

# On the Weyl Representation of Metaplectic Operators

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## Abstract

We study the Weyl representation of metaplectic operators associated to a symplectic matrix having no non-trivial fixed point, and justify a formula suggested in earlier work of Mehlig and Wilkinson. We give precise calculations of the associated Maslov-type indices; these indices intervene in a crucial way in Gutzwiller's formula of semiclassical mechanics, and are simply related to an index defined by Conley and Zehnder.

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## 1 Introduction

In a remarkable paper [14] Mehlig and Wilkinson propose a simple derivation of Gutzwiller's [9] approximation

$$\tilde{\rho}_{\text{Gutz}}(E) = \frac{1}{\pi\hbar} \operatorname{Re} \sum_{po} \frac{T_{po} i^{\nu_{po}}}{\sqrt{|\det(S_{po} - I)|}} e^{i\mathcal{A}_{po}/\hbar} \quad (1)$$

for the oscillating part of the semiclassical level density for chaotic systems whose periodic orbits “ $po$ ” are all isolated and non-degenerate ( $T_{po}$  is the prime period,  $\nu_{po}$  an integer related to the Maslov index,  $\mathcal{A}_{po}$  the action, and  $S_{po}$  the stability matrix). Mehlig and Wilkinson's derivation heavily relies upon

their observation that for any symplectic matrix  $S$  such that  $\det(S - I) \neq 0$  one has

$$\frac{i^\nu}{\sqrt{|\det(S - I)|}} = \text{Tr}[\widehat{R}_\nu(S)]$$

where  $\widehat{R}_\nu(S)$  is the operator  $L^2(\mathbb{R}_x^n) \rightarrow L^2(\mathbb{R}_x^n)$  defined by

$$\widehat{R}_\nu(S)\Psi(x) = \left(\frac{1}{2\pi}\right)^n \frac{i^{\nu(S)}}{\sqrt{|\det(S - I)|}} \int e^{\frac{i}{2}\langle M_S z_0, z_0 \rangle} \widehat{T}(z_0)\Psi(x) d^{2n} z_0$$

where:

- $\widehat{T}(z_0)$  is the Weyl–Heisenberg operator associated to  $z_0 = (x_0, p_0) \in \mathbb{R}^{2n}$ :

$$\widehat{T}(z_0)f(x) = e^{i\langle p_0, x \rangle - \frac{1}{2}\langle p_0, x_0 \rangle} f(x - x_0)$$

for any function  $f$  defined on  $\mathbb{R}^{2n}$ ;

- $M_S$  is a symmetric matrix, associated to  $S$  by the formula

$$M_S = \frac{1}{2}J(S + I)(S - I)^{-1} \quad (2)$$

$I$  being the  $2n \times 2n$  identity matrix and  $J$  the standard symplectic matrix;

- The integer  $\nu(S)$  corresponds to a choice of  $\arg \det(S - I)$ .

In what follows we will write Mehlig–Wilkinson’s formula as a Bochner integral

$$\widehat{R}_\nu(S) = \left(\frac{1}{2\pi}\right)^n \frac{i^{\nu(S)}}{\sqrt{|\det(S - I)|}} \int e^{\frac{i}{2}\langle M_S z_0, z_0 \rangle} \widehat{T}(z_0) d^{2n} z_0. \quad (3)$$

The validity of Mehlig and Wilkinson’s derivation requires that –as these authors claim–  $\widehat{R}_\nu(S)$  is one of the two metaplectic operators  $\pm\widehat{S}$  associated with the symplectic matrix  $S$ . To sustain this claim the authors refer to previous work of one of the authors [16]; they also claim that for all  $S, S'$  such that  $\det(S - I) \neq 0$ ,  $\det(S' - I) \neq 0$  and  $\det(SS' - I) \neq 0$  their operators satisfy

$$\widehat{R}_{\nu''}(SS') = \pm\widehat{R}_\nu(S)\widehat{R}_{\nu'}(S). \quad (4)$$

The purpose of this paper is to fully justify Mehlig and Wilkinson’s statements. There are actually several options available. The *a priori* most direct strategy would be to use Howe’s theory [10] of operators with Gaussian kernels (the “oscillator semigroup” theory: see [3] for a review); another approach would be to use Hannabuss’ character theory [8] for contact transformations

(also see the follow-up [1] to this paper for interesting applications to star-products). These two methods however both have, drawbacks. Howe's theory needs quite a lot of technical prerequisites and would lead here to unnecessarily long calculations; in addition it would not be very helpful for the study of the sign ambiguity in (4) since this point is not really addressed in Howe's work. On the other hand, Hannabuss' machinery is quite abstract (it makes a heavy use of group character theory) and the use of this approach would perhaps have a tendency to obscure things. For these reasons we prefer a more straightforward line of attack, using standard Weyl calculus together with the theory of the metaplectic group as developed in [4]. This approach moreover has, as we will see, the overwhelming advantage of producing simple formulae relating the integer  $\nu$  in (3) to the usual Maslov index of the metaplectic group (we emphasize that  $\nu$  is *not* the Maslov index!). This is important, because Gutzwiller's theory has been plagued since its very beginning by the question of how to calculate the indices  $\nu_{po}$  appearing in the trace formula (1), as witnessed by the abundant literature devoted to this delicate topic (see [7] where we discuss these issues and give a rather exhaustive list of references).

## Notations

We denote by  $\sigma$  the canonical symplectic form on the phase space  $\mathbb{R}_z^{2n} = \mathbb{R}_x^n \times \mathbb{R}_p^n$ :

$$\sigma(z, z') = \langle p, x' \rangle - \langle p', x \rangle \quad \text{if } z = (x, p), z' = (x', p')$$

that is, in matrix form

$$\sigma(z, z') = \langle Jz, z' \rangle \quad , \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The real symplectic group  $Sp(n)$  consists of all linear automorphisms  $S$  of  $\mathbb{R}_z^{2n}$  such that  $\sigma(Sz, Sz') = \sigma(z, z')$  for all  $z, z'$ . It is a connected Lie group and  $\pi_1(Sp(n))$  is isomorphic to  $(\mathbb{Z}, +)$ .  $\mathcal{S}(\mathbb{R}^n)$  is the Schwartz space of rapidly decreasing functions on  $\mathbb{R}^n$  and its dual  $\mathcal{S}'(\mathbb{R}^n)$  the space of tempered distributions.

We will denote by  $\text{Inert } R$  the number of negative eigenvalues of a real symmetric matrix  $R$ .

## 2 Prerequisites

In this Section we briefly recall the main definitions and properties of the metaplectic group and of Weyl calculus we will need in the rest of this paper.

## 2.1 Standard theory of $Mp(n)$ : Review

The material of this first subsection is quite classical; see for instance [3, 4, 5] and the references therein.

Every  $\widehat{S} \in Mp(n)$  is the product of two “quadratic Fourier transforms”, which are operators  $\widehat{S}_{W,m}$  defined on  $\mathcal{S}(\mathbb{R}^n)$  by

$$\widehat{S}_{W,m}f(x) = \left(\frac{1}{2\pi i}\right)^{n/2} i^m \sqrt{|\det L|} \int e^{iW(x,x')} f(x') d^n x' \quad (5)$$

where  $W$  is a quadratic form in the variables  $x, x'$  of the type

$$W(x, x') = \frac{1}{2}\langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2}\langle Qx', x' \rangle \quad (6)$$

with  $P = P^T$ ,  $Q = Q^T$ ,  $\det L \neq 0$ . The integer  $m$  (“Maslov index”) appearing in (5) corresponds to a choice of  $\arg \det L$ :

$$m\pi \equiv \arg \det L \pmod{2\pi}$$

and to every  $W$  there thus corresponds two different choices of  $m$  modulo 4: if  $m$  is one choice, then  $m + 2$  is the other, reflecting the fact that  $Mp(n)$  is a two-fold covering of  $Sp(n)$ . The projection  $\pi : Mp(n) \longrightarrow Sp(n)$  is entirely specified by the datum of each  $\pi(\widehat{S}_{W,m})$ , and we have  $\pi(\widehat{S}_{W,m}) = S_W$  where

$$(x, p) = S_W(x', p') \iff p = \partial_x W(x, x') \quad \text{and} \quad p' = -\partial_{x'} W(x, x'). \quad (7)$$

Rewriting these conditions in terms of  $P, L, Q$  we get  $p = Px - L^T x'$  and  $p' = Lx - Qx'$ ; solving these equations in  $x$  and  $p$  yields

$$x = L^{-1}(p' + Qx') \quad \text{and} \quad p = (PL^{-1}Q - L^T)x' + PL^{-1}p'$$

hence the projection  $S_W$  of  $\widehat{S}_{W,m}$  is just the free symplectic matrix

$$S_W = \begin{bmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^T & PL^{-1} \end{bmatrix} \quad (8)$$

generated by the quadratic form  $W$ . Note that if conversely  $S$  is a free symplectic matrix

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(n) \quad , \quad \det B \neq 0 \quad (9)$$

then  $S = S_W$  with  $P = DB^{-1}$ ,  $L = B^{-1}$ ,  $Q = B^{-1}A$ . Observe that the free symplectic  $2n \times 2n$  matrices form a *dense subset* of  $Sp(n)$ ; this property will be used in the proof of Proposition 10.

The inverse  $\widehat{S}_{W,m}^{-1} = (\widehat{S}_{W,m})^*$  of  $\widehat{S}_{W,m}$  is the operator  $S_{W^*,m^*}$  where  $W^*(x, x') = -W(x', x)$  and  $m^* = n - m, \pmod{4}$ .

## 2.2 Heisenberg–Weyl operators

The operators  $\widehat{T}(z_0)$  satisfy the metaplectic covariance formula:

$$\widehat{S}\widehat{T}(z) = \widehat{T}(Sz)\widehat{S} \quad (S = \pi(\widehat{S})) \quad (10)$$

for every  $\widehat{S} \in Mp(n)$  and  $z$ . In fact, the metaplectic operators are the only unitary operators (up to a factor in  $S^1$ ) satisfying (10):

*For every  $S \in Sp(n)$  there exists a unitary transformation  $\widehat{U}$  in  $L^2(\mathbb{R}^n)$  satisfying (10) and  $\widehat{U}$  is uniquely determined apart from a constant factor of modulus one.*

The Heisenberg–Weyl operators moreover satisfy the relations

$$\widehat{T}(z_0)\widehat{T}(z_1) = e^{i\sigma(z_0, z_1)}\widehat{T}(z_1)\widehat{T}(z_0) \quad (11)$$

$$\widehat{T}(z_0 + z_1) = e^{-\frac{i}{2}\sigma(z_0, z_1)}\widehat{T}(z_0)\widehat{T}(z_1) \quad (12)$$

as is easily seen from their definition.

## 2.3 Weyl operators

Let  $a^w = a^w(x, D)$  be the Weyl operator with symbol  $a$  (which we always assume to belong to some suitable class, allowing the integrals to be viewed as distributions):

$$a^w f(x) = \left(\frac{1}{2\pi}\right)^n \iint e^{i\langle p, x-y \rangle} a\left(\frac{1}{2}(x+y), p\right) f(y) d^n y d^n p;$$

where  $f \in \mathcal{S}(\mathbb{R}^n)$ ; equivalently

$$a^w = \left(\frac{1}{2\pi}\right)^n \int a_\sigma(z_0) \widehat{T}(z_0) d^{2n} z_0$$

where the “twisted symbol”  $a_\sigma$  is the symplectic Fourier transform  $F_\sigma a$  of  $a$ :

$$a_\sigma(z) = F_\sigma a(z) = \left(\frac{1}{2\pi}\right)^n \int e^{-i\sigma(z, z')} a(z') d^{2n} z'.$$

The compose  $c^w = a^w \circ b^w$  (when defined) is the Weyl operator with twisted Weyl symbol

$$c_\sigma = \left(\frac{1}{2\pi}\right)^n (a_\sigma *_\sigma b_\sigma) \quad (13)$$

where

$$a *_\sigma b(z) = \int e^{\frac{i}{2}\sigma(z, u)} a(z-u) b(u) d^{2n} u \quad (14)$$

(see for instance Littlejohn [11], Wong [17]).

## 2.4 Generalized Fresnel Formula

We will use the following formula, generalizing the usual Fresnel integral to complex Gaussians. Let  $M$  be a real symmetric  $m \times m$  matrix. If  $M$  is invertible then the Fourier transform of the exponential  $\exp(i \langle Mx, x \rangle / 2)$  is given by the formula

$$\left(\frac{1}{2\pi}\right)^{m/2} \int e^{-i \langle v, u \rangle} e^{\frac{i}{2} \langle Mu, u \rangle} d^m u = |\det M|^{-1/2} e^{\frac{i\pi}{4} \operatorname{sgn} M} e^{-\frac{i}{2} \langle M^{-1}v, v \rangle} \quad (15)$$

where  $\operatorname{sgn} M$ , the “signature” of  $M$ , is the number of  $> 0$  eigenvalues of  $M$  minus the number of  $< 0$  eigenvalues.

For a proof of this formula see for instance [3], Appendix A.

## 3 Discussion of the Mehlig–Wilkinson Formula

In this Section we show that the Mehlig–Wilkinson operators (3) indeed are metaplectic operators. We begin by giving two alternative expressions for these operators.

### 3.1 Equivalent formulations

We begin by remarking that the matrix

$$M_S = \frac{1}{2} J(S + I)(S - I)^{-1}$$

is symmetric; this immediately follows from the conditions

$$S \in Sp(n) \iff S^T J S = J \iff S J S^T = J.$$

Notice that for every  $M$  with  $\det(M - \frac{1}{2}J) \neq 0$  the equation

$$M = \frac{1}{2} J(S + I)(S - I)^{-1}$$

can be solved in  $S$ , yielding

$$S = (M - \frac{1}{2}J)^{-1}(M + \frac{1}{2}J);$$

the relation  $S \in Sp(n)$  is then equivalent to  $M$  being real and symmetric.

**Lemma 1** *Let  $S \in Sp(n)$  be such that  $\det(S - I) \neq 0$ . The operator*

$$\widehat{R}_\nu(S) = \left(\frac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{|\det(S - I)|}} \int e^{\frac{i}{2} \langle M_S z, z \rangle} \widehat{T}(z) d^{2n} z \quad (16)$$

can be written as

$$\widehat{R}_\nu(S) = \left(\frac{1}{2\pi}\right)^n i^\nu \sqrt{|\det(S-I)|} \int e^{-\frac{i}{2}\sigma(Sz,z)} \widehat{T}((S-I)z) d^{2n}z \quad (17)$$

that is, as

$$\widehat{R}_\nu(S) = \left(\frac{1}{2\pi}\right)^n i^\nu \sqrt{|\det(S-I)|} \int \widehat{T}(Sz) \widehat{T}(-z) d^{2n}z. \quad (18)$$

**Proof.** We have

$$\frac{1}{2}J(S+I)(S-I)^{-1} = \frac{1}{2}J + J(S-I)^{-1}$$

hence, in view of the antisymmetry of  $J$ ,

$$\langle M_S z, z \rangle = \langle J(S-I)^{-1}z, z \rangle = \sigma((S-I)^{-1}z, z)$$

Performing the change of variables  $z \mapsto (S-I)z$  we can rewrite the integral in the right-hand side of (16) as

$$\begin{aligned} \int e^{\frac{i}{2}\langle M_S z, z \rangle} \widehat{T}(z) d^{2n}z &= \sqrt{|\det(S-I)|} \int e^{\frac{i}{2}\sigma(z, (S-I)z)} \widehat{T}((S-I)z) d^{2n}z \\ &= \sqrt{|\det(S-I)|} \int e^{-\frac{i}{2}\sigma(Sz, z)} \widehat{T}((S-I)z) d^{2n}z \end{aligned}$$

hence (17). Taking into account the relation (12) we have

$$\widehat{T}((S-I)z) = e^{\frac{i}{2}\sigma(Sz, z)} \widehat{T}(Sz) \widehat{T}(-z)$$

and formula (18) follows. ■

**Remark 2** *Formulae (16) and (17)–(18) suggest that  $\nu\pi$  could be a choice of  $\pm \arg \det(S-I)$ . This is however not the case (see (23) in Proposition 6 below); formula (26) will identify the integer  $\nu$  with the Conley–Zehnder index.*

**Corollary 3** *We have  $\widehat{R}_\nu(S) = c_S \widehat{S}_{W,m}$  with  $|c_S| = 1$ .*

**Proof.** The operator  $\widehat{R}_\nu(S)$  satisfies the metaplectic covariance relation

$$\widehat{R}_\nu(S) \widehat{T}(z) = \widehat{T}(Sz) \widehat{R}_\nu(S)$$

as immediately follows from the alternative form (17) of  $\widehat{R}_\nu(S)$ . On the other hand, a straightforward calculation using formula (18) shows that  $\widehat{R}_\nu(S)$  is unitary, hence the claim. ■

Let us precise Corollary 3 by discussing the choice of the constant  $c_S$ .

### 3.2 The case $\widehat{S} = \widehat{S}_{W,m}$

We are going to show that the Mehlig–Wilkinson operators coincide with the metaplectic operators  $\widehat{S}_{W,m}$  when  $S = S_W$  and we will thereafter determine the correct choice for  $\nu$ ; we will see that it is related by a simple formula to the usual Maslov index as defined in [4].

Let us first prove the following technical result:

**Lemma 4** *Let  $S_W$  be a free symplectic matrix (9). We have*

$$\det(S_W - I) = (-1)^n \det(B) \det(B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}) \quad (19)$$

that is, when  $S$  is written in the form (8):

$$\det(S_W - I) = (-1)^n \det(L^{-1}) \det(P + Q - L - L^T). \quad (20)$$

**Proof.** Since  $B$  is invertible we can write  $S - I$  as

$$\begin{bmatrix} A - I & B \\ C & D - I \end{bmatrix} = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} C - (D - I)B^{-1}(A - I) & 0 \\ B^{-1}(A - I) & I \end{bmatrix} \quad (21)$$

and hence

$$\det(S_W - I) = \det(-B) \det(C - (D - I)B^{-1}(A - I)).$$

Since  $S$  is symplectic we have  $C - DB^{-1}A = -(B^T)^{-1}$  (use for instance the fact that  $S^T J S = S J S^T = J$ ) and hence

$$C - (D - I)B^{-1}(A - I) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1};$$

the Lemma follows since  $\det(-B) = (-1)^n \det B$ . ■

**Remark 5** *The factorization (21) shows in particular that  $\ker(S - I)$  is isomorphic to  $\ker(P + Q - L - L^T)$  (cf. [13], Lemma 2.8, and proof of Lemma 2.9).*

Let us denote by  $W_{xx}$  the Hessian matrix of the function  $x \mapsto W(x, x)$ , that is

$$W_{xx} = P + Q - L - L^T.$$

We have:

**Proposition 6** *Let  $S = S_W$  be a free symplectic matrix (9) and  $\widehat{R}_\nu(S)$  the corresponding Mehlig–Wilkinson operator. We have  $\widehat{R}_\nu(S) = \widehat{S}_{W,m}$  provided that  $\nu$  is chosen so that*

$$\nu \equiv m - \text{Inert } W_{xx} \pmod{4} \quad (22)$$

in which case we have

$$\frac{1}{\pi} \arg \det(S - I) \equiv -\nu + n \pmod{2} \quad (23)$$

**Proof.** Recall that we have shown that  $\widehat{R}_\nu(S) = c_S \widehat{S}_{W,m}$  where  $c_S$  is some complex constant with  $|c_S| = 1$ . Let us determine that constant. Let  $\delta \in \mathcal{S}'(\mathbb{R}^n)$  be the Dirac distribution centered at  $x = 0$ ; setting

$$C_{W,\nu} = \left(\frac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{|\det(S-I)|}}$$

we have, by definition of  $\widehat{R}_\nu(S)$ , writing  $z_0 = (x_0, p_0)$  in place of  $z = (x, p)$ :

$$\begin{aligned} \widehat{R}_\nu(S)\delta(x) &= C_{W,\nu} \int e^{\frac{i}{2}\langle M_S z_0, z_0 \rangle} e^{i(\langle p_0, x \rangle - \frac{1}{2}\langle p_0, x_0 \rangle)} \delta(x - x_0) d^{2n} z_0 \\ &= C_{W,\nu} \int e^{\frac{i}{2}\langle M_S(x, p_0), (x, p_0) \rangle} e^{\frac{i}{2}\langle p_0, x \rangle} \delta(x - x_0) d^{2n} z_0 \end{aligned}$$

hence, setting  $x = 0$ ,

$$\widehat{R}_\nu(S)\delta(0) = C_{W,\nu} \int e^{\frac{i}{2}\langle M_S(0, p_0), (0, p_0) \rangle} \delta(-x_0) d^{2n} z_0$$

that is, since  $\int \delta(-x_0) d^n x_0 = 1$ ,

$$\widehat{R}_\nu(S)\delta(0) = \left(\frac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{|\det(S-I)|}} \int e^{\frac{i}{2}\langle M_S(0, p_0), (0, p_0) \rangle} d^n p_0. \quad (24)$$

Let us calculate the scalar product

$$\langle M_S(0, p_0), (0, p_0) \rangle = \sigma((S-I)^{-1}0, p_0), (0, p_0).$$

The relation  $(x, p) = (S-I)^{-1}(0, p_0)$  is equivalent to  $S(x, p) = (x, p + p_0)$  that is to

$$p + p_0 = \partial_x W(x, x) \quad \text{and} \quad p = -\partial_{x'} W(x, x).$$

Using the explicit form (6) of  $W$  together with Lemma 4 these relations yield

$$x = (P + Q - L - L^T)^{-1} p_0 \quad \text{and} \quad p = (L - Q)(P + Q - L - L^T)^{-1} p_0$$

and hence

$$\langle M_S(0, p_0), (0, p_0) \rangle = -\langle W_{xx}^{-1} p_0, p_0 \rangle. \quad (25)$$

Applying Fresnel's formula (15) we get

$$\left(\frac{1}{2\pi}\right)^n \int e^{\frac{i}{2}\langle M_S(0, p_0), (0, p_0) \rangle} d^n p_0 = \left(\frac{1}{2\pi}\right)^{n/2} e^{-\frac{i\pi}{4} \operatorname{sgn} W_{xx}} |\det W_{xx}|^{1/2};$$

noting that

$$\frac{1}{\sqrt{|\det(S_W - I)|}} = |\det L|^{1/2} |\det W_{xx}|^{-1/2}$$

(formula (20) in Lemma 4) we thus have

$$\widehat{R}_\nu(S_W)\delta(0) = \left(\frac{1}{2\pi}\right)^{n/2} i^\nu e^{-\frac{i\pi}{4} \operatorname{sgn} W_{xx}} |\det L|^{1/2}.$$

Now, by definition (5) of  $\widehat{S}_{W,m}$ ,

$$\widehat{S}_{W,m}\delta(0) = \left(\frac{1}{2\pi}\right)^{n/2} i^{m-n/2} |\det L|^{1/2}$$

hence  $i^\nu e^{-\frac{i\pi}{4} \operatorname{sgn} W_{xx}} = i^{m-n/2}$ . It follows that we have

$$\nu - \frac{1}{2} \operatorname{sgn} W_{xx} \equiv m - \frac{1}{2}n \pmod{4}$$

which is the same thing as (22) since  $W_{xx}$  has rank  $n$ . In view of (19) we have

$$\frac{1}{\pi} \arg \det(S_W - I) = n + m + \arg \det W_{xx} \pmod{2};$$

formula (23) follows using (22). ■

Let us digress for a while on the integers  $m$  and  $\operatorname{Inert} W_{xx}$  appearing in formula (22) and discuss them from the point of view of calculus of variations. It is for this purpose useful to recall that in Gutzwiller's formula (of which Mehlig and Wilkinson precisely want to give a new approach using the operators  $\widehat{R}_\nu(S)$ ) the symplectic matrix  $S$  is obtained from the monodromy matrix of an isolated Hamiltonian periodic orbit. Let us go a little bit further. Consider a Hamiltonian flow  $\phi_t$  determined by some time-dependent Hamiltonian  $H = H(z, t)$  defined on  $\mathbb{R}_z^{2n} \times \mathbb{R}_t$ , and let  $z_0 = (x_0, p_0)$  be such that  $\phi_T(z_0) = z_0$  for some  $T > 0$ . The Jacobian matrices  $S(z_0, t) = D\phi_t(z_0)$  are symplectic and satisfy the ‘‘variational equation’’

$$\frac{d}{dt} S(z_0, t) = JH''(z_0, t)S(z_0, t)$$

where  $H''(z_0, t)$  is the Hessian matrix  $D^2H(\phi_t(z_0), t)$ . When  $t$  varies from 0 to  $T$  the matrices  $S(z_0, t)$  describe a path in  $Sp(n)$  originating at the identity and ending at  $S(z_0, T)$  (the ‘‘monodromy matrix’’). Suppose that  $S(z_0, T)$  is a free symplectic matrix  $S_W$ ; then there exist  $p$  and  $p'$  such that  $(x_0, p) = S_W(x_0, p')$ , that is, expressing  $W$  in terms of  $P, L, Q$  as in (8) and using (7),  $p = (P - L^T)x_0$  and  $p' = (L - Q)x_0$ , that is

$$p - p' = (P + Q - L - L^T)x_0.$$

It thus appears (see for instance [13, 15]) that  $\operatorname{Inert} W_{xx}$  is Morse's [12] *order of concavity* of the periodic orbit through  $z_0$ .

**Remark 7** In [13] Piccione and his collaborators use the order of concavity, which they identify with  $\text{Inert } W_{xx}$ , to investigate the Maslov and Morse indices for periodic geodesics. The topic is also discussed at some length in Muratore–Ginnaneschi [15] in connection with the study of Gutzwiller’s formulae using field-theoretical methods.

Perhaps even more interesting is the relation between the index  $\nu$  and an index defined by Conley and Zehnder in [2]. Let us denote by  $Sp_0(n)$  the set of all  $S \in Sp(n)$  such that  $\det(S - I) \neq 0$ . We have

$$Sp_0(n) = Sp_+(n) \cup Sp_-(n)$$

where  $S \in Sp_{\pm}(n)$  if and only if  $\pm \det(S - I) > 0$ . The sets  $Sp_{\pm}(n)$  are connected and every loop in  $Sp_0(n)$  is contractible in  $Sp(n)$ . Consider now a path  $\tilde{S} : [0, T] \mapsto Sp_0(n)$  going from the identity to  $S = S(T)$  ( $S(T)$  may be viewed, if one wants, as the monodromy matrix of a periodic Hamiltonian orbit corresponding to a *time-dependent* Hamiltonian). The index of Conley and Zehnder associates to the path  $\tilde{S}$  an integer  $\mu_{CZ}(\tilde{S})$  only depending on the homotopy class (with fixed endpoints) of that path, and such that

$$\text{sign } \det(S - I) = (-1)^{n - \mu_{CZ}(\tilde{S})}$$

that is, equivalently,

$$\frac{1}{\pi} \arg \det(S - I) \equiv n - \mu_{CZ}(\tilde{S}) \pmod{2}.$$

It follows from formula (23) in Proposition 6 that we have

$$\nu \equiv \mu_{CZ}(\tilde{S}) \pmod{2} \tag{26}$$

and from formula (22) in the same proposition that

$$\mu_{CZ}(\tilde{S}) \equiv m + n - \text{Inert } W_{xx} \pmod{2}. \tag{27}$$

### 3.3 The general case

Recall that we established in Lemma 4 the equality

$$\det(S_W - I) = (-1)^n \det L^{-1} \det(P + Q - L - L^T). \tag{28}$$

valid for all free matrices  $S_W \in Sp(n)$ . Also recall that every  $\hat{S} \in Mp(n)$  can be written (in infinitely many ways) as a product  $\hat{S} = \hat{S}_{W,m} \hat{S}_{W',m'}$ . We are going to show that  $\hat{S}_{W,m}$  and  $\hat{S}_{W',m'}$  always can be chosen such that  $\det(S_W - I) \neq 0$  and  $\det(S_{W'} - I) \neq 0$ . For that purpose we need the following straightforward factorization result, which we nevertheless glorify by putting it into italics:

**Lemma 8** *Let  $W$  be given by (6); then*

$$\widehat{S}_{W,m} = \widehat{V}_{-P} \widehat{M}_{L,m} \widehat{J} \widehat{V}_{-Q} \quad (29)$$

where

$$\widehat{V}_{-P} f(x) = e^{\frac{i}{2} \langle Px, x \rangle} f(x) \quad , \quad \widehat{M}_{L,m} f(x) = i^m \sqrt{|\det L|} f(Lx) \quad ,$$

and  $\widehat{J}$  is the modified Fourier transform given by

$$\widehat{J} f(x) = \left(\frac{1}{2\pi i}\right)^{n/2} \int e^{-i \langle x, x' \rangle} f(x') d^n x'.$$

**Proof.** It is obvious using the explicit expression (6) of the quadratic form  $W$  (see [4]). ■

Let us now state and prove the first result of this section:

**Proposition 9** (i) *Every  $\widehat{S} \in Mp(n)$  can be written as a product*

$$\widehat{S} = \widehat{R}_\nu(S_W) \widehat{R}_{\nu'}(S_{W'}). \quad (30)$$

(ii) *The Mehlig–Wilkinson operators thus generate  $Mp(n)$ .*

**Proof.** (ii) follows from (i) since the  $\widehat{S}_{W,m}$  generate  $Mp(n)$ . To prove (i) let us write  $\widehat{S} = \widehat{S}_{W,m} \widehat{S}_{W',m'}$  and apply (29) to each of the factors; letting  $P', L', Q'$  define  $W'$  just as (6) is defined by  $P, L, Q$  we have

$$\widehat{S} = \widehat{V}_{-P} \widehat{M}_{L,m} \widehat{J} \widehat{V}_{-(P'+Q)} \widehat{M}_{L',m'} \widehat{J} \widehat{V}_{-Q'}. \quad (31)$$

We claim that  $\widehat{S}_{W,m}$  and  $\widehat{S}_{W',m'}$  can be chosen in such a way that  $\det(S_W - I) \neq 0$  and  $\det(S_{W'} - I) \neq 0$  that is,

$$\det(P + Q - L - L^T) \neq 0 \quad \text{and} \quad \det(P' + Q' - L' - L'^T) \neq 0;$$

this will prove the assertion in view of (28). We first remark that the right hand-side of (31) obviously does not change if we replace  $P'$  by  $P' + \lambda$  and  $Q'$  by  $Q' - \lambda$  where  $\lambda \in \mathbb{R}$ . Choose now  $\lambda$  such that it is not an eigenvalue of  $P + Q - L - L^T$  and  $-\lambda$  is not an eigenvalue of  $P' + Q' - L' - L'^T$ ; then

$$\det(P + Q - \lambda I - L - L^T) \neq 0 \quad \text{and} \quad \det(P' + \lambda + Q' - L' - L'^T) \neq 0$$

and we have  $\widehat{S} = \widehat{S}_{W_1, m_1} \widehat{S}_{W'_1, m'_1}$  with

$$\begin{aligned} W_1(x, x') &= \frac{1}{2} \langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2} \langle (Q - \lambda)x', x' \rangle \\ W'_1(x, x') &= \frac{1}{2} \langle (P' + \lambda)x, x \rangle - \langle L'x, x' \rangle + \frac{1}{2} \langle Q'x', x' \rangle. \end{aligned}$$

■

So far, so good. But we haven't told the whole story yet: there remains to prove that  $\widehat{S} \in Mp(n)$  can be written in the form  $\widehat{R}_\nu(S)$  if  $\det(S - I) \neq 0$ .

**Proposition 10** Let  $\widehat{S} \in Mp(n)$  be such that  $\det(S - I) \neq 0$ . If  $\widehat{S} = \widehat{R}_\nu(S_W)\widehat{R}_{\nu'}(S_{W'})$  then  $\widehat{S} = \widehat{R}_{\nu(S)}(S)$  with

$$\nu(S) = \nu + \nu' + n - \text{Inert}(M + M') \quad (32)$$

the matrices  $M$  and  $M'$  being associated to  $S$  and  $S'$  by formula (2).

**Proof.** A straightforward calculation using the composition formula (14) and the Fresnel integral (15) shows that

$$\widehat{S} = \left(\frac{1}{2\pi}\right)^n \frac{i^{\nu+\nu'+\frac{1}{2}\text{sgn}(M+M')}}{\sqrt{|\det(S_W - I)(S_{W'} - I)(M + M')|}} \int e^{\frac{i}{2}\langle Nz, z \rangle} \widehat{T}(z) d^{2n}z \quad (33)$$

where  $M$  and  $M'$  correspond to  $S_W$  and  $S_{W'}$  by (2) and

$$N = M - (M + \frac{1}{2}J)(M + M')^{-1}(M - \frac{1}{2}J).$$

We claim that

$$\det[(S_W - I)(S_{W'} - I)(M + M')] = \det(S - I) \quad (34)$$

(hence  $M + M'$  is indeed invertible), and

$$N = \frac{1}{2}J(S + I)(S - I)^{-1} = M_S. \quad (35)$$

Formula (34) is easy to check by a direct calculation: by definition of  $M$  and  $M'$  we have, since  $\det J = 1$ ,

$$\begin{aligned} \det[(S_W - I)(S_{W'} - I)(M + M')] &= \\ \det[(S_W - I)(I + (S_W - I)^{-1} + (S_{W'} - I)^{-1})(S_{W'} - I)] \end{aligned}$$

that is

$$\det[(S_W - I)(S_{W'} - I)(M + M')] = \det(S_W S_{W'} - I)$$

which is precisely (34). Formula (35) is at first sight more cumbersome, and one might be tempted to use the oscillator semigroup calculations of Howe [10] at this stage. There is however an easier way out: assume that  $\widehat{S} = \widehat{S}_{W'', m''}$ ; we *know* by Proposition 9 that we *must* have in this case

$$N = \frac{1}{2}J(S_W S_{W'} + I)(S_W S_{W'} - I)^{-1}$$

and this algebraic identity then holds for all  $S = S_W S_{W'}$  since the free symplectic matrices are dense in  $Sp(n)$ . Formula (33) can thus be rewritten

$$\widehat{S} = \left(\frac{1}{2\pi}\right)^n \frac{i^{\nu+\nu'+\frac{1}{2}\text{sgn}(M+M')}}{\sqrt{|\det(S - I)|}} \int e^{\frac{i}{2}\langle M_S z, z \rangle} \widehat{T}(z) d^{2n}z$$

and formula (32) follows noting that if  $R$  is any real invertible  $2n \times 2n$  symmetric matrix with  $q$  negative eigenvalues we have  $\arg \det R = q\pi \bmod 2\pi$  and  $\frac{1}{2} \operatorname{sgn} R = n - q$  and hence

$$\frac{1}{2} \operatorname{sgn}(M + M') = n - \operatorname{Inert}(M + M').$$

■

## 4 Concluding Remarks

We have justified Mehlig and Wilkinson's claim that the metaplectic operators corresponding to symplectic matrices with no eigenvalues equal to one can be written in the form (3); we have in addition shown that every metaplectic operator can be written as the product of exactly two such operators. There are however still interesting open problems. It would be interesting to relate the index  $\nu$  appearing in (3) to the cohomological Maslov index on  $Mp(n)$  we constructed in [4, 5]: this would certainly lead to simpler –or at least more tractable– calculations for the indices intervening in Gutzwiller's formula, as already demonstrated in our previous paper [7] where we examined the Maslov index of the monodromy matrix associated to a periodic Hamiltonian orbit. As shown by (26) a related mathematical problem would be to express the Conley–Zehnder index in terms of Leray's index studied in de Gosson [6].

We hope to come back to these important and interesting questions in a near future, together with applications to various trace formulae.

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