge 0)$ possesses a uniform (with respect to $f \in \hat{\Sigma}_1$) compact attractor $\mathcal{A}_{\hat{\Sigma}_1}$ satisfying*

$$\bigcup_{f\in\hat{\Sigma}_1} \mathcal{A}^{(1)}_{\{f\}} \subseteq \mathcal{A}_{\hat{\Sigma}_1}$$

and the following assertions hold:

(1) *$\hat{f}_1$ is translation compact in $\hat{E}_1$ and any function $f \in \hat{\Sigma}_1 = H_+(\hat{f}_1)$ is also translation compact in $\hat{E}_1, H_+(f) \subseteq H_+(\hat{f}_1)$;*
(2) *the set $H_+(\hat{f}_1)$ is bounded in $L^2_b(\mathbb{R}^+, L^2(0,1))$ such that*

$$\eta_f(h) \le \eta_{\hat{f}_1}(h) < \infty, \forall f \in \hat{\Sigma}_1$$

where $\eta_f(h) = \sup_{t\in\mathbb{R}^+} \int_t^{t+h} \|f(s)\|^2 ds$;
(3) *the translation semigroup $T_1(t)\hat{\Sigma}_1 = \hat{\Sigma}_1, \forall t \ge 0$, where $T_1(t) : \hat{\Sigma}_1 \to \hat{\Sigma}_1, T_1(t)f(s) = f(s+t), \forall f \in \hat{\Sigma}_1$ and the semiprocess $\Big\{U^{(1)}_{f|\hat{\Sigma}_1}(t,\tau)\Big\}$ sat-*

isfies

$$U^{(1)}_{f|\hat{\Sigma}_1}(t+h,\tau+h) = U^{(1)}_{T_1(t)f(s)}(t,\tau), \forall t \geq \tau \geq 0, h \geq 0, f \in \hat{\Sigma}_1;$$

(4) *the translation semigroup* $\{T_1(t)\}$ *is continuous on* $\hat{\Sigma}_1$ *in the topology of* $\hat{E}_1$. *Moreover, there exists a semigroup* $\{S_1(t)\}$ *acting on* $H^1_+ \times \hat{\Sigma}_1$ *by the formula*

$$S_1(t)(U,f) = (U^{(1)}_{f|\hat{\Sigma}_1}(t,0)U, T_1(t)f), \forall U = (u,v) \in H^1_+$$

which possesses a compact attractor $\mathcal{A}_1$ *being strictly invariant with respect to* $S_1(t) : S_1(t)\mathcal{A}_1 = \mathcal{A}_1$ *for all* $t \geq 0$ *and satisfying*

(i) $\pi_1\mathcal{A}_1 = \mathcal{A}_1^+ = \mathcal{A}_{\hat{\Sigma}_1}$ *is the uniform (with respect to* $f \in \hat{\Sigma}_1$*) attractor of the family of semiprocess* $\left\{U^{(1)}_{f|\hat{\Sigma}_1}(t,\tau)\right\}(f \in \hat{\Sigma}_1, t \geq \tau \geq 0)$*;*

(ii) $\pi_2\mathcal{A}_1 = \mathcal{A}_1^- = \omega(\hat{\Sigma}_1)$ *is the attractor of the semigroup* $\{T_1(t)\}$ *acting on* $\hat{\Sigma}_1 = H_+(\hat{f}_1) : T_1(t)\hat{\Sigma}_1 = \hat{\Sigma}_1$ *for all* $t \geq 0$*;*

(iii) $\mathcal{A}_{\hat{\Sigma}_1} = \mathcal{A}_{\omega(\hat{\Sigma}_1)}$*, which is the uniform (with respect to* $f \in \omega(\hat{\Sigma}_1)$*) attractor of the family of semiprocesses* $\left\{U^{(1)}_{f|\hat{\Sigma}_1}(t,\tau)\right\}(f \in \omega(\hat{\Sigma}_1), t \geq \tau \geq 0)$*;*

(iv) $T_1(t)$ *is an isometric operator on* $\hat{\Sigma}_1$ *in the topology of* E_1*, i.e.,*

$$\|T_1(t)g_1 - T_1(t)g_2\|_{E_1} = \|g_1 - g_2\|_{E_1}, \forall g_1, g_2 \in \hat{\Sigma}_1$$

where π_1 *and* π_2 *are two projectors from* $H^1_+ \times \hat{\Sigma}_1$ *onto* H^1_+ *and* $\hat{\Sigma}_1$ *respectively, that is, for all* $U = (u,v) \in H^1_+$ *and for all* $f \in \hat{\Sigma}_1, \pi_1(U,f) = U, \pi_2(U,f) = f$.

Remark 5 *It is easy to know that when* $f \in \hat{\Sigma}_1 \subseteq E_1, U^{(1)}_f(t,\tau) = U^{(1)}_{f|\hat{\Sigma}_1}(t,\tau)$. *The next result concerns the existence of solutions and uniform compact semiprocess of two-parameter family of operators in* H^2_+.

Theorem 6 *Let*

$$\sigma(s) \in C^2(\mathbb{R}), \sigma(s) > 0, s \in \mathbb{R}, \tag{13}$$

and set

$$E_2 = L^2(\mathbb{R}^+, H^1(0,1)) \cap H^1(\mathbb{R}^+, L^2(0,1)) \cap L^1(\mathbb{R}^+, L^2(0,1)).$$

Then for any fixed $f \in E_2$ *and for any* $(u_0^\tau, v_0^\tau) \in H^2_+, \tau \in \mathbb{R}^+$*, problem* (4)-(7) *admits a unique global (regular) weak solution* $(u^\tau(t), v^\tau(t)) \in H^2_+$*, which generates a unique semiprocess* $\left\{U^{(2)}_f(t,\tau)\right\}$ *on* H^2_+ *of a two-parameter family of operators such that for any* $t \geq \tau \geq 0$,

$$U^{(2)}_f(t,\tau)(u_0^\tau, v_0^\tau) = (u^\tau(t), v^\tau(t)) \in H^2_+, \tag{14}$$

$$\|(u^\tau(t),v^\tau(t))\|^2_{H^2}+\int\limits_\tau^t(\|u^\tau\|^2_{H^2}+\|v^\tau\|^2_{H^3}+\|v^\tau_t\|^2_{H^1})(s)ds\le C_2(\tau) \tag{15}$$

here and hereafter $C_2(\tau)=C_2(\|(u_0^\tau,v_0^\tau)\|_{H^2_+},\|f\|_{E_2})>0$ *is a generic constant. If we further assume that*

$$\|f(t)\|_{H^1}+\|f_t(t)\|\le \hat{C}_2e^{-\gamma_2 t},\ \ t\ge\tau\ge 0 \tag{16}$$

for some positive constants γ_2 *and* $\hat{C}_2$*, then there exists a constant* $0<\beta_2=\beta_2(C_2(\tau))\le\beta_1$ *such that for any fixed* $\beta\in(0,\beta_2]$*, there holds that for any* $t\ge\tau\ge0$*,*

$$e^{\beta t}\|(u^\tau(t),v^\tau(t))\|^2_{H^2}+\int\limits_\tau^t e^{\beta s}(\|u^\tau\|^2_{H^2}+\|v^\tau\|^2_{H^3}+\|v^\tau_t\|^2_{H^1})(s)ds$$
$$\le C_2(\tau)e^{\beta\tau}. \tag{17}$$

The semiprocess $\left\{U_f^{(2)}(t,\tau)\right\}$ $(f\in E_2,t\ge\tau\ge0)$ *is* $\left(H^2_+\times E_2,H^2_+\right)$*-continuous in the sense that for all fixed* t *and* $\tau,t\ge\tau,\tau\in\mathbb{R}^+$*, the mapping* $\Big((u,v),f\Big)\to U_f^{(2)}(t,\tau)(u,v)$ *is continuous from* $H^2_+\times E_2$ *to* H^2_+*. Moreover, every* fixed *semiprocess* $\left\{U_f^{(2)}(t,\tau)\right\}$ *(i.e.,* $f\in E_2$ *fixed) possesses a (non-uniform) compact attractor* $\mathcal{A}^{(2)}_{\{f\}}$ *in* H^2_+*.*

Theorem 7 *Assume that* (13) *holds, and let*

$$\Sigma_2=H_+(f_2)=\Big[f_2(t+h)|h\in\mathbb{R}^+\Big]_{E_2}$$

where $f_2\in E_2$ *is an arbitrary but fixed function. Then for any* $f\in\Sigma_2$ *and for any* $(u_0^\tau,v_0^\tau)\in H^2_+,\tau\in\mathbb{R}^+$*, problem* (4)-(7) *admits a unique global (regular) weak solution* $(u^\tau(t),v^\tau(t))\in H^2_+$*, which generates a unique semiprocess* $\left\{U^{(2)}_{f|\Sigma_2}(t,\tau)\right\}$ *on* H^2_+ *of a two-parameter family of operators such that* (14) *and* (15) *hold where* $U_f^{(2)}(t,\tau)$ *should be replaced by* $U^{(2)}_{f|\Sigma_2}(t,\tau)$*; if* (16) *holds, then* (17) *still holds, where* $C_2(\tau)$ *should be replaced by* $C_2^*(\tau)=C_2^*(\|(u_0^\tau,v_0^\tau)\|_{H^2},\|f_2\|_{E_2})$*. Moreover, the semiprocess* $\left\{U^{(2)}_{f|\Sigma_2}(t,\tau)\right\}(f\in\Sigma_2,t\ge\tau\ge0)$ *possesses a uniform (with respect to* $f\in\Sigma_2$*) compact attractor* $\mathcal{A}_{\Sigma_2}$ *satisfying*

$$\bigcup_{f\in\Sigma_2}\mathcal{A}^{(2)}_{\{f\}}\subseteq\mathcal{A}_{\Sigma_2}.$$

Theorem 8 *Assume that* (13) *holds, and let*

$$\hat{\Sigma}_2=H_+(\hat{f}_2)=\Big[\hat{f}_2(t+h)|h\in\mathbb{R}^+\Big]_{\hat{E}_2},$$

$$\hat{E}_2 = L^2_{\mathrm{loc},w}(\mathbb{R}^+, H^1(0,1)) \cap H^1_{\mathrm{loc},w}(\mathbb{R}^+, L^2(0,1)),$$

where $\hat{f}_2 \in E_2$ *is an arbitrary but fixed function. Then for any* $f \in \hat{\Sigma}_2$ *and for any* $(u_0^\tau, v_0^\tau) \in H^2_+, \tau \in \mathbb{R}^+$, *problem* (4)-(7) *admits a unique global (regular) weak solution* $(u^\tau(t), v^\tau(t)) \in H^2_+$, *which generates a unique semiprocess* $\left\{U^{(1)}_{f|\hat{\Sigma}_2}(t,\tau)\right\}$ *on* H^2_+ *of a two-parameter family of operators such that* (14)-(15) *hold where* $\left\{U^{(2)}_f(t,\tau)\right\}$ *should be replaced by* $\left\{U^{(2)}_{f|\hat{\Sigma}_2}(t,\tau)\right\}$; *if* (16) *holds, then estimate* (17) *still holds where* $C_2(\tau)$ *should be replaced by* $\hat{C}_2^*(\tau) = \hat{C}_2^*(\|(u_0^\tau, v_0^\tau)\|_{H^2}, \|\hat{f}_2\|_{E_2})$. *Moreover, the semiprocess* $\left\{U^{(2)}_{f|\hat{\Sigma}_2}(t,\tau)\right\}$ $(f \in \hat{\Sigma}_2, t \geq \tau \geq 0)$ *possesses a uniform* (*with respect to* (*w.r.t*) $f \in \hat{\Sigma}_2$) *compact attractor* $\mathcal{A}_{\hat{\Sigma}_2}$ *satisfying*

$$\bigcup_{f\in\hat{\Sigma}_2} \mathcal{A}^{(2)}_{\{f\}} \subseteq \mathcal{A}_{\hat{\Sigma}_2}$$

and the following assertions hold:
(1) $\hat{f}_2$ *is translation compact in* $\hat{E}_2$ *and any function* $f \in \hat{\Sigma}_2 = H_+(\hat{f}_2)$ *is also translation compact in* $\hat{E}_2, H_+(f) \subseteq H_+(\hat{f}_2)$;
(2) *the set* $H_+(\hat{f}_2)$ *is bounded in* $L^2_b(\mathbb{R}^+, H^1(0,1)) \cap H^1_b(\mathbb{R}^+, L^2(0,1))$ *such that*

$$\hat{\eta}_f(h) \leq \hat{\eta}_{\hat{f}_2}(h) < +\infty, \forall f \in H_+(\hat{f}_2)$$

where $\hat{\eta}_f(h) = \sup_{t\in\mathbb{R}^+} \int_t^{t+h} \|f(s)\|^2_{H^1} ds + \sum_{i=0}^1 \sup_{t\in\mathbb{R}^+} \int_t^{t+h} \|\partial_s^i f(s)\|^2 ds$;
(3) *the translation semigroup* $T_2(t)\hat{\Sigma}_2 = \hat{\Sigma}_2, \forall t \geq 0$, *where* $T_2(t) : \hat{\Sigma}_2 \to \hat{\Sigma}_2, T_2(t)f(s) = f(s+t), \forall f \in \hat{\Sigma}_2$ *and the semiprocess* $\left\{U^{(2)}_{f|\hat{\Sigma}_2}(t,\tau)\right\}(f \in \hat{\Sigma}_2, t \geq \tau \geq 0)$ *satisfies*

$$U^{(2)}_{f|\hat{\Sigma}_2}(t+h, \tau+h) = U^{(2)}_{T_2(t)f(s)|\hat{\Sigma}_2}(t,\tau), \forall t \geq \tau \geq 0, h \geq 0, f \in \hat{\Sigma}_2;$$

(4) *the translation semigroup* $\{T_2(t)\}$ *is continuous on* $\hat{\Sigma}_2$ *in the topology of* $\hat{E}_2$.
Moreover, there exists a semigroup $\{S_2(t)\}$ *acting on* $H^2_+ \times \hat{\Sigma}_2$ *by the formula*

$$S_2(t)(U, f) = (U^{(2)}_{f|\hat{\Sigma}_2}(t,0)U, T_2(t)f), \forall U = (u,v) \in H^2_+$$

which possesses a compact attractor $\mathcal{A}_2$ *being strictly invariant with respect to* $S_2(t) : S_2(t)\mathcal{A}_2 = \mathcal{A}_2$ *for all* $t \geq 0$ *and satisfies*
(i) $\hat{\pi}_1\mathcal{A}_2 = \mathcal{A}_2^+ = \mathcal{A}_{\hat{\Sigma}_2}$ *is the uniform attractor of the family of semiprocesses* $\left\{U^{(2)}_{f|\hat{\Sigma}_2}(t,\tau)\right\}$
$(f \in \hat{\Sigma}_2, t \geq \tau \geq 0)$;
(ii) $\hat{\pi}_2\mathcal{A}_2 = \mathcal{A}_2^- = \omega(\hat{\Sigma}_2)$ *is the attractor of the semigroup* $\{T_2(t)\}$ *acting on* $\hat{\Sigma}_2 = H_+(\hat{f}_2) : T_2(t)\hat{\Sigma}_2 = \hat{\Sigma}_2$ *for all* $t \geq 0$;
(iii) $\mathcal{A}_{\hat{\Sigma}_2} = \mathcal{A}_{\omega(\hat{\Sigma}_2)}$, *which is the uniform* (*w.r.t.* $f \in \omega(\hat{\Sigma}_2)$) *attractor of the*

family of semiprocesses $\left\{U_f^{(2)}(t,\tau)\right\}(f \in \omega(\hat{\Sigma}_2), t \geq \tau \geq 0)$;
(iv) $T_2(t)$ *is an isometric operator on* $\hat{\Sigma}_2 = H_+(\hat{f}_2)$ *in the topology of* E_2, *i.e.,*

$$\|T_2(t)g_1 - T_2(t)g_2\|_{E_2} = \|g_1 - g_2\|_{E_2}, \forall g_1, g_2 \in H_+(\hat{f}_2).$$

It is obvious that the global solutions obtained in Theorems 1-8 are not classical solutions to problem (4)-(7). The following theorems concern the results in H_+^4 which yield the results of classical solutions by the standard embedding theorem.

Theorem 9 *Let*

$$\sigma(s) \in C^3(\mathbb{R}), \sigma(s) > 0, s \in \mathbb{R}, \tag{18}$$

and set

$$\begin{aligned} E_3 = &H^1(\mathbb{R}^+, H^1(0,1)) \cap H^2(\mathbb{R}^+, L^2(0,1)) \cap L^2(\mathbb{R}^+, H^3(0,1)) \\ &\cap L^1(\mathbb{R}^+, L^2(0,1)) \cap L^\infty(\mathbb{R}^+, H^2(0,1)) \cap W^{1,\infty}(\mathbb{R}^+, H^1(0,1)) \\ &\cap W^{2,\infty}(\mathbb{R}^+, L^2(0,1)). \end{aligned}$$

Then for any fixed $f \in E_3$ *and for any* $(u_0^\tau, v_0^\tau) \in H_+^4, \tau \in \mathbb{R}^+$, *problem* (4)-(7) *admits a unique global solution* $(u^\tau(t), v^\tau(t)) \in H_+^4$, *which generates a unique semiprocess* $\left\{U_f^{(3)}(t,\tau)\right\}$ *on* H_+^4 *of two-parameter family of operators such that for any* $t \geq \tau \geq 0$,

$$U_f^{(3)}(t,\tau)(u_0^\tau, v_0^\tau) = (u^\tau(t), v^\tau(t)) \in H_+^4, \tag{19}$$

$$\begin{aligned} \|(u^\tau(t), v^\tau(t))\|_{H^4}^2 + \int_\tau^t (&\|u^\tau\|_{H^4}^2 + \|v^\tau\|_{H^5}^2 + \|v_t^\tau\|_{H^3}^2 \\ &+ \|v_{tt}^\tau\|_{H^1}^2)(s)ds \leq C_4(\tau), \end{aligned} \tag{20}$$

here and hereafter $C_4(\tau) = C_4(\|(u_0^\tau, v_0^\tau)\|_{H^4}, \|f\|_{E_3}) > 0$ *is a generic constant. If we further assume that*

$$\|f(t)\|_{H^3} + \|f_t(t)\|_{H^1} + \|f_{tt}(t)\| \leq \hat{C}_3 e^{-\gamma_3 t}, \ \ t \geq \tau \geq 0 \tag{21}$$

for some positive constants γ_3 *and* $\hat{C}_3$, *then there exists a constant* $0 < \beta_3 = \beta_3(C_4(\tau)) \leq \beta_2$ *such that for any fixed* $\beta \in (0, \beta_3]$, *we have*

$$e^{\beta t}\|(u^\tau(t), v^\tau(t))\|_{H^4}^2 + \int_\tau^t e^{\beta s}(\|u^\tau\|_{H^4}^2 + \|v^\tau\|_{H^5}^2$$

$$+\|v_t^\tau\|_{H^3}^2+\|v_{tt}^\tau\|_{H^1}^2)(s)ds \le C_4(\tau)e^{\beta\tau}, \tag{22}$$

for any $t \ge \tau \ge 0$. *The semiprocess* $\left\{U_f^{(3)}(t,\tau)\right\}(f \in E_3, t \ge \tau \ge 0)$ *is* $\left(H_+^4 \times E_3, H_+^4\right)$*-continuous in the sense that for any fixed* t *and* $\tau, t \ge \tau, \tau \in \mathbb{R}^+$, *the mapping* $\left((u,v),f\right) \to U_f^{(3)}(t,\tau)(u,v)$ *is continuous from* $H_+^4 \times E_3$ *to* H_+^4. *Moreover, for any* fixed $f \in E_3$, *the semiprocess* $\left\{U_f^{(3)}(t,\tau)\right\}$ *possesses a (non-uniform) compact attractor* $\mathcal{A}_{\{f\}}^{(3)}$.

Theorem 10 *Assume that* (18) *holds, and let*

$$\begin{aligned}\hat{E}_3 =& L^2(\mathbb{R}^+, H^3(0,1)) \cap L^1(\mathbb{R}^+, L^2(0,1)) \cap H^1(\mathbb{R}^+, H^2(0,1))\\ &\cap H^2(\mathbb{R}^+, H^1(0,1)) \cap H^3(\mathbb{R}^+, L^2(0,1)),\\ \Sigma_3 =& H_+(f_3) = \left[f_3(t+h) | h \in \mathbb{R}^+\right]_{\hat{E}_3}\end{aligned}$$

where $f_3 \in \hat{E}_3$ *is an arbitrary but fixed function. Then for any* $f \in \Sigma_3$ *and for any* $(u_0^\tau, v_0^\tau) \in H_+^4, \tau \in \mathbb{R}^+$, *there exists a unique global solution* $(u^\tau(t), v^\tau(t)) \in H_+^4$ *to problem* (4)-(7), *which generates a unique semiprocess* $\left\{U_{f|\Sigma_3}^{(3)}(t,\tau)\right\}$ *on* H_+^4 *of a two-parameter family of operators such that* (19)-(20) *hold where* $\left\{U_f^{(3)}(t,\tau)\right\}$ *should be replaced by* $\left\{U_{f|\Sigma_3}^{(3)}(t,\tau)\right\}$; *if* (21) *holds, then* (22) *still holds where* $C_4(\tau)$ *should be replaced by* $C_4^*(\tau) = C_4^*(\|(u_0^\tau, v_0^\tau)\|_{H^4}, \|f_3\|_{\hat{E}_3})$. *Moreover, the semiprocess* $\left\{U_{f|\Sigma_3}^{(3)}(t,\tau)\right\}(f \in \Sigma_3, t \ge \tau \ge 0)$ *possesses a uniform (with respect to* $f \in \Sigma_3$) *compact attractor* $\mathcal{A}_{\Sigma_3}$ *satisfying*

$$\bigcup_{f\in\Sigma_3} \mathcal{A}_{\{f\}}^{(3)} \subseteq \mathcal{A}_{\Sigma_3}.$$

Theorem 11 *Assume that* (18) *holds, and let*

$$\begin{aligned}\tilde{E}_3 =& L^2_{\mathrm{loc},w}(\mathbb{R}^+, H^3(0,1)) \cap H^1_{\mathrm{loc},w}(\mathbb{R}^+, H^2(0,1))\\ &\cap H^2_{\mathrm{loc},w}(\mathbb{R}^+, H^1(0,1)) \cap H^3_{\mathrm{loc},w}(\mathbb{R}^+, L^2(0,1)),\\ \tilde{\Sigma}_3 =& H_+(\tilde{f}_3) = \left[\tilde{f}_3(t+h) | h \in \mathbb{R}^+\right]_{\tilde{E}_3},\end{aligned}$$

where $\tilde{f}_3 \in \hat{E}_3 \subseteq \tilde{E}_3$ *is an arbitrary but fixed function. Then for any* $f \in \tilde{\Sigma}_3$ *and for any* $(u_0^\tau, v_0^\tau) \in H_+^4, \tau \in \mathbb{R}^+$, *problem* (4)-(7) *admits a unique global solution* $(u^\tau(t), v^\tau(t)) \in H_+^4$, *which generates a unique semiprocess* $\left\{U_{f|\tilde{\Sigma}_3}^{(3)}(t,\tau)\right\}$ *on* H_+^4 *of two-parameter family of operators such that* (19)-(20) *hold where* $\left\{U_f^{(3)}(t,\tau)\right\}$ *should be replaced by* $\left\{U_{f|\tilde{\Sigma}_3}^{(3)}(t,\tau)\right\}$; *if* (21) *holds, then estimate* (22) *still holds where* $C_4(\tau)$ *should be replaced by* $\tilde{C}_4^*(\tau) = \tilde{C}_4^*(\|(u_0^\tau, v_0^\tau)\|_{H^4}, \|\tilde{f}_3\|_{\hat{E}_3})$. *Moreover, the semiprocess* $\left\{U_{f|\tilde{\Sigma}_3}^{(3)}(t,\tau)\right\}(f \in \tilde{\Sigma}_3, t \ge \tau \ge 0)$ *possesses* *a uniform (with respect to $f \in \tilde{\Sigma}_3$) compact attractor $\mathcal{A}_{\tilde{\Sigma}_3}$ satisfying*

$$\bigcup_{f\in\tilde{\Sigma}_3} \mathcal{A}^{(3)}_{\{f\}} \subseteq \mathcal{A}_{\tilde{\Sigma}_3}$$

and the following assertions hold:

(1) *$\tilde{f}_3$ is translation compact in $\tilde{E}_3$ and any function $f \in \tilde{\Sigma}_3 = H_+(\tilde{f}_3)$ is also translation compact in $\tilde{E}_3, H_+(f) \subseteq H_+(\tilde{f}_3)$;*

(2) *the set $H_+(\tilde{f}_3)$ is bounded in*

$$L^2_b(\mathbb{R}^+, H^3(0,1)) \cap H^1_b(\mathbb{R}^+, H^2(0,1)) \cap H^2_b(\mathbb{R}^+, H^1(0,1)) \cap H^3_b(\mathbb{R}^+, L^2(0,1))$$

such that

$$\tilde{\eta}_f(h) \leq \tilde{\eta}_{\tilde{f}_3}(h) < +\infty, \forall f \in H_+(\tilde{f}_3)$$

where

$$\begin{aligned}\tilde{\eta}_f(h) = &\sup_{t\in\mathbb{R}^+}\int_t^{t+h} \|f(s)\|^2_{H^3} ds + \sum_{i=0}^{1}\sup_{t\in\mathbb{R}^+}\int_t^{t+h} \|\partial^i_s f(s)\|^2_{H^2} ds \\ &+ \sum_{i=0}^{2}\sup_{t\in\mathbb{R}^+}\int_t^{t+h} \|\partial^i_s f(s)\|^2_{H^1} ds + \sum_{i=0}^{3}\sup_{t\in\mathbb{R}^+}\int_t^{t+h} \|\partial^i_s f(s)\|^2 ds;\end{aligned}$$

(3) *the translation semigroup $T_3(t)\tilde{\Sigma}_3 = \tilde{\Sigma}_3, \forall t \geq 0$, where $T_3(t) : \tilde{\Sigma}_3 \to \tilde{\Sigma}_3, T_3(t)f(s) = f(s+t), \forall f \in \tilde{\Sigma}_3$ and the semiprocess $\left\{U^{(3)}_{f|\tilde{\Sigma}_3}(t,\tau)\right\}(f \in \tilde{\Sigma}_3, t \geq \tau \geq 0)$ satisfies*

$$U^{(3)}_{f|\tilde{\Sigma}_3}(t+h, \tau+h) = U^{(3)}_{T_3(t)f(s)|\tilde{\Sigma}_3}(t,\tau), \forall t \geq \tau \geq 0, h \geq 0, f \in \tilde{\Sigma}_3;$$

(4) *the translation semigroup $\{T_3(t)\}$ is continuous on $\tilde{\Sigma}_3$ in the topology of $\tilde{E}_3$.*
Moreover, there exists a semigroup $\{S_3(t)\}$ acting on $H^4_+ \times \tilde{\Sigma}_3$ by the formula

$$S_3(t)(U,f) = (U^{(3)}_{f|\tilde{\Sigma}_3}(t,0)U, T_3(t)f), \forall U = (u,v) \in H^4_+$$

which possesses a compact attractor $\mathcal{A}_3$ being strictly invariant with respect to $S_3(t) : S_3(t)\mathcal{A}_3 = \mathcal{A}_3$ for all $t \geq 0$ and satisfying

(i) *$\tilde{\pi}_1\mathcal{A}_3 = \mathcal{A}^+_3 = \mathcal{A}_{\tilde{\Sigma}_3}$ is the uniform attractor of the family of semiprocess $\left\{U^{(3)}_{f|\tilde{\Sigma}_3}(t,\tau)\right\}(f \in \tilde{\Sigma}_3, t \geq \tau \geq 0)$;*

(ii) *$\tilde{\pi}_2\mathcal{A}_3 = \mathcal{A}^-_3 = \omega(\tilde{\Sigma}_3)$ is the attractor of the semigroup $\{T_3(t)\}$ acting on $\tilde{\Sigma}_3 = H_+(\tilde{f}_3) : T_3(t)\tilde{\Sigma}_3 = \tilde{\Sigma}_3$ for all $t \geq 0$;*

(iii) *$\mathcal{A}_{\tilde{\Sigma}_3} = \mathcal{A}_{\omega(\tilde{\Sigma}_3)}$, which is the uniform (with respect to $f \in \omega(\tilde{\Sigma}_3)$) attractor of the family of semiprocess $\left\{U^{(3)}_f(t,\tau)\right\}(f \in \omega(\tilde{\Sigma}_3), t \geq \tau \geq 0)$;*

(iv) $T_3(t)$ *is an isometric operator on* $\tilde{\Sigma}_3 = H_+(\tilde{f}_3)$ *in the norm of* $\hat{E}_3$, *i.e.,*

$$\|T_3(t)g_1 - T_3(t)g_2\|_{\hat{E}_3} = \|g_1 - g_2\|_{\hat{E}_3}, \forall g_1, g_2 \in \tilde{\Sigma}_3$$

where $\tilde{\pi}_1$ *and* $\tilde{\pi}_2$, *respectively, are two projectors from* $H_+^4 \times \tilde{\Sigma}_3$, *that is, for any* $U = (u, v) \in H_+^4$, $f \in \tilde{\Sigma}_3$, *we have* $\tilde{\pi}_1(U, f) = U$, $\tilde{\pi}_2(U, f) = f$.

Remark 12 *We shall study the case when* $\sigma(s)$ *changes sign (i.e., the model corresponds to phase transitions) in a forthcoming paper.*

The rest of this paper is organized as follows. In Section 2, we shall derive the estimates in H_+^1 and shall complete the proofs of Theorems 1-4. Section 3 will be concerned with the estimates in H_+^2, and the proofs of Theorems 6-8 will be finished. In Section 4, we shall derive the estimates in H_+^4 and shall complete the proofs of Theorems 9-11.

2 Estimates in H_+^1

Lemma 13 *Let* $-\tilde{A}$ *be the infinitesimal generator of an analytic semigroup* $S(t)$ *defined on a Banach space* E. *If* $0 \in \rho(\tilde{A})$ *(the resolvent set of* $\tilde{A}$*), then for every* $t > 0$ *the operator* $\tilde{A}^m S(t)$ *is bounded and*

$$\|\tilde{A}^m S(t)\|_{\mathcal{L}(E)} \leq C(m)t^{-m}e^{-\delta t}, \ \ \forall m \geq 0,$$

where $C(m)$ *and* δ *are positive constants independent of* $t > 0$.

Proof. See., e.g., Theorem 6.13 in [23]. □

We consider a closed linear operator $A = \frac{\partial^2}{\partial x^2} : L^2(0,1) \to L^2(0,1)$ with the domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$ which satisfies

$$Au(x) = u_{xx}(x) \in L^2(0,1), \forall u \in D(A). \tag{23}$$

It is well-known (see, e.g., [23, 36]) that A generates an analytic semigroup of bounded linear operators $\{\hat{T}(t)\}$ on $X_0 = L^2(0,1)$ such that

$$\|\hat{T}(t)u\| \leq \|u\|, \quad \forall u \in X_0$$

with $\|u\| = \|u\|_{X_0}$, and A has a bounded inverse A^{-1} given by

$$(A^{-1}u)(x) = \int_0^x (x - \xi)u(\xi)d\xi + x\int_0^1 (\xi - 1)u(\xi)d\xi, \quad \forall u \in X_0. \tag{24}$$

By (23)-(24), we can regard the function y in (1) as a map from $\mathbb{R}^+ = [0, +\infty)$ to X_0, formally rewrite (1)-(3) in the following abstract Cauchy problem:

$$\begin{aligned} &y_{tt}(t) - Ay_t(t) - By(t) = f(t), \quad t \geq \tau \geq 0 \\ &y(\tau) = y_0^\tau, y_t(\tau) = y_1^\tau, \tau \in \mathbb{R}^+ \end{aligned}$$

where $y = y^\tau(t), B$ is a nonlinear operator defined by

$$(By)(x) = \sigma(y_x(x))y_{xx}(x)$$

with a domain $D(B) = D(A)$.
Set $V(t) = (v_1^\tau(t), v_2^\tau(t))^T = (y_t^\tau, Ay^\tau)^T$. Then (1)-(3) can be converted into

$$\begin{aligned} &V_t(t) = \mathcal{A}V(t) + C(V(t)) + F(t), \quad t \geq \tau \geq 0, \\ &V(\tau) = V_0^\tau \equiv (y_1^\tau, Ay_0^\tau)^T \end{aligned} \tag{25}$$

where

$$\mathcal{A} = \begin{pmatrix} A & \sigma(0) \\ A & 0 \end{pmatrix}, \tag{26}$$

$$C(V) = \Big(B(A^{-1}v_2^\tau) - \sigma(0)v_2^\tau, 0\Big)^T, \ F(t) = \big(f(t), 0\big)^T. \tag{27}$$

It is easily see (see, e.g., [23, 36]) that $\mathcal{A}$ can generate an analytic semigroup of bounded linear operators $\{T(t)\} = \{e^{t\mathcal{A}}\}$ on $X_0 \times X_0$.
We now consider the following problem

$$z_{tt} - Az_t - \sigma(0)Az = 0, \tag{28}$$

$$z(\tau) = z_0^\tau, \ z_t(\tau) = z_1^\tau \tag{29}$$

where $(z_1^\tau, Az_0^\tau) \in X_0 \times X_0$ and $z = z^\tau(t)$. Then problem (28)-(29) can be changed into

$$W_t = \mathcal{A}W, t \geq \tau \geq 0; \ W(\tau) = W_0^\tau = (z_1^\tau, Az_0^\tau)^T \in X_0 \times X_0 \tag{30}$$

where

$$W(t) = (w_1^\tau, w_2^\tau)^T = (z_t^\tau, Az^\tau)^T = T(t-\tau)W_0^\tau, \quad t \geq \tau \geq 0. \tag{31}$$

Thus it follows from (30)-(31) that $V(t)$ satisfies

$$V(t) = T(t-\tau)V_0^\tau + \int_\tau^t T(t-s)[C(V(s)) + F(s)]ds, \quad t \geq \tau \geq 0. \tag{32}$$

The following lemma concerns the properties on the semigroup $\{T(t)\}$.

Lemma 14 *For any $t \geq \tau + 1, \tau \geq 0$, there exist some positive constants $K_1 = K_1(\tau)$ and α (an absolute constant) such that*

$$\|T(t-\tau)W_0^\tau\|_{H^6\times H^5} \leq K_1 e^{-\alpha t/2}\|W_0^\tau\|_{X_0\times X_0}.$$

for any $W_0^\tau \in X_0 \times X_0, \tau \in \mathbb{R}^+$.

PROOF. Without of loss of generality, we prove this lemma only for $\tau = 0$. We write $W_0 = W_0^0 = (z_1, Az_0)^T, W^0(t) = W(t) = (w_1(t), w_2(t))^T = (z_t, Az)^T$. Put

$$\hat{u} = z_x, \hat{v} = z_t. \tag{33}$$

Then (28)-(29) takes the form

$$\begin{aligned}
&\hat{u}_t - \hat{v}_x = 0, && (34)\\
&\hat{v}_t - \hat{v}_{xx} - \sigma(0)\hat{u}_x = 0, && (35)\\
&\hat{v}|_{x=0,1} = 0, && (36)\\
&\hat{u}|_{t=0} = \hat{u}_0 = z_{0,x}(x), \quad \hat{v}|_{t=0} = z_1(x)
\end{aligned}$$

where $W_0 = (z_1, Az_0)^T \in X_0 \times X_0$. First from (34) and (36), we know

$$\int_0^1 \hat{u}dx = \int_0^1 z_{0,x}dx = 0, \quad \|\hat{u}\| \leq \|\hat{u}_x\|, \quad \|\hat{v}\| \leq \|\hat{v}_x\|. \tag{37}$$

Multiplying (35) by $\hat{v}_t e^{\alpha t}$ in $L^2(0,1)$ yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big\{(\|\hat{v}(t)\|^2 + \sigma(0)\|\hat{u}(t)\|^2)e^{\alpha t}\Big\} + (1-\alpha/2)\|\hat{v}_x(t)\|^2 e^{\alpha t}&\\
\leq [\alpha\sigma(0)/2]\|\hat{u}(t)\|^2 e^{\alpha t}.&
\end{aligned} \tag{38}$$

Multiplying (35) by $\hat{u}_x e^{\alpha t}$ in $L^2(0,1)$ and using (34) we arrive at

$$\begin{aligned}
\frac{d}{dt}\Big\{[\frac{1}{2}\|\hat{u}_x(t)\|^2 - (\hat{v}, \hat{u}_x)]e^{\alpha t}\Big\} + (\sigma(0)-\alpha)\|\hat{u}_x(t)\|^2 e^{\alpha t}&\\
\leq [(\alpha/2)\|\hat{v}(t)\|^2 + \|\hat{v}_x(t)\|^2]e^{\alpha t}&
\end{aligned} \tag{39}$$

where $(\hat{v}, \hat{u}_x) = \int_0^1 \hat{v}\hat{u}_x dx$. Multiplying (39) by a parameter $\lambda \in (0,1)$ and adding the resulting inequality to (38), we readily derive

$$\frac{d}{dt}G_1(t) + (1-\alpha/2-\lambda\alpha/2-\lambda)\|\hat{v}_x(t)\|^2 e^{\alpha t}$$
$$+ [\lambda(\sigma(0)-\alpha) - \alpha\sigma(0)/2]\|\hat{u}_x(t)\|^2 e^{\alpha t} \le 0 \tag{40}$$

where

$$G_1(t) = \Big\{\frac{\sigma(0)}{2}\|\hat{u}(t)\|^2 + \frac{1}{2}\|\hat{v}(t)\|^2 + \frac{\lambda}{2}\|\hat{u}_x(t)\|^2 - \lambda(\hat{v},\hat{u}_x)\Big\}e^{\alpha t}.$$

Now fixing $\lambda \in (0,1)$ and picking $\alpha > 0$ so small that

$$0 < \alpha < \min\{\frac{2(1-\lambda)}{1+\lambda}, \frac{2\lambda\sigma(0)}{\sigma(0)+2\lambda}\} \equiv \alpha_0$$

and noting that

$$\frac{1}{2}\|\hat{v}(t)\|^2 + \frac{\alpha}{2}\|\hat{u}_x(t)\|^2 - \lambda(\hat{v},\hat{u}_x) \ge C_0(\lambda)(\|\hat{v}(t)\|^2 + \|\hat{u}_x(t)\|^2) \tag{41}$$

with $C_0(\lambda) = \frac{\lambda(1-\lambda)}{2}$, we finally deduce from (40)-(41) that for any $\alpha \in (0,\alpha_0)$ and for any $t \ge 1$,

$$e^{\alpha t}(\|\hat{u}(t)\|^2_{H^1} + \|\hat{v}(t)\|^2) + \int_1^t (\|\hat{v}_x\|^2 + \|\hat{u}_x\|^2)(s)e^{\alpha s}ds$$
$$\le C_\alpha(\|\hat{u}(1)\|^2_{H^1} + \|\hat{v}(1)\|^2) \tag{42}$$

here and hereafter $C_\alpha = C(\alpha) > 0$ is a generic constant depending only on α and λ.
Analogously, we have from (35) for $\alpha \in (0,\alpha_0)$,

$$\frac{d}{dt}(\|\hat{v}_x(t)\|^2 e^{\alpha t}) + \|\hat{v}_t(t)\|^2 e^{\alpha t} \le (\alpha/2)\|\hat{v}_x(t)\|^2 e^{\alpha t} + (\sigma^2(0)/2)\|\hat{u}_x(t)\|^2 e^{\alpha t}$$

which, along with (42), yields that for any $\alpha \in (0,\alpha_0), t \ge 1$,

$$e^{\alpha t}\|\hat{v}_x(t)\|^2 + \int_1^t \|\hat{v}_t(s)\|^2 e^{\alpha s}ds$$
$$\le C_\alpha\|\hat{v}_x(1)\|^2 + C_\alpha\int_1^t (\|\hat{v}_x\|^2 + \|\hat{u}_x\|^2)(s)e^{\alpha s}ds$$
$$\le C_\alpha(\|\hat{u}(1)\|^2_{H^1} + \|\hat{v}(1)\|^2_{H^1}). \tag{43}$$

Thus it follows from (42) and (43) that for any $\alpha \in (0,\alpha_0)$ and for any $t \ge 1$,

$$e^{\alpha t}(\|\hat{u}(t)\|_{H^1}^2+\|\hat{v}(t)\|_{H^1}^2)+\int_1^t(\|\hat{v}\|_{H^2}^2+\|\hat{u}\|_{H^1}^2)(s)e^{\alpha s}ds$$
$$\le C_\alpha(\|\hat{u}(1)\|_{H^1}^2+\|\hat{v}(1)\|_{H^1}^2). \quad (44)$$

Differentiating (35) with respect to t, multiplying the resulting equation by $e^{\alpha t}\hat{v}_t$ in $L^2(0,1)$, and using (43) and (44), we immediately obtain that for any $\alpha\in(0,\alpha_0)$ and for any $t\ge 1$,

$$\begin{aligned}
e^{\alpha t}(\|\hat{v}_{xx}(t)\|^2+\|\hat{v}_t(t)\|^2)+\int_1^t\|\hat{v}_{tx}(s)\|^2e^{\alpha s}(s)ds&\\
\le C_\alpha\|\hat{v}_t(1)\|^2+C_\alpha\int_1^t(\|\hat{v}_t\|^2+\|\hat{v}_{xx}\|^2)(s)e^{\alpha s}ds&\\
\le C_\alpha(\|\hat{u}(1)\|_{H^1}^2+\|\hat{v}(1)\|_{H^1}^2+\|\hat{v}_t(1)\|^2).& \quad (45)
\end{aligned}$$

Obviously, we have from (34)-(35),

$$\hat{u}_{txx}+\sigma(0)\hat{u}_{xx}=\hat{v}_{tx}. \quad (46)$$

Multiplying (46) by $e^{\alpha t}\hat{u}_{xx}$ in $L^2(0,1)$, we get

$$\frac{1}{2}\frac{d}{dt}[\|\hat{u}_{xx}(t)\|^2e^{\alpha t}]+(\sigma(0)-\alpha)\|\hat{u}_{xx}(t)\|^2e^{\alpha t}\le(2\alpha)^{-1}\|\hat{v}_{tx}(t)\|^2e^{\alpha t}. \quad (47)$$

Combined with (43)-(45) it follows that for any $\alpha\in(0,\alpha_0)$ and for any $t\ge 1$

$$\begin{aligned}
e^{\alpha t}(\|\hat{u}(t)\|_{H^2}^2+\|\hat{v}(t)\|_{H^2}^2+\|\hat{v}_t(t)\|^2)&\\
+\int_1^t(\|\hat{v}\|_{H^3}^2+\|\hat{u}\|_{H^2}^2+\|\hat{v}_t\|_{H^1}^2)(s)e^{\alpha s}ds&\\
\le C_\alpha(\|\hat{u}(1)\|_{H^2}^2+\|\hat{v}(1)\|_{H^1}^2)+\|\hat{v}_t(1)\|^2).& \quad (48)
\end{aligned}$$

Clearly, differentiating (35) with respect to x and t respectively, and using (34), we have

$$\begin{aligned}
\|\hat{v}_{xxx}(t)\|&\le C(\|\hat{v}_{tx}(t)\|+\|\hat{u}_{xx}(t)\|),\\
\|\hat{v}_{xxxx}(t)\|&\le C(\|\hat{v}_{tt}(t)\|+\|\hat{v}_{xx}(t)\|+\|\hat{u}_{xx}(t)\|). \quad (49)
\end{aligned}$$

Differentiating (35) with respect to t, multiplying the resulting equation by $e^{\alpha t}\hat{v}_{tt}$ in $L^2(0,1)$, we derive

$$\frac{d}{dt}\Big\{\frac{1}{2}(\|\hat{v}_{tt}(t)\|^2+\sigma(0)\|\hat{v}_{tx}(t)\|^2)e^{\alpha t}\Big\}+(1-\alpha/2)\|\hat{v}_{ttx}(t)\|^2e^{\alpha t}$$
$$\leq(\alpha/2)\|\hat{v}_{ttx}(t)\|^2e^{\alpha t}+(\sigma(0)\alpha/2)\|\hat{v}_{tx}(t)\|^2e^{\alpha t}$$

which, using (49) and (50), gives us for any $\alpha\in(0,\alpha_1),\alpha_1=\min(\alpha_0,1/2)$ and for any $t\geq 1$,

$$e^{\alpha t}(\|\hat{v}_{tt}(t)\|^2+\|\hat{v}_{tx}(t)\|^2+\|\hat{v}_{xxx}(t)\|^2)+\int_1^t\|\hat{v}_{ttx}(s)\|e^{\alpha s}ds$$
$$\leq C_\alpha(\|\hat{v}_{tt}(1)\|^2+\|\hat{u}(1)\|^2_{H^2}+\|\hat{v}(1)\|^2_{H^1}+\|\hat{v}_t(1)\|^2_{H^1}). \qquad (50)$$

Differentiating (46) with respect to x, multiplying the resulting equation by $e^{\alpha t}\hat{u}_{xxx}$ in $L^2(0,1)$, using (43), (44) and (50), for any $\alpha\in(0,\alpha_1)$ and any $t\geq 1$ we obtain

$$\|\hat{u}_{xxx}(t)\|^2e^{\alpha t}+\int_1^t\|\hat{u}_{xxx}(s)\|^2e^{\alpha s}ds$$
$$\leq C_\alpha\|\hat{u}_{xxx}(1)\|^2+C_\alpha\int_1^t e^{\alpha s}\|\hat{v}_{txx}(s)\|^2ds$$
$$\leq C_\alpha(\|\hat{u}(1)\|^2_{H^3}+\|\hat{v}(1)\|^2_{H^1}+\|\hat{v}_t(1)\|^2_{H^1}+\|\hat{v}_{tt}(1)\|^2). \qquad (51)$$

Using (48) and (49), gives us for any $\alpha\in(0,\alpha_1),\alpha_1=\min(\alpha_0,1/2)$ and for any $t\geq 1$,

$$e^{\alpha t}\|\hat{v}_{xxxx}(t)\|^2+\int_1^t\|\hat{v}_{xxxx}(s)\|e^{\alpha s}ds$$
$$\leq C_\alpha(\|\hat{v}_{tt}(1)\|^2+\|\hat{u}(1)\|^2_{H^3}+\|\hat{v}(1)\|^2_{H^1}+\|\hat{v}_t(1)\|^2_{H^1}). \qquad (52)$$

Similarly us (51), we differentiate (46) with respect to x twice to obtain

$$\frac{1}{2}\frac{d}{dt}[\|\hat{u}_{xxxx}(t)\|^2e^{\alpha t}]+(\sigma(0)-\alpha)\|\hat{u}_{xxxx}(t)\|^2e^{\alpha t}=\int_0^1\hat{v}_{txxx}\hat{u}_{xxxx}dxe^{\alpha t}$$
$$\leq(\alpha/2)\|\hat{u}_{xxxx}(t)\|^2e^{\alpha t}+(2\alpha)^{-1}\|\hat{v}_{txxx}(t)\|^2e^{\alpha t}$$
$$\leq(\alpha/2)\|\hat{u}_{xxxx}(t)\|^2e^{\alpha t}+C\alpha^{-1}(\|\hat{v}_{ttx}(t)\|^2+\|\hat{v}_{xxx}(t)\|^2)e^{\alpha t}.$$

Thus the relations(43)-(44), (50) and (52) together with(35) yield

$$\|\partial_x^5\hat{v}(t)\|\leq C(\|\hat{v}_{ttx}(t)\|+\|\hat{v}_{xxx}(t)\|+\|\hat{u}_{xxxx}(t)\|), \qquad (53)$$

and we obtain for any $\alpha \in (0, \alpha_1)$ and $t \geq 1$,

$$\begin{aligned}
&\|\hat{u}_{xxxx}(t)\|^2 e^{\alpha t} + \int_1^t \|\hat{u}_{xxxx}(s)\|^2 e^{\alpha s} ds \\
&\quad \leq C_\alpha \|\hat{u}_{xxxx}(1)\|^2 + C(\alpha)^{-1} \int_1^t e^{\alpha s}(\|\hat{v}_{ttx}(s)\|^2 + \|\hat{v}_{xxx}(s)\|^2) ds \\
&\quad \leq C_\alpha (\|\hat{u}(1)\|_{H^4}^2 + \|\hat{v}(1)\|_{H^1}^2 + \|\hat{v}_t(1)\|_{H^1}^2 + \|\hat{v}_{tt}(1)\|^2).
\end{aligned}$$

Thus from (43)-(44), (50)-(53) it follows that for any $\alpha \in (0, \alpha_1), t \geq 1$,

$$\begin{aligned}
&e^{\alpha t}\{\|\hat{u}(t)\|_{H^4}^2 + \|\hat{v}(t)\|_{H^4}^2 + \|\hat{v}_t(t)\|_{H^2}^2 + \|\hat{v}_{tt}(t)\|^2\} \\
&\quad + \int_1^t e^{\alpha s}(\|\hat{u}\|_{H^4}^2 + \|\hat{v}\|_{H^5}^2 + \|\hat{v}_t\|_{H^3}^2 + \|\hat{v}_{tt}\|_{H^1}^2)(s) e^{\alpha s} ds \\
&\quad \leq C_\alpha (\|\hat{u}(1)\|_{H^4}^2 + \|\hat{v}(1)\|_{H^1}^2 + \|\hat{v}_t(1)\|_{H^1}^2 + \|\hat{v}_{tt}(1)\|^2).
\end{aligned} \tag{54}$$

Similarly as (50),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{(\|\hat{v}_{ttt}(t)\|^2 + \sigma(0)\|\hat{v}_{ttx}(t)\|^2) e^{\alpha t}\} \\
&\qquad + (1 - \alpha/2)\|\hat{v}_{tttx}(t)\|^2 e^{\alpha t} \leq (\sigma(0)\alpha/2)\|\hat{v}_{ttx}(t)\|^2 e^{\alpha t}
\end{aligned} \tag{55}$$

which with (54) and (38) implies for any $\alpha \in (0, \alpha_1)$ and for any $t \geq 1$,

$$\begin{aligned}
&\{\|\hat{v}_{ttt}(t)\|^2 + \|\hat{v}_{ttx}(t)\|^2 + \|\hat{v}_{txx}(t)\|^2\} e^{\alpha t} + \int_1^t e^{\alpha s}(\|\hat{v}_{tttx}\|^2 + \|\hat{v}_{ttt}\|^2)(s) ds \\
&\leq C_\alpha\{\|\hat{u}(1)\|_{H^2}^2 + \|\hat{v}(1)\|_{H^1}^2 + \|\hat{v}_t(1)\|_{H^1}^2 + \|\hat{v}_{tt}(1)\|_{H^1}^2 + \|\hat{v}_{ttt}(1)\|^2\}.
\end{aligned} \tag{56}$$

From (35), we get

$$\hat{v}_{txxxx} = \hat{v}_{ttt} - \sigma(0)\hat{v}_{xxx} - \sigma(0)\hat{v}_{tt} + \sigma^3(0)\hat{v}_{xx}, \tag{57}$$

which implies that for any $\alpha \in (0, \alpha_1)$ and for any $t \geq 1$,

$$\begin{aligned}
&\int_1^t \|\hat{v}_{txxxx}(s)\|^2 e^{\alpha s} ds \\
&\leq C_\alpha\{\|\hat{u}(1)\|_{H^2}^2 + \|\hat{v}(1)\|_{H^1}^2 + \|\hat{v}_t(1)\|_{H^1}^2 + \|\hat{v}_{tt}(1)\|_{H^1}^2 + \|\hat{v}_{ttt}(1)\|^2\}.
\end{aligned} \tag{58}$$

Similarly to (47),

$$\frac{1}{2}\frac{d}{dt}(\|\hat{u}_{xxxxx}(t)\|^2 e^{\alpha t}) + (\sigma(0)-\alpha)\|\hat{u}_{xxxxx}(t)\|^2 e^{\alpha t} \le (2\alpha)^{-1}\|\hat{v}_{txxxx}(t)\|^2 e^{\alpha t}$$

whence, by (55)-(57),

$$\|\hat{u}_{xxxx}(t)\|^2 e^{\alpha t} + \int_1^t \|\hat{u}_{xxxxx}(s)\|^2 e^{\alpha s} ds$$
$$\le C_\alpha\{\|\hat{u}(1)\|^2_{H^5} + \|\hat{v}(1)\|^2_{H^1} + \|\hat{v}_t(1)\|^2_{H^1} + \|\hat{v}_{tt}(1)\|^2_{H^1} + \|\hat{v}_{ttt}(1)\|^2\} \quad (59)$$

In an analogous manner, from (35), (55)-(58) it follows that for any $t \ge 1$ and for any $\alpha \in (0, \alpha_1)$,

$$\begin{aligned}\|\partial_x^6 \hat{v}(t)\|^2 &\le C_\alpha(\|\hat{v}_{ttt}(t)\| + \|\hat{v}_{xx}(t)\|_{H^1} + \|\partial_x^5 \hat{u}(t)\| + \|\hat{v}_{tt}(t)\|)\\ &\le C_\alpha[\|\hat{u}(1)\|^2_{H^5} + \|\hat{v}(1)\|^2_{H^1} + \|\hat{v}_t(1)\|^2_{H^1}\\ &+ \|\hat{v}_{tt}(1)\|^2_{H^1} + \|\hat{v}_{ttt}(1)\|^2]e^{-\alpha t}. \end{aligned} \quad (60)$$

We get from (35)

$$\partial_x^5 \hat{v}_t = \hat{v}_{tttx} - 2\sigma(0)\hat{v}_{ttx} + 2\sigma^2(0)\hat{v}_{xxx} + \sigma^2(0)\hat{u}_{xxxx}$$

which, along with (55)-(59), gives for any $t \ge 1$,

$$\begin{aligned}\int_1^t \|\partial_t^5 \hat{v}(s)\|^2 e^{\alpha s} ds &\le C_\alpha\{\|\hat{u}(1)\|^2_{H^5} + \|\hat{v}(1)\|^2_{H^1} + \|\hat{v}_t(1)\|^2_{H^1}\\ &+ \|\hat{v}_{tt}(1)\|^2_{H^1} + \|\hat{v}_{ttt}(1)\|^2\}. \end{aligned} \quad (61)$$

Applying the operator ∂_x^5 to (35) and using (34), we obtain

$$\frac{1}{2}\frac{d}{dt}\{\|\partial_x^6 \hat{u}(t)\|^2 e^{\alpha t}\} + (\sigma(0)-\alpha)\|\partial_x^6 \hat{u}(t)\|^2 e^{\alpha t} \le (2\alpha)^{-1}\|\partial_x^5 \hat{v}_t(t)\|^2 e^{\alpha t}$$

which, together with (61), yields for any $\alpha \in (0, \alpha_1)$ and any $t \ge 1$,

$$\|\partial_x^6 \hat{u}(t)\|^2 e^{\alpha t} + \int_1^t \|\partial_x^6 \hat{u}(s)\|^2 e^{\alpha s}$$
$$\le C_\alpha\{\|\hat{u}(1)\|^2_{H^5} + \|\hat{v}(1)\|^2_{H^1} + \|\hat{v}_t(1)\|^2_{H^1} + \|\hat{v}_{tt}(1)\|^2_{H^1} + \|\hat{v}_{ttt}(1)\|^2\}. \quad (62)$$

Thus it follows from (34)-(35),(43)-(44), (50)-(62) that

$$\{\|\hat{u}(t)\|_{H^6}^2 + \|\hat{v}(t)\|_{H^6}^2 + \|\hat{v}_t(t)\|_{H^2}^2 + \|\hat{v}_{tt}(t)\|_{H^1}^2 + \|\hat{v}_{ttt}(t)\|^2\}e^{\alpha t} \leq G_0(1), \tag{63}$$

$$\int_1^t [\|\hat{u}\|_{H^6}^2 + \|\hat{v}\|_{H^7}^2 + \|\hat{v}_t\|_{H^5}^2 + \|\hat{v}_{tt}\|_{H^3}^2 + \|\hat{v}_{ttt}\|_{H^1}^2](s)e^{\alpha s}ds \leq G_0(1) \tag{64}$$

for any $\alpha \in (0, \alpha_1), t \geq 1$ where

$$G_0(1) = \|\hat{u}(1)\|_{H^6}^2 + \|\hat{v}(1)\|_{H^1}^2 + \|\hat{v}_t(1)\|_{H^1}^2 + \|\hat{v}_{tt}(1)\|_{H^1}^2 + \|\hat{v}_{ttt}(1)\|^2.$$

Since $\{T(t)\}$ is an analytic semigroup on $X_0 \times X_0$ with its infinitesimal generator $\mathcal{A}$, it is easy to prove by the regularity of elliptic equations that

$$0 \in \rho(\mathcal{A}) \tag{65}$$

where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$. Thus from Lemma 13 and the relation (65) it follows that for any $t > 0, m \geq 0$,

$$\|\mathcal{A}^m T(t)\|_{\mathcal{L}(X_0)} \leq C(m)t^{-m}e^{-\delta t}, \tag{66}$$

where $\delta > 0$ and $C(m)$ are positive constants depending only on the operator $\mathcal{A}$ and m, but independent of t. Noting that

$$\hat{u}_x(t) = w_2(t), \quad \hat{v} = w_1(t),$$

from (33), (37) and (66) we easily obtain

$$\begin{aligned}
\|\hat{u}(1)\|_{H^6} \leq & C\|w_2(1)\|_{H^5} \\
\leq & C\{\|w_2(1)\| + \|Aw_2(1)\| + \|A^2w_2(1)\|\|A^3w_2(1)\|\} \\
\leq & C\{\|W(1)\| + \|\mathcal{A}W(1)\| + \|\mathcal{A}^2W(1)\| + \|\mathcal{A}^3W(1)\|\} \\
\leq & C\{\|T(1)\|_{\mathcal{L}(X_0)} + \|\mathcal{A}T(1)\|_{\mathcal{L}(X_0)} + \|\mathcal{A}^2T(1)\|_{\mathcal{L}(X_0)} \\
& + \|\mathcal{A}^3T(1)\|_{\mathcal{L}(X_0)}\}\|W_0\| \leq C\|W_0\|, && (67) \\
\|\hat{v}(1)\|_{H^1} = & \|w_1(1)\|_{H^1} \leq C(\|T(1)\|_{\mathcal{L}(X_0)} \\
& + \|\mathcal{A}T(1)\|_{\mathcal{L}(X_0)})\|W_0\| \leq C\|W_0\|, && (68) \\
\|\hat{v}_t(1)\| = & \|w_{1t}(1)\| \leq \|W_t(1)\| \leq \|\mathcal{A}T(1)\|_{\mathcal{L}(X_0)}\|W_0\| \leq C\|W_0\|, && (69) \\
\|\hat{v}_{ttt}(1)\| = & \|w_{1ttt}(1)\| \leq \|W_{ttt}(1)\| \leq \|\mathcal{A}W_{tt}(1)\| \\
\leq & \|\mathcal{A}T(1)\|_{\mathcal{L}(X_0)}\|W_0\| \leq C\|W_0\|, && (70) \\
\|\hat{v}_{ttx}(1)\| = & \|w_{1ttx}(1)\| \leq C(\|w_{1tt}(1)\| + \|Aw_{1tt}(1)\|) && (71) \\
\leq & C(\|W_{ttt}(1)\| + \|\mathcal{A}W_{ttt}(1)\|) \\
\leq & C\{\|\mathcal{A}^2T(1)\|_{\mathcal{L}(X_0)} + \|\mathcal{A}^3T(1)\|_{\mathcal{L}(X_0)}\}\|W_0\| \leq C\|W_0\|, && (72) \\
\|\hat{v}_{tt}(1)\| = & \|w_{1tt}(1)\| \leq \|W_{tt}(1)\| \leq \|\mathcal{A}^2T(1)\|_{\mathcal{L}(X_0)}\|W_0\| \leq C\|W_0\|, && (73)
\end{aligned}$$

$$
\begin{aligned}
\|\hat{v}_{tx}(1)\| = \|w_{1tx}(1)\| &\le C(\|w_{1t}(1)\| + \|w_{1t}(1)\|) \\
&\le C\{\|\mathcal{A}T(1)\|_{\mathcal{L}(X_0)} + \|\mathcal{A}^2T(1)\|_{\mathcal{L}(X_0)}\}\|W_0\| \le C\|W_0\|, \qquad (74)
\end{aligned}
$$

where $C > 0$ is a generic constant depending only on the operators $\mathcal{A}$ and $T(t)$, but independent of $t > 0$. Thus we finally deduce from (64), (67)-(73) that

$$
G_0(1) \le C\|W_0\|
$$

which, together with (63), gives us

$$
\begin{aligned}
\|T(t)W_0\|_{H^6\times H^5} = \|W(t)\|_{H^6\times H^5} \\
\le \|(\hat{u},\hat{v})\|_{H^6\times H^6} \le G_0(1) \le K_1\|W_0\|e^{-\alpha t}.
\end{aligned}
$$

for any $\alpha \in (0, \alpha_1)$ and for any $t \ge 1$. This proves Lemma 14. □

Lemma 15 *If a family of semiprocesses* $\Big\{U_g(t,\tau)\Big\}(g \in \Sigma, t \ge \tau \ge 0)$ *(symbol set) is uniformly asymptotically (with respect to* $g \in \Sigma$*) compact, then it possesses the uniform compact attractor* $\mathcal{A}_\Sigma$.

PROOF. See, e.g., p.131, Theorem 1.1 in [4]. □

Lemma 16 *If a family of semiprocesses* $\Big\{U_g(t,\tau)\Big\}(g \in \Sigma, t \ge \tau \ge 0)$*(symbol set) is uniformly asymptotically (with respect to* $g \in \Sigma$*) compact, then for any subset* $\Sigma' \subseteq \Sigma$ *there exists a uniform (with respect to* $g \in \Sigma'$*) attractor* $\mathcal{A}_{\Sigma'}$ *of the family* $\Big\{U_g(t,\tau)\Big\}(g \in \Sigma')$ *such that* $\mathcal{A}_{\Sigma'} \subseteq \mathcal{A}_\Sigma$*. The inclusion can be proper.*

PROOF. See, e.g., p.131, Corollary 1.1 in [4]. □

Lemma 17 *Let* E *be a reflexive separable Banach space and suppose that* $p > 1$*. A function* $g(s) \in L^p_{\text{loc}}(\mathbb{R}^+, E)$ *is translation compact in* $L^p_{\text{loc},w}$ *if and only if* $g(s)$ *is translation bounded in* $L^p_{\text{loc}}(\mathbb{R}^+, E)$,

$$
\|g\|^p_{L^p_b(\mathbb{R}^+,E)} = \sup_{t\in\mathbb{R}^+} \int_t^{t+1} \|g(s)\|^p_E ds < +\infty.
$$

PROOF. See, e.g., p.105, Proposition 4.1 in [4]. □

Lemma 18 *Let* $g(s)$ *be translation compact in* $L^p_{\text{loc},w}(\mathbb{R}^+, E)$*. Then*

(1) *any function* $g_1 \in H_+(g)$ *is also translation compact in* $L^p_{\text{loc},w}(\mathbb{R}^+, E)$*; moreover,* $H_+(g_1) \subseteq H_+(g)$*;*

(2) *the set* $H_+(g)$ *is bounded in* $L^p_b(\mathbb{R}^+, E)$, *and* $\eta_{g_1}(h) \leq \eta_g(h)$ *for any* $g_1 \in H_+(g)$;
(3) *the translation group* $\left\{\tilde{T}(t)\right\}$ *is continuous on* $H_+(g)$ *in the topology of* $L^p_{\mathrm{loc},w}(\mathbb{R}^+, E)$;
(4) $\tilde{T}(t)H_+(g) = H_+(g)$ *for all* $t \in \mathbb{R}^+$.

PROOF. See, e.g., p.106, Proposition 4.2 in [4]. □

Lemma 19 *Let a family of semiprocess* $\left\{U_g(t,\tau)\right\}(g \in \Sigma, t \geq \tau \geq 0)$ (*symbol set*) *acting on the space* E *be uniformly asymptotically* (*with respect to* $g \in \Sigma$) *compact and* $(E \times \Sigma, E)$*-continuous. Moreover, let* Σ *be a bounded complete metric space, and let a continuous asymptotically compact semigroup* $\left\{\hat{T}(t)\right\}$ *satisfying the translation identity*

$$U_g(t+h, \tau+h) = U_{\hat{T}(t)g}(t,\tau),\ t \geq \tau \geq 0, h \geq 0, g \in \Sigma$$

act on Σ, $\left\{\hat{T}(t)\right\} : \Sigma \to \Sigma, t \geq 0$. *Then the semigroup* $\left\{S(t)\right\}$ *acting on* $E \times \Sigma$ *by the formula*

$$S(t)(u,g) = (U_g(t,0)u, \hat{T}(t)g), \forall t \geq 0, (u,g) \in E \times \Sigma_1$$

possesses the compact attractor $\mathcal{A}$, *which is strictly invariant with respect to* $\left\{S(t)\right\} : S(t)\mathcal{A} = \mathcal{A}$ *for any* $t \geq 0$; *moreover,*

(1) $\pi_1\mathcal{A} = \mathcal{A}_1 = \mathcal{A}_\Sigma$ *is the uniform* (*with respect to* $g \in \Sigma$) *attractor of the family of semiprocess;*
(2) $\pi_2\mathcal{A} = \mathcal{A}_2 = \omega(\Sigma)$ *is the attractor of the semigroup* $\left\{\hat{T}(t)\right\}$ *acting on* Σ : $\hat{T}(t)\omega(\Sigma) = \omega(\Sigma)$, *for any* $t \geq 0$;
(3) $\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)}$, *where* $\mathcal{A}_{\omega(\Sigma)}$ *is the uniform attractor of the family of semiprocesses* $\left\{U_g(t,\tau)\right\}(g \in \omega(\Sigma), t \geq \tau \geq 0)$.

PROOF. See, e.g., p.134, Theorem 2.1 in [4]. □

Proof of Theorem 1 Multiplying (5) by v in $L^2(0,1)$, we obtain

$$\frac{d}{dt}\|v(t)\|^2 + \|v_x(t)\|^2 + 2\frac{d}{dt}\int_0^1\int_\tau^{u(t)}\int_0^s \sigma(\xi)d\xi ds dx \leq \|f(t)\|^2$$

which yields for any $t \geq \tau$

$$\|v(t)\|^2+\int_\tau^t\|v_x(s)\|^2ds+2\int_0^1\int_\tau^{u(t)}\int_0^s\sigma(\xi)d\xi dsdx$$
$$\leq\|v_0^\tau(x)\|^2+2\int_0^1\int_\tau^{u_0^\tau}\int_0^s\sigma(\xi)d\xi dsdx+\int_\tau^t\|f(s)\|^2ds. \tag{75}$$

Analogously, multiplying (5) by u_x, using (4), it follows that

$$\frac{d}{dt}\Big\{\frac{1}{2}\|u_x\|^2-(v,u_x)\Big\}+\int_0^1\sigma(u)u_x^2dx=\|v_x\|^2-(f,u_x) \tag{76}$$

which, together with (75), implies

$$\frac{1}{2}\|u_x(t)\|^2+\int_\tau^t\int_0^1\sigma(u)u_x^2dxds\leq\frac{1}{4}\|u_x(t)\|^2+\|v(t)\|^2+\frac{1}{2}\|u_{0,x}^\tau\|^2$$
$$-(v_0^\tau,u_{0,x}^\tau)+\int_\tau^t\|v_x(s)\|^2ds+\int_\tau^t\|f(s)\|\|u_x(s)\|ds$$
$$\leq\frac{1}{4}\|u_x(t)\|^2+\int_\tau^t\|f\|^2ds+\int_\tau^t\|f(s)\|\|u_x(s)\|ds+H_1 \tag{77}$$

for

$$H_1=\frac{1}{2}\|u_0^\tau\|^2+\|v_0^\tau\|\|u_0^\tau\|+\|v_0^\tau\|^2+2\int_0^1\int_\tau^{u_0^\tau}\int_0^s\sigma(\xi)d\xi dsdx\leq C_1(\tau).$$

Thus,

$$F^2(t)\leq H(t)+\int_\tau^t F(s)G(s)ds,\quad t\geq\tau\geq0$$

with $F(t)=\frac{1}{2}\|u_x(t)\|, G(t)=2\|f(t)\|, H(t)=\int_\tau^t\|f(s)\|^2ds+H_1$, which gives us

$$\frac{1}{4}\|u_x(t)\|=\frac{1}{2}F(t)\leq\frac{1}{2}\int_\tau^t G(s)ds+\sup_{\tau\leq s\leq t}H^{1/2}(s)\leq C_1(\tau). \tag{78}$$

From (4) we have

$$\int_0^1 u_0^\tau dx=\int_0^1 udx=0 \tag{79}$$

which yields, together with Poincaré's inequality,

$$\|u(t)\|_{L^\infty} \le C\|u_x(t)\| \le C_1(\tau), \tag{80}$$
$$C_1^{-1}(\tau) \le \sigma(u) \le C_1(\tau). \tag{81}$$

From (77) and (81), we obtain

$$\int_\tau^t \|u_x(s)\|^2 ds \le C_1(\tau), \quad t \ge \tau \ge 0. \tag{82}$$

Now multiplying (5) by v_t in $L^2(0,1)$, we have

$$\frac{d}{dt}\|v_x(t)\|^2 + \|v_t(t)\|^2 \le 2\|\sigma(u)\|_{L^\infty}\|u_x(t)\|^2.$$

In view of (78) and (80)-(82), we get for $t \ge \tau \ge 0$,

$$\|v_x(t)\|^2 + \int_\tau^t \|v_t(s)\|^2 ds \le C_1(\tau)\int_\tau^t \|u_x(s)\|^2 ds + C_1(\tau) \le C_1(\tau)$$

which, according to (6), (35) and (78)-(82), gives us for $t \ge \tau \ge 0$,

$$\|v_x(t)\|^2 + \int_\tau^t (\|u\|_{H^1}^2 + \|v\|_{H^2}^2 + \|v_t\|^2)(s) ds \le C_1(\tau). \tag{83}$$

Therefore (10) follows from (78), (80)-(82) and (83) and hence the solution $(u(t), v(t))$ exists globally in time in H_+^1 for any given initial datum $(u_0^\tau, v_0^\tau) \in H_+^1, \tau \in \mathbb{R}^+$.
Multiplying (5) by $e^{\beta t}v$ in $L^2(0,1)$, and using (4), we derive

$$\begin{aligned}
&\|v(t)\|^2 e^{\beta t} + \int_\tau^t \|v_x(s)\|^2 e^{\beta s} ds \\
&\qquad \le C_1(\tau)e^{\beta\tau} + C_1(\tau)\beta e^{\beta t}(\|v(t)\|^2 + \|u(t)\|^2) + \int_\tau^t \|f(s)\|^2 e^{\beta s} ds \\
&\qquad \le C_1(\tau)e^{\beta\tau} + C_1(\tau)\beta e^{\beta t}(\|v(t)\|^2 + \|u_x(t)\|^2) + \int_\tau^t \|f(s)\|^2 e^{\beta s} ds
\end{aligned}$$

which implies that there is a constant $\beta^1 = \beta^1(C_1(\tau)) > 0$ such that for any $\beta \in (0, \beta^1]$,

$$\|v(t)\|^2 e^{\beta t} + \int_\tau^t \|v_x(s)\|^2 e^{\beta s} ds$$

$$\leq C_1(\tau)e^{\beta\tau} + C_1(\tau)\beta e^{\beta t}\|u_x(t)\|^2 + \int_\tau^t \|f(s)\|^2 e^{\beta s} ds \tag{84}$$

where we have used the following estimate by (79)-(81):

$$\left| \int_0^1 \int_\tau^{u(t)} \int_0^s \sigma(\xi) d\xi ds dx \right| \leq C_1(\tau) \int_0^1 \int_0^{|u(t)|} s ds dx \leq C_1(\tau)\|u(t)\|^2$$

$$\leq C_1(\tau)\|u_x(t)\|^2. \tag{85}$$

Analogously, multiplying (76) by $e^{\beta t}$, we obtain

$$\left\{ \frac{1}{2}\|u_x(t)\|^2 - \ (v, u_x) \right\} e^{\beta t} + C_1^{-1}(\tau) \int_\tau^t \|u_x(s)\|^2 e^{\beta s} ds$$

$$\leq C_1(\tau)e^{\beta\tau} + C_1(\tau)\beta \int_\tau^t e^{\beta s}(\|u_x\|^2 + \|v_x\|^2)(s) ds$$

$$+ \int_\tau^t e^{\beta s}\|v_x(s)\|^2 ds + \int_\tau^t \|f(s)\|\|u_x(s)\| e^{\beta s} ds$$

which, combined with (84), gives us for any $\beta \in (0, \beta^1]$,

$$\left\{ \frac{1}{2}\|u_x\|^2 - \ (v, u_x) \right\} e^{\beta t} + C_1^{-1}(\tau) \int_\tau^t \|u_x(s)\|^2 e^{\beta s} ds$$

$$\leq C_1(\tau)e^{\beta\tau} + C_1(\tau)\beta \int_\tau^t e^{\beta s}(\|u_x\|^2 + \|v_x\|^2)(s) ds$$

$$+ C_1(\tau) \int_\tau^t e^{\beta s}\|f(s)\|\|u_x(s)\| ds$$

$$+ \ C_1(\tau)\beta e^{\beta t}(\|v(t)\|^2 + \|u_x(t)\|^2) + \int_\tau^t \|f(s)\|^2 ds. \tag{86}$$

Noting that

$$\frac{1}{2}\|u_x(t)\|^2 - \|v(t)\|\|u_x(t)\| + \|v(t)\|^2 \geq \frac{1}{8}(\|u_x(t)\|^2 + \|v(t)\|^2)$$

and summing up (84) and (86), we see that there exists a positive constant $\beta_1 = \beta_1(C_1(\tau)) \leq \min[\beta^1, \gamma_1/2, 1/(2C_1(\tau)^2)]$ such that for $\beta \in (0, \beta_1]$,

$$[\|v(t)\|^2 + \|u_x(t)\|^2]e^{\beta t} + \int_\tau^t (\|u_x\|^2 + \|v_x\|^2)(s)e^{\beta s}ds$$
$$\leq C_1(\tau)e^{\beta\tau} + C_1(\tau)\int_\tau^t \|f(s)\|^2 e^{\beta s}ds \leq C_1(\tau)e^{\beta\tau}. \tag{87}$$

Multiplying (5) by $e^{\beta t}v_t$ in $L^2(0,1)$ and using (87), we get for any $\beta \in (0, \beta_1]$,

$$\|v_x(t)\|^2 e^{\beta t} + \int_\tau^t e^{\beta s}\|v_t(s)\|^2 ds \leq C_1(\tau)e^{\beta\tau}. \tag{88}$$

Thus the combination of (87) and (88) gives (12). In the sequel, we will show the uniqueness of the global solution $(u(t), v(t))$ in H_+^1. To this end, we assume that $(u_i^\tau(t), v_i^\tau(t)) = (u_i(t), v_i(t))$ $(i = 1, 2)$ are two global solutions corresponding to the initial data $(u_{0,i}^\tau, v_{0,i}^\tau)$ and external forces $f^i(t) \in E_1 (i = 1, 2)$, respectively. We write

$$u = u_1 - u_2, v = v_1 - v_2, f = f^1 - f^2.$$

Then (u, v) satisfies

$$u_t = v_x, \qquad t \geq \tau, \tag{89}$$
$$v_t = \sigma(u_1)u_x + v_{xx} + \big(\sigma(u_1) - \sigma(u_2)\big)u_{2x} + f, \qquad t \geq \tau, \tag{90}$$
$$v(0,t) = v(1,t) = 0, \qquad t \geq \tau, \tag{91}$$
$$u(0,x) = u_{0,1}^\tau - u_{0,2}^\tau \equiv u_0^\tau, v(0,x) = v_{0,1}^\tau - v_{0,2}^\tau \equiv v_0^\tau. \tag{92}$$

Multiplying (89) and (90) by u and v_t in $L^2(0,1)$ respectively, and using the embedding theorem and the mean value theorem, we derive

$$\frac{d}{dt}\|u(t)\|^2 \leq \frac{1}{2}(\|v_x(t)\|^2 + \|u(t)\|^2), \tag{93}$$
$$\frac{d}{dt}\|v(t)\|^2 + 2\|v_x(t)\|^2 \leq C_1(\tau)(\|u(t)\|_{H^1}^2 + \|v(t)\|^2 + \|f(t)\|^2), \tag{94}$$
$$\frac{1}{2}\frac{d}{dt}\|v_x(t)\|^2 + \|v_t(t)\|^2 \leq \frac{1}{2}\|v_t(t)\|^2 + C_1(\tau)(\|u_x(t)\|^2 + \|f(t)\|^2)$$
$$+ C\|\sigma(u_2) - \sigma(u_1)\|_{L^\infty}^2 \|u_{2x}(t)\|^2$$
$$\leq \frac{1}{2}\|v_t(t)\|^2 + C_1(\tau)(\|u(\tau)\|_{H^1}^2 + \|f(t)\|^2)$$

or

$$\frac{d}{dt}\|v_x(t)\|^2 + \|v_t(t)\|^2 \leq C_1(\tau)(\|u_x(t)\|^2 + \|f(t)\|^2). \tag{95}$$

Here and in the following in the proof $C_1(\tau)$ denotes the generic positive constant depending only on $\|(u_{0,i}^\tau, v_{0,i}^\tau)\|_{H^1}$ and $\|f^i\|_{E_1}$ $(i = 1, 2)$. Multiplying (90) by u_x and using (89), we derive

$$\begin{aligned}\frac{d}{dt}\Big\{\frac{1}{2}\|u_x(t)\|^2 - (v, u_x)\Big\} + \int_0^1 \sigma(u_1)u_x^2 dx \\ \leq C_1(\tau)(\|u(t)\|_{H^1}^2 + \|v_x(t)\|^2 + \|f(t)\|^2).\end{aligned} \tag{96}$$

Noting that

$$\frac{7}{16}(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2) \leq F_1(t) \leq \frac{5}{4}(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2) \tag{97}$$

where

$$F_1(t) = \|u(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2 - (v, u_x) + \|v(t)\|^2 + \|v_x(t)\|^2,$$

and adding up (93)-(96), we obtain

$$\begin{aligned}\frac{d}{dt}F_1(t) &\leq C_1(\tau)\Big\{\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|f(t)\|^2\Big\} \\ &\leq C_1(\tau)(F_1(t) + \|f(t)\|^2), \quad t \geq \tau \geq 0\end{aligned} \tag{98}$$

which, by the Gronwall's inequality, gives us

$$F_1(t) \leq \Big\{F_1(\tau) + C_1(\tau)\int_\tau^\infty \|f(s)\|^2 ds\Big\}e^{C_1(\tau)(t-\tau)}, \;\; t \geq \tau \geq 0. \tag{99}$$

Thus it follows from (97) and (99) that

$$\begin{aligned}\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 \leq\, & e^{C_1(\tau)(t-\tau)t}\{\|u_{0,1}^\tau - u_{0,2}^\tau\|_{H^1}^2 + \|v_{0,1}^\tau - v_{0,2}^\tau\|_{H^1}^2 \\ & + C_1(\tau)\|f\|_{E_1}^2\}.\end{aligned} \tag{100}$$

First, the estimate (100) readily implies the uniqueness of the global solution $(u(t), v(t)) \in H_+^1$ to problem (4)-(7) for any arbitrary but fixed external force $f \in E_1$, and hence by the uniqueness there exists a unique semiprocess $\Big\{U_f^{(1)}(t,\tau)\Big\}$, which is generated by the global weak (regular) solution

$(u(t),v(t)) = (u^\tau(t),v^\tau(t)) \in H^1_+$ with the initial datum $(u^\tau_0,v^\tau_0) \in H^1_+$ such that (9) holds. Second, the estimate (100) also implies that the semiprocess $\left\{U^{(1)}_f(t,\tau)\right\}$ is $(H^1_+ \times E_1, H^1_+)$-continuous. In what follows, we shall show that the semiprocess $\left\{U^{(1)}_f(t,\tau)\right\}$(for fixed $f \in E_1, t \geq \tau \geq 0$) possesses a (non-uniform) compact attractor $\mathcal{A}^{(1)}_{\{f\}}$. To this end, for any given bounded set $B_1 \subseteq H^1_+$, we assume that $(u^\tau_0,v^\tau_0) \in B_1$, i.e., $\|(u^\tau_0,v^\tau_0)\|_{H^1} \leq \hat{B}_1$, $\hat{B}_1$ is a positive constant. Consequently, we derive from (12) that there exists a positive constant $\beta_1 = \beta_1(C_1(\tau),\hat{B}_1)$ such that for any fixed $\beta \in (0,\beta_1]$, we obtain

$$\begin{aligned} e^{\beta t}\|(u^\tau(t),v^\tau(t))\|^2_{H^1} + \int_\tau^t e^{\beta s}(\|(u^\tau\|^2 + \|v^\tau\|^2_{H^2} + \|v^\tau_t\|^2)(s)ds \\ \leq C_1(\tau,\hat{B}_1)e^{\beta\tau}. \end{aligned} \tag{101}$$

for any $t \geq \tau \geq 0$. Now fix $\beta \in (0,\beta_1(C_1(\tau),\hat{B}_1)], \tau \in \mathbb{R}^+$, and let

$$R_1 = 1, t_1 = \max\{\beta^{-1}, \log[C_1(\tau,\hat{B}_1)e^{\beta\tau}], \tau\}.$$

Then it follows from (101) that, when $t \geq t_1 \geq \tau$, for any fixed $f \in E_1$,

$$(u^\tau(t),v^\tau(t)) = U^{(1)}_f(u^\tau_0,v^\tau_0) \in B(0,R_1) \equiv \{(u,v) \in H^1_+ : \|(u,v)\|_{H^1} \leq 1\}.$$

Hence

$$\bigcup_{f\in\Sigma_1} U^{(1)}_f(t,\tau)B_1 \subseteq B(0,R_1)$$

which implies that $B(0,R_1)$ is a (non-uniform) absorbing set in H^1_+ for the semiprocess $\left\{U^{(1)}_f(t,\tau)\right\}(t \geq \tau \geq 0, f \in E_1$ fixed). In the sequel, we shall prove that this semiprocess is compact.

Lemma 14, the mean value theorem and (27),(101), together with $u^\tau_x = Ay^\tau = v^\tau_2(t)(u(t) = u^\tau(t))$, easily yield for fixed $\beta \in (0,\beta_1]$,

$$\begin{aligned} \|B(A^{-1}v^\tau_2) - \sigma(v^\tau_2)\| &= \|(\sigma(w) - \sigma(0))v^\tau_2\| \leq C_1(\tau,\hat{B}_1)\|w\|_{L^\infty}\|v^\tau_2\| \\ &\leq C_1(\tau,\hat{B}_1)\|w\|_{L^\infty}\|u^\tau_x\| \leq C_1(\tau,\hat{B}_1)e^{-\beta t}\|V(t)\| \\ &\leq C_1(\tau,\hat{B}_1)e^{-\beta t}\|V(t)\|_{H^6\times H^5} \end{aligned} \tag{102}$$

where

$$w(t) = \int_0^x v^\tau_2(\xi,t)d\xi + \int_0^1 (\xi-1)v^\tau_2(\xi,t)d\xi.$$

By the mean value theorem there is a point $x_0 \in [0,1]$ such that

$$w(t) = \int_{x_0}^x v^\tau_2(\xi,t)d\xi, \quad x \in [0,1]$$

satisfying, by (101),

$$\|w\|_{L^\infty} \leq \|v_2^\tau\| = \|u_x^\tau\| \leq C_1(\tau, \hat{B}_1)e^{-\beta t}.$$

Thus from Lemma 14, (27), (32) and (102) it follows that

$$\begin{aligned}
&\|V(t)\|_{H^6\times H^5}\\
&\leq \|T(t-\tau)V_0^\tau\|_{H^6\times H^5} + \int_\tau^t \|T(t-s)\Big(C(V(s)) + F(s)\Big)\|_{H^6\times H^5} ds\\
&\leq K_1 e^{-\alpha t/2}\|V_0^\tau\| + K_1 e^{-\alpha\tau/2}\int_\tau^t e^{-\alpha(t-s)/2}\|C(V(s)) + F(s)\| ds\\
&\leq K_1 e^{-\alpha t/2}\|V_0^\tau\| + C_1(\tau, \hat{B}_1)\int_\tau^t e^{-\alpha(t-s)/2}e^{-\beta s}\|V(s)\|_{H^6\times H^5} ds\\
&+C\int_\tau^t e^{-\alpha(t-s)/2}e^{-\gamma_0 s} ds
\end{aligned}$$

for any $t \geq \tau + 1, \tau \geq 0$, which implies that for any fixed $\alpha \in (0, \alpha_1), \gamma_0 < \min[\alpha/2, \gamma_1]$,

$$\begin{aligned}
e^{\alpha t/2}\|V(t)\|_{H^6\times H^5} \leq & K_1\|V_0^\tau\| + C_\alpha e^{(\alpha/2-\gamma_0)t}\\
&+ C_1(\tau, \hat{B}_1)\int_\tau^t e^{-\beta s}e^{\alpha s/2}\|V(s)\|_{H^6\times H^5} ds.
\end{aligned} \tag{103}$$

Therefore by the Gronwall inequality, we obtain from (103) for $t \geq \tau+1, \tau \geq 0$,

$$\begin{aligned}
e^{\alpha t/2}\|V(t)\|_{H^6\times H^5} \leq & \{K_1\|V_0^\tau\| + C_\alpha e^{(\alpha/2-\gamma_0)t}\}e^{C_1(\tau,\hat{B}_1)\int_\tau^t e^{-\beta s}ds}\\
\leq & C_1(\tau, \hat{B}_1, \beta)\{K_1\|V_0^\tau\| + C_\alpha e^{(\alpha/2-\gamma_0)t}\}.
\end{aligned} \tag{104}$$

Noting that $v_1^\tau = y_t^\tau = v^\tau, v_2^\tau = Ay^\tau = u_x^\tau(u = u^\tau(t), v = v^\tau(t))$, we readily deduce from (104) that for $t \geq \tau + 1, \tau \geq 0$,

$$\|(u^\tau(t), v^\tau(t))\|_{H^6\times H^6} \leq C_1(\tau, \hat{B}_1, \beta)e^{-\alpha t/2}\Big(\|(u_0^\tau, v_0^\tau)\|_{H^1} + e^{(\alpha/2-\gamma_0)t}\Big) \tag{105}$$

which implies that there exists some time $t_1' = t_1'(C_1(\tau, \hat{B}_1, \beta), \alpha) \geq t_1$ such that as $t \geq t_1'$,

$$\|(u^\tau(t), v^\tau(t))\|_{H^6\times H^6} \leq 1. \tag{106}$$

Since the embedding $H^6 \times H^6 \hookrightarrow H^1_+$ is compact, it follows from (106) that $B(0, R_1)$ is a (non-uniform) compact absorbing set in H^1_+ and furthermore $\left\{U_f^{(1)}(t,\tau)\right\}$ is a (non-uniform) compact family of semiprocess, and accordingly in view of Lemma 15, we derive that for any fixed $f \in E_1$, the non-uniform compact family of semiprocesses $\left\{U_f^{(1)}(t,\tau)\right\}(t \geq \tau \geq 0)$ possesses a (non-uniform) attractor $\mathcal{A}^{(1)}_{\{f\}}$ in H^1_+. The proof is complete. □

Proof of Theorem 2 Since $f_1 \in E_1$, a direct computation yields that for any $h > 0$,

$$\|f_1(t+h)\|_{E_1} = \|f_1(t)\|_{E_1} < +\infty.$$

Thus for any $f \in \Sigma_1$, there exists a sequence $\{h_n\} \subseteq \mathbb{R}^+$ such that

$$f_1(t + h_n) \longrightarrow f(t) \;\; in \;\; E_1 \;\; as \;\; n \to \infty. \tag{107}$$

Since E_1 is a Banach space and $\{f_1(t + h_n)\} \subseteq E_1$, we have $f \in E_1$. That is,

$$\Sigma_1 \subseteq E_1, \; \|f_1(t + h_n)\|_{E_1} = \|f_1(t)\|_{E_1} < +\infty \tag{108}$$

which, according to (107), implies

$$\sup_{f\in\Sigma_1} \|f\|_{E_1} \leq \|f_1\|_{E_1} < +\infty \tag{109}$$

where

$$\|f_1\|_{E_1} = \Big(\int\limits_{\mathbb{R}^+} \|f_1(t)\|^2_{L^2(0,1)}dt\Big)^{1/2} + \int\limits_{\mathbb{R}^+} \|f_1(t)\|_{L^2(0,1)}dt.$$

Note that the estimate (109) implies that the generic constants $C_1(\tau)$ and β_1 obtained in Theorem 1 eventually depend only on $C_1^*(\tau)$(i.e., $C_1(\tau) \leq C_1^*(\tau) = C_1^*(\|(u_0^\tau, v_0^\tau)\|_{H^1}, \|f_1\|_{E_1}$, but are independent of $f \in \Sigma_1$). Thus it follows from Theorem 1 (noting that $\Sigma_1 \subseteq E_1$) that for any $(u_0^\tau, v_0^\tau) \in H^1_+$, there exists a unique global (regular) weak solution $(u^\tau(t), v^\tau(t)) \in H^1_+$ to problem (4)-(7), which generates a unique semiprocess $\left\{U^{(1)}_{f|\Sigma_1}(t,\tau)\right\}$ on H^1_+ of a two parameter family of operators such that (9)-(10) and (12) hold if (11) holds where $C_1(\tau)$ should be replaced by $C_1^*(\tau)$ and hence β_1 should be replaced by $\beta^1_* = \beta^1_*(C_1^*(\tau))$. Moreover, in view of $\Sigma_1 \subseteq E_1$, (100) still holds with $\|f\|_{E_1}$ being replaced by $\|f\|_{\Sigma_1}$, this implies that the semiprocess $\left\{U^{(1)}_{f|\Sigma_1}(t,\tau)\right\}$ is $(H^1_+ \times \Sigma_1, H^1_+)$-continuous. In what follows, we show that the semiprocess $\left\{U^{(1)}_{f|\Sigma_1}(t,\tau)\right\}(f \in \Sigma_1, t \geq \tau \geq 0)$ possesses a uniform compact attractor $\mathcal{A}_{\Sigma_1}$. Similarly, for any given bounded set $\tilde{B}_1 \subseteq H^1_+$, we assume that $(u_0^\tau, v_0^\tau) \in \tilde{B}_1$, i.e., $\|(u_0^\tau, v_0^\tau)\|_{H^1} \leq \hat{B}_1^*$, for a positive constant $\hat{B}_1^*$. Then $C_1^*(\tau) \leq C_1^*(\tau, \hat{B}_1^*)$ where $C_1^*(\tau, \hat{B}_1^*)$ is a positive constant depending only on

$\hat{B}_1^*$ and $\|f_1\|_{E_1}$(noting (109)). Consequently, by (9)-(12), from that we see that there exists a positive constant $\beta_1^* = \beta_1^*(C_1^*)$ such that for any fixed $\beta \in (0, \beta_1^*]$

$$e^{\beta t}\|(u^\tau(t), v^\tau(t))\|_{H^1}^2 + \int_\tau^t e^{\beta s}(\|(u^\tau\|^2 + \|v^\tau\|_{H^2}^2 + \|v_t^\tau\|^2)(s)ds \\ \leq C_1^*(\tau, \hat{B}_1^*)e^{\beta\tau}. \tag{110}$$

for any $t \geq \tau \geq 0$. Now fix $\beta \in (0, \beta_1^*], \tau \in \mathbb{R}^+$ and let

$$R_1 = 1, t_1^* = \max\left\{\beta^{-1}, \log[C_1^*(\tau, \hat{B}_1^*)e^{\beta\tau}], \tau\right\}.$$

Then it follows from (110) that as $t \geq t_1^* \geq \tau$, for any $f \in \Sigma_1$,

$$(u^\tau(t), v^\tau(t)) = U_{f|\Sigma_1}^{(1)}(t, \tau)(u_0^\tau, v_0^\tau) \\ \in B(0, R_1) \equiv \{(u, v) \in H_+^1 : \|(u, v)\|_{H^1} \leq 1\}.$$

Hence

$$\bigcup_{f\in\Sigma_1} U_{f|\Sigma_1}^{(1)}(t, \tau)\tilde{B}_1 \subseteq B(0, R_1)$$

which implies that $B(0, R_1)$ is a uniform absorbing set in H_+^1 for the semiprocess $\left\{U_{f|\Sigma_1}^{(1)}(t, \tau)\right\}(t \geq \tau \geq 0, f \in \Sigma_1)$. Similarly, in this case, (105) is still valid with $C_1(\tau, \hat{B}_1, \beta)$ being replaced by $C_1^*(\tau, \hat{B}_1^*, \beta)$. This implies that there exists some time $t_1^{**} = \max[t_1^*, t_1']$ such that for $t \geq t_1^{**}$,

$$\|(u^\tau(t), v^\tau(t))\|_{H^6\times H^6} \leq 1$$

which implies that $B(0, R_1)$ is a uniform compact absorbing set in H_+^1 and hence $\left\{U_{f|\Sigma_1}^{(1)}(t, \tau)\right\}$ is a uniformly compact family of semiprocess and further it is a uniformly asymptotically compact family of semiprocess. Thus it follows from Lemma 15 that $\left\{U_{f|\Sigma_1}^{(1)}(t, \tau)\right\}$ possesses a uniform (with respect to $f \in \Sigma_1$) compact attractor $\mathcal{A}_{\Sigma_1}$, and evidently $\bigcup_{f\in\Sigma_1} \mathcal{A}_{\{f\}}^{(1)} \subseteq \mathcal{A}_{\Sigma_1}$. The proof is complete. □

Proof of Theorem 4 First, since

$$\hat{f}_1 \in E_1 = L^2(\mathbb{R}^+, L^2(0, 1)) \cap L^1(\mathbb{R}^+, L^2(0, 1)) \subseteq L_{\text{loc},w}^2(\mathbb{R}^+, L^2(0, 1)) = \hat{E}_1,$$

the corresponding conclusions except for (1)-(4) and (i)-(iv) of Theorem 2 follow for $\hat{\Sigma}_1$ and $\hat{E}_1$ in place of Σ_1 and E_1 respectively. Second, we readily get

$$\|\hat{f}_1\|_{L_b^2(\mathbb{R}^+, L^2(0,1))} \leq \|\hat{f}_1\|_{E_1} < +\infty,$$

which, by Lemma 17, implies that $\hat{f}_1$ is translation compact in $\hat{E}_1$ and hence the conclusions (1)-(4) and (i)-(iii) in Theorem 4 follow from Lemmas 18-19. A direct computation of a transform of variables gives us (iv) in Theorem 4. The proof is now complete. $\square$

3 Estimates in H_+^2

In this section we shall complete the proofs of Theorems 6-7.

Proof of Theorem 6 First, by the embedding theorem, we have

$$f(t) \subseteq E_2 \subseteq H^1(\mathbb{R}^+, L^2(0,1)) \subseteq L^\infty(\mathbb{R}^+, L^2(0,1))$$

whence

$$\|f(t)\|_{L^\infty(\mathbb{R}^+, L^2(0,1))} \leq C\|f(t)\|_{E_2} \tag{111}$$

with $C > 0$ being an absolute constant independent of t.

Differentiating (5) with respect to t, multiplying the resulting equation by v_t in $L^2(0,1)$ and using Theorem 1, we have for $t \geq \tau$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_t(t)\|^2 + \|v_{xt}(t)\|^2 \\
&\quad \leq C_1(\tau)(\|v_x(t)\|_{L^\infty}\|u_x(t)\| + \|v_{xx}(t)\|)\|v_t(t)\| + \|f_t(t)\|\|v_t(t)\| \\
&\quad \leq \frac{1}{2}\|v_{tx}(t)\|^2 + C_1(\tau)(\|v_x(t)\|^2_{L^\infty} + \|f_t(t)\|^2 + \|v_{xx}(t)\|^2)
\end{aligned} \tag{112}$$

which implies

$$\begin{aligned}
\|v_t(t)\|^2 + \int_\tau^t \|v_{tx}(s)\|^2 ds \leq \|v_t(\tau)\|^2 \\
+ C_1(\tau)\int_\tau^t (\|v_x\|^2 + \|v_{xx}\|^2 + \|f_t\|^2)(s) ds \leq C_2(\tau)
\end{aligned} \tag{113}$$

Here we have used

$$\|v_t(\tau)\| \leq C_1(\tau)(\|u_0^\tau\|_{H^1} + \|v_0^\tau\|_{H^2}) + \|f(\tau)\| \leq C_2(\tau).$$

By virtue of (5), (113) and Theorem 1, we easily obtain

$$\|v_{xx}(t)\| \le C(\|v_t(t)\| + \|u_x(t)\| + \|f(t)\|) \le C_2(\tau), \tag{114}$$
$$\|v_{xxx}(t)\| \le C_1(\tau)(\|v_{tx}(t)\| + \|u(t)\|_{H^1} + \|v(t)\|_{H^1}) \tag{115}$$

which, together with (113) and Theorem 1, yields

$$\|v(t)\|^2_{H^2} + \int_\tau^t (\|v\|^2_{H^3} + \|v_t\|^2_{H^1})(s)ds \le C_2(\tau).$$

Differentiating (5) with respect to x, and using (4), we deduce from Theorem 1 and the interpolation inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_{xx}(t)\|^2 + C_1^{-1}(\tau)\|u_{xx}(t)\|^2 \le (4C_1(\tau))^{-1}\|u_{xx}(t)\|^2 \\
&\quad + C_1(\tau)(\|v_{tx}(t)\|^2 + \|f_x(t)\|^2 + \|u_x(t)\|_{L^\infty}\|u_x(t)\|\|u_{xx}(t)\|) \\
&\quad \le (4C_1(\tau))^{-1}\|u_{xx}(t)\|^2 + C_1(\tau)(\|v_{tx}(t)\|^2 + \|f_x(t)\|^2) \\
&\quad + C_1(\tau)(\|u_x(t)\|^{1/2}\|u_{xx}(t)\|^{1/2} + \|u_x(t)\|)\|u_x(t)\|\|u_{xx}(t)\| \\
&\quad \le (2C_1(\tau))^{-1}\|u_{xx}(t)\|^2 + C_1(\tau)(\|v_{tx}(t)\|^2 + \|f_x(t)\|^2 + \|u_x(t)\|^2)
\end{aligned} \tag{116}$$

which, combined with Theorem 1, gives us

$$\|u_{xx}(t)\|^2 + \int_\tau^t \|u_{xx}(s)\|^2 ds \le C_2(\tau). \tag{117}$$

In view of (113)-(115) and (117), we have

$$\|u(t)\|^2_{H^2} + \|v(t)\|^2_{H^2} + \int_\tau^t (\|u\|^2_{H^2} + \|v\|^2_{H^3} + \|v_{tx}\|^2)(s)ds \le C_2(\tau) \tag{118}$$

which implies that there exists the global solution $(u(t), v(t)) = (u^\tau(t), v^\tau(t)) \in H^2_+$ for any given initial datum $(u_0^\tau, v_0^\tau) \in H^2_+$ and for fixed $f \in E_2$.
Now multiplying (112) and (116) by $e^{\beta t}$, we derive that there exists a constant $\beta_2 \le \min[\beta_1, \gamma_2/2]$ such that for any $\beta \in (0, \beta_2]$

$$e^{\beta t}(\|v_t(t)\|^2 + \|u_{xx}(t)\|^2) + \int_\tau^t e^{\beta s}(\|v_{tx}\|^2 + \|u_{xx}\|^2)(s)ds \le C_2(\tau)e^{\beta\tau} \tag{119}$$

for any $t \ge \tau \ge 0$ which, together with (114), (16) and Theorem 1, yields for any $\beta \in (0, \beta_2], t \ge \tau \ge 0$,

$$e^{\beta t}\|v_{xx}(t)\|^2 + \int_\tau^t e^{\beta s}\|v_{xx}(s)\|^2 ds \le C_2(\tau)e^{\beta\tau}. \tag{120}$$

Thus (17) follows from (12) and (119)-(120). In the sequel, we will show the uniqueness of the global solution $(u(t), v(t))$ in H_+^2. In fact, noting that $E_2 \subseteq E_1$, the global solution in H_+^2 is also a solution in H_+^1, so the uniqueness of solution in H_+^1 implies that of the solution in H_+^2. By the global existence of solution in H_+^2, we know that there exists a semiprocess of a two-parameter family of operators $\left\{U_f^{(2)}(t,\tau)\right\}$ which is generated by the global solution in H_+^2 such that (14) holds. We shall show that the semiprocess is $(H_+^2 \times E_2, H_+^2)$-continuous. To this end, we assume that $(u_i^\tau(t), v_i^\tau(t)) = (u_i(t), v_i(t))(i = 1, 2)$ are two global solutions in H_+^2 corresponding to the initial data $(u_{0,i}^\tau, v_{0,i}^\tau) \in H_+^2$ and external forces $f^i(t) \in E_2(i = 1, 2)$, respectively. Writing

$$u = u_1 - u_2, v = v_1 - v_2, f = f^1 - f^2,$$

then $(u(t), v(t))$ satisfies (89)-(92). Since $H_+^2 \subseteq H_+^1, E_2 \subseteq E_1$, we know that estimate (98) still holds. Similarly, we can deduce from (89)-(90)

$$\frac{d}{dt}\|v_t(t)\|^2 + \|v_{tx}(t)\|^2 \leq C_2(\tau)(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^2}^2 + \|f_t(t)\|^2) \quad (121)$$

and

$$\frac{d}{dt}\|u_{xx}(t)\|^2 + C_1^{-1}(\tau)\|u_{xx}(t)\|^2 \\ \leq C_2(\tau)(\|u(t)\|_{H^1}^2 + \|v_{tx}(t)\|^2 + \|f_x(t)\|^2). \quad (122)$$

Multiplying (121) by $2C_2(\tau)(C_2(\tau) > 1)$ and adding the resulting equation to (122), we get

$$\frac{d}{dt}\{2C_2(\tau)\|v_t(t)\|^2 + \|u_{xx}(t)\|^2\} + 2C_2(\tau)\|v_{tx}(t)\|^2 + C_1^{-1}(\tau)\|u_{xx}(t)\|^2 \\ \leq C_2(\tau)(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^2}^2 + \|f_t(t)\|^2 + \|f_x(t)\|^2). \quad (123)$$

Put

$$F_2(t) = F_1(t) + 2C_2(\tau)\|v_t(t)\|^2 + \|u_{xx}(t)\|^2. \quad (124)$$

We easily derive from (90),

$$\|v_{xx}(t)\| \leq C_1(\tau)(\|u(t)\|_{H^1} + \|v_t(t)\| + \|f(t)\|) \quad (125)$$
$$\|v_t(t)\| \leq C_1(\tau)(\|u(t)\|_{H^1} + \|v_{xx}(t)\| + \|f(t)\|) \quad (126)$$

which, together with (97) and (124), gives us,

$$\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 \leq C_2(\tau)(F_2(t) + \|f(t)\|^2), \tag{127}$$
$$F_2(t) \leq C_2(\tau)(\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \|f(t)\|^2). \tag{128}$$

Therefore, it follows from (123)-(128) that

$$\begin{aligned} F_2'(t) &\leq C_2(\tau)\Big\{F_2(t) + \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^2}^2 + \|f(t)\|_{H^1}^2 + \|f_t(t)\|^2\Big\} \\ &\leq C_2(\tau)(F_2(t) + \|f(t)\|_{H^1}^2 + \|f_t(t)\|^2) \end{aligned} \tag{129}$$

which, by the Gronwall inequality, results in

$$F_2(t) \leq \Big\{F_2(\tau) + C_2(\tau)\int_\tau^\infty (\|f\|_{H^1}^2 + \|f_t\|^2)(s)ds\Big\}e^{C_2(\tau)(t-\tau)}, \quad t \geq \tau \geq 0.$$

Thus, by (111) and noting that

$$\|f(t)\| + \|f(\tau)\| \leq 2\|f(t)\|_{L^\infty(\mathbb{R}^+, L^2(0,1))} \leq C\|f(t)\|_{E_2},$$

we conclude

$$\begin{aligned} \|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 &\leq C_2(\tau)(F_2(t) + \|f(t)\|^2) \\ &\leq C_2(\tau)e^{\alpha(t-\tau)}\Big\{\|u_0^\tau\|_{H^2}^2 + \|v_0^\tau\|_{H^2}^2 + \|f(\tau)\|^2 + \|f(t)\|^2 \\ &+ \int_\tau^\infty (\|f\|_{H^1}^2 + \|f_t\|^2)(s)ds\Big\} \\ &\leq C_2(\tau)e^{\alpha(t-\tau)}\Big\{\|u_0^\tau\|_{H^2}^2 + \|v_0^\tau\|_{H^2}^2 + \|f(t)\|_{E_2}^2\Big\} \end{aligned} \tag{130}$$

which implies the uniqueness of the solution $(u(t), v(t)) = (u^\tau(t), v^\tau(t)) \in H_+^2$. Thus it follows from (118) and (130) that there exists a semiprocess of a two-parameter family of operators $\Big\{U_f^{(2)}(t,\tau)\Big\}(f \in E_2, t \geq \tau \geq 0)$ such that (14) holds. On the other hand, (130) also implies that $\Big\{U_f^{(2)}(t,\tau)\Big\}(f \in E_2, t \geq \tau \geq 0)$ is $(H_+^2 \times E_2, H_+^2)-$continuous.
For any given bounded set $B_2 \subseteq H_+^2(\subseteq H_+^1)$, we assume that $(u_0^\tau, v_0^\tau) \in B_2$, i.e., $\|(u_0^\tau, v_0^\tau)\|_{H^2} \leq \hat{B}_2$, $\hat{B}_2(\geq \hat{B}_1)$ is a positive constant. Now fix $\beta \in (0, \beta_2], \tau \in \mathbb{R}^+$, and let

$$R_2 = 1, t_2 = \max\Big\{t_1, \log[C_2(\tau, \hat{B}_2)e^{\beta\tau}]\Big\}.$$

Thus it follows from (17) that as $t \geq t_2 \geq \tau$, for any fixed $f \in E_2$,

$$(u^\tau(t), v^\tau(t)) = U_f^{(2)}(u_0^\tau, v_0^\tau) \subseteq B(0, R_2) \equiv \{(u,v) \in H_+^2 : \|(u,v)\|_{H^2} \leq 1\}$$

which implies that $B(0, R_2)$ is a (non-uniform) absorbing set in H_+^2 for the semiprocess $\Big\{U_f^{(2)}(t,\tau)\Big\}(t \geq \tau \geq 0, f \in E_2$ fixed). In the sequel, we shall prove that this semiprocess is compact. On the other hand, picking $t_2' = \max[t_1', t_2]$, we conclude from (106) that for $t \geq t_2'$,

$$\|(u^\tau(t), v^\tau(t))\|_{H^6 \times H^6} \leq 1$$

which implies that $B(0, R_2)$ is a (non-uniform) compact absorbing set in H_+^2 noting that the embedding $H^6 \times H^6 \hookrightarrow H_+^2$ is compact. Hence the semiprocess $\left\{U_f^{(2)}(t, \tau)\right\}(t \geq \tau \geq 0, f \in E_2$ fixed) is a (non-uniform) compact family of operators and is further an asymptotically compact family of semiprocesses. Thus it follows from Lemma 15 that the semiprocess $\left\{U_f^{(2)}(t, \tau)\right\}(t \geq \tau \geq 0, f \in E_2$ fixed) possesses a (non-uniform) compact attractor $\mathcal{A}_{\{f\}}^{(2)}$. The proof is now complete. $\square$

Proof of Theorem 7 Since $f_2 \in E_2$, similarly to the arguments which gave us (108)-(109), we can derive

$$\Sigma_2 \subseteq E_2, \sup_{f \in \Sigma_2} \|f(t)\|_{E_2} \leq \|f_2(t)\|_{E_2} < +\infty. \tag{131}$$

On the other hand, we have by the embedding theorem,

$$E_2 \subseteq H^1(\mathbb{R}^+, L^2(0,1)) \subseteq L^\infty(\mathbb{R}^+, L^2(0,1)).$$

Hence it follows from Theorem 6 that for any $(u(t), v(t)) = (u_0^\tau, v_0^\tau) \in H_+^2$, there exists a unique global (regular) weak solution $(u^\tau(t), v^\tau(t)) \in H_+^2$ to problem (4)-(7) which generates a unique semiprocess $\left\{U_{f|\Sigma_2}^{(2)}(t, \tau)\right\}$ on H_+^2 of a two parameter family of operators such that (14)-(15) and (17) hold if (16) holds where $U_f^{(2)}(t, \tau)$ and $C_2(\tau)$ should be replaced by $U_{f|\Sigma_2}^{(2)}(t, \tau)$ and $C_2^*(\tau) = C_2^*(\|(u_0^\tau, v_0^\tau)\|_{H^2}, \|f_2\|_{E_2})$ (but independent of $f \in \Sigma_2$), respectively. In what follows, we show that the semiprocess $\left\{U_{f|\Sigma_2}^{(2)}(t, \tau)\right\}(f \in \Sigma_2, t \geq \tau \geq 0)$ possesses a uniform compact attractor $\mathcal{A}_{\Sigma_2}$. Note that estimate (131) implies that the generic constant $C_2(\tau)$ and hence β_2 obtained in Theorem 4 eventually depend only $C_2^*(\tau)$ (i.e., $C_2(\tau) \leq C_2^*(\tau)$). For any given bounded set $\tilde{B}_2 \subseteq H_+^2(\subseteq H_+^1)$, we assume that $(u_0^\tau, v_0^\tau) \in \tilde{B}_2$, i.e., $\|(u_0^\tau, v_0^\tau)\|_{H^2} \leq \hat{B}_2^*$, $\hat{B}_2^*(\geq \hat{B}_1^*)$ is a positive constant. Then $C_2^*(\tau) \leq C_2^*(\tau, \hat{B}_2^*)$, where $C_2^*(\tau, \hat{B}_2^*)$ is a positive constant depending only on $\hat{B}_2^*$ and $\|f_2\|_{E_2}$. Consequently, it follows that there exist positive constants $C_2^*(\tau, \hat{B}_2^*)$ and $\beta_2^* = \beta_2^*(C_2^*)$ such that for any fixed $\beta \in (0, \beta_2^*]$, estimate (17) still holds where $C_2(\tau)$ should be replaced by $C_2^*(\tau, \hat{B}_2^*)$, i.e.,

$$e^{\beta t}\|(u^\tau(t), v^\tau(t))\|_{H^2}^2 + \int_\tau^t e^{\beta s}(\|u^\tau\|_{H^2}^2 + \|v^\tau\|_{H^3}^2 + \|v_t^\tau\|_{H^1}^2)(s)ds \leq C_2^*(\tau, \hat{B}_2^*)e^{\beta\tau} \tag{132}$$

for any $t \geq \tau \geq 0$. Now fix $\beta \in (0, \beta_2^*], \tau \in \mathbb{R}^+$, and let

$$R_2 = 1, t_2^* = \max\left\{t_1^*, \log[C_2^*(\tau, \hat{B}_2^*)e^{\beta\tau}]\right\}.$$

Thus it follows from (132) that as $t \geq t_2^* \geq \tau$, for any $f \in \Sigma_2$,

$$\begin{aligned}(u^\tau(t), v^\tau(t)) = & U^{(2)}_{f|\Sigma_2}(t,\tau)(u_0^\tau, v_0^\tau) \\ & \in B(0, R_2) \equiv \{(u,v) \in H^2_+ : \|(u,v)\|_{H^2} \leq 1\}.\end{aligned}$$

Hence we have

$$\bigcup_{f\in\Sigma_1} U^{(2)}_{f|\Sigma_2}(t,\tau)\tilde{B}_2 \subseteq B(0, R_2)$$

and thus $B(0, R_2)$ is a uniform absorbing set in H^2_+ for the semiprocess $\left\{U^{(2)}_{f|\Sigma_2}(t,\tau)\right\}(t \geq \tau \geq 0, f \in \Sigma_2)$. In the sequel, we shall prove this semiprocess is compact. On the other hand, picking $t_2^{**} = \max[t_1', t_2^*]$, we conclude from (106) that as $t \geq t_2^{**}$,

$$\|(u^\tau(t), v^\tau(t))\|_{H^6\times H^6} \leq 1 \tag{133}$$

which implies that $B(0, R_2)$ is a uniform compact absorbing set in H^2_+, noting that the embedding $H^6 \times H^6 \hookrightarrow H^2_+$ is compact. Hence the semiprocess $\left\{U^{(2)}_{f|\Sigma_2}(t,\tau)\right\}(t \geq \tau \geq 0, f \in \Sigma_2)$ is a uniformly compact family of operators and is further asymptotically compact family of semiprocess. Thus it follows from Lemma 15 that $\left\{U^{(2)}_{f|\Sigma_2}(t,\tau)\right\}(t \geq \tau \geq 0, f \in \Sigma_2)$ possesses a uniform (with respect to $f \in \Sigma_2$) compact attractor $\mathcal{A}_{\Sigma_2}$, and evidently $\bigcup_{f\in\Sigma_2} \mathcal{A}^{(2)}_{\{f\}} \subseteq \mathcal{A}_{\Sigma_2}$. The proof is complete. $\square$

Proof of Theorem 8 The proof is basically same as that of Theorem 4. Noting that $f_2 \in E_2 \subseteq \hat{E}_2$, the conclusions of Theorem 8 follows from Theorem 7, where Σ_2 and E_2 should be replaced by $\hat{\Sigma}_2$ and $\hat{E}_2$, respectively. On the other hand, we easily deduce from Lemma 18 that any $f(s) \in \hat{\Sigma}_2$ and $f_s(s)$ are translation compact in $L^2_{\mathrm{loc},w}(\mathbb{R}^+, H^1(0,1))$ and $L^2_{\mathrm{loc},w}(\mathbb{R}^+, L^2(0,1))$, respectively, and that

$$\sup_{t\in\mathbb{R}^+} \int_t^{t+h} \|f(s)\|^2_{H^1} ds \leq \sup_{t\in\mathbb{R}^+} \int_t^{t+h} \|\hat{f}_2(s)\|^2_{H^1} ds, \tag{134}$$

$$\sum_{i=0}^{1} \sup_{t\in\mathbb{R}^+} \int_t^{t+h} \|\partial_s^i f(s)\|^2 ds \leq \sum_{i=0}^{1} \sup_{t\in\mathbb{R}^+} \int_t^{t+h} \|\partial_s^i \hat{f}_2(s)\|^2 ds. \tag{135}$$

Thus (134)-(135) and Lemma 17 give rise to

$$\|\hat{f}_2\|^2_{L^2_b(\mathbb{R}^+,H^1(0,1))\cap H^1_b(\mathbb{R}^+,L^2(0,1))}$$
$$= \sup_{t\in\mathbb{R}^+}\int_t^{t+1}\|\hat{f}_2(s)\|^2_{H^1} + \sum_{i=0}^{1}\sup_{t\in\mathbb{R}^+}\int_t^{t+1}\|\partial_s^i\hat{f}_2(s)\|^2 ds \le \|\hat{f}_2\|^2_{E_2} < +\infty$$

which, by Lemma 17, yields that $\hat{f}_2$ is translation compact in $\hat{E}_2$. Moreover, conclusions (1)-(4) and (i)-(iii) follow from Lemma 17, (134)-(135) and Lemmas 18-19, respectively. The proof of (iv) is similar to that in Theorem 4. The proof is now complete. □

4 Estimates in H_+^4

In this section we derive the estimates in H_+^4 and complete the proofs of Theorems 9-11.

Proof of Theorem 9 Applying ∂_t^2 on (5), multiplying the resulting equation by v_{tt} in $L^2(0,1)$, we readily get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_{tt}(t)\|^2 + \|v_{ttx}(t)\|^2\\
&\le \frac{1}{4}\|v_{ttx}(t)\|^2 + C_2(\tau)\Big\{\|v(t)\|_{H^2} + \|v_{tx}(t)\| + \|f_{tt}(t)\|\Big\}\|v_{ttx}(t)\|\\
&+ C_2(\tau)\|v_{tx}(t)\|^2\\
&\le \frac{1}{2}\|v_{ttx}(t)\|^2 + C_2(\tau)\Big\{\|v(t)\|^2_{H^2} + \|v_t(t)\|^2_{H^1} + \|f_{tt}(t)\|^2\Big\} \qquad (136)
\end{aligned}$$

whence, by Theorem 6,

$$\|v_{tt}(t)\|^2 + \int_\tau^t \|v_{ttx}(s)\|^2 ds \le C_2(\tau)\|v_{tt}(\tau)\|^2 + C_4(\tau) \le C_4(\tau) \qquad (137)$$

for $t \ge \tau \ge 0$. Here, from (5), we have used the following estimates

$$\|v_{tt}(t)\|^2 \le C_2(\tau)(\|v(t)\|^2_{H^2} + \|u(t)\|^2_{H^3} + \|f_t(t)\|^2), \qquad (138)$$
$$\|v_{xxxx}(t)\|^2 \le C_2(\tau)(\|v(t)\|^2_{H^2} + \|u(t)\|^2_{H^3} + \|v_{tt}(t)\|^2 + \|f_t(t)\|^2). \qquad (139)$$

Applying $\partial_t\partial_x$ on (5), multiplying the resulting equation by v_{tx}, we derive from Young's inequality that for $\varepsilon > 0$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_{tx}(t)\|^2+\|v_{txx}(t)\|^2=v_{txx}v_{tx}|_{x=0}^{x=1}\\
&-\int_0^1[\sigma''(u)v_xu_x^2+2\sigma'(u)u_xv_{xx}+\sigma'(u)v_xu_{xx}+\sigma(u)v_{xxx}+f_{tx}]v_{tx}dx\\
&\le C_1(\tau)\{\|v_{txx}(t)\|^{1/2}\|v_{txxx}(t)\|^{1/2}+\|v_{txx}(t)\|\}\\
&\times\{\|v_{tx}(t)\|^{1/2}\|v_{txx}(t)\|^{1/2}+\|v_{tx}(t)\|\}\\
&+C_2(\tau)\{\|v(t)\|_{H^3}^2+\|v_{tx}(t)\|^2+\|u_{xx}(t)\|^2\}\\
&\le\frac{1}{2}\|v_{txx}(t)\|^2+\varepsilon\|v_{txxx}(t)\|^2\\
&+C_2(\tau)(\varepsilon)\{\|v_{tx}(t)\|^2+\|v(t)\|_{H^3}^2+\|u(t)\|_{H^2}^2+\|f_{tx}(t)\|^2\} \qquad (140)
\end{aligned}
$$

which gives for any $\varepsilon>0$,

$$
\|v_{tx}(t)\|^2+\int_\tau^t\|v_{txx}(s)\|^2ds\le C_3(\tau)+\varepsilon\int_\tau^t\|v_{txxx}(s)\|^2ds,\quad t\ge\tau\ge 0. \qquad (141)
$$

Here we have employed the following estimates from (5)

$$
\begin{aligned}
\|v_{tx}(t)\|&\le C_1(\tau)(\|v_{xxx}(t)\|+\|u_x^2(t)\|+\|u_{xx}(t)\|+\|f_x(t)\|)\\
&\le C_2(\tau)(\|v(t)\|_{H^3}+\|u(t)\|_{H^2}+\|f_x(t)\|), \qquad (142)\\
\|v_{tx}(\tau)\|&\le C_3(\tau)=C_3(\|(u_0^\tau,v_0^\tau)\|_{H^3},\|f(\tau)\|_{H^1}), \qquad (143)\\
\|v_{xxx}(t)\|&\le C_2(\tau)(\|u(t)\|_{H^2}+\|v_{tx}(t)\|+\|f_x(t)\|). \qquad (144)
\end{aligned}
$$

Applying $\partial_t\partial_x$ on (5), we derive

$$
\|v_{txxx}(t)\|\le C_2(\tau)(\|v(t)\|_{H^3}+\|v_{ttx}(t)\|+\|u(t)\|_{H^2}+\|f_{tx}(t)\|) \qquad (145)
$$

which, inserted into (4.6) and by taking ε small enough, implies

$$
\|v_{tx}(t)\|^2+\int_\tau^t\|v_{txx}(s)\|^2ds\le C_4(\tau),\quad t\ge\tau\ge 0. \qquad (146)
$$

Thus it follows from (137)-(146) and Theorems 1 and 6 that

$$
\|v(t)\|_{H^4}^2+\int_\tau^t(\|v_t\|_{H^3}^2+\|v_{tt}\|_{H^1}^2)(s)ds\le C_4(\tau),\quad t\ge\tau\ge 0. \qquad (147)
$$

Similarly to (116), it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_{xxx}(t)\|^2 + C_1^{-1}(\tau)\|u_{xxx}(t)\|^2$$
$$\leq C_2(\tau)(\|u(t)\|_{H^2}^2 + \|v_{txx}(t)\|^2 + \|f_{xx}(t)\|^2) \tag{148}$$

which, according to Theorem 1, gives us

$$\|u_{xxx}(t)\|^2 + \int_\tau^t \|u_{xxx}(s)\|^2 ds \leq C_4(\tau), \ \ t \geq \tau \geq 0. \tag{149}$$

Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\|u_{xxxx}(t)\|^2 + C_1^{-1}(\tau)\|u_{xxxx}(t)\|^2$$
$$\leq C_4(\tau)(\|u(t)\|_{H^3}^2 + \|v_{txxx}(t)\|^2 + \|f_{xxx}(t)\|^2) \tag{150}$$

which, combined with (144), (146), (4.13) and Theorem 1, yields

$$\|u_{xxxx}(t)\|^2 + \int_\tau^t \|u_{xxxx}(s)\|^2 ds \leq C_4(\tau), \ \ t \geq \tau \geq 0. \tag{151}$$

Thus estimate (20) follows from Theorems 1 and 6, (147), (149) and (151), and (20) also implies that there exists a global solution $(u(t), v(t)) = (u^\tau(t), v^\tau(t)) \in H_+^4$ for any given datum $(u_0^\tau, v_0^\tau) \in H_+^4$. Noting that $E_3 \subseteq E_2 \subseteq E_1$, the global solution $(u(t), v(t))$ in H_+^4 is also a global solution in H_+^1 and H_+^2, so the uniqueness in H_+^4 follows from that in H_+^1 (or in H_+^2). Therefore the global solution $(u(t), v(t))$ in H_+^4 generates a semiprocess $\left\{U_f^{(3)}(t,\tau)\right\}(f \in E_3, t \geq \tau \geq 0)$ of a two-parameter family of operators such that (19)-(20) hold.

Now multiplying (136), (140), (148) and (150) by $e^{\beta t}$ respectively, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\Big\{e^{\beta t}[\|v_{tt}(t)\|^2 + \|v_{tx}(t)\|^2 + \frac{1}{2}(\|u_{xxx}(t)\|^2 + \|u_{xxxx}(t)\|^2)]\Big\} \\
&\quad + e^{\beta t}\Big\{\frac{1}{2}(\|v_{ttx}(t)\|^2 + \|v_{txx}(t)\|^2) + C_1^{-1}(\tau)(\|u_{xxx}(t)\|^2 + \|u_{xxxx}(t)\|^2)\Big\} \\
&\quad \leq \beta e^{\beta t}\Big\{\|v_{tt}(t)\|^2 + \|v_{tx}(t)\|^2 + \frac{1}{2}(\|u_{xxx}(t)\|^2 + \|u_{xxxx}(t)\|^2)\Big\} \\
&\quad + C_2(\tau,\varepsilon)e^{\beta t}\Big(\|v(t)\|_{H^3}^2 + \|u(t)\|_{H^2}^2 + \|v_t(t)\|^2 + \|f_{tt}(t)\|^2 + \|f_{tx}(t)\|^2\Big) \\
&\quad + C_2(\tau)\varepsilon e^{\beta t}\Big(\|v(t)\|_{H^3}^2 + \|v_{ttx}(t)\|^2 + \|u(t)\|_{H^2}^2 + \|f_{tx}(t)\|^2\Big)
\end{aligned}$$

which, combined with Theorem 1, implies that there exists a constant $\beta_3 = \beta_3(C_4(\tau)) \leq \beta_2$ such that for any fixed $\beta \in (0, \beta_3], \varepsilon \in (0,1)$ small enough, for $t \geq \tau \geq 0$,

$$
\begin{aligned}
e^{\beta t}\Big\{&\|v_{tt}(t)\|^2+\|v_{tx}(t)\|^2+\|u_{xxx}(t)\|^2+\|u_{xxxx}(t)\|^2\Big\}\\
&+\int_\tau^t e^{\beta s}(\|v_{ttx}\|^2+\|v_{txx}\|^2+\|u_{xxx}\|^2+\|u_{xxxx}\|^2)(s)ds\\
&\le C_4(\tau)e^{\beta\tau}. \qquad (152)
\end{aligned}
$$

Thus estimate (22) follows from (12), (17) and (152).

In what follows, we shall show that $\left\{U_f^{(3)}(t,\tau)\right\}$ is $(H_+^4\times E_3, H_+^4)$-continuous. To this end, we assume that $(u_i(t),v_i(t))=U_f^{(3)}(t,0)(u_{0,i}^\tau,v_{0,i}^\tau)$ are two global solutions corresponding to the initial datum $(u_{0,i}^\tau,v_{0,i}^\tau)\in H_+^4$ and the external force $f^i\in E_3(i=1,2)$, respectively. We set $u=u_1-u_2, v=v_1-v_2, f=f^1-f^2$. Noting that $E_3\subseteq E_2\subseteq E_1$, we still have equations (89)-(92) and estimates (97), (98) and (125)-(129).
We now differentiate (90) with respect to x and use Theorems 1 and 6 to derive

$$
\begin{aligned}
\|v_{tx}(t)\|&\le C_4(\tau)(\|u(t)\|_{H^2}+\|v_{xxx}(t)\|+\|f_x(t)\|), \qquad (153)\\
\|v_{xxx}(t)\|&\le C_4(\tau)(\|u(t)\|_{H^2}+\|v_{tx}(t)\|+\|f_x(t)\|)\\
&\le C_4(\tau)(\|u(t)\|_{H^2}+\|v(t)\|_{H^1}+\|v_t(t)\|\\
&\qquad+\|v_{tx}(t)\|+\|f(t)\|_{H^1}), \qquad (154)\\
\|v_{txx}(t)\|&\le C_4(\tau)(\|v_{tt}(t)\|+\|u(t)\|_{H^1}+\|v(t)\|_{H^2}\\
&\qquad+\|v_{tx}(t)\|+\|f_t(t)\|)\\
&\le C_4(\tau)(\|v_{tt}(t)\|+\|u(t)\|_{H^1}+\|v(t)\|_{H^1}\\
&\qquad+\|v_{tx}(t)\|+\|f(t)\|+\|f_t(t)\|), \qquad (155)\\
\|v_{tt}(t)\|&\le C_4(\tau)(\|v_{txx}(t)\|+\|u(t)\|_{H^1}+\|v(t)\|_{H^1}\\
&\qquad+\|v_t(t)\|+\|f_t(t)\|). \qquad (156)
\end{aligned}
$$

Next we differentiate (90) with respect to x twice, use the mean value theorem to obtain

$$
\begin{aligned}
\|v_{txx}(t)\|&\le C_4(\tau)(\|u(t)\|_{H^3}+\|v_{xxx}(t)\|+\|f_{xx}(t)\|), \qquad (157)\\
\|v_{xxxx}(t)\|&\le C_4(\tau)(\|u(t)\|_{H^3}+\|v(t)\|_{H^3}+\|v_{txx}(t)\|+\|f_{xx}(t)\|)\\
&\le C_4(\tau)(\|u(t)\|_{H^3}+\|v(t)\|_{H^1}+\|v_{tx}(t)\|\\
&\qquad+\|v_{tt}(t)\|+\|f(t)\|_{H^2}+\|f_t(t)\|).
\end{aligned}
$$

On the other hand, by differentiating (90) with respect to t, and using the embedding theorem and Theorems 1-6 we get

$$
\begin{aligned}
\|v_{txxx}(t)\|&\le C_4(\tau)(\|v_{ttx}(t)\|+\|u(t)\|_{H^2}+\|v(t)\|_{H^3}+\|v_{tx}(t)\|+\|f_{tx}(t)\|)\\
&\le C_4(\tau)(\|v_{ttx}(t)\|+\|u(t)\|_{H^2}+\|v(t)\|_{H^1}+\|v_t(t)\|\\
&\qquad+\|v_{tx}(t)\|+\|f(t)\|_{H^1}+\|f_{tx}(t)\|). \qquad (158)
\end{aligned}
$$

Thus inserting (126) and (157) into (156) gives us

$$\|v_{tt}(t)\| \le C_4(\tau)(\|u(t)\|_{H^3} + \|v(t)\|_{H^4} + \|f(t)\|_{H^2} + \|f_t(t)\|). \tag{159}$$

Differentiating (90) with respect to x twice, and using (89), yields

$$u_{txxx} + \sigma(u_1)u_{xxx} = v_{txx} + R(t), \tag{160}$$

where

$$\begin{aligned} R(t) = &- \Big\{\sigma''(u_1)u_{1x}^2 u_x + \sigma'(u_1)u_{1xx}u_x + 2\sigma'(u_1)u_{1x}u_{xx} \\ &+ [(\sigma(u_1) - \sigma(u_2))u_{2x}]_{xx} + f_{xx}\Big\} \end{aligned}$$

satisfies, by Theorems 1 and 6,

$$\|R(t)\| \le C_4(\tau)(\|u(t)\|_{H^2} + \|f_{xx}(t)\|). \tag{161}$$

By virtue of (157) and (160), we deduce

$$\begin{aligned} \frac{d}{dt}\|u_{xxx}(t)\|^2 + &C_1^{-1}(\tau)\|u_{xxx}(t)\|^2 \\ &\le C_1(\tau)(\|R(t)\|^2 + \|v_{txx}(t)\|^2) \\ &\le C_4(\tau)(\|u(t)\|_{H^3}^2 + \|v(t)\|_{H^3}^2 + \|f(t)\|_{H^2}^2). \end{aligned} \tag{162}$$

In view of (161), and using Theorems 1 and 6, and the mean value theorem and embedding theorem, we obtain

$$\|R_x(t)\| \le C_4(\tau)(\|u(t)\|_{H^3} + \|f_{xxx}(t)\|). \tag{163}$$

Analogously, we get from (158), (160) and (163) that

$$\begin{aligned} \frac{d}{dt}\|u_{xxxx}(t)\|^2 + C_1^{-1}(\tau)\|u_{xxxx}(t)\|^2 &\le C_1(\tau)(\|R_x(t)\|^2 + \|v_{txxx}(t)\|^2) \\ &\le C_4(\tau)\Big\{\|v_{ttx}(t)\|^2 + \|u(t)\|_{H^2}^2 + \|v(t)\|_{H^3}^2 + \|f_{tx}(t)\|^2\Big\} \\ &\le C_4(\tau)\Big\{\|v_{ttx}(t)\|^2 + \|u(t)\|_{H^3}^2 + \|v(t)\|_{H^1}^2 + \|v_t(t)\|^2 \\ &+ \|v_{tx}(t)\|^2 + \|f(t)\|_{H^3}^2 + \|f_{tx}(t)\|^2\Big\}. \end{aligned} \tag{164}$$

Now differentiating (90) with respect to t twice, multiplying the resulting equation by v_{tt} in $L^2(0,1)$, integrating by parts, and employing Theorems 1 and 6, estimate (155), the mean value theorem and the embedding theorem, finally give rise to

$$\begin{aligned}
&\frac{d}{dt}\|v_{tt}(t)\|^2 + C_1^{-1}(\tau)\|v_{ttx}(t)\|^2 \\
&\le C_4(\tau)(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^2}^2 + \|v_{tx}(t)\|^2 + \|v_{txx}(t)\|^2 + \|f_{tt}(t)\|^2) \\
&\le C_4(\tau)(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|v_t(t)\|^2 + \|v_{tx}(t)\|^2 \\
&+\|v_{tt}(t)\|^2 + \|f(t)\|^2 + \|f_t(t)\|^2 + \|f_{tt}(t)\|^2). \qquad (165)
\end{aligned}$$

Differentiating (90) with respect to t and x respectively, perfoming an integration by parts, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|v_{tx}(t)\|^2 + \|v_{txx}(t)\|^2 = H_0 + H_1 \qquad (166)$$

where

$$\begin{aligned}
H_0 &= v_{txx}v_{tx}|_{x=0}^{x=1}, \\
H_1 &= \int_0^1 (\sigma(u_1)u_x + (\sigma(u_1) - \sigma(u_2))u_{2x} + f)_{tx} v_{tx} dx. \qquad (167)
\end{aligned}$$

Using Sobolev's interpolation inequality, we infer from Theorems 1 and 6, (158) that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
H_0 &\le C_4(\tau)\Big\{\|v_{txx}(t)\|^{1/2}\|v_{txxx}(t)\|^{1/2} + \|v_{txx}(t)\|\Big\}\|v_{tx}(t)\|^{1/2}\|v_{txx}\|^{1/2} \\
&\le \varepsilon(\|v_{txxx}(t)\|^2 + \|v_{txx}(t)\|^2) + C_4(\varepsilon)\|v_{tx}(t)\|^2 \\
&\le C_4(\tau)\varepsilon\Big\{\|v_{ttx}(t)\|^2 + \|u(t)\|_{H^2}^2 + \|v(t)\|_{H^1}^2 + \|v_t(t)\|^2 + \|v_{tx}(t)\|^2 \\
&\quad + \|f(t)\|_{H^1}^2 + \|f_{tx}(t)\|^2\Big\} + \varepsilon\|v_{txx}(t)\|^2 + C_4(\tau,\varepsilon)\|v_{tx}(t)\|^2, \qquad (168) \\
H_1 &\le C_4(\tau)(\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^3}^2 + \|v_{tx}(t)\|^2 + \|f_{tx}(t)\|^2). \qquad (169)
\end{aligned}$$

Thus the combination of (168)-(169) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_{tx}(t)\|^2 + (1-\varepsilon)\|v_{txx}(t)\|^2 \\
&\qquad \le C_4(\varepsilon)\|v_{txxx}(t)\|^2 + C_4(\tau,\varepsilon)(\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^1}^2 \\
&\qquad + \|v_t(t)\|^2 + \|v_{tx}(t)\|^2 + \|f(t)\|_{H^1}^2 + \|f_{tx}(t)\|^2). \qquad (170)
\end{aligned}$$

Set

$$F_3(t) = \|v_{tt}(t)\|^2 + \frac{1}{2}\|v_{tx}(t)\|^2 + \varepsilon(\|u_{xxx}(t)\|^2 + \|u_{xxxx}(t)\|^2).$$

Then multiplying (162) and (164) by ε respectively, adding up the resulting equations, (165) and (169), and picking $\varepsilon > 0$ small enough, we get

$$\frac{d}{dt}F_3(t) + C_4^{-1}(\tau)(\|v_{ttx}(t)\|^2 + \|v_{txx}(t)\|^2 + \|u_{xxx}(t)\|^2 + \|u_{xxxx}(t)\|^2) \\ \leq C_4(\tau)G(t) \tag{171}$$

where

$$G(t) = \|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^4} + \|v_t(t)\|^2 + \|v_{tx}(t)\|^2 + \|v_{tt}(t)\|^2 \\ + f(t)\|^2_{H^3} + \|f_t(t)\|^2_{H^1} + \|f_{tt}(t)\|^2.$$

On the other hand, we derive from (126), (153) and (159),

$$\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^4} \leq G(t) \\ \leq C_4(\tau)\Big\{\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^4} + \|f(t)\|^2_{H^3} + \|f_t(t)\|^2 + \|f_{tt}(t)\|^2\Big\}. \tag{172}$$

Set

$$M(t) = F_2(t) + F_3(t).$$

Obviously, fixing $\varepsilon > 0$, we have from (126), (153) and (159)

$$M(t) \geq C_4^{-1}(\tau)\Big\{\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^1} + \|v_t(t)\|^2 \\ + \|v_{tx}(t)\|^2 + \|v_{tt}(t)\|^2\Big\} \tag{173}$$

which, by (126), (154) and (172), gives us

$$\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^4} \leq C_4(\tau)\Big\{M(t) + \|f(t)\|^2_{H^1}\Big\}. \tag{174}$$

We conclude from (125), (154), (157), (171) and (173),

$$G(t) \leq C_4(\tau)\Big\{\|u(t)\|^2_{H^4} + \|v(t)\|^2_{H^1} + \|v_t(t)\|^2 + \|v_{tx}(t)\|^2 \\ + \|v_{tt}(t)\|^2 + \|f(t)\|^2_{H^3} + \|f_t(t)\|^2_{H^1} + \|f_{tt}(t)\|^2\Big\} \\ \leq C_4(\tau)\Big\{M(t) + \|f(t)\|^2_{H^3} + \|f_t(t)\|^2_{H^1} + \|f_{tt}(t)\|^2\Big\}. \tag{175}$$

Thus adding up (129), (170) and using (176) yields

$$\frac{d}{dt}M(t) \leq C_4(\tau)\Big\{M(t) + \|f(t)\|^2_{H^3} + \|f_t(t)\|^2_{H^1} + \|f_{tt}(t)\|^2\Big\}. \tag{176}$$

By virtue of Gronwall's inequality and (176), we have that for any $t \geq \tau \geq 0$,

$$M(t) \leq e^{C_4(\tau)(t-\tau)}\Big\{M(\tau) + \int_\tau^\infty (\|f\|_{H^3}^2 + \|f_t\|_{H^1}^2 + \|f_{tt}\|^2)(s)ds\Big\}$$

which, together with (171) and (174)-(175), gives us for $t \geq \tau \geq 0$,

$$\begin{aligned}
&\|u(t)\|_{H^4}^2 + \|v(t)\|_{H^4}^2 \leq G(t) \\
&\leq C_4(\tau)\Big\{M(t) + \|f(t)\|_{H^3}^2 + \|f_t(t)\|_{H^1}^2 + \|f_{tt}(t)\|^2\Big\} \\
&\leq C_4(\tau)e^{C_4(\tau)(t-\tau)}\Big\{M(\tau) + \|f(t)\|_{H^3}^2 + \|f_t(t)\|_{H^1}^2 + \|f_{tt}(t)\|^2 \\
&\qquad + \int_\tau^\infty (\|f\|_{H^3}^2 + \|f_t\|_{H^1}^2 + \|f_{tt}\|^2)(s)ds\Big\} \\
&\leq C_4(\tau)e^{C_4(\tau)(t-\tau)}\Big\{\|u_0^\tau\|_{H^4}^2 + \|v_0^\tau\|_{H^4}^2 + \|f(\tau)\|_{H^2}^2 + \|f_t(\tau)\|^2 \\
&\qquad + \|f(t)\|_{H^3}^2 + \|f_t(t)\|^2 + \|f_{tt}(t)\|^2 \\
&\qquad + \int_\tau^\infty (\|f\|_{H^3}^2 + \|f_t\|_{H^1}^2 + \|f_{tt}\|^2)(s)ds\Big\}. \qquad (177)
\end{aligned}$$

This implies that the semiprocess $\Big\{U_f^{(3)}(t,\tau)\Big\}$ is $(H_+^4 \times E_3, H_+^4)$-continuous. For any given bounded set $B_3 \subseteq H_+^4 (\subseteq H_+^2 \subseteq H_+^1)$, we assume that $(u_0^\tau, v_0^\tau) \in B_3$, i.e., $\|(u_0^\tau, v_0^\tau)\|_{H^4} \leq \hat{B}_3$, $\hat{B}_3 (\geq \hat{B}_2)$ is a positive constant. Now let us fix $\beta \in (0, \beta_3], \tau \in \mathbb{R}^+$, and let

$$R_3 = 1, t_3 = \max\Big\{t_2, \log[C_4(\tau, \hat{B}_3)e^{\beta\tau}]\Big\}.$$

Then it follows from (22) that for $t \geq t_3 \geq \tau$, for any fixed $f \in E_3$,

$$(u^\tau(t), v^\tau(t)) = U_f^{(3)}(u_0^\tau, v_0^\tau) \in B(0, R_3) \equiv \{(u,v) \in H_+^4 : \|(u,v)\|_{H^4} \leq 1\}$$

which implies that $B(0, R_3)$ is a (non-uniform) absorbing set in H_+^4 for the semiprocess $\Big\{U_f^{(3)}(t,\tau)\Big\}(t \geq \tau \geq 0, f \in E_3 \text{ fixed})$. In the sequel, we shall prove that this semiprocess is compact. In fact, picking $t_3' = \max[t_2', t_3]$, we conclude from (106) that as $t \geq t_3'$,

$$\|(u^\tau(t), v^\tau(t))\|_{H^6 \times H^6} \leq 1$$

which implies that $B(0, R_3)$ is a (non-uniform) compact absorbing set in H_+^4 noting that the embedding $H^6 \times H^6 \hookrightarrow H_+^4$ is compact. Hence the semiprocess $\Big\{U_f^{(3)}(t,\tau)\Big\}(t \geq \tau \geq 0, f \in E_3 \text{ fixed})$ is a (non-uniform) compact family of operators and is further asymptotically a compact family of semiprocesses. Thus it follows from Lemma 15 that the semiprocess $\Big\{U_f^{(3)}(t,\tau)\Big\}(t \geq \tau \geq$

$0, f \in E_3$ fixed) possesses a (non-uniform) compact attractor $\mathcal{A}^{(3)}_{\{f\}}$. The proof is now complete. $\square$

Proof of Theorem 10 Obviously, by the embedding theorem,

$$\hat{E}_3 \subseteq E_3. \tag{178}$$

Similarly to (131), we conclude

$$\Sigma_3 \subseteq \hat{E}_3, \sup_{f\in\Sigma_3} \|f(t)\|_{\hat{E}_3} \le \|f_3\|_{\hat{E}_3} < +\infty. \tag{179}$$

Hence it follows from (178) and Theorem 9 that for any $(u_0^\tau, v_0^\tau) \in H_+^4$, there exists a unique global solution $(u^\tau(t), v^\tau(t))$ in H_+^4, which generates a semiprocess $\left\{U^{(3)}_{f|\Sigma_3}(t,\tau)\right\}(f \in \Sigma_3, t \ge \tau \ge 0)$ of two-parameter family of operators such that (19), (20) and (125) hold if (124) holds where $\left\{U^{(3)}_f(t,\tau)\right\}$ and $C_4(\tau)$ should be replaced by $\left\{U^{(3)}_{f|\Sigma_3}(t,\tau)\right\}$ and $C_4^*(\tau) = C_4^*(\|(u_0^\tau, v_0^\tau)\|_{H^4}, \|f_3\|_{\hat{E}_3})$, respectively. In view of $\Sigma_3 \subseteq \hat{E}_3 \subseteq E_3$, we deduce from (177) that the semiprocess $\left\{U^{(3)}_{f|\Sigma_3}(t,\tau)\right\}(f \in \Sigma_3, t \ge \tau \ge 0)$ is $(H_+^4 \times \Sigma_3, H_+^4)$-continuous. Similarly to the proof of Theorem 7, noting that estimate (179) implies that the generic constant $C_4(\tau)$ and hence β_3 obtained in Theorem 9 eventually depend on $C_4^*(\tau)(i.e., C_4(\tau) \le C_4^*(\tau))$. For any given bounded set $\tilde{B}_3 \subseteq H_+^4 (\subseteq H_+^2 \subseteq H_+^1)$, we assume that $(u_0^\tau, v_0^\tau) \in \tilde{B}_3, i.e., \|(u_0^\tau, v_0^\tau)\|_{H^4} \le \hat{B}_3^*, \hat{B}_3^* (\ge \hat{B}_2^* \ge \hat{B}_1^*)$ is a positive constant. Then $C_4^*(\tau) \le C_4^*(\tau, \hat{B}_3^*)$ where $C_4^*(\tau, \hat{B}_3^*)$ is a positive constant depending only on $\hat{B}_3^*$ and $\|f_3\|_{\hat{E}_3}$. Thus it follows that there exist positive constants $C_4^*(\tau, \hat{B}_3^*)$ and $\beta_3^* = \beta_3^*(C_4^*(\tau, \hat{B}_3^*))$ such that for any fixed $\beta \in (0, \beta_3^*]$, estimate (22) still holds where $C_4(\tau)$ should be replaced by $C_4^*(\tau, \hat{B}_3^*)$, i.e.,

$$\begin{aligned} &e^{\beta t}\|(u^\tau(t), v^\tau(t))\|^2_{H^4} \\ &+ \int_\tau^t e^{\beta s}(\|u^\tau\|^2_{H^4} + \|v^\tau\|^2_{H^5} + \|v_t^\tau\|^2_{H^3} + \|v_{tt}^\tau\|^2_{H^1} + \|v_{ttt}^\tau\|^2)(s)ds \\ &\le C_4^*(\tau, \hat{B}_3^*)e^{\beta\tau}. \end{aligned} \tag{180}$$

for any $t \ge \tau \ge 0$. Now fix $\beta \in (0, \hat{B}_3^*], \tau \in \mathbb{R}^+$, and let

$$R_3 = 1, t_3^* = \max\left\{t_2^*, \log[C_4^*(\tau, \hat{B}_3^*)]\right\}.$$

Thus it follows from (180) that for $t \ge t_3^* \ge \tau$, for any $f \in \Sigma_3$,

$$(u^\tau(t), v^\tau(t)) = U^{(3)}_{f|\Sigma_3}(t,\tau)(u_0^\tau, v_0^\tau) \in B(0, R_3).$$

That is,

$$\bigcup_{f\in\Sigma_3} U^{(3)}_{f|\Sigma_3}(t,\tau)\tilde{B}_3 \subseteq B(0,R_3)$$

which implies that $B(0,R_3)$ is a uniform absorbing set in H^4_+ for the semiprocess $\left\{U^{(3)}_{f|\Sigma_3}(t,\tau)\right\}(f\in\Sigma_3, t\geq\tau\geq 0)$. On the other hand, similar to (133), picking $t_3^{**}=\max[t_2^{**},t_3^{*}]$, we conclude from (106) that as $t\geq t_3^{**}$,

$$\|(u^\tau(t),v^\tau(t))\|_{H^6\times H^6}\leq 1 \tag{181}$$

which implies that $B(0,R_3)$ is a uniform compact absorbing set in H^4_+ by the compact embedding $H^6\times H^6\hookrightarrow H^4_+$. Hence the semiprocess $\left\{U^{(3)}_{f|\Sigma_3}(t,\tau)\right\}(t\geq\tau\geq 0, f\in\Sigma_3)$ is a uniformly compact family of operators and is further asymptotically compact family of semiprocess. Thus it fololws from Lemma 15 that $\left\{U^{(3)}_{f|\Sigma_3}(t,\tau)\right\}(t\geq\tau\geq 0, f\in\Sigma_3)$ possesses a uniform (with respect to $f\in\Sigma_3$) compact attractor $\mathcal{A}_{\Sigma_3}$, and evidently $\bigcup_{f\in\Sigma_3}\mathcal{A}^{(3)}_{\{f\}}\subseteq\mathcal{A}_{\Sigma_3}$. The proof is complete. □

Proof of Theorem 11 The proof is basically same as those of Theorems 4 and 8. First, noting that $f_3\in\hat{E}_3\subseteq\tilde{E}_3$, the conclusions except for (1)-(4) and (i)-(iii) of Theorem 11 follow from Theorem 10, where Σ_3 and $\hat{E}_3$ should be replaced by $\tilde{\Sigma}_3$ and $\tilde{E}_3$ respectively. Second, we infer from Lemma 18 that any $f(s)\in\tilde{\Sigma}_3, f_s(s), f_{ss}(s)$ and $f_{sss}(s)$ are translation compact in $L^2_{\mathrm{loc},w}(\mathbb{R}^+,H^3(0,1))$, $L^2_{\mathrm{loc},w}(\mathbb{R}^+,H^2(0,1))$, $L^2_{\mathrm{loc},w}(\mathbb{R}^+,H^1(0,1))$ and $L^2_{\mathrm{loc},w}(\mathbb{R}^+, L^2(0,1))$ respectively, and there holds that

$$\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|f(s)\|^2_{H^3}ds\leq\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\tilde{f}_3(s)\|^2_{H^3}ds, \tag{182}$$

$$\sum_{i=0}^{1}\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\partial_s^i f(s)\|^2_{H^2}ds\leq\sum_{i=0}^{1}\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\partial_s^i\tilde{f}_3(s)\|^2_{H^2}ds, \tag{183}$$

$$\sum_{i=0}^{2}\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\partial_s^i f(s)\|^2_{H^1}ds\leq\sum_{i=0}^{2}\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\partial_s^i\tilde{f}_3(s)\|^2_{H^1}ds, \tag{184}$$

$$\sum_{i=0}^{3}\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\partial_s^i f(s)\|^2ds\leq\sum_{i=0}^{3}\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\partial_s^i\tilde{f}_3(s)\|^2ds. \tag{185}$$

Thus by virtue of Lemma 17 and (182)-(185), we see that for any $f\in\tilde{\Sigma}_3$,

$$\|f\|^2_{L^2_b(\mathbb{R}^+,H^3(0,1))\cap H^1_b(\mathbb{R}^+,H^2(0,1))\cap H^2_b(\mathbb{R}^+,H^1(0,1))\cap H^3_b(\mathbb{R}^+,L^2(0,1))}$$

$$
\begin{aligned}
&= \sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|f(s)\|_{H^3}^2 ds + \sum_{i=0}^{1}\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\partial_s^i f(s)\|_{H^2}^2 ds \\
&+ \sum_{i=0}^{2}\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\partial_s^i f(s)\|_{H^1}^2 ds + \sum_{i=0}^{3}\sup_{t\in\mathbb{R}^+}\int_t^{t+h}\|\partial_s^i f(s)\|^2 ds \\
&\le \|\tilde{f}_3\|_{\hat{E}_3}^2 < +\infty
\end{aligned}
$$

which together with Lemma 17 implies that $\tilde{f}_3$ is translation compact in $\tilde{E}_3$. Furthermore, conclusions (1)-(4) and (i)-(iii) also follow from Lemma 17, (182)-(185) and Lemmas 18-19 respectively. The proof of (iv) is similar to that of Theorem 4 or Theorem 8. The proof is now complete. □

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