Unitary Solutions of Partial Differential Equations

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Abstract

We give an explicit construction of a fundamental solution for an arbitrary non-degenerate partial differential equation with smooth coefficients.

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Introduction

After J. Leray [14], J. von Neumann believed that computers need a very good theory of partial differential equations. On the one hand, they need general existence, uniqueness and continuity theorems, to make sure that what they compute does exist and to show that it can be estimated as they do. On the other hand, computing is efficiently aided by the discovery of explicit solutions of special problems and by the study of the special functions functions appearing there.

The purpose of this paper is to bring together two approaches in constructing explicit fundamental solutions for non-degenerate partial differential equations. The first approach is due to Hadamard [10], it is based on the far-reaching geometric optics asymptotics in mathematical physics. However, it applies only to second order equations and gives only local results. The second approach is due to Petrovskii [15], it is based on powerful arsenals of Fourier transform and distribution theory and is global in the very nature. However, it is applicable only for general partial differential equations with constant coefficients.

The plan of the paper is as follows. Section 1 is devoted to explicit formulas of J. Leray [14] for solutions of the Cauchy problem for Tricomi's general operators. In Section 2, we introduce new coordinates. These are geodesic normal coordinates in Hadamard's terminology, or projection caractéristique d'apres Leray. In Section 3, an elementary proof of J. Leray's result is presented which generalises the eiconal equation. In the very technical Section 4 we study the transport equation. Section 5 gives the convergence and asymptotic proofs of the formal solution constructed in § 4. In Section 6 we discuss the classification of fundamental solutions. The last Section 7 gives a counterexample showing the significance of the non-degeneracy condition.

1 Functional transformations

Let X denote an n-dimensional manifold, x a point of X, the coordinates of x being $(x_1, \ldots, x_n)$, and $v(x) = \rho(x)dx$ a volume element on X.

Consider a differential operator A of order m on X, in local coordinates A being

$$A(x, D) = \sum_{|\alpha| \le m} a_\alpha(x)\, D^\alpha,$$

where $D_j = \dfrac{1}{\imath} \dfrac{\partial}{\partial x_j}$. Its dual is an operator A' such that

$$\int_X v\, Au\, \rho dx = \int_X A'v\, u\, \rho dx \tag{1.1}$$

for all functions u and v with compact support. In the local chart it is given by

$$A'(x, D)v = \sum_{|\alpha| \le m} \frac{(-1)^{|\alpha|}}{\rho}\, D^\alpha \Big(\rho a_\alpha v\Big).$$

As usual, the dual is assigned to act in spaces of densities on X, but we avoid this for simplicity of notation.

Write

$$A(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x)\, \xi^\alpha$$

as a sum of homogeneous polynomials in $\xi = (\xi_1, \ldots, \xi_n)$, i.e.,

$$A(x, \xi) = \sum_{j=0}^{\infty} A_{m-j}(x, \xi)$$

where A_{m-j} are homogeneous of degree $m - j$. Think of ξ as a covector (for instance a gradient) at x, then the maps

$$\begin{aligned} A(x, D) &\mapsto A_m(x, \xi), \\ A(x, D) &\mapsto A_{m-1}(x, \xi) + \frac{\imath}{2} \sum_{j=1}^{n} \frac{1}{\rho(x)} \frac{\partial^2}{\partial x_j \partial \xi_j} (\rho(x) A_m(x, \xi)) \end{aligned}$$

do not depend on the choice of the coordinates.

The first polynomial is called the principal symbol of A, it is denoted by $\sigma^m(A)(x,\xi)$. The second polynomial depends on ρ, it is usually referred to as the subprincipal symbol of A.

The hypersurfaces $\mathcal{C} = \{x \in X : c(x) = 0\}$ in X with c satisfying the non-linear first order differential equation

$$\sigma^m(A)(x, -\imath\nabla c\,(x)) = 0 \tag{1.2}$$

are called characteristics of A. The well-known theory of first order differential equations shows that they are generated by curves satisfying the system of ordinary differential equations

$$\frac{dx_1}{\dfrac{\partial H}{\partial \xi_1}} = \ldots = \frac{dx_n}{\dfrac{\partial H}{\partial \xi_n}} = -\frac{d\xi_1}{\dfrac{\partial H}{\partial x_1}} = \ldots = -\frac{d\xi_n}{\dfrac{\partial H}{\partial x_n}}$$

under the additional condition $H(x,\xi) = 0$, the system being the characteristic system of the characteristic equation (1.2). These curves are called bicharacteristics of A.

From now on we assume X real and affine and $\rho = 1$. Operators with constant coefficients can be studied by Fourier or Laplace transforms which have simple and useful properties. But in many problems variable coefficients occur. For instance, the study of transsonic flow makes use of Tricomi's operator $x_2(\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$. Let us call Tricomi's general operators the operators whose coefficients of order m, $m-1$ and $< m-1$ are affine, constant and null, respectively, i.e.,

$$A(x, D) = A_{m,0}(D) + \sum_{j=1}^{n} x_j A_{m,j}(D) + A_{m-1}(D)$$

where $A_{m,0}, A_{m,1}, \ldots, A_{m,n}$ are homogeneous of order m, and A_{m-1} is of order $m-1$. Their interest lies in the fact that they constitute a first approximation of the operators with variable coefficients and, for them, the Cauchy problem can be explicitly solved.

Remark 1.1 *The commutator of any two Tricomi's general operators is also such an operator, i.e., they form a Lie algebra.*

Let $S = \{x \in X : \varrho(x) = 0\}$ be a smooth hypersurface in X. By the Cauchy problem for A with data on S is meant the problem of finding an unknown function u satisfying

$$\left\{ \begin{array}{rcll} Au & = & f & \text{near } S, \\ u & = & u_0 & \text{up to order } m-1 \text{ on } S, \end{array} \right. \tag{1.3}$$

where f and u_0 are given functions in a neighbourhood of S.

It is called the analytic Cauchy problem when u is to be analytic, the data A and S, f, u_0 being analytic. Such a problem was solved by Cauchy-Kovalevskaya's theorem.

Fifty years later Hadamard [10] pointed out that the problem occurring in wave propagation is not at all an analytic problem, but a problem with real, not necessarily analytic, data, A being hyperbolic and S space like. He called such a problem well-posed. Since that time, the analytic Cauchy problem has been out of date. However, Hadamard's warning does not mean that analytic Cauchy problems never occur. In fact, the explicit information which can be obtained about well-posed problems comes from similar information about the so-called unitary solution which is a solution of the simplest analytic Cauchy problem.

Pick an affine function ξ on X and write $\xi \cdot x = \xi_0 + \xi_1 x_1 + \ldots + \xi_n x_n$ for the value of ξ at x. Thus, ξ can be specified as a vector with coordinates $(\xi_0, \xi_1, \ldots, \xi_n)$ of a vector space Ξ of dimension $n+1$. A hyperplane in X is $\xi' = \{x \in X : \xi \cdot x = 0\}$. These constitute a projective space Ξ' of dimension n which is the image of $\Xi \setminus \{0\}$.

Let A be an analytic differential operator on X. By its unitary solution $U(x,\xi)$ is meant the solution of the analytic Cauchy problem

$$\left\{ \begin{array}{rcll} A(x,D)U(x,\xi) & = & 1 & \text{near } \xi', \\ U(x,\xi) & = & 0 & \text{up to order } m\!-\!1 \text{ on } \xi', \end{array} \right. \tag{1.4}$$

i.e., for $\xi \cdot x = 0$. The Cauchy-Kovalevskaya theorem shows that $U(x,\xi)$ exists and is unique at the non-characteristic points x of ξ', i.e., for all x satisfying $\sigma(A)(x,\xi) \neq 0$.

We often use its derivative

$$U_m(x,\xi) = \Big(- \frac{\partial}{\partial \xi_0} \Big)^m U(x,\xi)$$

which satisfies the equation $A(x,D)U_m(x,\xi) = 0$ and which is called a unitary wave.

The singularity of the solution of the analytic Cauchy problem was studied by Leray [13, I]. The result essentially simplifies when applied to the unitary solution. Let $u(x,\xi)$ be a multivalued function, homogeneous in ξ. This function is said to be uniformisable when $u(x,\xi(t,x,\upsilon))$ is analytic in t, x and υ for some analytic mapping $\xi(t,x,\upsilon)$, with t a complex variable of small modulus, $x \in X$ and $\upsilon \in \Xi$, such that $\upsilon \cdot x = 0$, $\xi(t,x,\upsilon) \neq 0$ for $\upsilon \neq 0$, $\xi(0,x,\upsilon) = \upsilon$ and

$$\xi(\lambda^{1-m} t, x, \lambda \upsilon) = \lambda\, \xi(t,x,\upsilon) \tag{1.5}$$

for all complex numbers λ, m being a given integer. We say that $\xi(t,x,\upsilon)$ uniformises $u(x,\xi)$.

Moreover, $U(x,\xi)$ and its derivatives up to order $m-1$ are uniformisable. A mapping uniformising them is explicitly known. The carrier of the singularity of $U(x,\xi)$ is the characteristic tangent to ξ'. The principal part of the singularity of $U(x,\xi)$ is also explicitly known. In order to state these results precisely, consider the solution $x(t,y,\upsilon)$, $\xi(t,y,\upsilon)$, $l(t,y,\upsilon)$ of the ordinary differential system

$$\begin{aligned} \frac{dx_j}{dt} &= \frac{\partial H}{\partial \xi_j}(x,\xi), & \frac{d\xi_j}{dt} &= -\frac{\partial H}{\partial x_j}(x,\xi) \qquad \text{for } j=1,\ldots,n; \\ & & \frac{d\xi_0}{dt} &= \sum_{j=1}^{n} x_j \frac{\partial H}{\partial x_j}(x,\xi) - H(x,\xi); \\ & & \frac{dl}{dt} &= A_{m-1}(x,-\imath\xi) - \frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2 H}{\partial x_j \partial \xi_j}(x,\xi) \end{aligned} \tag{1.6}$$

with the initial data

$$x(0,y,\upsilon)=y, \quad \xi(0,y,\upsilon)=\upsilon, \quad l(0,y,\upsilon)=0.$$

Theorem 1.2 (Uniformisation Theorem) *The mapping $\xi(t,x,\upsilon)$ uniformises both $U(x,\xi)$ and $U'(x,\xi)$ and their derivatives up to order $m-1$.*

Here, U' stands for the unitary solution corresponding to the dual operator A'.

Leray [13, I] gives also an expression for the principal singularity of $U(x,\xi)$. Namely, the difference

$$U_m(x,\xi) - \frac{(-1)^m}{H(x,\upsilon)}\, e^l \Big(\det \frac{\partial(\xi_1,\ldots,\xi_n)}{\partial(\upsilon_1,\ldots,\upsilon_n)}\Big)^{-1/2}, \tag{1.7}$$

for $\xi=\xi(t,x,\upsilon)$ and $l=l(t,x,\upsilon)$, is an analytic function of t, x and υ when t is small. Moreover,

$$\det \frac{\partial(\xi_1,\ldots,\xi_n)}{\partial(\upsilon_1,\ldots,\upsilon_n)} = 1$$

for $t=0$. This is still true with U, H, l replaced by U', $(-1)^m H$ and $-l$. respectively.

The first line of (1.6) is a Hamiltonian system. It follows that (1.6) admits the first integrals $H(x,\xi)$ and $\xi\cdot x-(m-1)\,tH(x,\xi)$ and the invariant differential form $H(y,\upsilon)\,dt+(d\xi)\cdot x$. Hence, for $x=x(t,y,\upsilon)$ and $\xi=\xi(t,y,\upsilon)$, we get

$$\begin{aligned} H(x,\xi) &= H(y,\upsilon), \\ \xi\cdot x &= (m-1)\,tH(y,\upsilon), \\ (d\xi)\cdot x &= -H(y,\upsilon)\,dt - (\upsilon_1 dy_1+\ldots+\upsilon_n dy_n)\,, \end{aligned}$$

the latter relation meaning

$$\frac{\partial \xi}{\partial t} \cdot x = -H, \quad \frac{\partial \xi}{\partial y} \cdot x = -\upsilon, \quad \frac{\partial \xi}{\partial \upsilon} \cdot x = 0.$$

This yields

$$\det \frac{\partial(\xi_0, \xi_1, \ldots, \xi_n)}{\partial(t, \upsilon_1, \ldots, \upsilon_n)} = -H(y, \upsilon) \det \frac{\partial(\xi_1, \ldots, \xi_n)}{\partial(\upsilon_1, \ldots, \upsilon_n)}$$

which proves that the carrier of the singularity of U and U' is the characteristic tangent to ξ'.

The bicharacteristics generating the characteristic tangent to ξ', and the second term of (1.7) containing a Jacobian have a mechanical interpretation as trajectories and mass-impulse density of particles, which can be associated with the Cauchy problem (1.4).

Example 1.3 When $A(D)$ has constant coefficients and is homogeneous of degree m, then

$$U(x, \xi) = \frac{(\xi \cdot x)^m}{m!} \frac{1}{H(\xi)}$$

is an integral function of $\xi \cdot x / H(\xi)$ and $\xi_1, \ldots, \xi_n$ in accordance with the uniformisation theorem.

For Tricomi's general operators the determination of $U(x, \xi)$ reduces to a first order Cauchy problem by a reciprocity theorem, cf. [14].

By a (right) fundamental solution of the equation $A(x, D)u = f$ is meant a solution $E(x, y)$ corresponding to $f = \delta_y$, i.e., Dirac's measure at y. An immediate application of (1.1) shows that $A'(y, D)E(x, y) = \delta_x$, which just amounts to saying that $E'(y, x) := E(x, y)$ is a (right) fundamental solution of $A'(y, D)$.

If the support of $E(x, y)$ belongs to the characteristic conoid with vertex at y, i.e., the union of all bicharacteristics emanating from y, then knowledge of E enables us to solve the Cauchy problem (1.3) with zero initial data by the formula

$$u(x) = \int_X E(x, y) f(y) dy.$$

Hence, the general well-posed Cauchy problem for a hyperbolic operator A reduces to the investigation of its fundamental solution $E(x, y)$. Existence and uniqueness theorems give no precise information about E. However, from what we know about $U(x, \xi)$, such information can be deduced by the generalised Laplace transform $\mathcal{L}$, introduced by Leray [13, IV] who developed the theory of analytic functionals of Fantappiè [7]. More precisely,

$$E'(y, x) = \mathcal{L}_{\xi \mapsto x} U'(y, \xi). \tag{1.8}$$

is valid. Formula (1.8) is a deep analogue of John's famous formula for a fundamental solution of an elliptic operator with non-constant coefficients, cf. [12].

As the domain of $\mathcal{L}$ consists of multivalued functions $f(y,\xi)$, the most delicate part of the definition of $\mathcal{L}$ lies in the choice of a cycle of integration, h. It depends on y. The generalised Laplace transform $\mathcal{L}$ possesses familiar properties which allows one to use $\mathcal{L}$ for quantisation of functions homogeneous in ξ. Namely,

$$\begin{aligned} \Big(n-d-1-\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\Big)\mathcal{L}(f) &= \mathcal{L}(\xi_0 f), \\ \frac{\partial}{\partial x_j}\mathcal{L}(f) &= \mathcal{L}(\xi_j f), \end{aligned}$$

d being the homogeneity degree of f in ξ, and

$$\begin{aligned} \mathcal{L}\Big(\frac{\partial f}{\partial \xi_0}\Big) &= -\mathcal{L}(f), \\ \mathcal{L}\Big(\frac{\partial f}{\partial \xi_j}\Big) &= -x_j\mathcal{L}(f) \end{aligned}$$

for $x \neq y$. Since $\mathcal{L}$ does not affect the variable y it commutes with differential operators in y. Finally, to recover a fundamental solution $\mathcal{L}$ has to satisfy $\mathcal{L}(1) = \delta_y$. The latter relation corresponds to the relation $\mathcal{L}(f(y)) = f(y)$ which is deduced from Cauchy-Fantappiè's formula, cf. [13, III].

Let us express $E(x,y)$, for $x-y$ small, as the $\mathcal{L}$-transform of some function $f(y,\xi)$. Since $\mathcal{L}(f)=0$ if and only if $f=0$, the equation $A'(y,D)E(x,y)=\delta_x$ is equivalent to $A'(y,D)f(y,\xi) = 1$ under the assumptions that f and its derivatives in y up to order $m-1$ are rationally uniformisable. A trivial property of rationally uniformisable functions is that they vanish for $\xi\cdot y = 0$. Hence, $f(y,\xi)$ should coincide with the unitary solution of $A'(y,D)$. Now, the uniformisation theorem of the unitary solution shows not only that its derivatives up to order $m-1$ are uniformisable, but also that they are rationally iniformisable. Hence $E(x,y) = \mathcal{L}_{\xi\to x}U'(y,\xi)$, which implies the fundamental formula

$$E(x,y) = \mathcal{L}_{\xi\to x}U'_m(y,\xi)$$

where U'_m is the unitary wave of $A'(y,D)$.

From this it follows, in particular, that $E(x,y)$ is holomorphic when x does not belong to the characteristic conoid with vertex at y.

2 Characteristic map

In this paper we study a general linear partial differential operator of order $m \geq 2$

$$A(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x)\, D^\alpha \tag{2.1}$$

in a neighborhood $U \subset \mathbb{R}^n$ of $x = 0$. Its coefficients may be either C^∞ or real analytic. Still, the results until § 5 is actually the same. Denote its full symbol by

$$A(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\, \xi^\alpha,$$

and its principal symbol by

$$H(x, \imath\xi) := \sum_{|\alpha| = m} a_\alpha(x)\, \xi^\alpha,$$

for $\xi \in \mathbb{R}^n$. In order to get asymptotic results, we always require the non-degeneracy condition

$$\begin{aligned} \operatorname{Hess}_\xi H(x, \xi) &:= \det \Big(\frac{\partial^2 H}{\partial \xi_i \partial \xi_j}(x, \xi) \Big) \\ &\neq 0 \end{aligned} \tag{2.2}$$

to be fulfilled for all $x \in U$ and $\xi \neq 0$.

The Hamiltonian field is defined by

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial \xi}(x, \xi), \\ \frac{d\xi}{dt} &= -\frac{\partial H}{\partial x}(x, \xi). \end{aligned} \tag{2.3}$$

Its orbit in $T^*U \cong U \times \mathbb{R}^n$ is the bicharacteristic strip, and the x-component (x-orbit for short) is a curve in the base space U. We now consider the hypersurface composed of all the bicharacteristic curves through a fixed point, say $x = 0$, hence we also add the initial conditions

$$\begin{aligned} x(0) &= 0, \\ \xi(0) &= \upsilon, \end{aligned} \tag{2.4}$$

where $|\upsilon| = 1$. Its orbits constitute the hypersurface we need, i.e., the characteristic conoid. Denote the solution to this Cauchy problem by $x(t, \upsilon)$ and $\xi(t, \upsilon)$.

Under the dilation

$$\begin{aligned} x &\mapsto x' = x, & t &\mapsto t' = \lambda^{1-m} t, \\ \xi &\mapsto \xi' = \lambda \xi, & \upsilon &\mapsto \upsilon' = \lambda \upsilon, \end{aligned}$$

both $x'(t',\upsilon')$ and $\xi'(t',\upsilon')$ still satisfy the equation (2.3), but the initial conditions become $x'(0)=0$ and $\xi'(0)=\upsilon'$. It follows that

$$\begin{array}{rcl} x(t,\upsilon) & = & x(\lambda^{1-m}t,\lambda\upsilon), \\ \lambda\xi(t,\upsilon) & = & \xi'(\lambda^{1-m}t,\lambda\upsilon). \end{array}$$

Taking $\lambda=\tau$ and $t=\tau^{m-1}$, we conclude that

$$\begin{array}{rcl} x(t,\upsilon) & = & x(1,\tau\upsilon), \\ \tau\xi(t,\upsilon) & = & \xi'(1,\tau\upsilon), \end{array}$$

i.e., both x and $\tau\xi$ are functions of $\eta=\tau\upsilon$. This justifies the notation

$$\begin{array}{rcl} x & = & x(\eta), \\ \tau\xi & = & \xi'(\eta). \end{array} \tag{2.5}$$

We have thus constructed a mapping $\eta\mapsto x$ which is C^∞ or real analytic. In order to describe its inverse mapping, we note that from (2.3) and (2.4) we get

$$\begin{array}{rcl} x & = & O(t), \\ \xi & = & \upsilon+O(t). \end{array}$$

Differentiating the first equations in (2.3) and (2.4) in υ yields

$$\begin{array}{rcl} \dfrac{d}{dt}\Big(\dfrac{\partial x_i}{\partial\upsilon_j}\Big) & = & \displaystyle\sum_{k=1}^{n}\frac{\partial^2 H}{\partial\xi_i\partial\xi_k}\frac{\partial\xi_k}{\partial\upsilon_j}+\sum_{k=1}^{n}\frac{\partial^2 H}{\partial\xi_i\partial x_k}\frac{\partial x_k}{\partial\upsilon_j}, \\ \dfrac{\partial x_i}{\partial\upsilon_j}\Big|_{t=0} & = & 0. \end{array}$$

Also we have

$$\frac{\partial x_i}{\partial\upsilon_j}=O(t)$$

and

$$\frac{\partial x_i}{\partial\upsilon_j}=t\,\frac{\partial^2 H}{\partial\upsilon_i\partial\upsilon_j}+o(t).$$

But (2.5) gives

$$\frac{\partial x_i}{\partial\upsilon_j}=\tau\,\frac{\partial x_i}{\partial\eta_j},$$

and taking into account that H as a function of υ is homogeneous of order m we get

$$\frac{\partial x_i}{\partial\eta_j}=\tau^{m-2}\,\frac{\partial^2 H}{\partial\upsilon_i\partial\upsilon_j}(0,\upsilon)+o(1)\,\tau^{m-2}.$$

It follows that

$$\Big(\tau^{2-m}\,\frac{\partial x_i}{\partial\eta_j}\Big)=(I+o(1))\,\Big(\mathrm{Hess}_\upsilon\,H(0,\upsilon)\Big)^{-1}.$$

Summing up, we have

Lemma 2.1 *The inverse map of (2.5) is C^∞ or real analytic away from $\eta = 0$, and*

$$\frac{\partial \eta_i}{\partial x_j} = \tau^{2-m}\, F_{i,j}(\tau, \upsilon)$$

where $F_{i,j}$ is C^∞ or real analytic near $\eta = 0$.

The coordinates η are called geodesic normal coordinates while (τ, υ) are called generalized polar coordinates. In the latter case, $\upsilon \in \mathbb{S}^{n-1}$, hence τ and $n-1$ components of υ are independent. We can find a domain on $\mathbb{S}^{n-1}_\upsilon$, in which one of the components of υ, say υ_1, depends on other components, e.g., $\upsilon_1 = (1 - \upsilon_2^2 - \ldots - \upsilon_n^2)^{1/2}$. Thus, for υ in this domain, $(\tau, \upsilon_2, \ldots, \upsilon_n)$ are local coordinates. We now consider the image of $[0, \delta) \times \mathbb{S}^{n-1}_\upsilon$ in the x-space, $\delta > 0$ being small enough.

Lemma 2.2 *Under the non-degeneracy condition (2.2), we have*

$$\nabla_\upsilon H(0, \upsilon) \neq 0$$

for $\upsilon \neq 0$.

Proof. If there were $\upsilon^0 \neq 0$ such that $\nabla_\upsilon H(0, \upsilon^0) = 0$, then Euler's formula would give

$$\begin{aligned} 0 &= (m-1)\, \frac{\partial H}{\partial \upsilon_i}(0, \upsilon^0) \\ &= \sum_{j=1}^{n} \frac{\partial^2 H}{\partial \upsilon_i \partial \upsilon_j}(0, \upsilon^0)\upsilon_j^0 \end{aligned}$$

for all $i = 1, \ldots, n$. Thus, υ^0 would be a non-zero solution of the linear system above which has a non-vanishing determinant $\mathrm{Hess}_\upsilon\, H(0, \upsilon^0)$. This is impossible.

Q.E.D.

Lemma 2.3 *For $\delta > 0$ small enough, there exists a positive constant C such that the image of $\{(\tau, \upsilon) :\ \tau \in [0, \delta]\}$ lies in the band*

$$C^{-1}\delta \le |x| \le C\delta.$$

Proof. By the previous Lemma 2.2, there exists a positive constant c such that

$$c^{-1} \le |\nabla H(0, \upsilon)| \le c$$

for all $\upsilon \in \mathbb{S}^{n-1}$. Our result follows directly from the first part of the Hamiltonian system (2.3) and Lemma 2.2.

Q.E.D.

We are now in a position to prove

Theorem 2.4 *Assume that (2.2) is satisfied. Then, for sufficiently small* $\delta > 0$*, the transformation* $x = x(\eta)$ *in (2.5) maps* $[0, \delta) \times \mathbb{S}_v^{n-1}$ *onto a neighborhood* U *of* $x = 0$ *in* $\mathbb{R}^n$ *continuously. Moreover, this is a local diffeomorphism for* $\eta \neq 0$.

Proof. The proof is immediate because for $\eta \neq 0$ the Jacobi matrix

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(\tau, \upsilon)}$$

is regular.

Q.E.D.

We denote this map by π_H and call it the characteristic mapping. It is very close to J. Leray's projection caractéristique, cf. [13, IV]. The x-orbit connecting the point x and the origin corresponds to the radius through the point (τ, υ), where $\upsilon \in \mathbb{S}_\upsilon^{n-1}$. Thus, each x-orbit starting from the origin is defined by υ (the initial value for ξ in (2.3)), and every point $x \in U$ can be connected to the origin by a unique x-orbit. When υ lies in a neighbourhood $V \subset \mathbb{S}_\upsilon^{n-1}$ of υ_0, we call the image of these radii in the x-space a conoidal neighbourhood of the x-orbit defined by υ_0.

From the discussion above we see that a conoidal neighbourhood $\mathcal{C}$ of any x-orbit through the origin is blown up by π_H^{-1} to a domain $[0, \delta) \times V$, and the conic point $x = 0$ is blown up to a domain V on the hypersurface $\tau = 0$. The space $C^\infty([0, \delta) \times V)$ is pulled back to what we will denote in the sequel by $C_H^\infty(\mathcal{C})$, a proper subspace of $C^\infty(\mathcal{C})$. For any $u \in C_H^\infty(\mathcal{C})$, its pull back $\pi_H^* u$ can take different values along different x-orbits, hence different values at $x = 0$, i.e., at $\tau = 0$.

In the real analytic case, $A_H(\mathcal{C})$ is defined similarly. We shall consider our problem in either of the two spaces C_H^∞ and A_H.

Remark 2.5 For $m = 2$, the Jacobi matrix

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(\eta_1, \ldots, \eta_n)}$$

is regular, hence we have a local diffeomorphism near $x = 0$. This is just the case studied by Hadamard.

Remark 2.6 We also need estimates of $\partial\tau/\partial x_j$ and $\partial\upsilon_i/\partial x_j$. By methods similar to those used in the proof of Lemma 2.1, we obtain

$$\begin{aligned} \frac{\partial \tau}{\partial x_j} &= \tau^{2-m} F_{0,j}(\tau, \upsilon), \\ \frac{\partial \upsilon_i}{\partial x_j} &= \tau^{1-m} F_{i,j}(\tau, \upsilon), \end{aligned} \tag{2.6}$$

where $F_{0,j}$ and $F_{i,j}$ are C^∞ or real analytic functions.

3 Eiconal equation

In the sequel, we look for a solution $u(x)$ to the equation

$$A(x,D)u = \frac{\chi^\lambda(x)}{\Gamma(\lambda+1)}$$

of the form

$$u(x) = \Big(\sum_{k=0}^{\infty} \tau^k u_k(\tau,\upsilon)\Big) \frac{\chi^p(x)}{\Gamma(p+1)}, \tag{3.1}$$

where p is to be determined later. Changing the form of the solution to the present form (3.1) means that we switch to the (τ,υ) coordinates.

For second order equations, Hadamard [10] starts from the construction of the characteristic conoid composed of the bicharacteristic curves through the origin. He actually uses Fermat's principle, i.e., he works in the Lagrangian framework. What is a substitute of this principle for general partial differential operators (2.1)? We simply switch to the Hamiltonian framework and thus we can give Leray's result an elementary proof.

Theorem 3.1 (Leray [13, IV]) *Let $A(x,D)$ be a linear partial differential operator of (2.1) with C^∞ or real analytic coefficients of order $m \geq 2$ which satisfies the non-degeneracy condition (2.2). Then the characteristic conoid with vertex at $x=0$ can be written in the form $\chi(x)=0$, where $\chi(x)$ is C^∞ or real analytic near $x=0$ but $x=0$, such that*

$$H(x,\nabla\chi) = \frac{\chi}{m-1} \tag{3.2}$$

and

$$\nabla\chi = \tau\xi. \tag{3.3}$$

Here, $x = x(\tau,\upsilon)$ and $\xi = \xi(\tau,\upsilon)$ are solutions of (2.3), (2.4).

Proof. Introduce the energy integral

$$E(x) = \int_0^x H(x,\xi)\,dt.$$

Since $H(x,\xi)$ is a first integral of the Hamiltonian system, we deduce that $H(x,\xi) \equiv H(0,\upsilon)$ along the extremals. Since the non-degeneracy condition is satisfied, we can apply Legendre's transform $\dot{x} = \nabla_\xi H$ and obtain the Lagrangian

$$\begin{aligned} L(x,\dot{x}) &= \dot{x}\xi - H(x,\xi) \\ &= (m-1)\,H(x,\xi) \end{aligned}$$

(Euler's formula). It is easy to see that $L(x, \dot{x})$ is homogeneous in $\dot{x}$ of order $m/(m-1)$. Hence we use another integral as the eiconal integral in our case, namely

$$\begin{aligned} e(x) &= \int_0^x L(x, \dot{x})^{\frac{m-1}{m}}\, dt \\ &= (m-1)^{\frac{m-1}{m}} \int_0^x H(x, \xi)^{\frac{m-1}{m}}\, dt. \end{aligned} \tag{3.4}$$

Since a fractional power appears in (3.4), it is necessary to distinguish the cases when $H(0, v) \geq 0$ and when $H(0, v) \leq 0$. Divide $\mathbb{S}_v^{n-1}$ into two parts $\mathbb{S}_\pm$, such that $H(0, v) \geq 0$ for all $v \in \mathbb{S}_+$ and $H(0, v) \leq 0$ for all $v \in \mathbb{S}_-$ Denote the corresponding parts of the integrals $E(x)$ and $e(x)$ by $E_\pm(x)$ and $e_\pm(x)$, respectively. Since the integrands of $e_\pm$ (denoted here as $F(x, \dot{x})$) are homogeneous in $\dot{x}$ of order 1, the results in the calculus of variation [6, Ch. 2, p. 111] give

$$\begin{aligned} \frac{\partial e_+}{\partial t} &= F(x, \dot{x}) - \sum_{i=1}^n \dot{x}_i\, F_{\dot{x}_i}(x, \dot{x}) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial e_+}{\partial x_i} &= F_{\dot{x}_i} \\ &= \frac{m-1}{m}\, L(x, \dot{x})^{-\frac{1}{m}}\, \frac{\partial L}{\partial \dot{x}_i} \\ &= \frac{m-1}{m}\, L(x, \dot{x})^{-\frac{1}{m}}\, \xi_i. \end{aligned} \tag{3.5}$$

Substituting these expressions into the Hamiltonian $H(x, \xi)$, we get

$$\begin{aligned} L(x, \dot{x}) &= (m-1)\, H(x, \xi) \\ &= m^{-m}\, (m-1)^{1-m}\, L(x, \dot{x})\, H\Big(x, \frac{\partial e_+}{\partial x}\Big) \end{aligned}$$

or

$$H\Big(x, \frac{\partial e_+}{\partial x}\Big) = m^m\, (m-1)^{m-1}.$$

Set

$$\chi_+(x) = e_+(x)^{\frac{m}{m-1}}, \tag{3.6}$$

then

$$\begin{aligned} e_+(x) &= \chi_+(x)^{\frac{m-1}{m}}, \\ \frac{\partial e_+}{\partial x_i} &= \frac{m-1}{m}\, \chi_+(x)^{-\frac{1}{m}}\, \frac{\partial \chi_+}{\partial x_i} \end{aligned}$$

whence

$$H\Big(x, \frac{\partial \chi_+}{\partial x}\Big) = \frac{\chi_+}{m-1}.$$

We want to see what is the hypersurface $\chi_+(x) = 0$. For short, we call it the "light cone" hereafter. Since along the extremals

$$\begin{aligned} e_+(x) &= (m-1)^{\frac{m-1}{m}}\, t\, H_+(0,\upsilon)^{\frac{m-1}{m}}, \\ \chi_+(x) &= (m-1)\, \tau^m\, H_+(0,\upsilon) \end{aligned}$$

are fulfilled, we readily conclude that $\chi_+ = 0$ is composed of null-bicharacteristics and thus is the equation of the part of the characteritic conoid where $\upsilon \in \mathbb{S}_+$.

Replace $H(x,\xi)$ by $-H(x,\xi) = |H(x,\xi)|$ for $\upsilon \in \mathbb{S}_-$, and denote the corresponding χ by χ_-. Then $\chi_-(x) = 0$ is the equation of the other part of the characteristic conoid corresponding to $\upsilon \in \mathbb{S}_-$.

Setting

$$\chi(x) = \begin{cases} \chi_+(x), & \text{if } \ \upsilon \in \mathbb{S}_+, \\ \chi_-(x), & \text{if } \ \upsilon \in \mathbb{S}_-, \end{cases}$$

establisches formula (3.2).

By (3.5), we obtain

$$\begin{aligned} \xi_i &= \frac{m}{m-1}\, L(x,\dot{x})^{\frac{1}{m}}\, \frac{\partial e_+}{\partial x_i} \\ &= m(m-1)^{\frac{1}{m}-1}\, H(x,\xi)^{\frac{1}{m}}\, \frac{\partial e_+}{\partial x_i} \\ &= m(m-1)^{\frac{1}{m}-1}\, H(0,\upsilon)^{\frac{1}{m}}\, \frac{\partial e_+}{\partial x_i}. \end{aligned}$$

Using (3.6) we arrive at (3.3), as desired.

That $\chi \in C^\infty_H$ or A_H is clear from the expression

$$\chi(x) = (m-1)\, \tau^m\, H(0,\upsilon).$$

Q.E.D.

Remark 3.2 For $m = 2$, it is easy to see that χ is C^∞ or real analytic even at $x = 0$, because we can use Morse's lemma to write

$$\chi(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=1}^{q} x_{p+i}^2$$

with $p + q = n$.

We thus deduce that although it is impossible to reduce a second order equation to its normal form, it is possible and very useful to reduce its characteristic conoid to a quadratic surface.

We now study the properties of the function $\chi(x)$. First, it has a unique singular point at $x = 0$.

Theorem 3.3 *Under the assumptions of Theorem 3.1, $x = 0$ is the unique singular point of $\chi(x)$.*

Proof. Since $\nabla\chi = \tau\xi$, $\xi = \upsilon + o(t)$ and $|\upsilon| = 1$, it follows that $\nabla\chi(x) = 0$ near $x = 0$ only for $\tau = 0$.

Q.E.D.

We will need the following technical result on the derivatives of $\chi(x)$. We actually have

Theorem 3.4 *Under the above assumptions, we have*

$$\partial^\alpha \chi(x) = \tau^{m-(m-1)|\alpha|}\, F_\alpha(\tau, \upsilon) \tag{3.7}$$

where F_α is a C^∞ or real analytic function of (τ, υ). In the x-coordinates, this just amounts to saying that

$$\partial^\alpha \chi(x) = |x|^{\frac{m}{m-1}}\, F_\alpha(x), \tag{3.8}$$

where $F_\alpha(x)$ is of class C^∞_H or A_H.

Proof. The equality (3.7) evidently holds for $\alpha = 0$. Assume that it is true for all α with $|\alpha| \le A$, then for $|\alpha| = A$ we obtain

$$\partial_{x_j}\partial_x^\alpha \chi(x) = \Big(\frac{\partial\tau}{\partial x_j}\frac{\partial}{\partial\tau} + \sum_{i=1}^n \frac{\partial\upsilon_i}{\partial x_j}\frac{\partial}{\partial\upsilon_i}\Big)\left(\tau^{m-(m-1)|\alpha|}\chi_\alpha(\tau,\upsilon)\right).$$

Now, (3.7) follows from (2.6), and (3.8) follows from (3.7) and Lemma 2.3.

Q.E.D.

The special case $|\alpha| = 2$ is particularly useful later.

Theorem 3.5 *Under the same assumptions,*

$$\sum_{j=1}^n \frac{\partial^2 H}{\partial\xi_i\partial\xi_j}(0,\eta)\,\frac{\partial^2\chi}{\partial x_j\partial x_k} = \delta_{i,k} + o(\tau).$$

Proof. From (2.5) we know that both x and $\tau\xi$ are C^∞ or real analytic functions of η. The Hamiltonian equation (2.3) together with initial condition (2.4) yields

$$x_i = \frac{\partial}{\partial \xi_i} H(x, \tau\xi) + o(1)\, t.$$

Since $\nabla\chi = \tau\xi$, we get

$$x_i = \frac{\partial}{\partial \xi_i} H(x, \nabla\chi) + o(1)\, \tau^{m-1}.$$

Differentiating both sides with respect to x_k gives the result.

Remark 3.6 In the proof above we should have replaced $o(1)$ by suitable C^∞ or real analytic functions. This is omitted for simplicity.

4 Transport equations

As mentioned above, we are looking for a solution of the form

$$u = \Big(\sum_{k=0}^{\infty} \tau^k u_k(\tau, \upsilon) \Big) \frac{\chi^p(x)}{\Gamma(p+1)}$$

for the equation

$$A(x, D) u = f(x) \frac{\chi^\lambda}{\Gamma(\lambda+1)}. \tag{4.1}$$

We use the (τ, υ) coordinates in (3.1), for the characteristic mapping is not smooth at $x = 0$, and so $\chi(x) = O(1)\, |x|^{\frac{m}{m-1}}$ fails to be smooth. However, in the (τ, υ)-coordinates $\chi(x) = \tau^m H(0, \upsilon)$ is smooth. This is why we introduced the spaces C^∞_H and A_H. These difficulties arise because the order m of the differential operator may be greater than 2. In fact, we will see in the sequel that the case $m > 2$ causes some modifications in the original procedure of Hadamard. Hence, we will work in the (τ, υ)-frame, on the one hand, and pay attention to such modifications, which we call "perturbations," on the other hand.

Let us decompose $A(x, D)$ into the sum of homogeneous parts $A_{m-j}(x, D)$, namely

$$A(x, D) = \sum_{j=0}^{m} A_{m-j}(x, D).$$

We will consider the action of diverse parts on a typical term $U_k = \tau^k u_k(\tau, \upsilon)$ in (4.1). To this end, we need two lemmas in the x-space.

Lemma 4.1 (Generalised Leibniz's Lemma) *Let $A(x,D)$ be an arbitrary linear partial differential operator with sufficiently smooth coefficients. Then*

$$A(x,D)\,(uv) = \sum_{\alpha\in\mathbb{Z}_+^n} \frac{1}{\alpha!}\, D^\alpha u\, (\partial_\xi^\alpha A)(x,D)v.$$

Applying this formula, we readily get

$$A_m(x,D)\,u(x)\frac{\chi^p(x)}{\Gamma(p+1)} = \sum_{\beta\in\mathbb{Z}_+^n} \frac{1}{\beta!}\, D^\beta u(x)\, \Big((\partial_\xi^\beta A_m)(x,D)\frac{\chi^p(x)}{\Gamma(p+1)}\Big). \quad (4.2)$$

In order to compute a factor of the type $B(x,D)\,\chi^p/\Gamma(p+1)$ we need a lemma which is a generalisation of Faa de Bruno's formula, cf. [8], Chap. II, Ex.16, p. 78.

Lemma 4.2 *Let $A(x,D)$ be a homogeneous differential operator of order m, f a function of one variable and $\varphi(x)$ a function of n variables, both f and φ being smooth enough. Then*

$$A\,f(\varphi) = \sum \frac{1}{k!}\,(\partial_\xi^{k_1\alpha^1+\ldots+k_n\alpha^n} A)\, f^{(|k|)}(\varphi)\Big(\frac{1}{\alpha^1!}D^{\alpha^1}\varphi\Big)^{k_1}\ldots\Big(\frac{1}{\alpha^n!}D^{\alpha^n}\varphi\Big)^{k_n},$$

where the sum is over all multi-indices $\alpha = k_1\alpha^1+\ldots+k_n\alpha^n$ of length $|\alpha| = m$ with $\alpha^1,\ldots,\alpha^n \in \mathbb{Z}_+^n$ and $k_1,\ldots,k_n$ non-negative integers whose sum $|k|$ does not exceed m.

The proof of this lemma is also omitted, since the main idea of the proof is the same as that in [8]. Although it is a little technical, we need not explicit coefficients therein.

Now apply this lemma to diverse terms in (4.2). We start with the case $\beta = 0$ to consider

$$\begin{aligned} A_m\,\frac{\chi^p}{\Gamma(p+1)} &= \sum_{|\alpha|=m} a_\alpha\, D^\alpha\,\frac{\chi^p}{\Gamma(p+1)} \\ &= \sum \frac{\alpha!}{k!}\, a_\alpha\, \frac{\chi^{p-|k|}}{\Gamma(p-|k|+1)}\Big(\frac{1}{\alpha^1!}D^{\alpha^1}\chi\Big)^{k_1}\ldots\Big(\frac{1}{\alpha^n!}D^{\alpha^n}\chi\Big)^{k_n}, \end{aligned} \quad (4.3)$$

the latter sum being over all multi-indices $\alpha = k_1\alpha^1+\ldots+k_n\alpha^n$ of length $|\alpha| = m$ with $\alpha^1,\ldots,\alpha^n \in \mathbb{Z}_+^n$ and $k_1,\ldots,k_n$ non-negative integers satisfying $|k| \le m$.

Pick any summand in (4.3) and denote by l_1 the number of multi-indices α^i with $|\alpha^i| = 1$, etc., by l_m the number of multi-indices α^i with $|\alpha^i| = m$. It is easy to see that

$$l_1+\ldots+l_m = |k|, \quad (4.4)$$

and since the total order of differentiation in x is m, we have

$$l_1 + 2l_2 + \ldots + ml_m = m.$$

When $|k| = m$, the only non-negative integral solution of (4.4) is obviously $(m, 0, \ldots, 0)$. Hence, the part of (4.3) corresponding to $|k| = m$ is

$$\begin{aligned}\Big(\sum_{|\alpha|=m} a_\alpha(x)(-\imath\nabla\chi)^\alpha\Big)\frac{\chi^{p-m}}{\Gamma(p-m+1)} &= \frac{\chi}{m-1}\,\frac{\chi^{p-m}}{\Gamma(p-m+1)}\\ &= \frac{p-m+1}{m-1}\,\frac{\chi^{p-m+1}}{\Gamma(p-m+2)}.\end{aligned}$$

For $|k| = m-1$, the only solution is $(m-2, 1, 0, \ldots, 0)$, and so the corresponding part of (4.3) is

$$\begin{aligned}&\Big(\sum_{i,j=1}^{n}\frac{1}{2!}\,(\partial\xi_i\partial\xi_j A_m)(x, -\imath\nabla\chi)\,D_{x_i}D_{x_j}\chi\Big)\frac{\chi^{p-m+1}}{\Gamma(p-m+2)}\\ = \;&\Big(\frac{n}{2} + \tau\,F(\eta)\Big)\,\frac{\chi^{p-m+1}}{\Gamma(p-m+2)}.\end{aligned}$$

Here we made use of Theorem 3.5. The term $\tau\,F(\eta)$ will be absorbed into higher order terms later.

As for the remaining parts corresponding to $|k| \le m-2$, we can estimate them by Theorem 3.4, thus obtaining

$$\tau^{m|k|-m(m-1)}\,C(\eta)\,\frac{\chi^{p-|k|}}{\Gamma(p-|k|+1)}.$$

These terms can be written as

$$C(\eta)\,(H(0,\upsilon))^{m-|k|-1}\,\frac{\chi^{p-m+1}}{\Gamma(p-m+1)}.$$

This is one of the perturbations caused by the assumption $m \ge 2$. When $m = 2$, it does not appear. It vanishes on the light cone surface, i.e., there is no such perturbation on the light cone surface. We can actually prove that $C(\eta)\,(H(0,\upsilon))^{m-|k|-1} \equiv 0$. Indeed, it is easy to see that it is homogeneous in η of fisrt order and smooth up to $\eta = 0$. So it must be a polynomial of first degree in η. But it possesses a factor $H(0,\upsilon)$, and so it can be only 0 identically.

We next consider the case $|\beta| = 1$ in (4.2), namely

$$\sum_{j=1}^{n} D_{x_j}u(x)\,\Big((\partial_{\xi_j}A_m)(x,D)\frac{\chi^p(x)}{\Gamma(p+1)}\Big).$$

Write

$$\begin{aligned}(\partial_{\xi_j} A_m)(x,D) &=: B(x,D)\\ &= \sum_{|\alpha|=m-1} b_\alpha(x) D^\alpha.\end{aligned}$$

It is easy to see that

$$\begin{aligned}B(x,D)\frac{\chi^p}{\Gamma(p+1)} &= \frac{\chi^{p-m+1}}{\Gamma(p-m+2)} \sum_{|\alpha|=m-1} b_\alpha(x)(D_1\chi)^{\alpha_1}\ldots(D_n\chi)^{\alpha_n} + R\\ &= \frac{\chi^{p-m+1}}{\Gamma(p-m+2)} (\partial_{\xi_j} A_m)(x,-\imath\nabla\chi) + R, \qquad (4.5)\end{aligned}$$

where

$$R = \sum \frac{\alpha!}{k!} b_\alpha \frac{\chi^{p-|k|}}{\Gamma(p-|k|+1)} \Big(\frac{1}{\alpha^1!} D^{\alpha^1}\chi\Big)^{k_1} \ldots \Big(\frac{1}{\alpha^n!} D^{\alpha^n}\chi\Big)^{k_n},$$

the sum is over all multi-indices $\alpha = k_1\alpha^1 + \ldots + k_n\alpha^n$ of length $|\alpha| = m-1$ with $\alpha^1,\ldots,\alpha^n \in \mathbb{Z}_+^n$ and $k_1,\ldots,k_n$ non-negative integers satisfying $|k| \le m-2$.

The first term can be written as

$$\frac{\chi^{p-m+1}}{\Gamma(p-m+2)}\, t\,(\partial_{\xi_j} A_m)(x,-\imath\xi),$$

for $\partial_{x_i}\chi = \tau\xi_i$, while the second term gives another perturbation. In fact, it is easily verified that

$$\begin{aligned}R &= \sum_{|\alpha|=m-1} b_\alpha(x) \sum_{k=1}^{m-2} C_{\alpha,k}(\eta)\, \tau^{m(p-m+1)} H(0,\upsilon)^{p-k}\\ &= \tau^{m-1}\Big(\sum_{k=1}^{m-2} C_k(\eta)\, H(0,\upsilon)^{m-k-1}\Big)\frac{\chi^{p-m+1}}{\Gamma(p-m+2)}\\ &= \tau^{m-1} F(\eta)\, \frac{\chi^{p-m+1}}{\Gamma(p-m+2)}.\end{aligned}$$

The first term of (4.5) is a differential operator along the x-orbit while the remainder R is a perturbation containing a factor τ^{m-1}. Still, one should note that there is also a factor $H(0,\upsilon)$, so there will be no perturbation on the light cone surface.

The remaining part of (4.2) corresponding to $|\beta| \ge 2$ is easy to evaluate. Analysis similar to that in treating R actually shows that

$$\frac{1}{\beta!}\,(\partial_\xi^\beta A_m)(x,D)\frac{\chi^p(x)}{\Gamma(p+1)}$$

$$
\begin{aligned}
&= \sum c_\alpha(x)\, \frac{\chi^{p-|k|}}{\Gamma(p-|k|+1)} \Big(\frac{1}{\alpha^1!} D^{\alpha^1}\chi\Big)^{k_1} \ldots \Big(\frac{1}{\alpha^n!} D^{\alpha^n}\chi\Big)^{k_n} \\
&= \tau^{|\beta|(m-1)}\, C(\eta)\, H(0,\upsilon)^{m-|\beta|+1} \frac{\chi^{p-m+1}}{\Gamma(p-m+2)},
\end{aligned}
$$

the sum being over all multi-indices $\alpha = k_1\alpha^1 + \ldots + k_n\alpha^n$ of length $m-|\beta|$ with $\alpha^1, \ldots, \alpha^n \in \mathbb{Z}_+^n$ and $k_1, \ldots, k_n$ non-negative integers satisfying $|k| \le m - |\beta|$. By Lemma 2.1,

$$
D_x^\beta u = \sum_{j=1}^{|\beta|} \tau^{j(2-m)}\, L_j(\eta, D_\eta) u.
$$

Combinging this with the former formula, we conclude that

$$
\sum_{\substack{\beta\in\mathbb{Z}_+^n \\ 2\le|\beta|\le m}} \frac{1}{\beta!}\, D^\beta u(x) \Big((\partial_\xi^\beta A_m)(x,D) \frac{\chi^p(x)}{\Gamma(p+1)} \Big) = \tau^{|\beta|} \Big(L(\eta, D_\eta) u \Big) \frac{\chi^{p-m+1}(x)}{\Gamma(p-m+2)}.
$$

We can treat in the same manner those parts $A_{m-k}(x,D)$ where $k > 0$, acting on $u\,\chi^p/\Gamma(p+1)$, and obtain our final formula

$$
\begin{aligned}
&A(x,D)\, u(x) \frac{\chi^p}{\Gamma(p+1)} \\
&= \frac{\chi^{p-m+1}}{\Gamma(p-m+2)} \Big(\frac{1}{m-1} \tau \frac{du}{d\tau} + \Big(\frac{p-m+1}{m-1} + \frac{n}{2} \Big) u + \sum_{j=1}^{m} \tau^j L_j(\eta, D_\eta) u \Big),
\end{aligned}
\tag{4.6}
$$

where $L_j(\eta, D_\eta)$ are differential operators of order j with C^∞ or real analytic coefficients.

Our goal is to solve (4.1) in the form (3.1), and we first replace u in (4.6) by u_0. Then, in order that (4.1) could be satisfied, we should take $p - m + 1 = \lambda$, or

$$
p = \lambda + m - 1.
$$

Next, comparing the terms of the same order in τ on both sides of (4.1), we should also take u_0 to be a solution of the equation

$$
\frac{1}{m-1} \tau \frac{d}{d\tau} u_0 + \Big(\frac{\lambda}{m-1} + \frac{n}{2} \Big) u_0 = f(\eta). \tag{4.7}
$$

For the other terms $\tau^k u_k(\eta)$ in (4.1) we have similar equations

$$
\tau \frac{d}{d\tau} u_k + \Big(\lambda + k + \frac{n}{2}(m-1) \Big) u_k = L_k(u_{k-1}, \ldots, u_{k-m}), \tag{4.8}
$$

where L_h is a linear expression of u_{k-j} and its derivatives up to order j with C^∞ or real analytic coefficients, and $u_{k-j} \equiv 0$ when $k < j$.

The equations (4.7) and (4.8) are transport equations for u_k's. We are going to look for their solutions which are "in general" bounded and non-vanishing near $\eta = 0$. This will be done in the next section. Here we sum up our result as

Theorem 4.3 *Equation (4.1) possesses a formal solution (3.1) in which $\tau^k u_k$ are determined by the transport equations (4.7) and (4.8).*

Remark 4.4 If $m = 2$, there is no perturbations. The powers τ^k can be combined with the factors $H(0, \upsilon)$ to give χ. In this case formula (3.1) becomes

$$u(x) = \sum_{k=0}^{\infty} u_k(x) \frac{\chi^{\lambda+k+m-1}}{\Gamma(\lambda + k + m)}$$

agreeing with Hadamard's original result.

5 Convergence and asymptotics of the formal solutions

In the sections above we derived the eiconal and transport equations, and hence the formal solutions of equation (4.1) once we can solve the transport equations (4.7) and (4.8). This can be done easily. For instance, for (4.7) we get

$$\begin{aligned} u_0(\eta) &= (m-1)\,\tau^{-A} \int_0^{\tau} \sigma^A f(\sigma\upsilon)\,\frac{d\sigma}{\sigma} \\ &\overset{\sigma=\tau s}{=} (m-1) \int_0^1 s^A f(s\eta)\,\frac{ds}{s}, \end{aligned} \tag{5.1}$$

where $A = \lambda + (n/2)(m-1)$. The integration is taken along the x-orbit defined by the initial data υ, and so υ is treated as a constant parameter. The integral diverges when

$$\begin{aligned} \Re A &= \Re\lambda + \frac{n}{2}(m-1) \\ &\leq 0. \end{aligned}$$

In this case, it should be understood in the distributional sense, i.e., as a Riemann-Liouville integral. The integral (5.1) obviously defines a C^∞ or real analytic function, since $f(\eta)$ is assumed to be of such kind. As for $u_k(\eta)$, we have

$$u_k(\eta) = \int_0^1 s^{A+k} L_k(u_{k-1}, \ldots, u_{k-m})(s\eta)\,\frac{ds}{s}$$

also in the sense of Riemann-Liouville. Each u_k is also a C^∞ or real analytic function.

It is worth pointing out that if $A \le 0$ is integer, there will appear terms containing $\log \tau$. We can treat such cases as in [3]. So, we will not handle these special cases hereafter.

In the sequel, we shall distinguish the C^∞ and real analytic cases. Consider first the real analytic case.

Lemma 5.1 *As defined above, the solution $u = u(\tau, \upsilon)$ of the Fuchsian equation*

$$\tau \frac{du}{d\tau} + Au = f(\tau, \upsilon)$$

is majorised by

$$C \sup_{s \in [0,1]} |f(s\tau, \upsilon)|,$$

where C is a suitable constant. In fact, $C = 1/\Re A$ if $\Re A > 0$.

Proof. This follows from the formula

$$u(\tau, \upsilon) = \int_0^1 s^A f(s\tau, \upsilon) \, \frac{ds}{s}$$

cf. (5.1), by an easy computation.

Q.E.D.

We apply this lemma to the transport equations (4.7) and (4.8). Induction in $k = 0, 1, \ldots$ yields in a familiar manner

$$|u_k(\eta)| \le \frac{C^{k+1} M}{(1 - \tau/r)^{km+1} (1 - |\upsilon - \upsilon^0|/r)^{km+1}},$$

with $r \le 1$ and sufficiently large constants M and C independent of (τ, υ). We thus obtain

Theorem 5.2 *If $|\tau|$ is small enough, then the formal solution (3.1) converges to a real analytic solution of equaton (4.1).*

We now turn to the C^∞ case. It is very unlikely that one might prove a rather general convergence result, since a large class of C^∞ partial differential equations is not locally solvable, while the formal solution (3.1), when converges, implies that a C^∞ solution exists provided the right-hand side is C^∞. But in this case we can prove at least the existence of an asymptotic solution. In fact, the formal solution now is not a formal Taylor series, since its coefficients depend on τ. But if we expand the coefficients $u_k(\eta)$ as a formal power series

$$u_k(\eta) \sim \sum_{j=0}^{\infty} \tau^j \, u_{k,j}(\upsilon),$$

then (3.1) becomes a formal Taylor series. Using Borel's technique, we can construct a C^∞ function $u(x)$ with (3.1) as its asymptotic expansion. We call it an asymptotic solution of (4.1).

Theorem 5.3 *In the C^∞ case, equation (4.1) possesses an asymptotic solution $u(x)$ with (3.1) as its asymptotic expansion.*

The advantage of using asymptotic solutions lies in the fact that if the right-hand side $f(\tau, \upsilon)$ is compactly supported in a conoidal neighbourhood, then so are the solutions of transport equations (4.7) and (4.8). Hence, the asymptotic solution $u(x)$ vanishes identically, not only asymptotically, outside the neighbourhood. This is of importance for geometric optical problems, cf. [4].

6 Classification of fundamental solutions

We make use of the solution (3.1) obtained above to construct distributional solutions to

$$A(x, D)u = \text{Dirac type distribution.} \tag{6.1}$$

By a Dirac type distribution is meant $\delta(x)$ or a distribution supported on the characteristic conoid $\chi(x) = 0$. The former case gives a response to a point charge or mass at the origin while the latter corresponds to the propagation of wave fronts according to the Huygens' principle. The right-hand side can even be linear combinations of distributions mentioned above, and their derivatives. We thus deduce that fundamental solutions actually constitute a rather large class of distributions different in their nature. Hence we give their classification as follows.

If the right-hand side of (6.1) is a linear combination of δ and its derivatives, we call the solution (3.1) a Hadamard-Dirac fundamental solution. If it is a linear combination of distributions supported on $\chi(x) = 0$, we call (3.1) a Hadamard-Huygens fundamental solution. And if it is a sum of both, we call (3.1) a Hadamard mixed fundamental solution.

The main idea for constructing such fundamental solutions is as follows. Replace $\chi^\lambda(x)$ on the right-hand side of (4.1) by $\chi_+^\lambda(x)$ which is supported in the product $[0, \delta) \times U_+$ in the (τ, υ) space. Then, (3.1) with $\chi(x)$ replaced by $k_+(x)$ still gives a solution supported in the same conoidal neighbourhood. But $\chi_+^\lambda(x)$ is a distribution-valued meromorphic function of λ, cf. [2]), then so is the solution (3.1). Here the non-degeneracy condition (2.2) plays a crucial role.

If regarded as a function, χ_+^λ just amounts to

$$\chi_+^\lambda(x) = (m-1)^\lambda \, \tau^{m\lambda} \, H_+^\lambda(0, \upsilon)$$

for $\upsilon \in U_+$. When thought of as a distribution in the (τ, υ) space, more precisely, if υ lies in a neighbourhood of $(1, 0, \ldots, 0)$, such that

$$\upsilon_1 = \sqrt{1 - \upsilon_2^2 - \ldots - \upsilon_n^2},$$

the independent variables being now $(\tau, \upsilon_2, \ldots, \upsilon_n)$ with $\upsilon_2^2 + \ldots + \upsilon_n^2 < \varepsilon^2 < 1$, the right-hand side of (4.1) should be written as

$$C(\tau, \upsilon)\, \tau^{m\lambda} \frac{H_+^\lambda(0, \upsilon)}{\Gamma(\lambda + 1)} \Big| \det \frac{\partial(x_1, \ldots, x_n)}{\partial(\tau, \upsilon_2, \ldots, \upsilon_n)} \Big|.$$

Here, the constant $(m-1)^\lambda$ is absorbed into $C(\tau, \upsilon)$. By the method used in the proof of Lemma 2.1, we see that

$$\Big| \det \frac{\partial(x_1, \ldots, x_n)}{\partial(\tau, \upsilon_2, \ldots, \upsilon_n)} \Big| = \Delta(\tau, \upsilon)\, \tau^{m-2+(n-1)(m-1)}$$

where $\Delta(\tau, \upsilon)$ is C^∞ or real analytic on $[0, \delta) \times U_+$ and $\Delta(\tau, \upsilon) > \Delta_0 > 0$. Hence, as a distribution, the right-hand side of (4.1) is

$$C(\tau, \upsilon)\, \tau^{m(\lambda+n)-(n+1)} \otimes \frac{H_+^\lambda(0, \upsilon)}{\Gamma(\lambda + 1)} \tag{6.2}$$

for $\upsilon \in U_+$, with $C(\tau, \upsilon)$ a C^∞ or real analytic multiplier.

Theorem 6.1 *The right-hand side of (4.1) with χ replaced by χ_+ is a distribution-valued meromorphic function of λ divided by another meromorphic function $\Gamma(\lambda+1)$, and*

1) The values λ of $\mathcal{P}_1 = \{\lambda \notin -\mathbb{N} : m(n+\lambda) - (n+1) \in -\mathbb{N}\}$ are simple poles with residues

$$C(\tau, \upsilon)\, \delta^{(k)}(\tau) \otimes H^{n(1-m)/m+(1-k)/m}.$$

2) The values λ of $\mathcal{P}_2 = \{\lambda \in -\mathbb{N} : m(n+\lambda) - (n+1) \notin \mathbb{N}\}$ are regular points with first Taylor coefficient

$$C(\tau, \upsilon)\, \tau^{m(n-k)-(n+1)} \otimes \delta^{(k-1)}(H).$$

3) The values λ of $\mathcal{P}_3 = \{\lambda \in -\mathbb{N} : \lambda = n(1-m)/m + (1-k)/m,\ k \in \mathbb{N}\}$ are regular points with first Taylor coefficient

$$C(\tau, \upsilon)\, \delta^{(-\lambda)}(\tau) \otimes \delta^{(k-1)}(H).$$

Proof. Consider diverse factors in (6.2). The first is a multiplier. Since $\tau \in [0, \delta)$, the second factor

$$\tau^{m(n+\lambda)-(n+1)} = \tau_+^{m(n+\lambda)-(n+1)}$$

has simple poles when $m(n+\lambda) - (n+1) \in -\mathbb{N}$, and its residue is

$$\frac{(-1)^k}{mk!}\,\delta^{(k)}(\tau).$$

The last factor is $H^\lambda(0, \upsilon) = H_+^\lambda(0, \upsilon)$ when $\upsilon \in U_+$. Just as in the proof of Lemma 2.2, we prove that the surface $H(0, \upsilon) = 0$ has no critical points. So we are in a position to introduce the distribution $H_+^\lambda(0, \upsilon)$ with simple poles at $\lambda = -k$, where $k \in \mathbb{N}$, and the residues $C(\tau, \upsilon)\,\delta^{(k-1)}(H)$ at these poles, respectively. But $\lambda = -k$ with $k \in \mathbb{N}$ are also simple poles for the denominator $\Gamma(\lambda + 1)$, hence they are regular points for the right-hand side, and the first Taylor coefficients are just the quotients of both residues. The theorem is proved.

Q.E.D.

For a thorough discussion of distribution x_+^λ we refer the reader to [9]. We now formulate our final theorem.

Theorem 6.2 *The procedure in Sections 3, 4 and 5 gives distributional fundamental solutions. More precisely,*

1) $\lambda \in \mathcal{P}_1$ *leads to Hadamard-Dirac fundamental solutions;*
2) $\lambda \in \mathcal{P}_2$ *leads to Hadamard-Huygens fundamental solutions;*
3) $\lambda \in \mathcal{P}_3$ *leads to Hadamard mixed fundamental solutions.*

Proof. By the constructions of Sections 3 and 4, the solutions (whether convergent or asymptotic) are also distribution-valued meromorphic functions of λ. We write $u(x, \lambda)$ for them. They have the same poles and regular points as the right-hand side.

If λ_0 is a regular point, we get

$$A(x, D)u(x, \lambda_0) = f(x)\,\frac{\chi_+^{\lambda_0}(x)}{\Gamma(\lambda_0 + 1)}.$$

This is just the cases 2) and 3).

If λ_0 is a pole, as in the case 1), we have

$$A(x, D)\,\mathrm{res}\Big(u(x, \lambda)\Big)|_{\lambda=\lambda_0} = f(x)\,\mathrm{res}\Big(\frac{\chi_+^\lambda(x)}{\Gamma(\lambda + 1)}\Big)|_{\lambda=\lambda_0}.$$

Theorem 5.3 gives the desired result. Q.E.D.

If A is elliptic then any fundamental solution of A gives us the kernel of an inverse A^{-1} as a pseudodifferential operator. The construction of a fundamental solution for a non-degenerate differential equation leads to a much larger operator calculus.

7 Condition of non-degeneracy

The condition of non-degeneracy plays a crucial role in the whole paper. But in a complex domain there always exist non-zero ξ, such that $\mathrm{Hess}_\xi\, H(0,\xi) = 0$, for $\mathrm{Hess}_\xi\, H(0,\xi)$ is a polynomial. One might conjecture that the non-degeneracy condition can be weakened to

$$\mathrm{Hess}_\xi\, H(0,\xi) \neq 0$$

for all $\xi \neq 0$ satisfying $H(0,\xi) = 0$. However, this fails, as the following enlightening counter-example shows.

Let

$$A(x,D) = \partial_1^4 - \partial_2^4.$$

Then,

$$\mathrm{Hess}_\xi\, H(0,\xi) = 144\, \xi_1^2 \xi_2^2,$$

which vanishes for $\xi_1 = 0$ or $\xi_2 = 0$. From $H(0,\xi) = 0$ it follows that $\xi_1 = \pm\xi_2$, and so

$$\mathrm{Hess}_\xi\, H(0,\xi) = 144\, \xi_1^4$$

is non-zero when $\xi \neq 0$.

The characteristic mapping now is

$$\begin{array}{rcl} x_1 & = & 4\eta_1^3, \\ x_2 & = & 4\eta_2^3, \end{array}$$

which obvioisly induces diffeomorphisms between the four quadrants of the (x_1,x_2)-plane and those of the (η_1,η_2)-plane. The neighbourhood U in Theorem 2.4 should now be replaced by these quadrants, and we can obtain the eiconal integral $e(x)$ and $\chi(x)$ only in each of them. We get

$$\begin{array}{rcl} \chi(x) & = & 3\,(\eta_1^4 - \eta_2^4) \\ & = & 3 \cdot 4^{-4/3}\,(x_1^{4/3} - x_2^{4/3}) \end{array}$$

which is multi-valued, and so it is necessary to consider its uniformisation [13, IV]. We thus conclude that the theory above fails when the non-degeneracy condition is violated.

This phenomenon is closely related to the structure of the algebraic variety $\mathrm{Hess}_\xi\, H(0,\xi) = 0$. Atiyah in [1] pointed out the significance of Hironaka's desingularisation theorem in the study of partial differential equations. Since most of the difficulties in the general theory of partial differential operators arise from the singularities of the characteristic variety, it is quite natural to expect Hironaka's theorem to be relevant. Hironaka's result, in the version of

Atiyah, reads roughly as follows: For a real analytic algebraic variety $f(y) = 0$, one can always find local coordinates $y = \varphi(x)$, such that

$$\varphi^* f\,(x) = k(x) \prod_{j=1}^{n} x_j^{N_j},$$

where N_j are non-negative integers and $k(x)$ is a non-vanishing real analytic function. Therefore, a neighbourhood of $f^{-1}(0)$ possesses a stratification with an n-dimensional stratum $\cap_{j=1}^{n}\{x_j \neq 0\}$, an $(n-1)$-dimensional stratum $\cup_{i=1}^{n} \cap_{j=1}^{n} \{x_i = 0,\, x_j \neq 0 \text{ for } j \neq i\}$, etc., and a 0-dimensional stratum $\{x = 0\}$. For the non-degenerate case, this variety $H(0,\eta) = \tau^m\, H(0,\upsilon) = 0$ does not contain k-dimensional strata for $k = 1, \ldots, n-2$.

Hence, a deep-going development of differential analysis on real algebraic varieties is required if we wish to extend the theory above to include degenerate cases.

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