

# Operator Algebras Related to the Bochner-Martinelli Integral <sup>\*†</sup>

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## Abstract

We describe a general method of computing the square of the singular integral of Bochner-Martinelli. Any explicit formula for the square applies in a familiar way to describe the  $C^*$ -algebra generated by this integral.

## Introduction

The Cauchy type integral  $\mathcal{C}u$  in a planar domain  $\mathcal{D} \subset \mathbb{C}$  is completely defined by its limit values on the boundary curve. The Sokhotskii-Plemelj formulas  $\mathcal{C}u^\pm = \pm 1/2 u + \mathcal{C}u$  reduce thus the study of the  $C^*$ -algebra generated by the Cauchy type integral in a domain to the study of the  $C^*$ -algebra generated by the singular Cauchy type integral on its boundary. Since  $\mathcal{C}u$  is a holomorphic function in  $\mathcal{D}$ , the iterated application of  $\mathcal{C}$  shows that  $\mathcal{C}$  is a projection on functions in  $\mathcal{D}$  or its complement. This property is actually inherited by the boundary values  $\mathcal{C}u^\pm$ , and so the square of the singular Cauchy type integral just amounts to  $1/4$ .

Setting  $P^\pm = 1/2 \pm \mathcal{C}$  we obtain two orthogonal projections on  $L^2(\partial\mathcal{D})$ , whose sum is 1 and the difference  $2\mathcal{C}$ . It follows that the algebra generated by the singular Cauchy type integral coincides with the algebra generated by two orthogonal projections. The structure of the  $C^*$ -algebra with identity generated by two orthogonal projections is well understood, see for instance [Hal69], [Ped68], [VS81]. All irreducible representations of this algebra are either two- or one-dimensional.

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The situation is much subtler in the case of  $C^*$ -algebras generated by three and more orthogonal projections, cf. [Sun88], [RV96], [Vas98], etc. Even in the particular case when two of the projections commute or, equivalently, when two of the projections are mutually orthogonal, the algebra in question may have infinite dimensional irreducible representations. An algebra generated by two concrete non-commuting orthogonal projections was first treated in [GK70].

The theory of one-dimensional singular integral equations with Cauchy kernel in the complex plane has essentially enriched the general Fredholm theory, cf. [GK78]. It actually initiated the study of pseudodifferential operators on manifolds with singularities. However, singularities in higher dimensions are much more tricky than those in the complex plane. This motivates the study of concrete  $C^*$ -algebras generated by classical singular integral operators on higher-dimensional surfaces, which could give a powerful source of intuition to forecast the behaviour of general pseudodifferential operators. As but one example we show the  $C^*$ -algebra generated by the singular Bochner-Martinelli integral.

In [SV89] the quaternionic analysis is applied to derive an explicit formula for the square of the singular Bochner-Martinelli integral  $M$  over a smooth hypersurface  $S$  in  $\mathbb{C}^2$ . It reads  $M^2 = 1/4 + \bar{a}a$  where  $a$  is a certain singular integral operator on  $S$  and  $\bar{a}$  the operator with complex conjugate kernel. Moreover,  $a$  satisfies  $M\bar{a} + \bar{a}M = 0$ . Had we a solution  $X$  of the operator equations  $X^2 = -\bar{a}a$  and  $, the operators  $P^\pm = 1/2 \pm M + X$  would be two independent projections on  $L^2(S)$ , whose sum is  $1 + 2X$  and the difference  $2M$ . Since  $M^2 + X^2 = 1/4$  we would not change drastically the  $C^*$ -algebra generated by  $M$ , by adding the operator  $X \approx \sqrt{1/4 - M^2}$  to the generators.$

Then, one can look for a  $C^*$ -algebra generated by these projections or, equivalently, by  $M$  and  $X$ . This problem fits the framework of [RV96] or more general paper [Vas98] if one chooses three projections  $P^+$  and  $P^-$ ,  $1 - P^-$ , the latter two being mutually orthogonal.

The paper [RSS98] made use of Clifford analysis to evaluate the square of the singular Bochner-Martinelli integral  $M$  over a smooth hypersurface  $S$  in  $\mathbb{C}^n$ , for  $n > 2$ . It gives  $M^2 = 1/4 + \sum_j \bar{a}_j a_j$ , where  $a_j$  are certain singular integral operators on  $S$  and the index  $j$  runs from 1 to  $n(n-1)/2$ . One might argue as above, when having granted a solution  $X$  to the operator equations  $X^2 = -\sum_j \bar{a}_j a_j$  and  $MX + XM = 0$  modulo compact operators. Still, the additional relations for  $a_j$  mentioned in [RSS98] do not allow one to construct explicitly any solution  $X$ . Moreover, there is suggestive evidence to the contrary.

The aim of this paper is to bring together two areas in which elliptic theory plays an important role. The first area is the multidimensional complex anal-

ysis studying qualitative properties of solutions to the overdetermined elliptic Cauchy-Riemann system. The second one is the geometric analysis which deals with Dirac operators, i.e., first order matrix factorisations of the Laplace operators. If compared with [RSS98], our approach invokes the diversity of Clifford algebra structures in  $\mathbb{C}^n$  to find an adequate representation of  $M^2$ . In this way we produce some concrete realisations of the algebra generated by the singular Bochner-Martinelli integral.

## 1 Dirac operators

Let  $\mathbb{C}^n$  be the standard  $n$ -dimensional complex space obtained from the underlying real space  $\mathbb{R}^{2n}$  of variables  $x = (x_1, \dots, x_{2n})$  by introducing the complex structure

$$z_j = x_j + ix_{n+j},$$

for  $j = 1, \dots, n$ . As usual, we define complex derivatives by

$$\begin{aligned} \frac{\partial}{\partial z_j} &= \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial x_{n+j}} \right), \\ \frac{\partial}{\partial \bar{z}_j} &= \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial x_{n+j}} \right). \end{aligned}$$

Dirac, wanting to quantise the electron, looked for a first order differential operator with square the d'Alembertian. One can as well do the same thing for the ordinary Laplacian  $\Delta$  in  $\mathbb{R}^{2n}$ .

The question is whether there is a matrix-valued constant coefficient operator

$$D \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) = \sum_{j=1}^n \gamma_j \frac{\partial}{\partial z_j} + \delta_j \frac{\partial}{\partial \bar{z}_j}$$

such that

$$D^* D = -\frac{1}{4} \Delta E \tag{1.1}$$

where  $D^*$  is the formal adjoint of  $D$  and  $E$  an identity matrix. It is easily seen that the identity (1.1) reduces to a system of identities for the coefficients, namely

$$\begin{aligned} \gamma_j^* \gamma_k + \delta_k^* \delta_j &= \delta_{j,k} E, \\ \gamma_j^* \delta_k + \gamma_k^* \delta_j &= 0 \end{aligned} \tag{1.2}$$

for all  $j, k = 1, \dots, n$ .

It is well known that there is a solution of (1.2) amongst matrices of type  $(2^{n-1} \times 2^{n-1})$ , cf. for instance Chapter 3 in [Mel93]. Of course, having found a solution for any one  $n$ , it works for smaller values, simply by dropping some of the matrices.

First order differential operators with constant coefficients factorising the Laplacian in the sense of (1.1) are called Dirac operators.

**Example 1.1** The Cauchy-Riemann operator  $D = \partial/\partial\bar{z}$  is a Dirac operator in the complex plane.

**Example 1.2** The operator

$$D = \begin{pmatrix} \partial/\partial\bar{z}_1 & -\partial/\partial z_2 \\ \partial/\partial\bar{z}_2 & \partial/\partial z_1 \end{pmatrix}$$

is a Dirac operator in  $\mathbb{C}^2$ .

**Example 1.3** The operator

$$D = \begin{pmatrix} 0 & \partial/\partial\bar{z}_1 & \partial/\partial\bar{z}_2 & \partial/\partial\bar{z}_3 \\ \partial/\partial\bar{z}_1 & 0 & \partial/\partial z_3 & -\partial/\partial z_2 \\ \partial/\partial\bar{z}_2 & -\partial/\partial z_3 & 0 & \partial/\partial z_1 \\ \partial/\partial\bar{z}_3 & \partial/\partial z_2 & -\partial/\partial z_1 & 0 \end{pmatrix}$$

is a Dirac operator in  $\mathbb{C}^3$ .

Dirac operators in  $\mathbb{C}^4$  are available amongst differential operators mit values in  $(8 \times 8)$ -matrices, and so on. It is worth pointing out that one can choose a Dirac operator  $D$  in  $\mathbb{C}^n$  in such a way that the only non-zero entries of the first column of  $D$  are  $\partial/\partial\bar{z}_1, \dots, \partial/\partial\bar{z}_n$ . As but one example we mention the Dirac operator  $\bar{\partial} + \bar{\partial}^*$  associated with the Dolbeault complex in  $\mathbb{C}^n$ . This is actually a formally self-adjoint operator taking its values in matrices of type  $(2^n \times 2^n)$ .

## 2 Fundamental solutions

To construct an explicit fundamental solution  $\Phi$  for a Dirac operator  $D$  we make use of relation (1.1). Namely, denote by  $e$  the standard fundamental solution of convolution type for the Laplace operator in  $\mathbb{R}^{2n}$ . In the coordinates of  $\mathbb{C}^n$  it reads

$$e(z) = \frac{(n-1)!}{2\pi^n} \frac{1}{2-2n} \frac{1}{|z|^{2n-2}},$$

if  $n > 1$ , and  $e(z) = \frac{1}{2\pi} \ln |z|$ , if  $n = 1$ .

Set

$$\begin{aligned} \Phi(z - \zeta) &= 4D\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\bar{\zeta}}\right)^* e(z - \zeta) \\ &= \frac{(n-1)!}{\pi^n} \frac{D(\bar{z} - \bar{\zeta}, z - \zeta)^*}{|z - \zeta|^{2n}} \end{aligned}$$

for  $z \neq \zeta$ , where by  $D(\bar{z} - \bar{\zeta}, z - \zeta)^*$  is meant the adjoint of the matrix  $D(\bar{z} - \bar{\zeta}, z - \zeta)$ , i.e.,

$$D(\bar{z} - \bar{\zeta}, z - \zeta)^* = \sum_{j=1}^n \gamma_j^* (z_j - \zeta_j) + \delta_j^* (\bar{z}_j - \bar{\zeta}_j). \quad (2.1)$$

**Lemma 2.1** *As defined above,  $\Phi(z - \zeta)$  is a fundamental solution of the Dirac operator  $D$ , i.e.,  $\Phi \circ D = I$  and  $D \circ \Phi = I$  on  $C_{\text{comp}}^\infty(\mathbb{C}^n, \mathbb{C}^k)$ , where  $(k \times k)$  is the type of  $E$ .*

**Proof.** The first relation  $\Phi \circ D = I$  is fulfilled by the very construction of  $\Phi$ . Since  $D$  is a square matrix, it follows from (1.1) that  $DD^* = -(1/4)\Delta E$  whence  $\Phi^* \circ D^* = I$  on  $C_{\text{comp}}^\infty(\mathbb{C}^n, \mathbb{C}^k)$ . The latter equality is equivalent to  $D \circ \Phi = I$ , as desired.  $\square$

Substituting (2.1) into the expression for  $\Phi(z - y)$  we thus get an explicit formula

$$\Phi(z - \zeta) = \frac{(n-1)!}{\pi^n} \sum_{j=1}^n \gamma_j^* \frac{z_j - \zeta_j}{|z - \zeta|^{2n}} + \delta_j^* \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}}.$$

### 3 Green formula

Let  $\mathcal{D}$  be a bounded domain with smooth boundary  $S = \partial\mathcal{D}$  in  $\mathbb{C}^n$ . Write  $\nu(y) = (\nu_1(y), \dots, \nu_{2n}(y))$  for the unit outward normal vector of  $S$  at a point  $y \in S$ .

If  $\varrho(x)$  is a defining function of  $S$  then

$$\nu(y) = \frac{\nabla\varrho(y)}{|\nabla\varrho(y)|}$$

for  $y \in S$ , where  $\nabla\varrho(y)$  stands for the real gradient of  $\varrho$  at  $y$ . The complex vector  $\nu_c = (\nu_{c,1}, \dots, \nu_{c,n})$  with coordinates  $\nu_{c,j} = \nu_j + i\nu_{n+j}$  is called the complex normal of the hypersurface  $S$ . In the coordinates of  $\mathbb{C}^n$  we obviously have

$$\nu_{c,j}(\zeta) = \frac{\frac{\partial\varrho}{\partial\bar{\zeta}_j}}{|\nabla_{\bar{\zeta}}\varrho(\zeta)|}$$

for  $j = 1, \dots, n$ .

Denote by  $d\zeta$  the wedge product  $d\zeta_1 \wedge \dots \wedge d\zeta_n$ , and by  $d\bar{\zeta}[j]$  the wedge product of all  $d\bar{\zeta}_1, \dots, d\bar{\zeta}_n$  but  $d\bar{\zeta}_j$ .

**Lemma 3.1** *For each  $j = 1, \dots, n$ , the pull-back of the differential form  $d\zeta \wedge d\bar{\zeta}[j]$  under the embedding  $S \hookrightarrow \mathbb{C}^n$  is equal to  $(-1)^{j-1}(2i)^{n-1}i\nu_{c,j}ds$ , where  $ds$  is the area form on the hypersurface  $S$  induced by the Hermitian metric of  $\mathbb{C}^n$ .*

**Proof.** An easy computation shows that the pull-back of the differential form  $dy[j]$  under the embedding  $S \hookrightarrow \mathbb{C}^n$  is equal to  $(-1)^{j-1}\nu_j ds$ , for every  $j = 1, \dots, 2n$ . From this the lemma follows immediately.  $\square$

We are now in a position to specify the restriction of the Green operator  $G_D(g, u)$  of  $D$  to the boundary. By a Green operator of  $D$  is meant a bilinear operator  $G_D$  from  $C^\infty(\mathbb{C}^n, (\mathbb{C}^k)^*) \times C^\infty(\mathbb{C}^n, \mathbb{C}^k)$  to differential forms of degree  $2n-1$  on  $\mathbb{C}^n$ , such that  $dG_D(g^*, u) = ((Du, g)_y - (u, D^*g)_y) dy$  holds pointwise in  $\mathbb{C}^n$ .

By [Tar95, § 2.4.2], there is a unique Green operator for  $D$ , and its pull-back under the embedding  $S \hookrightarrow \mathbb{C}^n$  is

$$\begin{aligned} G_D(g, u) &= \frac{1}{i} g D\left(\frac{i}{2}(\nu_j - w_{n+j}), \frac{i}{2}(\nu_j + w_{n+j})\right) u ds \\ &= \frac{1}{2} g D\left(\bar{\nu}_{c,j}, \nu_{c,j}\right) u ds \end{aligned}$$

whence

$$G_D(\Phi(z - \zeta), E) = \frac{(n-1)!}{2\pi^n} \frac{D(\bar{z} - \bar{\zeta}, z - \zeta)^* D(\bar{\nu}_{c,j}(\zeta), \nu_{c,j}(\zeta))}{|z - \zeta|^{2n}} ds. \quad (3.1)$$

**Lemma 3.2** *Every vector-valued function  $u \in C^1(\bar{\mathcal{D}}, \mathbb{C}^k)$  has the integral representation*

$$\chi_{\mathcal{D}} u = - \int_S G_D(\Phi(z - \cdot), u) + \int_{\mathcal{D}} \Phi(z - \cdot) Du dv,$$

where  $dv$  is the Lebesgue measure in  $\mathbb{R}^{2n}$  and  $\chi_{\mathcal{D}}$  the characteristic function of  $\mathcal{D}$ .

**Proof.** This is a very special case of a general Green formula related to an elliptic system of differential equations, see for instance [Tar95, § 2.5.4] and elsewhere.  $\square$

Needless to say this formula extends to the case of Sobolev class functions  $u \in H^1(\mathcal{D}, \mathbb{C}^k)$  as well as more general functions on  $\mathcal{D}$ .

## 4 Structure of Green operator

Combining (3.1) with Lemma 3.1 enables us to recover the Green operator in all of  $\mathbb{C}^n$ , namely

$$\begin{aligned}
& G_D(\Phi(z - \zeta), E) \\
&= \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^n (-1)^{n+k-1} \left( \sum_{j=1}^n \gamma_j^* \gamma_k \frac{z_j - \zeta_j}{|z - \zeta|^{2n}} + \delta_j^* \gamma_k \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} \right) d\zeta[k] \wedge d\bar{\zeta} \\
& \quad \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^n (-1)^{k-1} \left( \sum_{j=1}^n \gamma_j^* \delta_k \frac{z_j - \zeta_j}{|z - \zeta|^{2n}} + \delta_j^* \delta_k \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} \right) d\zeta \wedge d\bar{\zeta}[k]
\end{aligned} \tag{4.1}$$

away from the diagonal of  $\mathbb{C}^n \times \mathbb{C}^n$ .

Write  $K(z, \zeta)$  for the first summand on the right-hand side of this formula, i.e.,

$$K(z, \zeta) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^n (-1)^{n+k-1} \left( \sum_{j=1}^n \gamma_j^* \gamma_k \frac{z_j - \zeta_j}{|z - \zeta|^{2n}} + \delta_j^* \gamma_k \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} \right) d\zeta[k] \wedge d\bar{\zeta}.$$

**Lemma 4.1** *Under the above notation, the Green operator of  $D$  splits into the sum*

$$G_D(\Phi(z - \zeta), E) = K(z, \zeta) - K(z, \zeta)^* - U(\zeta, z)E$$

for  $z \neq \zeta$ , where  $U(z, \zeta)$  is the Bochner-Martinelli kernel in  $\mathbb{C}^n$ .

The Bochner-Martinelli kernel is of crucial importance in multidimensional complex analysis, cf. [Kyt95]. The designation  $U(\zeta, z)$  is used in many papers and monographs on complex analysis, hence we keep it here. Recall that the Bochner-Martinelli kernel is given away from the diagonal of  $\mathbb{C}^n \times \mathbb{C}^n$  by the formula

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^n (-1)^{j-1} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} d\bar{\zeta}[j] \wedge d\zeta.$$

**Proof.** Using relations (1.2) yields

$$\begin{aligned}
& K(z, \zeta)^* \\
&= \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^n (-1)^{k-1} \left( \sum_{j=1}^n \gamma_k^* \gamma_j \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} + \gamma_k^* \delta_j \frac{z_j - \zeta_j}{|z - \zeta|^{2n}} \right) d\zeta \wedge d\bar{\zeta}[k] \\
&= -\frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^n (-1)^{k-1} \left( \sum_{j=1}^n (\delta_j^* \delta_k - \delta_{k,j} E) \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} + \gamma_j^* \delta_k \frac{z_j - \zeta_j}{|z - \zeta|^{2n}} \right) d\zeta \wedge d\bar{\zeta}[k]
\end{aligned}$$

for  $z \neq \zeta$ . Multiplying out  $\delta_j^* \delta_k - \delta_{k,j} E$  we obtain two sums on the right-hand side. The first sum just amounts to the second summand in (4.1) with the

opposite sign. And the second sum is equal to  $-U(\zeta, z)E$ , which establishes the formula.  $\square$

## 5 Plemelj formulas

Let  $u$  be a continuous function on  $S$  with values in  $\mathbb{C}^k$ ,  $k$  being the rank of  $E$ . The integral

$$\mathcal{C}u(z) = - \int_S G_D(\Phi(z - \cdot), u) \quad (5.1)$$

is well defined for all  $z$  away from  $S$ . Moreover, it satisfies the Dirac system  $D\mathcal{C}u = 0$  in  $\mathbb{C}^n \setminus S$  and vanishes at  $z = \infty$ . The potential  $\mathcal{C}u$  has usually been referred to as a Cauchy type integral related to the Dirac operator  $D$ , cf. [Tar95, § 3.2].

If  $z \in S$  the integral on the right side of (5.1) no longer exists in the usual sense. Still, for any function  $u \in C^{0,\lambda}(S, \mathbb{C}^k)$  of Hölder class with  $\lambda > 0$ , the integral can be interpreted in the sense of the Cauchy principal value. Namely, denote by  $B(z, \varepsilon)$  the ball with centre  $z$  and radius  $\varepsilon > 0$  in  $\mathbb{C}^n$ . Then the limit

$$\mathcal{C}u(z) = - \lim_{\varepsilon \rightarrow 0} \int_{S \setminus B(z, \varepsilon)} G_D(\Phi(z - \cdot), u) \quad (5.2)$$

exists for each  $z \in S$ , cf. *ibid.* It is called the singular Cauchy type integral of  $u$ .

In fact, it is a standard result of the modern harmonic analysis that the limit (5.2) exists almost everywhere on  $S$  for any function  $u \in L^1(S, \mathbb{C}^k)$ . Moreover, this is still true for all Lipschitz surfaces  $S$ . In the case of Hölder densities  $u$  the singular Cauchy type integral can be quite easily evaluated, namely

$$\mathcal{C}u(z) = \frac{1}{2} u(z) - \int_S G_D(\Phi(z - \zeta), u(\zeta) - u(z)) \quad (5.3)$$

for  $z \in S$ .

Given a function  $f$  on  $\mathbb{C}^n \setminus S$  with vector values, we denote by  $f^\pm$  the restrictions of  $f$  to  $\mathcal{D}$  and  $\mathbb{C}^n \setminus \overline{\mathcal{D}}$ , respectively.

**Lemma 5.1** *Suppose  $u \in C^{0,\lambda}(S, \mathbb{C}^k)$ , where  $0 < \lambda < 1$ . Then the restrictions  $\mathcal{C}u^\pm$  extend to functions of class  $C^{0,\lambda}$  in the closures of  $\mathcal{D}$  and  $\mathbb{C}^n \setminus \overline{\mathcal{D}}$ , respectively, and*

$$\mathcal{C}u^\pm(z) = \pm \frac{1}{2} u(z) + \mathcal{C}u(z) \quad (5.4)$$

for all  $z \in S$ .



The jump formulas (5.4) are usually referred to as Sokhotskii-Plemelj formulas, see [Tar95, § 3.2] for a thorough treatment.

**Proof.** We drop the proof of Hölder continuity, for it is a general property of potential operators whose kernels have the transmission property with respect to  $S$ .

Pick a point  $z_0 \in S$ . For  $z \in \mathcal{D}$ , we get

$$\mathcal{C}u(z) = - \int_S G_D(\Phi(z - \zeta), u(\zeta) - u(z_0)) + u(z_0),$$

which is due to the Green formula of Lemma 3.2. Letting  $z \rightarrow z_0$  obviously yields

$$\mathcal{C}u^+(z_0) = - \int_S G_D(\Phi(z_0 - \zeta), u(\zeta) - u(z_0)) + u(z_0),$$

which establishes the first formula of (5.4) when combined with (5.3). Similar arguments apply to  $\mathcal{C}u^-$ . □

Adding equalities (5.4) and subtracting them from each other, we arrive at the formulas

$$\begin{aligned} \mathcal{C}u^+ - \mathcal{C}u^- &= u, \\ \mathcal{C}u^+ + \mathcal{C}u^- &= 2\mathcal{C}u \end{aligned} \tag{5.5}$$

on  $S$ .

## 6 Rearrangement formula

The Poincaré-Bertrand rearrangement formula for singular integrals is of great importance in the theory of one-dimensional singular integral equations. Such a formula is also known for the singular Bochner-Martinelli integral in  $\mathbb{C}^n$ , cf. [KPT92]. However, this latter does not allow one to evaluate the square of the singular Bochner-Martinelli integral. For singular Cauchy type integrals (5.1) an analogue of the Poincaré-Bertrand rearrangement formula reads as follows, cf. [Tar95, 3.2.8].

**Theorem 6.1** *Assume that  $u \in C^{0,\lambda}(S \times S, \mathbb{C}^k)$ , where  $\lambda > 0$ . Then, for all  $w \in S$ ,*

$$\begin{aligned} & \int_{S_z} G_D(\Phi(w - z), E) \int_{S_\zeta} G_D(\Phi(z - \zeta), u(\zeta, z)) \\ &= \frac{1}{4}u(w, w) + \int_{S_\zeta} \int_{S_z} G_D(\Phi(w - z), E) G_D(\Phi(z - \zeta), u(\zeta, z)). \end{aligned} \tag{6.1}$$

**Proof.** We fix an arbitrary point  $w_0 \in S$  and prove (6.1) for  $w = w_0$ . To this end, consider

$$\begin{aligned} F(w) &= \int_{S_z} G_D(\Phi(w-z), E) \int_{S_\zeta} G_D(\Phi(z-\zeta), u(\zeta, z)), \\ G(w) &= \int_{S_\zeta} \int_{S_z} G_D(\Phi(w-z), E) G_D(\Phi(z-\zeta), u(\zeta, z)) \end{aligned}$$

for  $w \in \mathbb{C}^n \setminus S$ .

If  $w \notin S$  then the integral over  $z \in S$  is non-singular, and so we can change the order of integration. Hence it follows that  $F(w) = G(w)$  for all  $w \in \mathbb{C}^n \setminus S$ , and so  $F^\pm = G^\pm$ .

By (5.5), we have

$$F^+(w_0) + F^-(w_0) = 2 \int_{S_z} G_D(\Phi(w_0-z), E) \int_{S_\zeta} G_D(\Phi(z-\zeta), u(\zeta, z)). \quad (6.2)$$

On the other hand, let us rewrite the expression for the integral  $G(w)$  in the form

$$\begin{aligned} G(w) &= \int_{S_\zeta} \int_{S_z} G_D(\Phi(w-z), E) G_D(\Phi(z-\zeta), u(\zeta, z) - u(\zeta, \zeta)) \\ &+ \int_{S_\zeta} \left( \int_{S_z} G_D(\Phi(w-z), E) G_D(\Phi(z-\zeta) - \Phi(w-\zeta), E) \right) u(\zeta, \zeta) \\ &+ \int_{S_\zeta} \left( \int_{S_z} G_D(\Phi(w-z), E) \right) G_D(\Phi(w-\zeta), u(\zeta, \zeta)). \end{aligned} \quad (6.3)$$

Using the Green formula and (5.4), we deduce from (6.3) that

$$\begin{aligned} G^+(w_0) + G^-(w_0) &= u(w_0, w_0) + 2 \int_{S_\zeta} \int_{S_z} G_D(\Phi(w_0-z), E) G_D(\Phi(z-\zeta), u(\zeta, z)). \end{aligned} \quad (6.4)$$

Equating (6.2) and (6.4), we establish formula (6.1), which completes the proof.  $\square$

Theorem 6.1 applies to evaluate the square of the singular Cauchy type integral. Indeed, take  $u$  in (6.1) which merely depends on one variable  $\zeta \in S$ , i.e.,  $u \in C^{0,\lambda}(S, \mathbb{C}^k)$ . Then the iterated integral on the right-hand side vanishes, for

$$\begin{aligned} &\int_{S_\zeta} \int_{S_z} G_D(\Phi(w-z), E) G_D(\Phi(z-\zeta), u(\zeta)) \\ &= \int_{S_\zeta} \left( \int_{S_z} G_D(\Phi(w-z), \Phi(z-\zeta)) \right) G_D(E, u(\zeta)) \end{aligned}$$

and the inner integral is equal to zero for all  $w$  and  $\zeta$  away from the diagonal of  $S \times S$ .

**Corollary 6.2** *The square of the doubled singular Cauchy type integral  $\mathcal{C}$  is equal to the identity operator on  $C^{0,\lambda}(S, \mathbb{C}^k)$ .*

Since  $\mathcal{C}$  maps  $L^p(S, \mathbb{C}^k)$  continuously to  $L^p(S, \mathbb{C}^k)$ , for  $1 < p < \infty$ , and  $C^{0,\lambda}(S, \mathbb{C}^k)$  is dense in  $L^p(S, \mathbb{C}^k)$ , Corollary 6.2 actually shows that  $(2\mathcal{C})^2 = I$  on  $L^p(S, \mathbb{C}^k)$ .

## 7 $C^*$ algebra

As usual, we identify  $\mathbb{C}^k \otimes (\mathbb{C}^k)^*$  with the space of  $(k \times k)$ -matrices of complex numbers. By the Schwartz kernel theorem, for any continuous linear operator  $T : \mathcal{D}(S, \mathbb{C}^k) \rightarrow \mathcal{D}'(S, \mathbb{C}^k)$  there is a distribution  $\mathcal{K} \in \mathcal{D}'(S \times S, \mathbb{C}^k \otimes (\mathbb{C}^k)^*)$ , such that

$$Tu(z) = \int_S \mathcal{K}(z, \zeta)u(\zeta) ds(\zeta)$$

for all  $u \in \mathcal{D}(S, \mathbb{C}^k)$ , and this correspondence is one-to-one. The distribution  $\mathcal{K}$  is called the kernel of  $T$ .

Write  $\mathcal{K}_T$  for the kernel of  $T$  and, vice versa,  $T_{\mathcal{K}}$  for the operator with kernel  $\mathcal{K}$ .

If  $T$  restricts to a continuous mapping of  $L^2(S, \mathbb{C}^k)$  then the kernel of the adjoint operator is well defined. It is easy to check that  $\mathcal{K}_{T^*}(z, \zeta) = \mathcal{K}_T(\zeta, z)^*$  holds a.e. on  $S \times S$ .

Using Lemma 3.1 we describe the pull-back of the differential form  $K(z, \zeta)$  of Lemma 4.1 under the embedding  $S \hookrightarrow \mathbb{C}^n$ . It has the form  $\mathcal{K}(z, \zeta)ds(\zeta)$ , where

$$\mathcal{K}(z, \zeta) = \frac{(n-1)!}{2\pi^n} \sum_{k=1}^n \left( \sum_{j=1}^n \gamma_j^* \gamma_k \frac{z_j - \zeta_j}{|z - \zeta|^{2n}} + \delta_j^* \gamma_k \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} \right) \bar{v}_{c,k}(\zeta) \quad (7.1)$$

for  $z \neq \zeta$ .

Combining (5.1) with Lemma 4.1 and (7.1), we rewrite the Cauchy type integral as

$$\mathcal{C}u = -T_{\mathcal{K}}u + T_{\mathcal{K}^*}u + Mu$$

where  $M$  is the Bochner-Martinelli integral. In what follows we restrict our attention to operators on the boundary, i.e., all the integrals are thought of as principal value singular integrals on  $S$ . Needless to say every of these singular integral operators is a  $(k \times k)$ -matrix of scalar-valued singular integral operators on  $S$ .

**Example 7.1** If  $D$  is the Dirac operator of Example 1.2 then an easy verification shows that

$$\mathcal{C} = \begin{pmatrix} M & -\bar{a} \\ a & \bar{M} \end{pmatrix},$$

where  $a$  is a scalar-valued singular integral operator on  $S$  and  $\bar{a} := cac$ ,  $c$  being complex conjugation. In fact,

$$au(z) = \int_S \frac{1}{2\pi^2} \frac{(z_2 - \zeta_2)\nu_{c,1}(\zeta) - (z_1 - \zeta_1)\nu_{c,2}(\zeta)}{|z - \zeta|^4} u(\zeta) ds(\zeta),$$

cf. [SV89].

This example demonstrates rather strikingly that the operator matrix  $\mathcal{C}$  in general looks like

$$\mathcal{C} = \begin{pmatrix} a_{11} & -\bar{a}_{21} & \dots & -\bar{a}_{k1} \\ a_{21} & a_{22} & \dots & -\bar{a}_{k2} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix} + ME, \quad (7.2)$$

where  $a_{ij}$  are certain singular integral operators on  $S$ . Moreover, the diagonal elements  $a_{ii}$  are purely imaginary, i.e.,  $\Re a_{ii} = 0$  for all  $i = 1, \dots, k$ . Note that  $a_{ij}$  can be written explicitly.

If one chooses a Dirac operator  $D$  in  $\mathbb{C}^n$  in such a way that the only non-zero entries of the first column of  $D$  are  $\partial/\partial\bar{z}_1, \dots, \partial/\partial\bar{z}_n$ , then  $a_{11} = 0$ , as is easy to check.

**Theorem 7.2** *The  $C^*$ -algebra generated by the singular Cauchy type integral  $\mathcal{C}$  coincides with that generated by the projections  $P^\pm = (1/2)E \pm \mathcal{C}$ .*

**Proof.** If we prove that  $P^\pm$  are projections, the assertion follows. Using Corollary 6.2 we get

$$\begin{aligned} (P^\pm)^2 &= \frac{1}{4}E \pm \mathcal{C} + \mathcal{C}^2 \\ &= P^\pm, \end{aligned}$$

as desired. □

We can now return to the original question on the algebra generated by  $M$ . To describe this algebra it is sufficient to have a solution  $X$  to the operator equations  $M^2E + X^2 = 1/4 E$  and  $in  $\mathcal{L}(L^2(S, \mathbb{C}^k))$ . Then the operators  $P^\pm = 1/2 E \pm ME + X$  are independent projections on  $L^2(S, \mathbb{C}^k)$ , and the  $C^*$ -algebra generated by  $P^\pm$  just amounts to that generated by  $ME$$

and  $X$ . The problem can be treated in the framework of [RV96] or [Vas98] if one chooses three projections  $P^+$  and  $P^-$ ,  $E - P^-$ , the latter two being mutually orthogonal.

Observe that  $P^+$  coincides with the projection  $P^+$  of Theorem 7.2 for  $\mathcal{C} = ME + X$ , i.e., for  $X = A$  given by the first term on the right-hand side of (7.2). On the other hand, from the equality  $(A + ME)^2 = 1/4 E$  it follows that

$$\begin{aligned} (P^-)^2 &= \frac{1}{4}E - ME + A + M^2E + A^2 - (MA + AM) \\ &= P^- - 2(MA + AM) \end{aligned}$$

is a projection if and only if  $MA + AM = 0$ .

One question still unanswered is whether  $MA + AM = 0$  holds for a suitable choice of  $D$ .

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