The Hypoellipticity of Differential Forms on Closed Manifolds

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Abstract

In this paper we consider the hypo-ellipticity of differential forms on a closed manifold.The main results show that there are some topological obstruct for the existence of the differential forms with hypo-ellipticity.

Key Words Hypoellipticity, Form, Integrability, Diophantine Approximation.

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§1 Introduction

Let L be a differential operator acts on the Schwartz distribution based on a manifold $\mathcal{M}$. The (local) hypo-ellipticity of the operator L means that one can claim a distribution φ is smooth near a point $p \in \mathcal{M}$ whenever the action of the operator L on the distribution φ, $L\varphi$ is smooth near the point . The typical examples of the hypo-elliptic differential operators are the Laplacian, or generally the elliptical operator of constant coefficients. An important class of hypo-elliptic operators which are not elliptical is the Hörmander's sum of square[9]. All of those examples are of even order. Indeed, a differential operator of real coefficients must be order of even provided it has local hypo-ellipticity, see[14].

The first example of hypo-elliptical operator with real coefficient of order one was given by S.J.Greenfield and N.R.Wallach in 1972. Of cause, we

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have a different meaning of the hypo-ellipticity this time, the global hypo-ellipticity. For the details see definition below. After that there are many works concerned the global hypo-ellipticity of the vector fields on manifolds, see[1, 2, 3, 4, 6, 7, 8, 12] and references cited therein. These works shows that the hypo-ellipticity of differential operators of first order is involved into the Diophantine (and simultaneous Diophantine) approximation, the integrability and ergodicity of the dynamic systems and the topological properties.

In this paper, we consider hypo-ellipticity of differential operators which act on the current. Let Ω^p be the (smooth) $p-$form on a manifold $\mathcal{M}$ of dimension n and $\mathcal{D}_p$ be the dual space of Ω^p. The element of $\mathcal{D}_p$ is called $p-$current. For the theory of current see, for example [5] or [11]. Fixed a $p-$form ω, we have a nature differential operator L_ω from Ω^{n-p-1} to Ω^n given by

$$L_\omega(\varphi) = \omega \wedge d\varphi. \tag{1}$$

Correspondingly, we have an under-determined system $L_\omega(\varphi) = \mu$ if $p < n-1$. The dual operator of $(-1)^p L$ is

$$T_\omega(\eta) = \partial\eta \wedge w - \eta \wedge d\omega, \quad \forall \eta \in \mathcal{D}_n(M) \tag{2}$$

which is a differential operator of first order from $\mathcal{D}_n$ to $\mathcal{D}_{n-p-1}$. We then have an over-determined system $T_\omega = \psi$.

Definition 1 *Call a form ω be hypo-elliptic if there is a smooth function $\widetilde{\eta}$ on $\mathcal{M}$ such that $\eta = \widetilde{\eta}[\mathcal{M}]$ whenever $T_\omega(\eta)$ is smooth.*

Remark 1: If ω is a closed $1-$form, then the definition of hypo-ellipticity here is same as in [12].

Let us mention some examples which motivates us to consider the hypoellipticity of forms.

Example 1: (Greenfield and Wallach 1972) On torus T^2, let $\omega = dx + \Lambda dy$ with constant Λ. Then

$$T_\omega : \mathcal{D}_2 \Rightarrow \mathcal{D}'.$$

The form ω is hypoelliptic if and only if the real number Λ is irrational and non-Liouville, i.e. there are constant C_0, N_0 such that the inequality

$$|k + l\Lambda| \geq \frac{C_0}{(|k| + |l|)^{N_0}}$$

hold for every $(k, l) \in Z^2 \setminus \{0\}$.

Example 2: (A. Bergamasco, P. Cordaro, P. Malagutti, 1993) Let α be a closed form on N, and $\omega = d\theta + \alpha \in \Omega^1(S^1 \times N)$, then ω is hypoelliptic if and only if the form α is irrational and non-Liouville.

Example 3: Also on torus T^2, $\omega = dx + \lambda(x, y)dy$. Then ω is hypoelliptic if and only if the rotation number Λ of the system

$$\frac{dx}{dy} = -\lambda(x, y)$$

is an irrational and non-Liouville number.

Example 4: On a contact manifold $\mathcal{M}$, the contact form α is a hypoelliptic form because of maximal non-integrability of the form α.

We will discuss the integrability of the hypo-elliptic forms in section 2. Some extensions of Greenfield and Wallach's results to higher order both in the dimension and the forms are given in section 3.

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§2 The Integrability of the Hypoelliptic Differential Form

The top order of the operator L_ω or T_ω is $p = n - 1$. In this case the hypoelliptic theory is easier than general case. We first mention a fact that the differential operator L_ω acts on the smooth function and then the image of T_ω belongs to the Schwartz distribution. To be concise, assume that the manifold $\mathcal{M}$ is oriented. Otherwise we take its double covering. Let V_0 be a $n-$form which is non-vanishing every where. Notice that for any $n-$form ν there is a smooth function f on $\mathcal{M}$ such that $\nu = fV_0$ and hence the $n-$current η can be written as $\tilde{\eta}[\mathcal{M}]$ with a Schwartz distribution $\tilde{\eta}$, i.e.,

$$\eta(\nu) = \tilde{\eta}(f) \int_{\mathcal{M}} V_0. \tag{3}$$

Proposition 2 *Let ω be a closed hypoelliptic $(n-1)-$form, then the equation*

$$\omega \wedge df = 0 \tag{4}$$

has only trivial solutions.

Proof: Let f be a solution of (4). Take an $n-$current $\eta = K(f)[\mathcal{M}]$ with $K \in C^1(R^1)$. We then have for any smooth function φ

$$T_\omega(\eta)(\varphi) = \int_{\mathcal{M}} K(f)\omega \wedge d\varphi = -\int_{\mathcal{M}} \varphi K'(f) df \wedge \omega = 0.$$

Hence $T_\omega(\eta) = 0$ but the current η smooth for any K if and only if the function f is a constant.

The second one should be a relationship with the Lie derivative of the form with respective to a vector field.

Let X be a (smooth) vector field on the manifold $\mathcal{M}$ satisfying

$$\omega \wedge df = X(f)V_0. \tag{5}$$

It is easy to see that the vector field X is uniquely determined by the $(n-1)-$form ω and V_0.

By this corresponding, we have

$$T_\omega(\eta)(f) = \tilde{\eta}(X(f)) \int_{\mathcal{M}} V_0.$$

Let X^* be the dual operator which acts on the Schwartz distribution, then

$$T_\omega(\eta)(f) = X^*(\tilde{\eta})(f) \int_{\mathcal{M}} V_0,$$

Choose V_0 so that $\int_{\mathcal{M}} V_0 = 1$, we obtain

$$T_\omega(\eta) = X^*(\tilde{\eta}). \tag{6}$$

Remark 2: Consider the dynamical system generated by the vector field X. Actually proposition 2 says this system is ergodic, i.e., every trajectory of the vector field X is dense in $\mathcal{M}$.

Recall the hypoellipticity of a vector field X defined by Greenfield and Wallach [6]:

> X is hypoelliptic if $\tilde{\eta}$ is smooth whenever $\tilde{\eta}$ is a Schwartz distribution and $X^*(\tilde{\eta})$ is smooth.

Notice that the hypoellipticity of the form ω and the vector field X are equivalent by (6). Hence we are able to rewrite a theorem in [6] as follow:

Lemma 3 *Let ω be a hypoelliptic $(n-1)-$form, then there exists unique $n-$form μ_0 with*

$$X^*(\mu_0) = 0;$$
$$\int_{\mathcal{M}} \mu_0 = 1.$$

The uniqueness means that any other distribution μ with $X^*(\mu_0) = 0$ is different from μ_0 by a constant factor. Furthermore the form μ_0 does vanishes everywhere.

Now we state one of main results in this section

Theorem 4 *Let M be oriented and ω an $n-1$ form. If ω is hypoelliptic, then there is a smooth function g which is non-zero everywhere such that $g\omega$ is closed form.*

Proof: Let μ_0 as in Lemma 3. By a formula of E. Cartan's

$$L_X(\mu_0) = di_X(\mu_0) + i_X d\mu_0 = X^*(\mu_0),$$

we see that the $(n-1)-$form $i_X\mu_0$ is close. On the other hand

$$df \wedge i_X\mu_0 = X(f)\mu_0.$$

Let g be the function so that

$$\mu_0 = gV_0,$$

then

$$df \wedge i_X\mu_0 = X(f)\mu_0 = gX(f)V_0 = g\omega \wedge df.$$

Integrating, this identity over the manifold $\mathcal{M}$ we see that

$$\int_{\mathcal{M}} g\omega \wedge df = 0.$$

Therefore the form $g\omega$ is closed and the function g non-vanished everywhere.

Remark 3: Indeed $g\omega = i_X\mu_0$.

Remark 4: On M^{2n+1} with a contact form α, i.e., $\alpha \wedge (d\alpha)^n \neq 0$ everywhere. It is easy to verify that the contact form α is hypoelliptic. So we see that above theorem does not valid in the case of $p < n-1$.

A natural question is wether or not the closed hypoelliptic form is non-trivial. We give a partial answer for this problem here. An important ingredient is the solvability for the case of $p = n-1$.

Lemma 5 *Let ω be a closed hypoelliptic $(n-1)-$form, then the equation*

$$\omega \wedge df = \mu \tag{7}$$

is solvable if and only if the $n-$form μ is exact.

Proof: As in the proof of Theorem 4, the left hand side of (7) is just a differential operator X of first order which acts on $C^\infty(\mathcal{M})$, the space of smooth functions. By the theorem of Greenfield and Wallach mentioned above, its dual operator X^* has kernel of dimensional one. So the sufficiency follows from the proof of theorem in [10]. The necessary is trivial.

Theorem 6 *There is no hypoelliptic and exact $(n-1)-$form on M provided the first Betti number of M is non-zero.*

Proof: Suppose $\omega = d\beta$ were a hypoelliptic $(n-1)-$form. Choosing a nontrivial $1-$form α on $\mathcal{M}$, we then have an exact $n-$form $d\beta \wedge \alpha$. Let $f \in C^\infty(\mathcal{M})$ satisfying

$$\beta \wedge df = \beta \wedge \alpha.$$

Set $\alpha_0 = \alpha - df$. The $1-$form α_0 is nontrivial, hence nonzero. Using universal covering $\pi : \mathcal{M} \to \widetilde{\mathcal{M}}$, we can find a function $G \in C^\infty(\mathcal{M})$ such that $\alpha_0 = dG$ and $G(p_1) = G(p_2) + m, m \in Z$ whenever $\pi(p_1) = \pi(p_2)$. Now $\sin(2\pi G) \in C^\infty(\mathcal{M})$ and

$$d\sin(2\pi G) \wedge \omega = 2\pi \cos(2\pi G)\alpha \wedge \omega = 0$$

which is a contradiction because of Proposition 2.

Theorem 7 *A manifold $\mathcal{M}$ with a hypoelliptic $(n-1)-$form has first Betti number at most n.*

Proof: Let α_0 a closed, nonexact $1-$form. Choose $f \in C^\infty$ such that

$$\omega \wedge df = \lambda V_0 - \omega \wedge \alpha$$

where

$$\lambda = \int_{\mathcal{M}} \omega \wedge \alpha.$$

Set $\alpha = \alpha_0 + df$, then $\omega \wedge \alpha = \lambda V_0$. Therefore $i_X \alpha = \lambda$. E. Cartan's formula gives

$$L_X \alpha = 0. \tag{8}$$

This identity means that the closed 1−form α is invariant under the 1−parameter group induced by X. More precisely, we obtain an $X-$invariant representation in every cohomology. Now, by the ergodicity mentioned in Remark 2, the form α is totally determined by the behavior of itself at any fixed point. So the number of the $X-$invariant closed 1−forms is at most n.

§3. Hypoelliptic Differential Form on the Torus

On the torus T^n, we can characterize the $p-$form by the Fourier series as Paley-Wiener theorem pointed. Indeed for a $n-$form φ we have following Fourier representation

$$\varphi = \sum_{k \in \mathbf{Z}^n} \varphi_k \exp\{\sqrt{-1}kx\} dx^1 \wedge dx^2 \cdots \wedge dx^n$$

where the constants φ_k are evaluated by the Fourier formula

$$\varphi_k = \int_{T^n} \exp\{2\pi\sqrt{-1}kx\}\varphi.$$

By the smoothness of the form φ, the Fourier coefficients satisfy the *rapidly decay condition:* for any $N > 0$, there exists $C > 0$, such that

$$|\varphi_k| \leq \frac{C_N}{(1+|k|)^N}, \qquad \forall k \in Z^n.$$

It is easy to show that the rapidly decay condition is also sufficient for the Fourier series converging to a form.

By the Fourier representation of $n-$forms, we can write an $n-$current η as

$$\eta = \sum_{k \in Z^n} \eta_k \exp\{2\pi\sqrt{-1}kx\}[T^n] \tag{9}$$

with $[T^n]$ an $n-$current deduced by T^n and

$$\eta_k = \eta(\exp\{-2\pi\sqrt{-1}kx\} dx^1 \wedge dx^2 \cdots \wedge dx^n)$$

The coefficients η_k are slowly increasing, i.e., there are constants C_0 and N_0 so that

$$|\eta_k| \leq C_0(1+|k|)^{N_0}.$$

If, in additional, the coefficients η_k satisfy the rapidly decay condition, then the current η is smooth. Now let a $p-$current ζ on T^n is given by a Fourier series

$$\zeta = \sum_{k\in Z^n} \exp\{2\pi\sqrt{-1}kx\}\zeta_k[T^n]$$

with η_k $p-$forms of constant coefficients, i.e.,

$$\zeta_k = \sum_{1\le i_1<i_2<\cdots<i_p\le n} \Lambda^k_{i_1 i_2\cdots i_p} dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_p}.$$

Define the normal of the $p-$form ζ_k by

$$|\zeta_k| = \{\sum_{1\le i_1<i_2<\cdots<i_p\le n} |\Lambda^k_{i_1<i_2<\cdots<i_p}|^2\}^{\frac{1}{2}}.$$

Lemma 8 *(Paley-Wiener type theorem) A $p-$current ζ as above is smooth if and only if the normal of the coefficients ζ_k satisfy the rapidly decay condition, i.e., for any $N>0$ there exits C_N so that*

$$|\zeta_k| \le \frac{C_N}{(1+|k|)^N}, \qquad \forall k\in Z^n.$$

The main results of this section is the following

Theorem 9 *Let ω be a $p-$form on the torus T^n with constant coefficient, then ω is hypoelliptic if and only if $\exists C_0, N_0$ such that*

$$|k\cdot dx\wedge\omega| \ge \frac{C_0}{(1+|k|)^{N_0}} \qquad \forall k\in Z^n\setminus\{0\}. \tag{10}$$

A form ω is said to satisfy Diophantine condition if the inequality (10) hold.

Proof: For a closed $p-$form ω of constant coefficient, one has

$$T_\omega(\eta) = -2\pi\sqrt{-1}\sum_{k\in Z^n} \eta_k \exp\{2\pi\sqrt{-1}kx\}[T^n]k\cdot dx\wedge\omega[T^n]$$

for any $n-$current η with Fourier representation. If $T_\omega(\eta)$ is smooth then, by Lemma 8, $|\eta_k k\cdot dx\wedge\omega|$ decay rapidly and $|\eta_k|$ do. So the current η is smooth.

On the other hand, if a $p-$form ω violates the Diophantine condition (10), i.e., there are a subset $\{k_j\}_{j=1}^{+\infty}$ of Z^n such that

$$|k_j \cdot dx \wedge \omega| \leq \frac{1}{(1+|K_j|)^j} \qquad \forall k \in Z^n \setminus \{0\}. \tag{11}$$

Set

$$\eta_k = \begin{cases} 1, & k = k_j; \\ 0, & k \neq k_j. \end{cases}$$

We have an $n-$current given by

$$\eta = \sum_{j=1}^{+\infty} \exp\{2\pi\sqrt{-1}k_j x\}[T^n].$$

Hence

$$T_\omega \eta = -2\pi\sqrt{-1}\sum_{j=1}^{+\infty} \exp\{2\pi\sqrt{-1}k_j x\} k_j \cdot dx \wedge \omega[T^n]$$

which is smooth by the decay property of (11). Therefore the $p-$form ω is not hypoelliptic.

Remark 5: Let

$$\omega = \sum_{j=1}^{n} \Lambda_j dx^j$$

be a $1-$form, then

$$k \cdot dx = \sum_{1\leq i<j\leq n} \{k_i\Lambda_j - k_j\Lambda_i\} dx^i \wedge dx^j.$$

The inequality (10) becomes

$$\{\sum_{1\leq i<j\leq n} |k_i\Lambda_j - k_j\Lambda_i|^2\}^{\frac{1}{2}} \geq \frac{C_0}{(1+|k|)^{N_0}} \qquad \forall k \in Z^n \setminus \{0\}.$$

This is the simultaneous Diophantine approximation, see [13]. So the inequality (10) is a general form of the simultaneous Diophantine approximation.

Theorem 10 *Suppose that $\omega \in \Omega^{n-1}(T^n)$ be hypoelliptic and closed, then there is a diffeomorphism $y = \tau(x)$ of T^n to itself so that the form ω can be written as*

$$\omega = \sum_{j=1}^{n} (-1)^{j+1} \lambda^j dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$

with the constants Λ_j satisfy the Diophantine condition (10)

Proof: Let

$$\Lambda^j = \int_{\mathcal{M}} \omega \wedge dx^j,$$

then the equations

$$\omega \wedge df^j = \lambda^j \mu_0 - \omega \wedge dx^j$$

have solutions $f = (f_1, \cdots, f^n)$. Set $y^j = x^j + f^j$ then

$$dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n + d\beta$$

for some $(n-1)-$form β. Therefore the transformation of variables

$$\tau : T^n \to T^n,$$
$$\tau : x \mapsto y = x + f(x)$$

is diffeomorphism. Furthermore $\omega \wedge dy^j = \lambda^j \mu_0$. This means that $dy^j(X) = \lambda^j$. Therefore

$$X = \sum_{j=1}^{n} \lambda^j \frac{\partial}{\partial y^j}. \qquad (12)$$

Notice that the unique $n-$form which satisfying Lemma 3 for the vector field of (12) is $dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n$. Hence

$$\mu_0 = dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n$$

and

$$\omega = i_X \mu_0.$$

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