Asymptotic behavior of solutions to multidimensional nonisentropic hydrodynamic model for semiconductors*

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Abstract

In this paper, a global existence result of smooth solutions to the multidimensional nonisentropic hydrodynamic model for semiconductors is proved, under the assumption that the initial data is a perturbation of the stationary solutions for the thermal equilibrium state. The resulting evolutionary solutions converge to the stationary solutions in time asymptotically exponentially fast.

Keywords: Multidimensional nonisentropic hydrodynamic model, semiconductors, asymptotic behavior, global solutions.

1 Introduction

The multidimensional nonisentropic hydrodynamic model for semiconductors is given by

$$\begin{cases} n_t + \nabla\cdot(n\mathbf{u}) = 0 \\ \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{n}\nabla(nT) = \nabla\Phi - \frac{\mathbf{u}}{\tau_p} \\ T_t + \mathbf{u}\cdot\nabla T + \frac{2}{3}T\mathrm{div}\mathbf{u} - \frac{2}{3n}\nabla(\kappa\nabla T) = \frac{2\tau_w-\tau_p}{3\tau_w\tau_p}|\mathbf{u}|^2 - \frac{T-T^0}{\tau_w} \\ \triangle\Phi = n - b(x) \end{cases} \tag{1.1}$$

for $(x,t)\in\mathbb{R}^N\times[0,+\infty), N=2,3$. The system is supplemented with the initial data

$$n(x,0)=n_0(x),\quad \mathbf{u}(x,0)=\mathbf{u}_0(x),\quad T(x,0)=T_0(x)\quad x\in\mathbb{R}^N. \tag{1.2}$$

where $n, \mathbf{u}=(u^1,u^2,\cdots,u^N), \Phi$ and T denote the electron density, the electron velocity, the electrostatic potential and the electron temperature, respectively. The coefficients κ, τ_p and τ_w are the thermal conductivity, the momentum relaxation time and energy relaxation time, respectively. In general, the thermal conductivity κ is governed by the

*This work is supported by NSFC 10271108 and DFG.

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Wiedemann Franz law [21] and depends on n and T; τ_p and τ_w have the form $\tau_p = C_p(\frac{T}{T^0})^\vartheta$ and $\tau_w = C_w(\frac{T}{T+T^0}) + \frac{1}{2}\tau_p$, respectively. Here C_p and C_w are physical constants and the standard choice for ϑ is -1. The positive constant T^0 is ambient device temperature. The function $b(x)$ stands for the density of fixed, positively charged background ions.

The system (1.1) was introduced about thirty years ago to describe the electron flow in semiconductor devices when the transport of energy plays a crucial role, as in submicron devices or in the occurrence of high field phenomena([1],[2]). These modes all make up of a set of balance laws for the moments of the electron distribution density, derived from the infinite hierarchy of moment equations of the semiclassical Boltzmann equation for semiconductors, coupled with the electric potential through a Possion equation([3],[4]). If we substitute $T = T(n) = T^0 n^{\alpha-1} (T^0 > 0, \alpha \geqslant 1)$ for $(1.1)_3$, the system (1.1) is so-called isentropic hydrodynamic models. In the isentropic case, the system has been extensively studied for the Cauchy problem and the initial-boundary value problem in the one dimensional or multidimensional case by many authors(see [5]-[11],[14]-[18]). Degond-Markowich, Gamba proved the existence and uniqueness of steady-state solution in subsonic case and in transonic case, respectively. In the dynamic case, Zhang and Marcati-Natalini investigated the global existence of weak solutions of the one-dimensional initial-boundary value problem and Cauchy problem, respectively, by using the tools of compensated compactness. The corresponding results on the zero relaxation limit have been also obtained. Luo-Natalini-Xin and Hsiao-Yang investigated the asymptotic behavior of smooth solutions for the Cauchy problem and the initial-boundary value problem, respectively, which proved the solutions converged to the unique stationary solution time asymptotically. In the multidimensional case, it is more difficult to establish the global existence of weak or smooth solutions than in the one dimensional case due to overcome the geometrical structure caused by the multidimensional unbounded domain. In this field, Hsiao-Wang *et al.* have already gotten many results systemically, we cite [14]-[18].

Here, we are interested in the nonisentropic case. For the one dimensional case, the Cauchy problem and the initial boundary value problem of (1.1) have been also largely studied by many authors in the literature (see, *e.g.*[22],[24]-[26]). Ali-Bini-Natalini [22] studied the system (1.1) with $\kappa = 0$ discussing that under the assumption that the initial data was a perturbation of a stationary solution of the Drift-Diffusion equations, then the resulting evolutionary solutions converged asymptotically in time to the unperturbed state. Cheng-Jerome-Zhang [24] gave the existence of solutions to the initial-boundary value problem for (1.1) and the convergence to a constant state, moreover, also discussed the zero relaxation time problem. Hsiao-Wang [26] investigated the asymptotic behavior of global smooth solutions to the initial-boundary problem for (1.1) and established the exponential convergence rate of the solutions to the problem. As far as weak solutions are concerned, Gasser and Natalini [25] studied the Cauchy problem (1.1) with $\kappa = 0$ and the zero relaxation convergence of weak solutions to the corresponding Drift-Diffusion equations.

Physically, it is more important and more interesting to study the system (1.1) in the multidimensional case, but very little is known so far. One elementary difficulty is that the one dimensional problem of (1.1) can be reduced to the wave equation of second order coupled by a parabolic equation when $\kappa > 0$ or the pure symmetric hyperbolic systems when $\kappa = 0$, however, these methods do not work for the multidimensional problem (1.1), which is a strong coupled hyperbolic–elliptic (or hyperbolic–parabolic–elliptic) when $\kappa = 0$ (or $\kappa > 0$). Another elementary difficulty, alike the isentropic case, is to overcome the geometrical structure which is caused by the multidimensional unbounded domain. For example, the technical one is caused by the difference of the $Sobolev's$ embedding results between the one dimension and the multidimension. Recently, Hsiao-Jiang-Zhang [27] discussed the asymptotic behavior of the smooth solution to the initial-boundary value problem of (1.1) and proved that the solutions of the problem converged to a constant steady state exponentially asymptotically as time tended to infinity for small solutions. G.Ali [23] relied essentially on the extended thermodynamic model and proved that the initial data was a perturbation of the corresponding Drift-Diffusion equation, and the resulting evolutionary solutions converged to the stationary solutions time asymptotically exponentially fast. Hsiao-Wang [28] dealt with the large time behavior of the globally smooth solutions to the Cauchy problem for (1.1) under the assumption $b(x) = positive\ constant$. In present paper, we establish the global existence and asymptotic behavior of smooth solutions to the Cauchy problem of (1.1) in $\mathbb{R}^N$ $(N = 2, 3)$ without the restriction of $b(x) = positive\ constant$. Consider $b(x)$ satisfying the following general conditions:

$$\lim_{|x|\to+\infty} b(x) = B > 0, \tag{1.3}$$

$$b(x) > 0,\ b(x) \in C^4(\mathbb{R}^N) \ \text{ and } \ \nabla b(x) \in H^3(\mathbb{R}^N). \tag{1.4}$$

Which is also to extend the results in [18] for the isentropic case. We shall study the system (1.1)-(1.2) with or without heat flux term.

For simplicity, we assume that κ, τ_p, τ_w are all constants. In this section, we can take $\kappa = \tau_p = \tau_w = 1$. As for another case $\kappa = 0, \tau_p = \tau_w = 1$, main results are given in Section 3. Now, we consider the stationary solution $(\mathcal{N}, \mathcal{U}, \mathcal{E}, T)$ of the thermal equilibrium state for (1.1) with $\mathcal{U} \equiv 0$ and $T \equiv T^0$. That is, we want to look for the solutions of the system

$$\begin{cases} T^0 \nabla \mathcal{N} = \mathcal{N}\mathcal{E} \\ \mathrm{div}\mathcal{E} = \mathcal{N} - b(x) \end{cases} \tag{1.5}$$

under the condition

$$\mathcal{N} - b(x) \in H^4(\mathbb{R}^N). \tag{1.6}$$

In [18], Hsiao-Ju-Wang proved the existence and uniqueness of solutions to a slightly more general system than (1.5)-(1.6) by the standard iteration technique and $Lerry-Schauder's$ fixed point principle. Here, by applying those results directly in [18], we can obtain the following results:

Theorem 1.1

Suppose $b(x)$ satisfies the condition (1.3),(1.4),(1.5) and (1.6), then the system (1.5)-(1.6) has an unique classical solution $(\mathcal{N}, \mathcal{E})$.

Theorem 1.2
Suppose $b(x)$ satisfying the condition (1.3),(1.4). Let $(\mathcal{N},\mathcal{E})$ be the solutions of (1.5)-(1.6) given by Theorem 1.1. Then,

$$\inf_{x\in\mathbb{R}^N} b(x) \leqslant \mathcal{N} \leqslant \sup_{x\in\mathbb{R}^N} b(x)\ . \tag{1.7}$$

Furthermore, if $\|\nabla b\|_{H^3}$ is small enough, then

$$\|\nabla\mathcal{N}\|_{H^3} \leqslant C\|\nabla b\|_{H^3}\ , \tag{1.8}$$

where C depends on T^0.
Remark 1.2.1: (1.3), (1.4) and (1.7) ensure the strict positivity of $\mathcal{N}(x)$.

The main purpose of this paper is to investigate the global existence and large time behavior of smooth solutions to (1.1)-(1.2). The following results are proved in Section 2.
Theorem 1.3 (main results)
Suppose that $b(x)$ satisfies the condition (1.3),(1.4) and $n(\cdot,0)-\mathcal{N}\in H^3(\mathbb{R}^N), \mathbf{u}(\cdot,0)\in H^3(\mathbb{R}^N), \nabla\Phi(\cdot,0)-\mathcal{E}\in H^3(\mathbb{R}^N)$ and $T(\cdot,0)-T^0\in H^4(\mathbb{R}^N)$. Then there exists sufficiently small constant $\delta_0>0$, depending only on $b(x)$, such that if

$$\begin{aligned}\|(n(\cdot,0)-\mathcal{N},\mathbf{u}(\cdot,0),\nabla\Phi(\cdot,0)-\mathcal{E})\|_{H^3(\mathbb{R}^N)}+\|T(\cdot,0)-T^0\|_{H^4(\mathbb{R}^N)}\\ +\|(n_t,\mathbf{u}_t,\nabla\Phi_t,T_t)(\cdot,0)\|_{H^2(\mathbb{R}^N)}+\|\nabla b\|_{H^3(\mathbb{R}^N)}\leqslant\delta_0\end{aligned}$$

Then the Cauchy problem (1.1)-(1.2) exists an unique global smooth solution $(n(x,t),\mathbf{u}(x,t),\Phi(x,t),T(x,t))$ for all $t\geqslant 0$. Moreover,

$$\begin{aligned}&\|(n(\cdot,t)-\mathcal{N},\mathbf{u}(\cdot,t),\nabla\Phi(\cdot,t)-\mathcal{E})\|^2_{H^3(\mathbb{R}^N)}+\|T(\cdot,t)-T^0\|^2_{H^4(\mathbb{R}^N)}+\|(n_t,\mathbf{u}_t,\nabla\Phi_t,T_t)(\cdot,t)\|^2_{H^2(\mathbb{R}^N)}\\ &\leqslant C_0[\|(n(\cdot,0)-\mathcal{N},\mathbf{u}(\cdot,0),\nabla\Phi(\cdot,0)-\mathcal{E})\|^2_{H^3(\mathbb{R}^N)}+\|T(\cdot,0)-T^0\|^2_{H^4(\mathbb{R}^N)}\\ &\quad+\|(n_t,\mathbf{u}_t,\nabla\Phi_t,T_t)(\cdot,0)\|^2_{H^2(\mathbb{R}^N)}]\exp(-\alpha_0 t)\end{aligned} \tag{1.9}$$

for some positive constants α_0 and C_0.
Remark 1.3.1
The estimate (1.9) is proved by the careful energy method. Throughout introducing an suitable function, we can divide the nonisentropic case into two parts in energy estimates. One part is dealt with as the isentropic case similarly; As for another part, we are to give new estimates in order to obtain (1.9), for detail, see Section 2.

We list the following notations used in this paper: C denotes some generic constants. $H^m(\mathbb{R}^N), m\in\mathbb{Z}_+\cup\{0\}$, denotes the usual *Sobolev* space of order m equipped with the norm

$$\|g\|_{H^m(\mathbb{R}^N)}=\sum_{0\leqslant|\alpha|\leqslant m}\|\partial_x^\alpha g\|$$

where $\|\cdot\|=\|\cdot\|_{L^2(\mathbb{R}^N)}$ and $\partial_x^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_N^{\alpha_N}$ with $|\alpha|=\sum_{i=1}^N\alpha_i$ and $\partial_i=\partial_{x_i}$. We also label $\|(a,b,c,d)\|^2_{H^m(\mathbb{R}^\mathbb{N})}=\|a\|^2_{H^m(\mathbb{R}^\mathbb{N})}+\|b\|^2_{H^m(\mathbb{R}^\mathbb{N})}+\|c\|^2_{H^m(\mathbb{R}^\mathbb{N})}+\|d\|^2_{H^m(\mathbb{R}^\mathbb{N})}$, where $a,b,c,d\in$

$H^m(\mathbb{R}^{\mathbb{N}})$. The Euclidean norm and inner product for $\mathbb{R}^N$ are denoted by $|\cdot|$ and $a\cdot b$ for $a,b\in\mathbb{R}^N$, respectively. For a vector valued function $f=(f_1,f_2,\cdots,f_N)$ and a norm space X of scalar functions with the norm $|||\cdot|||$, $f\in X$ means that each component of f is in X; we put $|||f|||=|||f_1|||+|||f_2|||+\cdots+|||f_N|||$ and $\partial f=\partial_x f=(\partial_i f_j)_{N\times N}, \partial_x^k f=\partial_x(\partial_x^{k-1}f)$. For instance, $\partial_x V_t=(\partial_1 V_t,\partial_2 V_t,\cdots,\partial_N V_t),\ \partial_x^2 V_t=(\partial_1^2 V_t,\partial_1\partial_2 V_t,\cdots,\partial_N^2 V_t), etc.$ Moreover, $\int f$ means $\int_{\mathbb{R}^{\mathbb{N}}} f dx$ without any ambiguity. We shall also make use of some inequalities repeatedly as follows:

$Young's$ inequality:

$$|ab|\leqslant \epsilon a^2+C(\epsilon)b^2, \epsilon>0,$$

where $C(\epsilon)$ is some positive constant depending on ϵ;

$Gagliardo-Nirenberg's$ inequality:

$$\|u\|_{L^q}\leqslant C(N,q)\|u\|^{N/q-N/2+1}\|\nabla u\|^{N/2-N/q}$$

for $u\in H^1(\mathbb{R}^{\mathbb{N}}), q\geqslant 2$ when $N=2$ and $q\in[2,6]$ when $N=3$. And

$$\|u\|_{L^\infty}\leqslant C(N)\|u\|^{(4-N)/4}\|\partial_x^2 u\|^{N/4}$$

for $u\in H^2(\mathbb{R}^{\mathbb{N}})$. $C(N,q), C(N)$ are some positive constants depending on N,q and N, respectively.

2 Global existence and asymptotic behavior

In this section, we shall prove Theorem 1.3 by using the energy method. Set

$$n(x,t)=\mathcal{N}+V(x,t), \tag{2.1}$$

$$T(x,t)=T^0+y(x,t), \tag{2.2}$$

$$\nabla\Phi=\mathcal{E}+\mathbf{e}(x,t). \tag{2.3}$$

Then the function $(V,\mathbf{u},y,\mathbf{e})$ satisfies the following system:

$$\begin{cases} V_t+\mathrm{div}((\mathcal{N}+V)\mathbf{u})=0\\ \mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla(h(\mathcal{N}+V)-h(\mathcal{N}))+\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V}=\mathbf{e}-\mathbf{u}\\ y_t+\mathbf{u}\cdot\nabla y+\frac{2}{3}(T^0+y)\mathrm{div}\mathbf{u}-\frac{2}{3(\mathcal{N}+V)}\triangle y+y-\frac{1}{3}|\mathbf{u}|^2=0\\ \mathrm{div}\mathbf{e}=V \end{cases} \tag{2.4}$$

with the initial data

$$V(x,0)=n(x,0)-\mathcal{N},\quad \mathbf{u}(x,0)=\mathbf{u}_0(x),\quad y(x,0)=T_0(x)-T^0. \tag{2.5}$$

where the function $h(s)=T^0\ln s(s>0)$.

To prove Theorem 1.3, we need the local solution results and *a priori* estimate, which are given by Lemma 2.1 and Lemma 2.2, respectively.
Lemma 2.1 (local existence)
Suppose that $b(x)$ satisfies the condition (1.3),(1.4) and $n(\cdot,0)-\mathcal{N}\in H^3(\mathbb{R}^N), \mathbf{u}(\cdot,0)\in H^3(\mathbb{R}^N), \nabla\Phi(\cdot,0)-\mathcal{E}\in H^3(\mathbb{R}^N)$ and $T(\cdot,0)-T^0\in H^4(\mathbb{R}^N)$. Then there exists an unique smooth solution $(n(x,t),\mathbf{u}(x,t),\Phi(x,t),T(x,t))$ of the system (1.1)-(1.2) satisfying

$$n(x,t),\mathbf{u}(x,t),\nabla\Phi(x,t)\in C^1(\mathbb{R}^N\times[0,T_{\max})),$$

$$T(x,t)\in C^1(\mathbb{R}^N\times[0,T_{\max})),\quad T_{xx}\in C(\mathbb{R}^N\times[0,T_{\max}))$$

and

$$n(x,t)-\mathcal{N},\ \mathbf{u}(x,t),\ \nabla\Phi(x,t)-\mathcal{E}\in L^\infty(0,T;H^3(\mathbb{R}^N)),\ T(x,t)-T^0\in L^\infty(0,T;H^4(\mathbb{R}^N))$$

defined on a maximal interval of existence $[0,T_{\max})$. Moreover, if $T_{\max}<+\infty$, then

$$\|(n(\cdot,t)-\mathcal{N},\mathbf{u}(\cdot,t),\nabla\Phi(\cdot,t)-\mathcal{E})\|^2_{H^3(\mathbb{R}^N)}+\|T(\cdot,t)-T^0\|^2_{H^4(\mathbb{R}^N)}+\|(n_t,\mathbf{u}_t,\nabla\Phi_t,T_t)(\cdot,t)\|^2_{H^2(\mathbb{R}^N)}$$

$$+\int_0^t[\|(n(\cdot,\tau)-\mathcal{N},\mathbf{u}(\cdot,\tau),\nabla\Phi(\cdot,\tau)-\mathcal{E})\|^2_{H^3(\mathbb{R}^N)}+\|T(\cdot,\tau)-T^0\|^2_{H^4(\mathbb{R}^N)}$$
$$+\|(n_t,\mathbf{u}_t,\nabla\Phi_t,T_t)(\cdot,\tau)\|^2_{H^2(\mathbb{R}^N)}]d\tau\to\infty$$

as $t\to T_{\max}-$.
Remark 2.1.1
Using Green's formulation, the system (1.1) can be reduced to a strong coupled hyperbolic-parabolic system and the proof of the local solution can be established by a standard contraction mapping principle, which will be omitted here, see *e.g.* [12],[13].
Lemma 2.2 (*a priori* estimate)
Suppose that $(V,\mathbf{u},y,\mathbf{e})$ satisfies the system (2.4)-(2.5) for $(x,t)\in\mathbb{R}^N\times[0,T_{\max})$. Then there exists sufficiently small constant $\delta_1>0$, depending only on $b(x)$, such that for $0<S<T_{\max}$, if

$$\sup_{0\leqslant t\leqslant S}(\|(V,\mathbf{u},\mathbf{e})(\cdot,t)\|_{H^3(\mathbb{R}^N)}+\|y(\cdot,t)\|_{H^2(\mathbb{R}^N)}+\|(V_t,\mathbf{u}_t,y_t,\mathbf{e}_t)(\cdot,t)\|_{H^2(\mathbb{R}^N)}+\|\nabla b\|_{H^3(\mathbb{R}^N)})\leqslant\delta_1, \tag{2.6}$$

then

$$\begin{aligned}&\|(V,\mathbf{u},\mathbf{e})(\cdot,t)\|^2_{H^3(\mathbb{R}^N)}+\|y(\cdot,t)\|^2_{H^2(\mathbb{R}^N)}+\|(V_t,\mathbf{u}_t,y_t,\mathbf{e}_t)(\cdot,t)\|^2_{H^2(\mathbb{R}^N)}\\&\leqslant C_1(\|(V,\mathbf{u},\mathbf{e})(\cdot,0)\|^2_{H^3(\mathbb{R}^N)}+\|y(\cdot,0)\|^2_{H^2(\mathbb{R}^N)}\\&+\|(V_t,\mathbf{u}_t,y_t,\mathbf{e}_t)(\cdot,0)\|^2_{H^2(\mathbb{R}^N)})\exp(-\alpha_1 t)\end{aligned} \tag{2.7}$$

for any $t\in[0,S]$ and some positive constants α_1 and C_1.
Proof: From (1.8), *a priori* assumption (2.6) and *Sobolev's* inequality, we have

$$\sup_{x\in\mathbb{R}^N}|(V,\partial_x V,V_t,\mathbf{u},\partial_x\mathbf{u},\mathbf{u}_t,\mathbf{e},\partial_x\mathbf{e},\mathbf{e}_t,y,y_t,\partial_x\mathcal{N},\partial_x^2\mathcal{N})|$$

$$\leqslant C(\|(V,\mathbf{u},\mathbf{e})(\cdot,t)\|_{H^3(\mathbb{R}^N)}+\|y(\cdot,t)\|_{H^2(\mathbb{R}^N)}$$
$$+\|(V_t,\mathbf{u}_t,y_t,\mathbf{e}_t)(\cdot,t)\|_{H^2(\mathbb{R}^N)}+\|\nabla b\|_{H^3(\mathbb{R}^N)})\leqslant C\delta_1. \tag{2.8}$$

From the equation $(2.4)_3$, we can get

$$\triangle y=\frac{3}{2}(\mathcal{N}+V)f(x,t), \tag{2.9}$$

where

$$f(x,t)=y_t+\mathbf{u}\cdot\nabla y+\frac{2}{3}y\mathrm{div}\mathbf{u}+y+\frac{2}{3}T^0\mathrm{div}\mathbf{u}-\frac{1}{3}|\mathbf{u}|^2. \tag{2.10}$$

Then, with the help of (2.8) and the L^2-theory of elliptic operators, we have

$$\|\partial_x^3 y\|^2\leqslant C(\|f\|^2+\|\partial_x f\|^2)\leqslant C\delta_1^2$$

and

$$\|y(\cdot,t)\|_{H^4}^2\leqslant C\delta_1^2,$$

by using $Sobolev's$ inequality which gives

$$\|(\partial_x y,\partial_x^2 y)\|_{L^\infty}\leqslant C\delta_1. \tag{2.11}$$

On the other hand, by $(2.4)_1$, (2.8) and $Young's$ inequality, it is easy to get

$$\|\partial_x^i V_t\|\leqslant C\sum_{k=0}^{i+1}(\|\partial_x^k\mathbf{u}\|+\|\partial_x^k V\|)\quad i=0,1,2. \tag{2.12}$$

Take $\partial_x^j,\ j=1,2,3$ on both the sides of $(2.4)_4$ and multiply the resulting equation by $\partial_x^j\Phi$, then integrate it over $\mathbb{R}^{\mathbb{N}}$ to get

$$\int|\partial_x^j\mathbf{e}|^2=\int\partial_x^{j-1}V\partial_x^{j-1}(\mathrm{div}\mathbf{e})\leqslant\frac{1}{2}\int|\partial_x^j\mathbf{e}|^2+C\int|\partial_x^{j-1}V|^2.$$

i.e.

$$\|\partial_x^j\mathbf{e}\|^2\leqslant C\|\partial_x^{j-1}V\|^2,\quad j=1,2,3. \tag{2.13}$$

Similarly, we also have

$$\|\mathbf{e}_t\|^2\leqslant C(\|\mathbf{u}\|^2+\|V\|^2), \tag{2.14}$$

$$\|\partial_x^m\mathbf{e}_t\|^2\leqslant C\sum_{k=0}^{m}(\|\partial_x^k\mathbf{u}\|^2+\|\partial_x^k V\|^2),\quad m=1,2. \tag{2.15}$$

Now, multiplying $(2.4)_2$ by $\mathcal{N}\mathbf{u}$ and integrating it over $\mathbb{R}^N$, after integrating by parts, we have

$$\frac{d}{dt}\int\frac{\mathcal{N}}{2}|\mathbf{u}|^2+\int\mathcal{N}|\mathbf{u}|^2-\int(h(\mathcal{N}+V)-h(\mathcal{N}))\mathrm{div}(\mathcal{N}\mathbf{u})$$
$$+\int\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V}\mathcal{N}\mathbf{u}+\int((\mathbf{u}\cdot\nabla)\mathbf{u})\mathcal{N}\mathbf{u}-\int\mathcal{N}\mathbf{e}\mathbf{u}=0. \tag{2.16}$$

First, we choose δ_1 is sufficiently small such that

$$0 < \frac{\mathcal{N}}{2} \leqslant \mathcal{N} + V \leqslant 2\mathcal{N},$$

then there exist positive constants D_1, D_2 and D_3 (depending on T^0) such that

$$0 < D_1 \leqslant h'(\mathcal{N} + V) \leqslant D_2 < +\infty,$$

$$0 < |h^{(k)}(\mathcal{N} + V)| \leqslant D_3 < +\infty,$$

for any integer $k > 0$.
Thus, in estimating the third integration of (2.16)

$$\begin{aligned} - & \int (h(\mathcal{N}+V) - h(\mathcal{N}))\mathrm{div}(\mathcal{N}\mathbf{u}) \\ = & \frac{d}{dt}\int_{\mathbb{R}^N}\int_0^V (h(\mathcal{N}+s) - h(\mathcal{N}))dsdx + \int h'(\mathcal{N}+\theta V)V\mathrm{div}(\mathbf{u}V) \\ \geqslant & \frac{d}{dt}\int_{\mathbb{R}^N}\int_0^V (h(\mathcal{N}+s) - h(\mathcal{N}))dsdx - C\delta_1\int(|\mathbf{u}|^2 + |V|^2 + |\nabla V|^2), \end{aligned}$$

for some positive constant $\theta : 0 < \theta < 1$, with the smallness of $|V|$ and $|\nabla \mathbf{u}|$.
By (2.3),(2.4)$_4$ and (2.8), we have

$$\begin{aligned} -\int \mathcal{N}\mathbf{e}\mathbf{u} &= \int \mathbf{e}\mathbf{e}_t + \int V\mathbf{e}\mathbf{u} + \int \mathcal{E}(\mathbf{e}_t + (\mathcal{N}+V)\mathbf{u}) \\ &\geqslant \frac{1}{2}\frac{d}{dt}\int |\mathbf{e}|^2 - C\delta_1\int(|\mathbf{u}|^2 + |V|^2), \end{aligned}$$

where we have used the fact that

$$\int \partial_x^i \mathcal{E}\partial_x^i(\mathbf{e}_t + (\mathcal{N}+V)\mathbf{u}) = 0, \quad i = 0,1,2,3. \tag{2.17}$$

In fact, from (2.4)$_1$ and (2.4)$_4$, we can get

$$\partial_x^i \mathrm{div}(\mathbf{e}_t + (\mathcal{N}+V)\mathbf{u}) = 0.$$

Multiply the above equality by $\partial_x^i h(\mathcal{N})$ and integrate it over $\mathbb{R}^N$ to have

$$\int \partial_x^i h(\mathcal{N})\partial_x^i \mathrm{div}(\mathbf{e}_t + (\mathcal{N}+V)\mathbf{u}) = 0,$$

then integrating by parts and $\nabla h(\mathcal{N}) = T^0\frac{\nabla\mathcal{N}}{\mathcal{N}} = \mathcal{E}$, (2.17) is followed.
It is easy to get

$$\int((\mathbf{u}\cdot\nabla)\mathbf{u})\mathcal{N}\mathbf{u} \geqslant -C\delta_1\int(|\mathbf{u}|^2 + |\nabla\mathbf{u}|^2).$$

By (2.8), we have

$$\int \frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V}\mathcal{N}\mathbf{u} \geqslant - \int \mathcal{N}y\mathrm{div}\mathbf{u} - C\delta_1\int(|\mathbf{u}|^2+|y|^2)$$

Therefore, (2.16) together with these estimates implies

$$\frac{d}{dt}\int[\frac{\mathcal{N}}{2}|\mathbf{u}|^2+\int_0^V(h(\mathcal{N}+s)-h(\mathcal{N}))ds+\frac{1}{2}|\mathbf{e}|^2]+C\int|\mathbf{u}|^2$$
$$\leqslant C\delta_1\int(|V|^2+|\nabla V|^2+|\nabla\mathbf{u}|^2+|y|^2)+\int\mathcal{N}y\mathrm{div}\mathbf{u}. \tag{2.18}$$

Taking ∂_t on both the sides of $(2.4)_2$, then multiplying the resulting equation by $\mathcal{N}\mathbf{u}_t$ and integrating it over $\mathbb{R}^N$, we have

$$\frac{d}{dt}\int\frac{\mathcal{N}}{2}|\mathbf{u}_t|^2+\int\mathcal{N}|\mathbf{u}_t|^2+\int\nabla(h(\mathcal{N}+V)-h(\mathcal{N}))_t\partial_t(\mathcal{N}\mathbf{u})$$
$$+\int[\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V}]_t\mathcal{N}\mathbf{u}_t+\int((\mathbf{u}\cdot\nabla)\mathbf{u})_t\mathcal{N}\mathbf{u}_t-\int\mathcal{N}\mathbf{e}_t\mathbf{u}_t=0. \tag{2.19}$$

First, integrating by parts and (2.8) give

$$\begin{aligned}
&\int \nabla(h(\mathcal{N}+V)-h(\mathcal{N}))_t\partial_t(\mathcal{N}\mathbf{u})\\
=&-\int\mathcal{N}(h(\mathcal{N}+V)-h(\mathcal{N}))_t\mathrm{div}\mathbf{u}_t-\int\nabla\mathcal{N}(h(\mathcal{N}+V)-h(\mathcal{N}))_t\mathbf{u}_t\\
\geqslant&\ \frac{d}{dt}\int\frac{\mathcal{N}h'(\mathcal{N}+\theta V)}{2(\mathcal{N}+V)}|V_t|^2+\int\frac{\mathcal{N}h'(\mathcal{N}+\theta V)}{\mathcal{N}+V}\mathbf{u}\cdot\nabla(\frac{|V_t|^2}{2})-C\delta_1\int(|V_t|^2+|\mathbf{u}|^2)\\
\geqslant&\ \frac{d}{dt}\int\frac{\mathcal{N}h'(\mathcal{N}+\theta V)}{2(\mathcal{N}+V)}|V_t|^2-C\delta_1\int(|V_t|^2+|\mathbf{u}|^2),
\end{aligned}$$

for some the positive constant $\theta : 0<\theta<1$.
From (2.8), we have

$$\int[\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V}]_t\mathcal{N}\mathbf{u}_t \geqslant - \int\mathcal{N}y_t\mathrm{div}\mathbf{u}_t-C\delta_1\int(|V_t|^2+|\nabla V_t|^2+|\mathbf{u}_t|^2+|y_t|^2)$$

with

$$\int((\mathbf{u}\cdot\nabla)\mathbf{u})_t\mathcal{N}\mathbf{u}_t\geqslant -C\delta_1\int(|\mathbf{u}_t|^2+|\nabla\mathbf{u}|^2+|\nabla\mathbf{u}_t|^2),$$
$$-\int\mathcal{N}\mathbf{e}_t\mathbf{u}_t\geqslant\frac{1}{2}\frac{d}{dt}\int|\mathbf{e}_t|^2-C\delta_1\int(|V|^2+|\mathbf{u}_t|^2+|\mathbf{e}_t|^2).$$

So, (2.19) together with these estimates implies

$$\frac{d}{dt}\int[\frac{\mathcal{N}}{2}|\mathbf{u}_t|^2+\frac{\mathcal{N}h'(\mathcal{N}+\theta V)}{2(\mathcal{N}+V)}|V_t|^2+\frac{1}{2}|\mathbf{e}_t|^2]+C\int|\mathbf{u}_t|^2\leqslant C\delta_1(\|(V_t,\mathbf{u},\mathbf{u}_t)\|_{H^1}^2$$

$$+\|(V,\mathbf{e}_t,y_t)\|^2)+\int\mathcal{N}y_t\mathrm{div}\mathbf{u}_t. \tag{2.20}$$

Take $\partial_t^l(l=0,1)$ on the both side of $(2.4)_3$ and multiply the resulting equation by $\mathcal{N}\partial_t^l y$, then integrate in over $\mathbb{R}^N$ to get

$$\frac{d}{dt}\int\frac{\mathcal{N}}{2}|\partial_t^l y|^2+\int\mathcal{N}|\partial_t^l y|^2+\frac{2}{3}T^0\int\mathcal{N}\partial_t^l y\partial_t^l\mathrm{div}\mathbf{u}-\frac{2}{3}\int\partial_t^l(\frac{\triangle y}{\mathcal{N}+V})\mathcal{N}\partial_t^l y$$

$$+\int\partial_t^l(\mathbf{u}\cdot\nabla y+\frac{2}{3}y\mathrm{div}\mathbf{u}-\frac{1}{3}|\mathbf{u}|^2)\mathcal{N}\partial_t^l y=0. \tag{2.21}$$

It is easy to estimate that the last integral on the left side of (2.21) is no less than

$$-C\delta_1\|(\mathbf{u},\mathbf{u}_t,y,y_t)\|_{H^1}^2\quad,$$

and the fourth integral is estimated as follows:

$$-\frac{2}{3}\int\partial_t^l(\frac{\triangle y}{\mathcal{N}+V})\mathcal{N}\partial_t^l y\geqslant\int\frac{2\mathcal{N}}{3(\mathcal{N}+V)}(|\partial_x y|^2+|\partial_x y_t|^2)-C\delta_1(\|(y,y_t)\|_{H^1}^2+\int|V_t|^2).$$

(2.21) together the two estimates, we have

$$\frac{d}{dt}\int\frac{\mathcal{N}}{2}(|y|^2+|y_t|^2)+C\|(y,y_t)\|_{H^1}^2+\frac{2}{3}T^0\int(\mathcal{N}y\mathrm{div}\mathbf{u}+\mathcal{N}y_t\mathrm{div}\mathbf{u}_t)$$

$$\leqslant C\delta_1(\|(\mathbf{u},\mathbf{u}_t)\|_{H^1}^2+\int|V_t|^2). \tag{2.22}$$

Combining (2.18), (2.20) and (2.22), we have

$$\frac{d}{dt}\int[\frac{\mathcal{N}}{2}(|\mathbf{u}|^2+|\mathbf{u}_t|^2)+\int_0^V(h(\mathcal{N}+s)-h(\mathcal{N}))ds+\frac{\mathcal{N}h'(\mathcal{N}+\theta V)}{2(\mathcal{N}+V)}|V_t|^2+\frac{1}{2}(|\mathbf{e}|^2+|\mathbf{e}_t|^2)$$

$$+\frac{3\mathcal{N}}{4T^0}(|y|^2+|y_t|^2)]+C(\|(y,y_t)\|_{H^1}^2+\int(|\mathbf{u}|^2+|\mathbf{u}_t|^2))$$

$$\leqslant C\delta_1(\|(V,V_t)\|_{H^1}^2+\|(\nabla\mathbf{u},\nabla\mathbf{u}_t,\mathbf{e}_t)\|^2). \tag{2.23}$$

Taking div to $(2.4)_2$ and multiplying $\mathrm{div}\mathbf{u}$ on the both side of $(2.4)_2$, then integrating it over $\mathbb{R}^N$, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{div}\mathbf{u}\|^2+\|\mathrm{div}\mathbf{u}\|^2=-\int\mathrm{div}(\nabla(h(\mathcal{N}+V)-h(\mathcal{N})))\mathrm{div}\mathbf{u}-\int\mathrm{div}(\mathbf{u}\cdot\nabla\mathbf{u})\mathrm{div}\mathbf{u}$$

$$-\int\mathrm{div}(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\mathrm{div}\mathbf{u}+\int\mathrm{div}\mathbf{e}\mathrm{div}\mathbf{u}. \tag{2.24}$$

By $(2.4)_1$ and (2.8), we have

$$
\begin{aligned}
-\int \operatorname{div}(\nabla(h(\mathcal{N}+V)-h(\mathcal{N})))\operatorname{div}\mathbf{u} &= \int\{[\nabla(h(\mathcal{N}+V)-h(\mathcal{N}))]\cdot\nabla(\frac{V_t+\nabla(\mathcal{N}+V)\mathbf{u}}{\mathcal{N}+V})\}\\
&\leqslant -\frac{d}{dt}\int\frac{h'(\mathcal{N}+V)}{2(\mathcal{N}+V)}|\nabla V|^2\\
&+ C\delta_1\int(|V|^2+|V_t|^2+|\nabla V|^2+|\mathbf{u}|^2+|\nabla\mathbf{u}|^2).
\end{aligned}
$$

Using (2.8),(2.9),(2.10) and $Young's$ inequality, we get

$$
\begin{aligned}
&-\int \operatorname{div}\ (\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\operatorname{div}\mathbf{u}\\
&\leqslant -\frac{1}{2}\frac{d}{dt}\int\frac{y}{(\mathcal{N}+V)^2}|\nabla V|^2-\int\frac{y\mathbf{u}}{(\mathcal{N}+V)^2}\nabla(\frac{|\nabla(\mathcal{N}+V)|^2}{2})\\
&+ \epsilon\int|\nabla V_t|^2+C(\epsilon)\int|\nabla y|^2+C\delta_1(\|(V,V_t,\mathbf{u},y)\|_{H^1}^2+\int|y_t|^2)\\
&\leqslant -\frac{1}{2}\frac{d}{dt}\int\frac{y}{(\mathcal{N}+V)^2}|\nabla V|^2+C\delta_1(\|(V,V_t,\mathbf{u},y)\|_{H^1}^2+\int|y_t|^2)+C\int|\nabla y|^2,
\end{aligned}
$$

with the help of the smallness of ϵ and δ_1.
By the equation $(2.4)_1$ and $(2.4)_4$, we have

$$
\begin{aligned}
\int \operatorname{div}e\operatorname{div}\mathbf{u} &= -\int\frac{V}{\mathcal{N}}(V_t+\mathbf{u}\nabla\mathcal{N}+\operatorname{div}(\mathbf{u}V))\\
&\leqslant -\frac{d}{dt}\int\frac{|V|^2}{2\mathcal{N}}+C\delta_1\int(|V|^2+|\mathbf{u}|^2+|\nabla\mathbf{u}|^2).
\end{aligned}
$$

By integrating by parts and (2.8) to get

$$
\begin{aligned}
-\int\operatorname{div}(\mathbf{u}\cdot\nabla\mathbf{u})\operatorname{div}\mathbf{u} &= -\int\partial_i(u^j\partial_j u^i)\partial_k u^k\\
&= -\int(\partial_i u^j\partial_j u^i\operatorname{div}\mathbf{u}-\frac{1}{2}(\operatorname{div}\mathbf{u})^3)\\
&\leqslant C\delta_1\int|\nabla\mathbf{u}|^2.
\end{aligned}
$$

(2.24) together these estimates implies

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\|\operatorname{div}\mathbf{u}\|^2+\frac{h'(\mathcal{N}+V)}{\mathcal{N}+V}|\nabla V|^2+\frac{y}{(\mathcal{N}+V)^2}|\nabla V|^2+\frac{|V|^2}{\mathcal{N}})+\|\operatorname{div}\mathbf{u}\|^2\\
&\leqslant C\delta_1(\|(V,V_t,\mathbf{u},y)\|_{H^1}^2+\int|y_t|^2)+C\int|\nabla y|^2. \qquad (2.25)
\end{aligned}
$$

Taking curl to $(2.4)_2$ and multiplying curl$\mathbf{u}$ on the both side of $(2.4)_2$, then integrating it over $\mathbb{R}^N$, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{curl}\mathbf{u}\|^2+\|\mathrm{curl}\mathbf{u}\|^2+\int\mathrm{curl}(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\mathrm{curl}\mathbf{u}=-\int\mathrm{curl}(\mathbf{u}\cdot\nabla\mathbf{u})\mathrm{curl}\mathbf{u}. \quad (2.26)$$

The direct calculation gives

$$\begin{aligned}\int\mathrm{curl}(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\mathrm{curl}\mathbf{u} &= \int[\nabla(\frac{1}{\mathcal{N}+V})\times\nabla((\mathcal{N}+V)y)]\mathrm{curl}\mathbf{u}\\ &= -\int\frac{1}{\mathcal{N}+V}\nabla(\mathcal{N}+V)\times\nabla y\ \mathrm{curl}\mathbf{u}\\ &= -\int\frac{1}{\mathcal{N}+V}(\partial_i(\mathcal{N}+V)\partial_j y-\partial_j(\mathcal{N}+V)\partial_i y)\mathrm{curl}\mathbf{u}\\ &\leqslant C\delta_1\int(|\nabla V|^2+|\nabla\mathbf{u}|^2+|\nabla y|^2)\end{aligned}$$

and

$$\begin{aligned}-\int\mathrm{curl}(\mathbf{u}\cdot\nabla\mathbf{u})\mathrm{curl}\mathbf{u} &= -\int(\partial_k(u^j\partial_j u^i)-\partial_i(u^j\partial_j u^k))(\partial_k u^i-\partial_i u^k)\\ &\leqslant C\delta_1\|\nabla\mathbf{u}\|^2.\end{aligned}$$

Then, (2.26) with two estimates to give

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{curl}\mathbf{u}\|^2+\|\mathrm{curl}\mathbf{u}\|^2\leqslant C\delta_1\int(|\nabla V|^2+|\nabla\mathbf{u}|^2+|\nabla y|^2). \quad (2.27)$$

Together with (2.25) and (2.27), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(\|\nabla\mathbf{u}\|^2+\frac{h'(\mathcal{N}+V)}{\mathcal{N}+V}|\nabla V|^2+\frac{y}{(\mathcal{N}+V)^2}|\nabla V|^2+\frac{|V|^2}{\mathcal{N}})+C\|\nabla\mathbf{u}\|^2\\ &\leqslant C\delta_1[\|(V,V_t,y)\|_{H^1}^2+\int(|\mathbf{u}|^2+|y_t|^2)]+C\int|\nabla y|^2. \quad (2.28)\end{aligned}$$

On the other hand, multiplying ∇V on the both sides of $(2.4)_2$ and integrating in over $\mathbb{R}^N$, we can get

$$\begin{aligned}\int[(h'(\mathcal{N}+V)-h'(\mathcal{N}))\nabla\mathcal{N}+h'(\mathcal{N}+V)\nabla V]\nabla V &= \int\mathbf{e}\nabla V-\int(\mathbf{u}_t+\mathbf{u})\nabla V\\ &-\int(\mathbf{u}\cdot\nabla\mathbf{u})\nabla V-\int\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V}\nabla V. \quad (2.29)\end{aligned}$$

It is easy to obtain the following estimates,

$$\int\mathbf{e}\nabla V=-\int V\mathrm{div}\mathbf{e}=-\|V\|^2,$$

$$\int (\mathbf{u}_t + \mathbf{u})\nabla V \leqslant \epsilon \int |\nabla V|^2 + C(\epsilon)\int (|\mathbf{u}|^2 + |\mathbf{u}_t|^2),$$

$$-\int (\mathbf{u}\cdot\nabla\mathbf{u})\nabla V \leqslant C\delta_1 \int (|\nabla V|^2 + |\nabla \mathbf{u}|^2),$$

$$\begin{aligned} -\int \frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V}\nabla V &= -\int \nabla y \nabla V - \int \frac{y}{\mathcal{N}+V}\nabla(\mathcal{N}+V)\nabla V \\ &\leqslant \epsilon \int |\nabla y|^2 + C(\epsilon)\int |\nabla V|^2 + C\delta_1 \int (|\nabla V|^2 + |y|^2). \end{aligned}$$

(2.29) together with these estimates, we have

$$\int (|V|^2 + |\nabla V|^2) \leqslant C\delta_1 \int (|y|^2 + |\nabla y|^2 + |\nabla \mathbf{u}|^2) + C\int (|\mathbf{u}|^2 + |\mathbf{u}_t|^2), \tag{2.30}$$

with the help of the smallness of ϵ and δ_1.
From the equation $(2.4)_2$, we can get

$$\|\mathbf{e}\|^2 \leqslant C\|(\mathbf{u}, \mathbf{u}_t, \nabla\mathbf{u}, \nabla V, y, \nabla y)\|^2. \tag{2.31}$$

On the other hand, using $(2.4)_1$, $(2.4)_4$ and (2.17) with $i = 1$ we have

$$\begin{aligned} -\int \partial_x \mathbf{e}\partial_x(\mathcal{N}\mathbf{u}) &= \int \partial_x \mathbf{e}\partial_x \mathbf{e}_t + \int \partial_x \mathbf{e}\partial_x(V\mathbf{u}) + \int \partial_x \mathcal{E}\partial_x(\mathbf{e}_t + (\mathcal{N}+V)\mathbf{u}) \\ &= \frac{d}{dt}\int \frac{1}{2}|\partial_x \mathbf{e}|^2 + \int \partial_x \mathbf{e}\partial_x(V\mathbf{u}), \end{aligned}$$

which implies

$$\frac{d}{dt}\int \frac{1}{2}|\partial_x \mathbf{e}|^2 \leqslant C\delta_1 \int (|\partial_x V|^2 + |\partial_x \mathbf{u}|^2 + |\mathbf{u}|^2 + |\partial_x \mathbf{e}|^2) + \epsilon \int |\partial_x \mathbf{e}|^2 + C(\epsilon)\int |\partial_x \mathbf{u}|^2.$$

Therefore, combining (2.23),(2.28),(2.30) and (2.31), furthermore, noticing that (2.14),(2.12) with $i = 0$ and (2.13) with $j = 1$, we have

$$\begin{aligned} &\frac{d}{dt}\int [\frac{\mathcal{N}}{2}(|\mathbf{u}|^2 + |\mathbf{u}_t|^2) + \int_0^V (h(\mathcal{N}+s) - h(\mathcal{N}))ds + \frac{\mathcal{N}h'(\mathcal{N}+\theta V)}{2(\mathcal{N}+V)}|V_t|^2 + \frac{1}{2}(|\mathbf{e}|^2 + |\mathbf{e}_t|^2) \\ &+ \frac{3\mathcal{N}}{4T^0}(|y|^2 + |y_t|^2) + \frac{1}{2}|\nabla \mathbf{u}|^2 + \frac{h'(\mathcal{N}+V)}{2(\mathcal{N}+V})|\nabla V|^2 + \frac{y}{2(\mathcal{N}+V)^2}|\nabla V|^2 + \frac{|V|^2}{2\mathcal{N}} + \frac{1}{2}|\partial_x \mathbf{e}|^2] \\ &+ C\|(V, \nabla V, V_t, \mathbf{u}, \nabla\mathbf{u}, \mathbf{u}_t, y, \nabla y, y_t, \nabla y_t, \mathbf{e}, \nabla \mathbf{e}, \mathbf{e}_t)\|^2 \leqslant C\delta_1 \int (|\nabla \mathbf{u}_t|^2 + |\nabla V_t|^2). \end{aligned} \tag{2.32}$$

The next step is to get the estimates of the second derivatives, due to similar to the estimating progress of the first derivatives, so we give a brief form.

Take ∂_x^2 on the both sides of $(2.4)_2$ and multiply the resulting equation by $\partial_x^2(\mathcal{N}\mathbf{u})$, then integrate it over $\mathbb{R}^N$ to get

$$\int \partial_x^2\mathbf{u}_t\partial_x^2(\mathcal{N}\mathbf{u})+\int \partial_x^2\mathbf{u}\partial_x^2(\mathcal{N}\mathbf{u})+\int \partial_x^2\nabla(h(\mathcal{N}+V)-h(\mathcal{N}))\partial_x^2(\mathcal{N}\mathbf{u})+\int \partial_x^2(\mathbf{u}\cdot\nabla\mathbf{u})\partial_x^2(\mathcal{N}\mathbf{u})$$
$$+\int \partial_x^2(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\partial_x^2(\mathcal{N}\mathbf{u})-\int \partial_x^2\mathbf{e}\partial_x^2(\mathcal{N}\mathbf{u})=0. \tag{2.33}$$

The terms in (2.33) can be estimated to the following forms:

$$\int \partial_x^2\mathbf{u}_t\partial_x^2(\mathcal{N}\mathbf{u}) \geqslant \frac{d}{dt}\int(\frac{\mathcal{N}}{2}|\partial_x^2\mathbf{u}|^2+\partial_x^2\mathcal{N}\mathbf{u}\partial_x^2\mathbf{u}+2\partial_x\mathcal{N}\partial_x\mathbf{u}\partial_x^2\mathbf{u})-C\delta_1\int(|\partial_x^2\mathbf{u}|^2+|\mathbf{u}_t|^2+|\partial_x\mathbf{u}_t|^2),$$

$$\int \partial_x^2\mathbf{u}\partial_x^2(\mathcal{N}\mathbf{u}) \geqslant \int \mathcal{N}|\partial_x^2\mathbf{u}|^2 - C\delta_1\int(|\mathbf{u}|^2+|\partial_x\mathbf{u}|^2+|\partial_x^2\mathbf{u}|^2),$$

$$\int \partial_x^2(\mathbf{u}\cdot\nabla\mathbf{u})\partial_x^2(\mathcal{N}\mathbf{u}) \geqslant -C\delta_1\int|\partial_x^2\mathbf{u}|^2,$$

$$-\int \partial_x^2\mathbf{e}\partial_x^2(\mathcal{N}\mathbf{u}) \geqslant \frac{d}{dt}\int\frac{1}{2}|\partial_x^2\mathbf{e}|^2 - C\delta_1(\|(\partial_xV,\partial_x\mathbf{u})\|_{H^1}^2+\int|\partial_x^2\mathbf{e}|^2),$$

$$\begin{aligned}
\int &\ \partial_x^2\nabla(h(\mathcal{N}+V)-h(\mathcal{N}))\partial_x^2(\mathcal{N}\mathbf{u})\\
\geqslant &\ \frac{d}{dt}\int(\frac{\mathcal{N}\partial_x^2(h(\mathcal{N}+V)-h(\mathcal{N}))\partial_x^2V}{\mathcal{N}+V}-\frac{\mathcal{N}h'(\mathcal{N}+V)|\partial_x^2V|^2}{2(\mathcal{N}+V)})\\
- &\ \epsilon\int|\partial_x^2V|^2 - C\delta_1(\|(V,\mathbf{u})\|_{H^2}^2+\int|\partial_xV_t|^2),
\end{aligned}$$

$$\begin{aligned}
\int &\ \partial_x^2(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\partial_x^2(\mathcal{N}\mathbf{u})\\
\geqslant &\ \frac{d}{dt}\int\frac{\mathcal{N}y}{2(\mathcal{N}+V)^2}|\partial_x^2V|^2 - C\delta_1(\|(\mathbf{u},y)\|_{H^2}^2\\
+ &\ \int(|V_t|^2+|\partial_xV_t|^2+|\partial_x^2V|^2)) - \int\mathcal{N}\partial_x^2y\partial_x^2(\mathrm{div}\mathbf{u}).
\end{aligned}$$

(2.33) together these estimates implies

$$\frac{d}{dt}\int[\frac{\mathcal{N}}{2}|\partial_x^2\mathbf{u}|^2+\partial_x^2\mathcal{N}\mathbf{u}\partial_x^2\mathbf{u}+2\partial_x\mathcal{N}\partial_x\mathbf{u}\partial_x^2\mathbf{u}+\frac{1}{2}|\partial_x^2\mathbf{e}|^2+\frac{\mathcal{N}\partial_x^2(h(\mathcal{N}+V)-h(\mathcal{N}))\partial_x^2V}{\mathcal{N}+V}$$
$$-\frac{\mathcal{N}h'(\mathcal{N}+V)|\partial_x^2V|^2}{2(\mathcal{N}+V)}+\frac{\mathcal{N}y}{2(\mathcal{N}+V)^2}|\partial_x^2V|^2]+C\int|\partial_x^2\mathbf{u}|^2 \leqslant C\delta_1(\|(V,y)\|_{H^2}$$
$$+\|(V_t,\mathbf{u},\mathbf{u}_t)\|_{H^1}^2+\int|\partial_x^2\mathbf{e}|^2)+\int\mathcal{N}\partial_x^2y\partial_x^2(\mathrm{div}\mathbf{u}). \tag{2.34}$$

Similar to (2.30), we have

$$\int(|\partial_x V|^2+|\partial_x^2 V|^2)\leqslant C\delta_1\int(|\partial_x^2\mathbf{u}|^2+|\partial_x\mathbf{u}|^2+|\partial_x^2 y|^2+|\partial_x y|^2+|y|^2)$$
$$+C\int(|\partial_x\mathbf{u}|^2+|\partial_x\mathbf{u}_t|^2). \tag{2.35}$$

Taking $\partial_x\partial_t$ to $(2.4)_2$ and multiplying $\partial_x(\mathcal{N}\mathbf{u}_t)$, then integrating it over $\mathbb{R}^N$, we have

$$\int\partial_x\mathbf{u}_{tt}\partial_x(\mathcal{N}\mathbf{u}_t)+\int\partial_x\mathbf{u}_t\partial_x(\mathcal{N}\mathbf{u}_t)+\int\partial_{xt}\nabla(h(\mathcal{N}+\mathbf{u})-h(\mathcal{N}))\partial_x(\mathcal{N}\mathbf{u}_t)-\int\partial_x\mathbf{e}_t\partial_x(\mathcal{N}\mathbf{u}_t)$$
$$+\int\partial_{xt}(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\partial_x(\mathcal{N}\mathbf{u}_t)+\int\partial_{xt}(\mathbf{u}\cdot\nabla\mathbf{u})\partial_x(\mathcal{N}\mathbf{u}_t)=0. \tag{2.36}$$

Employing the equation$(2.4)_2$ and (2.8), we get

$$\begin{aligned}\int\partial_x\mathbf{u}_{tt}\partial_x(\mathcal{N}\mathbf{u}_t) &\geqslant \frac{d}{dt}\int(\frac{\mathcal{N}}{2}|\partial_x\mathbf{u}_t|^2+\partial_x\mathcal{N}\mathbf{u}_t\partial_x\mathbf{u}_t)-C\delta_1\int(|\partial_x\mathbf{u}_t|^2+|\mathbf{u}_{tt}|^2)\\ &\geqslant \frac{d}{dt}\int(\frac{\mathcal{N}}{2}|\partial_x\mathbf{u}_t|^2+\partial_x\mathcal{N}\mathbf{u}_t\partial_x\mathbf{u}_t)-C\delta_1(\|(V_t,\mathbf{u}_t,y_t)\|_{H^1}^2+\int(|\mathbf{u}|^2+|\mathbf{e}_t|^2))\end{aligned}$$

After the direct calculation, we arrive at

$$\int\partial_x\mathbf{u}_t\partial_x(\mathcal{N}\mathbf{u}_t)\geqslant\int\mathcal{N}|\partial_x\mathbf{u}_t|^2-C\delta_1\int(|\mathbf{u}_t|^2+|\partial_x\mathbf{u}_t|^2),$$

$$-\int\partial_x\mathbf{e}_t\partial_x(\mathcal{N}\mathbf{u}_t)\geqslant\frac{d}{dt}\int\frac{1}{2}|\partial_x\mathbf{e}_t|^2-C\delta_1\int(|\partial_x\mathbf{e}_t|^2+|\partial_x V_t|^2+|\partial_x\mathbf{u}_t|^2),$$

$$\int\partial_{xt}(\mathbf{u}\cdot\nabla\mathbf{u})\partial_x(\mathcal{N}\mathbf{u}_t)\geqslant-C\delta_1\int(|\partial_x^2\mathbf{u}|^2+|\partial_x\mathbf{u}|^2+|\partial_x\mathbf{u}_t|^2+|\mathbf{u}_t|^2),$$

$$\int\partial_{xt}\nabla(h(\mathcal{N}+V)-h(\mathcal{N}))\partial_x(\mathcal{N}\mathbf{u}_t)=\int\partial_x(h'(\mathcal{N}+V)V_t)\partial_x(\nabla\mathcal{N}\mathbf{u}_t)$$
$$-\textstyle\int\partial_x(h'(\mathcal{N}+V)V_t)\partial_x(\mathcal{N}\mathrm{div}\mathbf{u}_t)$$

$$\begin{aligned}&\geqslant \int\frac{\mathcal{N}h'(\mathcal{N}+V)}{\mathcal{N}+V}\partial_x V_t(\partial_x V_{tt}+\partial_x(\mathbf{u}\cdot\nabla\mathbf{u})_t)\\ &- C\delta_1\int(|V_t|^2+|\partial_x V_t|^2+|V_{tt}|^2+|\nabla V|^2+|\nabla\mathbf{u}|^2+|\nabla\mathbf{u}_t|^2)\\ &\geqslant \frac{d}{dt}\int\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)}|\partial_x V_t|^2\\ &- C\delta_1\int(|V_t|^2+|\nabla V_t|^2+|\nabla V|^2+|\mathbf{u}_t|^2+|\nabla\mathbf{u}|^2+|\nabla\mathbf{u}_t|^2),\end{aligned}$$

We deal with the last second integral in (2.36) next.

$$\begin{aligned}\int \partial_{xt}(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\partial_x(\mathcal{N}\mathbf{u}_t) &\geqslant \int \partial_x\nabla y_t\partial_x(\mathcal{N}\mathbf{u}_t)+\int \frac{y}{\mathcal{N}+V}\partial_x\nabla V_t\partial_x(\mathcal{N}\mathbf{u}_t)\\ &- C\delta_1(\|(V_t,\mathbf{u}_t,y_t,y)\|^2_{H^1}+\int|\partial_x^2V|^2).\end{aligned}$$

$$\int \partial_x\nabla y_t(\partial_x\mathcal{N}\mathbf{u}_t+\mathcal{N}\partial_x\mathbf{u}_t)\geqslant - \int \mathcal{N}\partial_x y_t\partial_x\mathrm{div}\mathbf{u}_t - C\delta_1\int(|\mathbf{u}_t|^2+|\partial_x\mathbf{u}_t|^2+|\partial_x y_t|^2),$$

and

$$\int \frac{y}{\mathcal{N}+V}\partial_x\nabla V_t(\partial_x\mathcal{N}\mathbf{u}_t+\mathcal{N}\partial_x\mathbf{u}_t)\geqslant \frac{d}{dt}\int\frac{\mathcal{N}y}{2(\mathcal{N}+V)^2}|\partial_x V_t|^2 - C\delta_1(\|(V_t,\mathbf{u}_t,y)\|^2_{H^1}+\int|\partial_x^2V|^2),$$

so, combining two estimates, we arrive at

$$\begin{aligned}\int \partial_{xt}(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\partial_x(\mathcal{N}\mathbf{u}_t) &\geqslant \frac{d}{dt}\int\frac{\mathcal{N}y}{2(\mathcal{N}+V)^2}|\partial_x V_t|^2 - C\delta_1(\|(V_t,\mathbf{u}_t,y,y_t)\|^2_{H^1}+\int|\partial_x^2V|^2)\\ &- \int \mathcal{N}\partial_x y_t\partial_x\mathrm{div}\mathbf{u}_t.\end{aligned}$$

(2.36) together these estimates and noticing (2.15) with $m=1$ implies

$$\begin{aligned}&\frac{d}{dt}\int[\frac{\mathcal{N}}{2}|\partial_x\mathbf{u}_t|^2+\partial_x\mathcal{N}\mathbf{u}_t\partial_x\mathbf{u}_t+\frac{1}{2}|\partial_x\mathbf{e}_t|^2+\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)}|\partial_x V_t|^2+\frac{\mathcal{N}y}{2(\mathcal{N}+V)^2}|\partial_x V_t|^2]\\ &+C\int(|\partial_x\mathbf{u}_t|^2+|\partial_x\mathbf{e}_t|^2)\leqslant C\delta_1(\|(\partial_x V,V_t,y,y_t)\|^2_{H^1}+\|\mathbf{u}\|^2_{H^2}+\int(|\mathbf{u}_t|^2+|\mathbf{e}_t|^2))\\ &\qquad +C\int(|\partial_x V|^2+|\partial_x\mathbf{u}|^2)+\int\mathcal{N}\partial_x y_t\partial_x\mathrm{div}\mathbf{u}_t. \qquad (2.37)\end{aligned}$$

Taking $\partial_x\partial_t^l$ $(l=0,1)$ to $(2.4)_3$ and multiplying $\mathcal{N}\partial_x\partial_t^l y$, then integrating it over $\mathbb{R}^N$, we have

$$\begin{aligned}&\frac{d}{dt}\int\frac{\mathcal{N}}{2}|\partial_x\partial_t^l y|^2+\int\mathcal{N}|\partial_x\partial_t^l y|^2+\frac{2}{3}T^0\int\mathcal{N}\partial_x\partial_t^l y\partial_x\partial_t^l\mathrm{div}\mathbf{u}-\frac{2}{3}\int\partial_x\partial_t^l(\frac{\triangle y}{\mathcal{N}+V})\mathcal{N}\partial_x\partial_t^l y\\ &\qquad +\int\partial_x\partial_t^l(\mathbf{u}\cdot\nabla y+\frac{2}{3}y\mathrm{div}\mathbf{u}-\frac{1}{3}|\mathbf{u}|^2)\mathcal{N}\partial_x\partial_t^l y=0. \qquad (2.38)\end{aligned}$$

It is easy to estimate that the last integral on the left side of (2.38) is no less than

$$-C\delta_1(\|(\partial_x\mathbf{u},\partial_x y,\partial_x y_t)\|^2_{H^1}+\int|\partial_x\mathbf{u}_t|^2)\ \ ,$$

and the fourth integral is estimated as follows:

$$-\frac{2}{3}\int \partial_x\partial_t^l(\frac{\triangle y}{\mathcal{N}+V})\mathcal{N}\partial_x\partial_t^l y \geqslant \int \frac{2\mathcal{N}}{3(\mathcal{N}+V)}(|\partial_x^2 y|^2+|\partial_x^2 y_t|^2) - C\delta_1\|(\partial_x y,\partial_x y_t)\|_{H^1}^2.$$

(2.38) together with two estimates and using $Young's$ inequality imply

$$\frac{d}{dt}\int \frac{\mathcal{N}}{2}(|\partial_x y|^2+|\partial_x y_t|^2)+C\int(|\partial_x y|^2+|\partial_x^2 y|^2+|\partial_x y_t|^2+|\partial_x^2 y_t|^2)+\frac{2}{3}T^0\int \mathcal{N}\partial_x \mathrm{div}\mathbf{u}_t\partial_x y_t$$

$$\leqslant C\delta_1\int(|\partial_x\mathbf{u}|^2+|\partial_x^2\mathbf{u}|^2+|\partial_x\mathbf{u}_t|^2)+C\int|\partial_x^2\mathbf{u}|^2. \tag{2.39}$$

Therefore, Combining (2.34),(2.35),(2.37),(2.39) and noticing that (2.12) with $i=1$ and (2.13) with $j=2$, we have

$$\frac{d}{dt}\int[\frac{\mathcal{N}}{2}|\partial_x^2\mathbf{u}|^2+\partial_x^2\mathcal{N}\mathbf{u}\partial_x^2\mathbf{u}+2\partial_x\mathcal{N}\partial_x\mathbf{u}\partial_x^2\mathbf{u}+\frac{1}{2}|\partial_x^2\mathbf{e}|^2+\frac{\mathcal{N}}{2}|\partial_x\mathbf{u}_t|^2+\partial_x\mathcal{N}\mathbf{u}_t\partial_x\mathbf{u}_t+\frac{1}{2}|\partial_x\mathbf{e}_t|^2$$

$$+\frac{3\mathcal{N}}{4T^0}(|\partial_x y|^2+|\partial_x y_t|^2)+\frac{\mathcal{N}\partial_x^2(h(\mathcal{N}+V)-h(\mathcal{N}))\partial_x^2 V}{\mathcal{N}+V}-\frac{\mathcal{N}h'(\mathcal{N}+V)|\partial_x^2 V|^2}{2(\mathcal{N}+V)}$$

$$+(\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)}+\frac{\mathcal{N}y}{2(\mathcal{N}+V)^2})|\partial_x V_t|^2]+C\int(|\partial_x^2\mathbf{u}|^2+|\partial_x\mathbf{u}_t|^2+|\partial_x^2 V|^2$$

$$+|\partial_x V_t|^2+|\partial_x^2\mathbf{e}|^2+|\partial_x\mathbf{e}_t|^2+|\partial_x y|^2+|\partial_x^2 y|^2+|\partial_x y_t|^2+|\partial_x^2 y_t|^2)$$

$$\leqslant C\delta_1\|(V,\mathbf{u})\|_{H^1}^2+C\delta_1\int(|V_t|^2+|\mathbf{u}_t|^2+|y|^2+|y_t|^2+|\mathbf{e}_t|^2)$$

$$+C\int(|\partial_x\mathbf{u}|^2+|\partial_x V|^2)+\int\mathcal{N}\partial_x^2 y\partial_x^2(\mathrm{div}\mathbf{u}). \tag{2.40}$$

We now turn to the estimates of the third derivatives. However, the above arguments in estimating the first and second derivatives do not work for the third derivatives because we can not obtain the smallness of $|(\partial_x^2 V,\partial_x^2\mathbf{u},\partial_x^2\mathbf{e})|$ and $|(\partial_x V_t,\partial_x\mathbf{u}_t,\partial_x\mathbf{e}_t,\partial_x y_t)|$. We use $Young's$ inequality, $Sobolev's$ inequality, $Gagliardo-Nirenberg's$ inequality and the smallness of $\int|\partial_x^3\mathcal{N}|$ and $\int|\partial_x^4\mathcal{N}|$, frequently. Hence we give a detailed discussion.
Take ∂_x^3 on the both sides of $(2.4)_2$ and multiply the resulting equation by $\partial_x^3(\mathcal{N}\mathbf{u})$, then integrate it over $\mathbb{R}^N$ to get

$$\int\partial_x^3\mathbf{u}_t\partial_x^3(\mathcal{N}\mathbf{u})+\int\partial_x^3\mathbf{u}\partial_x^3(\mathcal{N}\mathbf{u})+\int\partial_x^3\nabla(h(\mathcal{N}+V)-h(\mathcal{N}))\partial_x^3(\mathcal{N}\mathbf{u})+\int\partial_x^3(\mathbf{u}\cdot\nabla\mathbf{u})\partial_x^3(\mathcal{N}\mathbf{u})$$

$$+\int\partial_x^3(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\partial_x^3(\mathcal{N}\mathbf{u})-\int\partial_x^3\mathbf{e}\partial_x^3(\mathcal{N}\mathbf{u})=0. \tag{2.41}$$

Employing $Young's$ inequality and $Sobolev's$ inequality, we can get

$$\begin{aligned}
\int \partial_x^3\mathbf{u}_t\partial_x^3(\mathcal{N}\mathbf{u}) &= \frac{d}{dt}\int (\frac{\mathcal{N}}{2}|\partial_x^3\mathbf{u}|^2 + 3\partial_x\mathcal{N}\partial_x^2\mathbf{u}\partial_x^3\mathbf{u} + 3\partial_x^2\mathcal{N}\partial_x\mathbf{u}\partial_x^3\mathbf{u} + \partial_x^3\mathcal{N}\mathbf{u}\partial_x^3\mathbf{u}) \\
&- 3\int \partial_x\mathcal{N}\partial_x^2\mathbf{u}_t\partial_x^3\mathbf{u} - 3\int \partial_x^2\mathcal{N}\partial_x\mathbf{u}_t\partial_x^3\mathbf{u} - \int \partial_x^3\mathcal{N}\mathbf{u}_t\partial_x^3\mathbf{u} \\
&\geqslant \frac{d}{dt}\int (\frac{\mathcal{N}}{2}|\partial_x^3\mathbf{u}|^2 + 3\partial_x\mathcal{N}\partial_x^2\mathbf{u}\partial_x^3\mathbf{u} + 3\partial_x^2\mathcal{N}\partial_x\mathbf{u}\partial_x^3\mathbf{u} + \partial_x^3\mathcal{N}\mathbf{u}\partial_x^3\mathbf{u}) \\
&- C\delta_1\int (|\partial_x^3\mathbf{u}|^2 + |\partial_x^2\mathbf{u}_t|^2 + |\partial_x\mathbf{u}_t|^2) - \epsilon\int |\partial_x^3\mathbf{u}|^2 - C(\epsilon)\|\mathbf{u}\|_{H^2}^2\int |\partial_x^3\mathcal{N}|^2,
\end{aligned}$$

and

$$\begin{aligned}
\int \partial_x^3\mathbf{u}\partial_x^3(\mathcal{N}\mathbf{u}) &\geqslant \int \mathcal{N}|\partial_x^3\mathbf{u}|^2 - C\delta_1\int (|\partial_x\mathbf{u}|^2 + |\partial_x^2\mathbf{u}|^2 + |\partial_x^3\mathbf{u}|^2) \\
&- \epsilon\int |\partial_x^3\mathbf{u}|^2 - C(\epsilon)\|\mathbf{u}\|_{H^2}^2\int |\partial_x^3\mathcal{N}|^2.
\end{aligned}$$

By (2.17) with $i=3$, $(2.4)_1$ and $(2.4)_4$, we arrive at

$$\begin{aligned}
-\int \partial_x^3\mathbf{e}\partial_x^3(\mathcal{N}\mathbf{u}) &= \int \partial_x^3(\mathcal{E}+\mathbf{e})\partial_x^3\mathbf{e}_t + \int \partial_x^3(\mathcal{E}+\mathbf{e})\partial_x^3(V\mathbf{u}) + \int \partial_x^3\mathcal{E}\partial_x^3(\mathcal{N}\mathbf{u}) \\
&\geqslant \frac{1}{2}\frac{d}{dt}\int |\partial_x^3\mathbf{e}|^2 - C\delta_1\int (|\partial_x^2\mathbf{u}|^2 + |\partial_x^3\mathbf{u}|^2 + |\partial_x^2V|^2 + |\partial_x^3V|^2 + |\partial_x^3\mathbf{e}|^2).
\end{aligned}$$

Using $Young's$ inequality, $Sobolev's$ inequality, (2.8) and integrating by parts, through some tedious but straightforward calculations, we can obtain

$$\begin{aligned}
\int \partial_x^3\nabla(h(\mathcal{N}+V) - h(\mathcal{N}))\partial_x^3(\mathcal{N}\mathbf{u}) \geqslant &- \int \mathcal{N}\partial_x^3(h(\mathcal{N}+V) - h(\mathcal{N}))\partial_x^3(\mathrm{div}\mathbf{u}) \\
&- \epsilon\|(\mathbf{u},V)\|_{H^3}^2 - \epsilon\|\mathbf{u}\|_{H^2}^2\int |\partial_x^3\mathcal{N}|^2 \\
&- C(\epsilon)(\|\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}\|_{H^3}^2 + \|V\|_{H^2}^2 + \|V\|_{H^3}^2)(\int |\partial_x^3\mathcal{N}|^2 \\
&+ \int |\partial_x^4\mathcal{N}|^2) - C\delta_1(\|V\|_{H^3}^2 + \int |\partial_x^3\mathbf{u}|^2),
\end{aligned}$$

and

$$\begin{aligned}
\int \partial_x^3(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})\partial_x^3(\mathcal{N}\mathbf{u}) \geqslant &- \int \frac{\mathcal{N}y}{\mathcal{N}+V}\partial_x^3V\partial_x^3(\mathrm{div}\mathbf{u}) + T^0\int \mathcal{N}(\mathcal{N}+V)\partial_x^3\mathbf{u}\cdot\partial_x^2\mathrm{div}\mathbf{u} \\
&- \epsilon\|(V,\mathbf{u},y)\|_{H^3}^2 - \epsilon(\|\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}\|_{H^2}^3)\int |\partial_x^3\mathcal{N}|^2 \\
&- \epsilon(\|\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}\|_{H^3}^2)\int |\partial_x^3V|^2 - C(\epsilon)\|y\|_{H^2}^2\int (|\partial_x^3\mathcal{N}|^2 + |\partial_x^4\mathcal{N}|^2) \\
&- C(\epsilon)\|(\partial_x^2y,\partial_x^2y_t)\|^2 - C\delta_1(\|\partial_xV,\partial_x\mathbf{u},y_t)\|_{H^2}^2 + \|y\|_{H^3}^2).
\end{aligned}$$

Set

$$\begin{aligned} J: &= -\int \mathcal{N}\partial_x^3(h(\mathcal{N}+V)-h(\mathcal{N}))\partial_x^3(\mathrm{div}\mathbf{u}) - \int \frac{\mathcal{N}y}{\mathcal{N}+V}\partial_x^3 V\partial_x^3(\mathrm{div}\mathbf{u}) \\ &= \int [\mathcal{N}\partial_x^3(h(\mathcal{N}+V)-h(\mathcal{N})) + \frac{\mathcal{N}y}{\mathcal{N}+V}\partial_x^3 V]\partial_x^3(\frac{V_t+\mathbf{u}\cdot\nabla(\mathcal{N}+V)}{\mathcal{N}+V}) \end{aligned}$$

To make our calculations more explicit, we are going to expand the integrated functions of J, for simplicity, omitting constant coefficients of terms.

$$\begin{aligned} J &= \int [\mathcal{N}(h'(\mathcal{N}+V)\partial_x^3(\mathcal{N}+V) + h''(\mathcal{N}+V)\partial_x(\mathcal{N}+V)\partial_x^2(\mathcal{N}+V) + h'''(\mathcal{N}+V)(\partial_x(\mathcal{N}+V))^3 \\ &- h'(\mathcal{N})\partial_x^3\mathcal{N} - h''(\mathcal{N})\partial_x\mathcal{N}\partial_x^2\mathcal{N} - h'''(\mathcal{N})(\partial_x\mathcal{N})^3) + \frac{\mathcal{N}y}{\mathcal{N}+V}\partial_x^3 V]\cdot[\frac{1}{\mathcal{N}+V}(\partial_x^3 V_t \\ &+ \mathbf{u}\cdot\nabla\partial_x^3(\mathcal{N}+V) + \partial_x\mathbf{u}\cdot\nabla\partial_x^2(\mathcal{N}+V) + \partial_x^2\mathbf{u}\cdot\nabla\partial_x(\mathcal{N}+V) + \partial_x^3\mathbf{u}\cdot\nabla(\mathcal{N}+V)) \\ &- \frac{\partial_x(\mathcal{N}+V)}{(\mathcal{N}+V)^2}(\partial_x^2 V_t + \mathbf{u}\cdot\nabla\partial_x^2(\mathcal{N}+V) + \partial_x\mathbf{u}\cdot\nabla\partial_x(\mathcal{N}+V) + \partial_x^2\mathbf{u}\cdot\nabla(\mathcal{N}+V)) \\ &- (\frac{\partial_x^2(\mathcal{N}+V)}{(\mathcal{N}+V)^2} - \frac{(\partial_x(\mathcal{N}+V))^2}{(\mathcal{N}+V)^3})(\partial_x V_t + \mathbf{u}\cdot\nabla\partial_x(\mathcal{N}+V) + \partial_x\mathbf{u}\cdot\nabla(\mathcal{N}+V)) \\ &- (\frac{\partial_x^3(\mathcal{N}+V)}{(\mathcal{N}+V)^2} - \frac{\partial_x(\mathcal{N}+V)\partial_x^2(\mathcal{N}+V)}{(\mathcal{N}+V)^3} + \frac{(\partial_x(\mathcal{N}+V))^3}{(\mathcal{N}+V)^4})(V_t + \mathbf{u}\cdot\nabla(\mathcal{N}+V))] \end{aligned}$$

Set

$$\begin{aligned} J_1 := \int [&\frac{\mathcal{N}h'(\mathcal{N}+V)}{\mathcal{N}+V}\partial_x^3 V\partial_x^3 V_t + \frac{\mathcal{N}y}{(\mathcal{N}+V)^2}\partial_x^3 V\partial_x^3 V_t + \frac{\mathcal{N}}{\mathcal{N}+V}\partial_x^3 V_t(h'(\mathcal{N}+V)\partial_x^3\mathcal{N} \\ &+h''(\mathcal{N}+V)\partial_x(\mathcal{N}+V)\partial_x^2(\mathcal{N}+V) + h'''(\mathcal{N}+V)(\partial_x(\mathcal{N}+V))^3 \\ &-h'(\mathcal{N})\partial_x^3\mathcal{N} - h''(\mathcal{N})\partial_x\mathcal{N}\partial_x^2\mathcal{N} - h'''(\mathcal{N})(\partial_x\mathcal{N})^3)], \end{aligned}$$

$$\begin{aligned} J_2 := \int [&\frac{\mathcal{N}h'(\mathcal{N}+V)}{\mathcal{N}+V)}\mathbf{u}\cdot\nabla(|\frac{\partial_x^3(\mathcal{N}+V)}{2}|^2) + \frac{\mathcal{N}y}{(\mathcal{N}+V)^2}\mathbf{u}\cdot\nabla(|\frac{\partial_x^3 V}{2}|^2) + \frac{\mathcal{N}}{\mathcal{N}+V}(h''(\mathcal{N}+V) \\ &\partial_x(\mathcal{N}+V)\partial_x^2(\mathcal{N}+V) + h'''(\mathcal{N}+V)(\partial_x(\mathcal{N}+V))^3 - h'(\mathcal{N})\partial_x^3\mathcal{N} \\ &-h''(\mathcal{N})\partial_x\mathcal{N}\partial_x^2\mathcal{N} - h'''(\mathcal{N})(\partial_x\mathcal{N})^3)\mathbf{u}\cdot\nabla\partial_x^3(\mathcal{N}+V)], \end{aligned}$$

$$J_3 := \int (f_1 f_2 f_3 f_4) = \int (\text{the sums of all terms without smallness of the maximum norm of exact three multipliers}),$$

J_R denotes the remained term integrals which are no less than:

$$J_R \geqslant -\epsilon\|(V,\mathbf{u})\|_{H^3}^2 - C(\epsilon)(\|V\|_{H^2}^2 + \|V\|_{H^3}^2 + \|V_t\|_{H^2}^2 + \|\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}\|_{H^3}^2)\int(|\partial_x^3\mathcal{N}|^2 + |\partial_x^4\mathcal{N}|^2)$$

$$-C\delta_1(\|(\partial_x V, \partial_x\mathbf{u})\|_{H^2}^2 + \|\partial_x V_t\|_{H^1}^2) \quad .$$

Now, we shall estimate J_1, J_2, J_3 as follows:

$$J_1 \geqslant \frac{d}{dt}\int[(\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)}+\frac{\mathcal{N}y}{2(\mathcal{N}+V)^2})|\partial_x^3V|^2+\frac{\mathcal{N}}{\mathcal{N}+V}\partial_x^3V(h'(\mathcal{N}+V)\partial_x^3\mathcal{N}$$
$$+h''(\mathcal{N}+V)\partial_x(\mathcal{N}+V)\partial_x^2(\mathcal{N}+V)+h'''(\mathcal{N}+V)(\partial_x(\mathcal{N}+V))^3$$
$$-h'(\mathcal{N})\partial_x^3\mathcal{N}-h''(\mathcal{N})\partial_x\mathcal{N}\partial_x^2\mathcal{N}-h'''(\mathcal{N})(\partial_x\mathcal{N})^3)]-\epsilon\int|\partial_x^3V|^2$$
$$-C(\epsilon)\int|y_t|^2-C\delta_1\|(\partial_xV,V_t)\|_{H^2}^2+J_{1R}\ ,$$

where $J_{1R}=\int\frac{\mathcal{N}h''(\mathcal{N}+V)}{\mathcal{N}+V}\partial_x^2V\partial_x^3V\partial_x^2V_t$.

$$J_2\geqslant-\epsilon\|V\|_{H^3}^2-\epsilon(\|V\|_{H^2}^2+\|V\|_{H^3}^2)\int|\partial_x^3\mathcal{N}|^2-C(\epsilon)(\|\mathbf{u}\|_{H^2}^2+\|\mathbf{u}\|_{H^3}^2)$$
$$\cdot\int(|\partial_x^3\mathcal{N}|^2+|\partial_x^4\mathcal{N}|^2)-\|\mathbf{u}\|_{H^2}^2\int(|\partial_x^3\mathcal{N}|^2+|\partial_x^4\mathcal{N}|^2)+J_{2R},$$

where $J_{2R}=\int\frac{\mathcal{N}h''(\mathcal{N}+V)}{\mathcal{N}+V}\mathbf{u}\cdot\partial_x(\nabla V)\partial_x^2V\partial_x^3V$. Put

$$\begin{aligned}
I: &= J_3+J_{1R}+J_{2R}=\int\{[\frac{\mathcal{N}h'(\mathcal{N}+V)}{\mathcal{N}+V}\partial_x^2\mathbf{u}\nabla\partial_xV\partial_x^3V-\frac{\mathcal{N}h'(\mathcal{N}+V)}{(\mathcal{N}+V)^2}\partial_x^2V\partial_x^3V(\partial_xV_t+\mathbf{u}\cdot\nabla\partial_xV)\\
&+\frac{\mathcal{N}h''(\mathcal{N}+V)}{\mathcal{N}+V}\partial_x^2\mathbf{u}\nabla\partial_xV\partial_x(\mathcal{N}+V)\partial_x^2V-\frac{\mathcal{N}\partial_x^2V}{(\mathcal{N}+V)^2}(\partial_xV_t+\mathbf{u}\cdot\nabla\partial_xV)\partial_x(\mathcal{N}+V)\partial_x^2V]\\
&+[\frac{\mathcal{N}y}{\mathcal{N}+V}\partial_x^3V\partial_x^2\mathbf{u}\nabla\partial_xV-\frac{\mathcal{N}y}{(\mathcal{N}+V)^3}\partial_x^3V\partial_x^2V(\partial_xV_t+\mathbf{u}\cdot\nabla\partial_xV)]\}\\
&+\int\frac{\mathcal{N}h''(\mathcal{N}+V)}{\mathcal{N}+V}\partial_x^2V\partial_x^3V\partial_xV_t+\int\frac{\mathcal{N}h''(\mathcal{N}+V)}{\mathcal{N}+V}\mathbf{u}\cdot\partial_x(\nabla V)\partial_x^2V\partial_x^3V\\
&\geqslant -\epsilon\|(\partial_x^2V,\partial_x^3V)\|^2-C(\epsilon)\int(|\partial_x^2V|^4+|\partial_x^2\mathbf{u}|^4+|\partial_xV_t|^4),
\end{aligned}$$

with the help of $Young's$ inequality. By (2.12),(2.8),(2.6) and $Gagliardo-Nirenberg's$ inequality with $N=2,3$, we have

$$\begin{aligned}
\int(|\partial_x^2V|^4+|\partial_x^2\mathbf{u}|^4+|\partial_xV_t|^4) &\leqslant C\int(|\partial_x^2V|^4+|\partial_x^2\mathbf{u}|^4)+C\delta_1\|\partial_x\mathbf{u}\|^2\\
&\leqslant C\|\partial_x^2\mathbf{u}\|^{4-N}\|\partial_x^3\mathbf{u}\|^N+C\|\partial_x^2V\|^{4-N}\|\partial_x^3V\|^N+C\delta_1\|\partial_x\mathbf{u}\|^2\\
&\leqslant C\|\partial_x^2\mathbf{u}\|^{4-N}\|\partial_x^3\mathbf{u}\|^{N-2}\|\partial_x^3\mathbf{u}\|^2\\
&+ C\|\partial_x^2V\|^{4-N}\|\partial_x^3V\|^{N-2}\|\partial_x^3V\|^2+C\delta_1\|\partial_x\mathbf{u}\|^2\\
&\leqslant C\delta_1\|\partial_x^3\mathbf{u}\|^2+C\delta_1\|\partial_x^3V\|^2+C\delta_1\|\partial_x\mathbf{u}\|^2
\end{aligned}$$

i.e.

$$I\geqslant-\epsilon\|(\partial_x^2V,\partial_x^3V)\|^2-C\delta_1\|\partial_x^3\mathbf{u}\|^2-C\delta_1\|\partial_x^3V\|^2-C\delta_1\|\partial_x\mathbf{u}\|^2.$$

Using $Young's$ inequality and $Gagliardo-Nirenberg's$ inequality, with the help of the smallness of ϵ,δ_1 and $\int|\partial_x^3\mathcal{N}|^2$, it is easy to get

$$\int\partial_x^3(\mathbf{u}\cdot\nabla\mathbf{u})\partial_x^3(\mathcal{N}\mathbf{u})\geqslant -C\delta_1\|\mathbf{u}\|_{H^3}^2.$$

(2.41) combining these estimates, we can get

$$\begin{aligned}&\frac{d}{dt}\int[\frac{\mathcal{N}}{2}|\partial_x^3\mathbf{u}|^2+3\partial_x\mathcal{N}\partial_x^2\mathbf{u}\partial_x^3\mathbf{u}+3\partial_x^2\mathcal{N}\partial_x\mathbf{u}\partial_x^3\mathbf{u}+\partial_x^3\mathcal{N}\mathbf{u}\partial_x^3\mathbf{u}+\frac{1}{2}|\partial_x^3\mathbf{e}|^2+(\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)}\\
&+\ \frac{\mathcal{N}y}{2(\mathcal{N}+V)^2})|\partial_x^3V|^2+\frac{\mathcal{N}}{\mathcal{N}+V}\partial_x^3V(h'(\mathcal{N}+V)\partial_x^3\mathcal{N}+h''(\mathcal{N}+V)\partial_x(\mathcal{N}+V)\partial_x^2(\mathcal{N}+V)\\
&+\ h'''(\mathcal{N}+V)(\partial_x(\mathcal{N}+V))^3-h'(\mathcal{N})\partial_x^3\mathcal{N}-h''(\mathcal{N})\partial_x\mathcal{N}\partial_x^2\mathcal{N}-h'''(\mathcal{N})(\partial_x\mathcal{N})^3)]\\
&+C\int|\partial_x^3\mathbf{u}|^2\leqslant C\delta_1\|(V,y)\|_{H^3}^2+C\delta_1\|(V_t,\mathbf{u},\mathbf{u}_t,y_t)\|_{H^2}^2+C\int(|y_t|^2+|\partial_x^2y|^2+|\partial_x^2y_t|^2),
\end{aligned}\tag{2.42}$$

with the help of the smallness of ϵ,δ_1, $\int|\partial_x^3\mathcal{N}|$ and $\int|\partial_x^4\mathcal{N}|$. Similar to (2.35), we have

$$\begin{aligned}\int(|\partial_x^2V|^2+|\partial_x^3V|^2)&\leqslant C\delta_1\int(|\partial_x^3\mathbf{u}|^2+|\partial_x^2\mathbf{u}|^2+|\partial_x^3y|^2+|\partial_x^2y|^2+|\partial_xy|^2)\\
&+C\int(|\partial_x\mathbf{u}|^2+|\partial_x^2\mathbf{u}|^2+|\partial_x\mathbf{u}_t|^2+|\partial_x^2\mathbf{u}_t|^2).\end{aligned}\tag{2.43}$$

Taking $\partial_x^2\partial_t$ to $(2.4)_2$ and multiplying $\partial_x^2(\mathcal{N}\mathbf{u}_t)$, then integrating it over $\mathbb{R}^N$, we have

$$\begin{aligned}&\int\partial_x^2\mathbf{u}_{tt}\partial_x^2(\mathcal{N}\mathbf{u}_t)+\int\partial_x^2\mathbf{u}_t\partial_x^2(\mathcal{N}\mathbf{u}_t)+\int\partial_x^2\nabla(h(\mathcal{N}+\mathbf{u})-h(\mathcal{N}))_t\partial_x^2(\mathcal{N}\mathbf{u}_t)-\int\partial_x^2\mathbf{e}_t\partial_x^2(\mathcal{N}\mathbf{u}_t)\\
&+\int\partial_x^2(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})_t\partial_x^2(\mathcal{N}\mathbf{u}_t)+\int\partial_x^2(\mathbf{u}\cdot\nabla\mathbf{u})_t\partial_x^2(\mathcal{N}\mathbf{u}_t)=0.\end{aligned}\tag{2.44}$$

Employing the equation$(2.4)_2$ and (2.8), we get

$$\begin{aligned}\int\partial_x^2\mathbf{u}_{tt}\partial_x^2(\mathcal{N}\mathbf{u}_t)&\geqslant\frac{d}{dt}\int(\frac{\mathcal{N}}{2}|\partial_x^2\mathbf{u}_t|^2+2\partial_x\mathcal{N}\partial_x\mathbf{u}_t\partial_x^2\mathbf{u}_t+\partial_x^2\mathcal{N}\mathbf{u}_t\partial_x^2\mathbf{u}_t)\\
&-\ C\delta_1\int(|\partial_x\mathbf{u}_t|^2+|\mathbf{u}_{tt}|^2+|\partial_x\mathbf{u}_{tt}|^2)\\
&\geqslant\frac{d}{dt}\int(\frac{\mathcal{N}}{2}|\partial_x^2\mathbf{u}_t|^2+2\partial_x\mathcal{N}\partial_x\mathbf{u}_t\partial_x^2\mathbf{u}_t+\partial_x^2\mathcal{N}\mathbf{u}_t\partial_x^2\mathbf{u}_t)\\
&-\ C\delta_1(\|(V_t,\mathbf{u},\mathbf{u}_t,y_t)\|_{H^2}^2+\|\mathbf{e}_t\|_{H^1}^2).\end{aligned}$$

After the direct calculation, we arrive at

$$\int\partial_x^2\mathbf{u}_t\partial_x^2(\mathcal{N}\mathbf{u}_t)\geqslant\int\mathcal{N}|\partial_x^2\mathbf{u}_t|^2-C\delta_1\int(|\mathbf{u}_t|^2+|\partial_x\mathbf{u}_t|^2+|\partial_x^2\mathbf{u}_t|^2),$$

$$-\int \partial_x^2\mathbf{e}_t\partial_x^2(\mathcal{N}\mathbf{u}_t) \geqslant \frac{d}{dt}\int \frac{1}{2}|\partial_x^2\mathbf{e}_t|^2 - C\delta_1(\|(\partial_x V_t, \partial_x\mathbf{u}_t)\|_{H^1}^2 + \int(|\partial_x^2\mathbf{e}_t|^2 + |\partial_x^2\mathbf{u}|^2)),$$

$$\int \partial_x^2(\mathbf{u}\cdot\nabla\mathbf{u})_t\partial_x^2(\mathcal{N}\mathbf{u}_t) \geqslant -C\delta_1\int(|\partial_x^2\mathbf{u}|^2 + |\partial_x^3\mathbf{u}|^2 + |\mathbf{u}_t|^2 + |\partial_x\mathbf{u}_t|^2 + |\partial_x^2\mathbf{u}_t|^2),$$

$$\begin{aligned}
\int &\quad \partial_x^2\nabla(h(\mathcal{N}+V) - h(\mathcal{N}))_t\partial_x^2(\mathcal{N}\mathbf{u}_t)\\
\geqslant &\quad -\int \mathcal{N}h'(\mathcal{N}+V)\partial_x^2 V_t\partial_x^2(\mathrm{div}\mathbf{u}_t) - \epsilon(\|V_t\|_{H^2}^2 + \int|\partial_x^2 V|^2) - C(\epsilon)(\|V_t\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^2}^2)\int|\partial_x^3\mathcal{N}|^2\\
- &\quad C\delta_1(\|V_t\|_{H^2}^2 + \|(\partial_x^2 V, \partial_x\mathbf{u}_t)\|_{H^1}^2) + \int \mathcal{N}h''(\mathcal{N}+V)\partial_x^2 V\partial_x V_t\partial_x^2\mathbf{u}_t,
\end{aligned}$$

We dealt with the last second integral in (2.44) as follows by using $Young's$ inequality:

$$\int \partial_x^2(\frac{\nabla((\mathcal{N}+V)y)}{\mathcal{N}+V})_t\partial_x^2(\mathcal{N}\mathbf{u}_t) = \int \partial_x^2(\nabla y_t + (\frac{\nabla(\mathcal{N}+V)}{\mathcal{N}+V}y)_t)\partial_x^2(\mathcal{N}\mathbf{u}_t).$$

$$\begin{aligned}
\int \partial_x^2\nabla y_t\partial_x^2(\mathcal{N}\mathbf{u}_t) \geqslant &\quad -\quad \epsilon\int|\partial_x^2 y_t|^2 - C(\epsilon)\|\mathbf{u}_t\|_{H^2}^2\int|\partial_x^3\mathcal{N}|^2\\
&\quad -\quad C\delta_1\int(|\partial_x\mathbf{u}_t|^2 + |\partial_x^2\mathbf{u}_t|^2 + |\partial_x^2 y_t|^2) - \int\mathcal{N}\partial_x^2 y_t\partial_x^2(\mathrm{div}\mathbf{u}_t)
\end{aligned}$$

and

$$\begin{aligned}
\int \partial_x^2(\frac{\nabla(\mathcal{N}+V)}{\mathcal{N}+V}y)_t\partial_x^2(\mathcal{N}\mathbf{u}_t) \geqslant &\quad -\quad \int\frac{\mathcal{N}y}{\mathcal{N}+V}\partial_x^2 V_t\partial_x^2(\mathrm{div}\mathbf{u}_t) - \epsilon(\|\mathbf{u}_t\|_{H^2}^2 + \int|\partial_x^2 V_t|^2)\\
&\quad -\quad C(\epsilon)\int|\partial_x^2\mathbf{u}_t|^2 - C(\epsilon)(\|y\|_{H^2}^2 + \|y_t\|_{H^2}^2)\int|\partial_x^3\mathcal{N}|^2\\
&\quad -\quad C\delta_1(\|(V_t, \mathbf{u}_t, y_t)\|_{H^2}^2 + \|\partial_x^2 V\|_{H^1}^2)\\
&\quad +\quad \int\frac{\mathcal{N}}{\mathcal{N}+V}\partial_x^2 V\partial_x y_t\partial_x^2\mathbf{u}_t + \int\frac{\partial_x\mathcal{N}}{\mathcal{N}+V}\partial_x^2 V\partial_x y_t\partial_x\mathbf{u}_t\\
&\quad -\quad \int\frac{\mathcal{N}y}{(\mathcal{N}+V)^2}\partial_x^2 V\partial_x V_t\partial_x^2\mathbf{u}_t - \int\frac{y\partial_x\mathcal{N}\partial_x\mathbf{u}_t}{(\mathcal{N}+V)^2}\partial_x^2 V\partial_x V_t\ .
\end{aligned}$$

Set

$$\begin{aligned}
J' : &= -\int\mathcal{N}h'(\mathcal{N}+V)\partial_x^2 V_t\partial_x^2(\mathrm{div}\mathbf{u}_t) - \int\frac{\mathcal{N}y}{\mathcal{N}+V}\partial_x^2 V_t\partial_x^2(\mathrm{div}\mathbf{u}_t)\\
&= \int(\mathcal{N}h'(\mathcal{N}+V) + \frac{\mathcal{N}y}{\mathcal{N}+V})\partial_x^2 V_t\partial_x^2(\frac{V_t + \mathbf{u}\cdot\nabla(\mathcal{N}+V)}{\mathcal{N}+V})_t\\
&\geqslant \frac{d}{dt}\int(\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)} + \frac{\mathcal{N}y}{2(\mathcal{N}+V)^2})|\partial_x^2 V_t|^2 - \epsilon\int|\partial_x^2 V_t|^2\\
&- C(\epsilon)(\|\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^2}^2)\int|\partial_x^3\mathcal{N}|^2 - C\delta_1(\|(\partial_x^2 V, \partial_x\mathbf{u}\|_{H^1}^2
\end{aligned}$$

$$
\begin{aligned}
+ & \quad \|(V_t, \mathbf{u}_t)\|_{H^2}^2) + \int (\frac{\mathcal{N}h'(\mathcal{N}+V)}{\mathcal{N}+V} + \frac{\mathcal{N}y}{(\mathcal{N}+V)^2})\partial_x^2 V_t \partial_x \mathbf{u}_t \partial_x(\nabla V) \\
- & \quad \int (\frac{\mathcal{N}h'(\mathcal{N}+V)}{(\mathcal{N}+V)^2} + \frac{\mathcal{N}y}{(\mathcal{N}+V)^3})\partial_x^2 V_t \partial_x V_t \partial_x V_t \\
- & \quad \int (\mathcal{N}h'(\mathcal{N}+V) + \frac{\mathcal{N}y}{\mathcal{N}+V})\mathbf{u}\partial_x^2 V_t \partial_x V_t \partial_x(\nabla V).
\end{aligned}
$$

As above estimates, we denotes I' the sums of all integral terms without smallness of the maximum norm of exact three multipliers, in the same way, with the help of $Young's$ inequality and $Gagliardo-Nirenberg's$ inequality, which are no more than

$$
I' \leqslant C\delta_1 \int (|\partial_x^3 V|^2 + |\partial_x^2 V|^2 + |\partial_x^2 V_t|^2 + |\partial_x^2 \mathbf{u}_t|^2 + |\partial_x^2 y_t|^2).
$$

(2.44) together these estimates, with the help of smallness of ϵ and δ_1, we have

$$
\frac{d}{dt}\int (\frac{\mathcal{N}}{2}|\partial_x^2 \mathbf{u}_t|^2 + 2\partial_x \mathcal{N}\partial_x \mathbf{u}_t \partial_x^2 \mathbf{u}_t + \partial_x^2 \mathcal{N}\mathbf{u}_t \partial_x^2 \mathbf{u}_t + \frac{1}{2}|\partial_x^2 \mathbf{e}_t|^2 + (\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)} + \frac{\mathcal{N}y}{2(\mathcal{N}+V)^2})|\partial_x^2 V_t|^2)
$$

$$
+C\int |\partial_x^2 \mathbf{u}_t|^2 \leqslant C\delta_1(\|(V,\mathbf{u})\|_{H^3}^2 + \|(V_t, y, y_t, \mathbf{e}_t)\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^1}^2) + \int \mathcal{N}\partial_x^2 y_t \partial_x^2(\mathrm{div}\mathbf{u}_t). \quad (2.45)
$$

Taking $\partial_x^2\partial_t^l(l=0,1)$ to $(2.4)_3$ and multiplying $\mathcal{N}\partial_x^2\partial_t^l y$, then integrating it over $\mathbb{R}^N$, we have

$$
\frac{d}{dt}\int \frac{\mathcal{N}}{2}|\partial_x^2\partial_t^l y|^2 + \int \mathcal{N}|\partial_x^2\partial_t^l y|^2 + \frac{2}{3}T^0\int \mathcal{N}\partial_x^2\partial_t^l y \partial_x^2\partial_t^l \mathrm{div}\mathbf{u} - \frac{2}{3}\int \partial_x^2\partial_t^l(\frac{\triangle y}{\mathcal{N}+V})\mathcal{N}\partial_x^2\partial_t^l y
$$

$$
+\int \partial_x^2\partial_t^l(\mathbf{u}\cdot\nabla y + \frac{2}{3}y\mathrm{div}\mathbf{u} - \frac{1}{3}|\mathbf{u}|^2)\mathcal{N}\partial_x^2\partial_t^l y = 0. \quad (2.46)
$$

It is easy to estimate that the last integral with $l=0$ on the left side of (2.46) is no less than

$$
-C\delta_1\int (|\partial_x \mathbf{u}|^2 + |\partial_x^2 \mathbf{u}|^2 + |\partial_x^3 \mathbf{u}|^2 + |\partial_x^2 y|^2 + |\partial_x^3 y|^2),
$$

and the last integral with $l=1$ is no less than, using $Gagliardo-Nirenberg's$ inequality

$$
-C\delta_1\int (|\partial_x^2 \mathbf{u}|^2 + |\partial_x^3 \mathbf{u}|^2 + |\partial_x \mathbf{u}_t|^2 + |\partial_x^2 \mathbf{u}_t|^2 + |\partial_x^3 y|^2 + |\partial_x^2 y_t|^2 + |\partial_x^3 y_t|^2).
$$

The last second integral with $l=0$ is estimated as follows:

$$
-\frac{2}{3}\int \partial_x^2(\frac{\triangle y}{\mathcal{N}+V})\mathcal{N}\partial_x^2 y \geqslant \frac{2}{3}\int \frac{\mathcal{N}}{\mathcal{N}+V}|\partial_x^2\nabla y|^2 - C\delta_1\int (|\partial_x^2 V|^2 + |\partial_x^2 y|^2 + |\partial_x^3 y|^2),
$$

and the last second integral with $l = 1$ is no less than, using integrating by parts and $Gagliardo - Nirenberg's$ inequality similarly,

$$
\begin{aligned}
-\frac{2}{3}\int \partial_x^2(\frac{\triangle y}{\mathcal{N}+V})_t\mathcal{N}\partial_x^2 y_t \geqslant & \frac{2}{3}\int \frac{\mathcal{N}}{\mathcal{N}+V}|\partial_x^2\nabla y_t|^2 - C\delta_1(\|\partial_x V_t, \partial_x^2 y, \partial_x^2 y_t)\|_{H^1}^2 + \int |\partial_x^2 V|^2). \\
+ & \int \frac{\mathcal{N}}{(N+V)^2}V_t\partial_x^2(\triangle y)\partial_x^2 y_t + \int \frac{\mathcal{N}}{(N+V)^2}\partial_x V_t\partial_x(\triangle y)\partial_x^2 y_t \\
+ & \int \frac{\mathcal{N}}{(N+V)^2}\partial_x^2 V\triangle y_t\partial_x^2 y_t \\
\geqslant & \frac{2}{3}\int \frac{\mathcal{N}}{\mathcal{N}+V}|\partial_x^2\nabla y_t|^2 - C\delta_1(\|\partial_x V_t, \partial_x^2 y, \partial_x^2 y_t)\|_{H^1}^2 + \int |\partial_x^2 V|^2).
\end{aligned}
$$

So, (2.46) together these estimates ($l = 0, 1$), we have

$$
\begin{aligned}
&\frac{d}{dt}\int \frac{\mathcal{N}}{2}(|\partial_x^2 y|^2 + |\partial_x^2 y_t|^2) + C\|(\partial_x^2 y, \partial_x^3 y, \partial_x^2 y_t, \partial_x^3 y_t)\|^2 + \frac{2}{3}T^0\int \mathcal{N}\partial_x^2\partial_t^l y\partial_x^2\partial_t^l \mathrm{div}\mathbf{u} \\
&\leqslant C\delta_1\int(|\partial_x^2 V|^2 + |\partial_x^2 V_t|^2 + |\partial_x V_t|^2 + |\partial_x^3\mathbf{u}|^2 + |\partial_x^2\mathbf{u}|^2 \\
&\quad +|\partial_x\mathbf{u}|^2 + |\partial_x^2\mathbf{u}_t|^2 + |\partial_x\mathbf{u}_t|^2). \qquad (2.47)
\end{aligned}
$$

Therefore, combining (2.42),(2.43),(2.45) and (2.47), we can get

$$
\begin{aligned}
\frac{d}{dt} & \int[\frac{\mathcal{N}}{2}|\partial_x^3\mathbf{u}|^2 + 3\partial_x\mathcal{N}\partial_x^2\mathbf{u}\partial_x^3\mathbf{u} + 3\partial_x^2\mathcal{N}\partial_x\mathbf{u}\partial_x^3\mathbf{u} + \partial_x^3\mathcal{N}\mathbf{u}\partial_x^3\mathbf{u} + \frac{1}{2}|\partial_x^3\mathbf{e}|^2 + (\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)} \\
+ & \frac{\mathcal{N}y}{2(\mathcal{N}+V)^2})|\partial_x^3 V|^2 + \frac{\mathcal{N}}{\mathcal{N}+V}\partial_x^3 V(h'(\mathcal{N}+V)\partial_x^3\mathcal{N} + h''(\mathcal{N}+V)\partial_x(\mathcal{N}+V)\partial_x^2(\mathcal{N}+V) \\
+ & h'''(\mathcal{N}+V)(\partial_x(\mathcal{N}+V))^3 - h'(\mathcal{N})\partial_x^3\mathcal{N} - h''(\mathcal{N})\partial_x\mathcal{N}\partial_x^2\mathcal{N} - h'''(\mathcal{N})(\partial_x\mathcal{N})^3) + \frac{\mathcal{N}}{2}|\partial_x^2\mathbf{u}_t|^2 \\
+ & 2\partial_x\mathcal{N}\partial_x\mathbf{u}_t\partial_x^2\mathbf{u}_t + \partial_x^2\mathcal{N}\mathbf{u}_t\partial_x^2\mathbf{u}_t + \frac{1}{2}|\partial_x^2\mathbf{e}_t|^2 + (\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)} + \frac{\mathcal{N}y}{2(\mathcal{N}+V)^2})|\partial_x^2 V_t|^2 \\
+ & \frac{3\mathcal{N}}{4T^0}(|\partial_x^2 y|^2 + |\partial_x^2 y_t|^2)] + C\int(|\partial_x^3 V|^2 + |\partial_x^2 V|^2 + |\partial_x^3\mathbf{u}|^2 + |\partial_x^2\mathbf{u}_t| + |\partial_x^3 y|^2 + |\partial_x^2 y|^2 \\
+ & |\partial_x^3 y_t|^2 + |\partial_x^2 y_t|^2) + \int \mathcal{N}\partial_x^2 y\partial_x^2(\mathrm{div}\mathbf{u}) \leqslant C\delta_1(\|(V_t, \mathbf{u}, \mathbf{e}_t)\|_{H^2}^2 + \|(V, \mathbf{u}_t, y, y_t)\|_{H^1}^2) \\
& +C\|(\partial_x^2 V_t, \partial_x V_t, \partial_x^2\mathbf{u}, \partial_x\mathbf{u}, \partial_x\mathbf{u}_t, \partial_x y, y_t)\|^2. \qquad (2.48)
\end{aligned}
$$

Noticing that (2.12) with $i = 2$, (2.13) with $j = 3$ and (2.15) with $m = 2$ and combining (2.32),(2.40) and (2.48), we have

$$
\frac{d}{dt}\int F + C(\|(V, \mathbf{u}, y, y_t, \mathbf{e})\|_{H^3}^2 + \|(V_t, \mathbf{u}_t, \mathbf{e}_t)\|_{H^2}^2) \leqslant 0. \qquad (2.49)
$$

Where

$$
\begin{aligned}
F = \quad & K_1 \quad [\frac{\mathcal{N}}{2}(|\mathbf{u}|^2+|\mathbf{u}_t|^2)+\int_0^V (h(\mathcal{N}+s)-h(\mathcal{N}))ds+\frac{\mathcal{N}h'(\mathcal{N}+\theta V)}{2(\mathcal{N}+V)}|V_t|^2+\frac{1}{2}(|\mathbf{e}|^2+|\mathbf{e}_t|^2)\\
+ \quad & \frac{3\mathcal{N}}{4T^0}(|y|^2+|y_t|^2)+\frac{1}{2}|\nabla\mathbf{u}|^2+(\frac{h'(\mathcal{N}+V)}{2(\mathcal{N}+V)}+\frac{y}{2(\mathcal{N}+V)^2})|\nabla V|^2+\frac{|V|^2}{2\mathcal{N}}+\frac{1}{2}|\partial_x\mathbf{e}|^2]\\
+ \quad & K_2[\frac{\mathcal{N}}{2}|\partial_x^2\mathbf{u}|^2+\partial_x^2\mathcal{N}\mathbf{u}\partial_x^2\mathbf{u}+2\partial_x\mathcal{N}\partial_x\mathbf{u}\partial_x^2\mathbf{u}+\frac{1}{2}|\partial_x^2\mathbf{e}|^2+\frac{\mathcal{N}}{2}|\partial_x\mathbf{u}_t|^2+\partial_x\mathcal{N}\mathbf{u}_t\partial_x\mathbf{u}_t+\frac{1}{2}|\partial_x\mathbf{e}_t|^2\\
+ \quad & \frac{3\mathcal{N}}{4T^0}(|\partial_x y|^2+|\partial_x y_t|^2)+(\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)}+\frac{\mathcal{N}y}{2(\mathcal{N}+V)^2})|\partial_x V_t|^2\\
+ \quad & \frac{\mathcal{N}\partial_x^2(h(\mathcal{N}+V)-h(\mathcal{N}))\partial_x^2 V}{\mathcal{N}+V}-\frac{\mathcal{N}h'(\mathcal{N}+V)|\partial_x^2 V|^2}{2(\mathcal{N}+V)})]+K_3[\frac{\mathcal{N}}{2}|\partial_x^3\mathbf{u}|^2+3\partial_x\mathcal{N}\partial_x^2\mathbf{u}\partial_x^3\mathbf{u}\\
+ \quad & 3\partial_x^2\mathcal{N}\partial_x\mathbf{u}\partial_x^3\mathbf{u}+\partial_x^3\mathcal{N}\mathbf{u}\partial_x^3\mathbf{u}+\frac{1}{2}|\partial_x^3\mathbf{e}|^2+(\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)}+\frac{\mathcal{N}y}{2(\mathcal{N}+V)^2})|\partial_x^3 V|^2\\
+ \quad & \frac{\mathcal{N}}{\mathcal{N}+V}\partial_x^3 V(h'(\mathcal{N}+V)\partial_x^3\mathcal{N}+h''(\mathcal{N}+V)\partial_x(\mathcal{N}+V)\partial_x^2(\mathcal{N}+V)\\
+ \quad & h'''(\mathcal{N}+V)(\partial_x(\mathcal{N}+V))^3-h'(\mathcal{N})\partial_x^3\mathcal{N}-h''(\mathcal{N})\partial_x\mathcal{N}\partial_x^2\mathcal{N}\\
- \quad & h'''(\mathcal{N})(\partial_x\mathcal{N})^3)+\frac{\mathcal{N}}{2}|\partial_x^2\mathbf{u}_t|^2+2\partial_x\mathcal{N}\partial_x\mathbf{u}_t\partial_x^2\mathbf{u}_t+\partial_x^2\mathcal{N}\mathbf{u}_t\partial_x^2\mathbf{u}_t\\
+ \quad & \frac{1}{2}|\partial_x^2\mathbf{e}_t|^2+(\frac{\mathcal{N}h'(\mathcal{N}+V)}{2(\mathcal{N}+V)}+\frac{\mathcal{N}y}{2(\mathcal{N}+V)^2})|\partial_x^2 V_t|^2+\frac{3\mathcal{N}}{4T^0}(|\partial_x^2 y|^2+|\partial_x^2 y_t|^2)],
\end{aligned}
$$

for some positive constants K_1, K_2 and K_3. It is easy to see that F satisfies

$$
\begin{aligned}
c(\|(V,\mathbf{u},\mathbf{e})\|_{H^3}^2+\|y\|_{H^2}^2 \quad + \quad & \|(V_t,\mathbf{u}_t,y_t,\mathbf{e}_t)\|_{H^2}^2) \leqslant \int F\\
\leqslant \quad & C(\|(V,\mathbf{u},\mathbf{e})\|_{H^3}^2+\|y\|_{H^2}^2+\|(V_t,\mathbf{u}_t,y_t,\mathbf{e}_t)\|_{H^2}^2),
\end{aligned}
$$

for some positive constants c and C. Therefore, (2.49) and the above estimate imply (2.7), which concludes the proof of Lemma 2.2. □

Based on the existence of local solutions and the *a priori* estimate, we can apply the standard continuous argument such as in [19]-[20] to show the global existence and uniqueness of smooth solutions to (1.1)-(1.2), satisfying (1.9), so Theorem 1.3 is proved. □

3 Main results for $\kappa = 0$

In this section, we study the system (1.1)-(1.2) without heat flux term (*i.e.* $\kappa = 0$), so the system (1.1)-(1.2) becomes a hyperbolic-elliptic system:

$$
\begin{cases}
n_t+\nabla\cdot(n\mathbf{u})=0\\
\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}+\frac{1}{n}\nabla(nT)=\nabla\Phi-\frac{\mathbf{u}}{\tau_p}\\
T_t+\mathbf{u}\cdot\nabla T+\frac{2}{3}T\mathrm{div}\mathbf{u}=\frac{2\tau_w-\tau_p}{3\tau_w\tau_p}|\mathbf{u}|^2-\frac{T-T^0}{\tau_w}\\
\triangle\Phi=n-b(x)
\end{cases}
\tag{3.1}
$$

for $(x,t)\in\mathbb{R}^N\times[0,+\infty), N=2,3$. The system is supplemented with the initial data

$$n(x,0)=n_0(x),\quad \mathbf{u}(x,0)=\mathbf{u}_0(x),\quad T(x,0)=T_0(x)\quad x\in\mathbb{R}^N. \tag{3.2}$$

Based on the similar energy argument, we can obtain the following main results:

Theorem 3.1

Suppose that $b(x)$ satisfies the condition (1.3),(1.4) and $n(\cdot,0)-\mathcal{N}\in H^3(\mathbb{R}^N), \mathbf{u}(\cdot,0)\in H^3(\mathbb{R}^N), \nabla\Phi(\cdot,0)-\mathcal{E}\in H^3(\mathbb{R}^N)$ and $T(\cdot,0)-T^0\in H^3(\mathbb{R}^N)$. Then there exists sufficiently small constant $\delta_0>0$, depending only on $b(x)$, such that if

$$\|(n(\cdot,0)-\mathcal{N},\mathbf{u}(\cdot,0),\nabla\Phi(\cdot,0)-\mathcal{E},T(\cdot,0)-T^0)\|_{H^3(\mathbb{R}^N)}$$
$$+\|(n_t,\mathbf{u}_t,\nabla\Phi_t,T_t)(\cdot,0)\|_{H^2(\mathbb{R}^N)}+\|\nabla b\|_{H^3(\mathbb{R}^N)}\leqslant\delta_0$$

Then the Cauchy problem (3.1)-(3.2) exists an unique global smooth solution $(n(x,t),\mathbf{u}(x,t),\Phi(x,t),T(x,t))$ for all $t\geqslant 0$. Moreover,

$$\|(n(\cdot,t)-\mathcal{N},\mathbf{u}(\cdot,t),\nabla\Phi(\cdot,t)-\mathcal{E},T(\cdot,t)-T^0)\|^2_{H^3(\mathbb{R}^N)}+\|(n_t,\mathbf{u}_t,\nabla\Phi_t,T_t)(\cdot,t)\|^2_{H^2(\mathbb{R}^N)}$$
$$\leqslant C_0[\|(n(\cdot,0)-\mathcal{N},\mathbf{u}(\cdot,0),\nabla\Phi(\cdot,0)-\mathcal{E},T(\cdot,0)-T^0)\|^2_{H^3(\mathbb{R}^N)}$$
$$+\|(n_t,\mathbf{u}_t,\nabla\Phi_t,T_t)(\cdot,0)\|^2_{H^2(\mathbb{R}^N)}]\exp(-\alpha_0 t)$$

for some positive constants α_0 and C_0.

In the same way, we need the local solution results and *a priori* estimate in order to prove Theorem 3.1, which are given by Lemma 3.2 and Lemma 3.3, respectively.

Lemma 3.2

Suppose that $b(x)$ satisfies the condition (1.3),(1.4) and $n(\cdot,0)-\mathcal{N}\in H^3(\mathbb{R}^N), \mathbf{u}(\cdot,0)\in H^3(\mathbb{R}^N), \nabla\Phi(\cdot,0)-\mathcal{E}\in H^3(\mathbb{R}^N)$ and $T(\cdot,0)-T^0\in H^3(\mathbb{R}^N)$. Then there exists an unique smooth solution $(n(x,t),\mathbf{u}(x,t),\Phi(x,t),T(x,t))$ of the system (3.1)-(3.2) satisfying

$$n(x,t),\mathbf{u}(x,t),\nabla\Phi(x,t),T(x,t)\in C^1(\mathbb{R}^N\times[0,T_{\max})),$$

and

$$n(x,t)-\mathcal{N},\ \mathbf{u}(x,t),\ \nabla\Phi(x,t)-\mathcal{E},T(x,t)-T^0\in L^\infty(0,T;H^3(\mathbb{R}^N)),$$

defined on a maximal interval of existence $[0,T_{\max})$. Moreover, if $T_{\max}<+\infty$, then

$$\|(n(\cdot,t)-\mathcal{N},\mathbf{u}(\cdot,t),\nabla\Phi(\cdot,t)-\mathcal{E},T(\cdot,t)-T^0)\|^2_{H^3(\mathbb{R}^N)}+\|(n_t,\mathbf{u}_t,\nabla\Phi_t,T_t)(\cdot,t)\|^2_{H^2(\mathbb{R}^N)}$$
$$+\int_0^t[\|(n(\cdot,\tau)-\mathcal{N},\mathbf{u}(\cdot,\tau),\nabla\Phi(\cdot,\tau)-\mathcal{E},T(\cdot,\tau)-T^0)\|^2_{H^3(\mathbb{R}^N)}$$
$$+\|(n_t,\mathbf{u}_t,\nabla\Phi_t,T_t)(\cdot,\tau)\|^2_{H^2(\mathbb{R}^N)}]d\tau\to\infty$$

as $t\to T_{\max}-$.

Lemma 3.3

Suppose that $(V,\mathbf{u},y,\mathbf{e})$ satisfies the system (2.4)-(2.5)($(2.4)_3$ without heat flux term) for

$(x,t) \in \mathbb{R}^N \times [0, T_{\max})$. Then there exists sufficiently small constant $\delta_1 > 0$, depending only on $b(x)$, such that for $0 < S < T_{\max}$, if

$$\sup_{0 \leqslant t \leqslant S} (\|(V, \mathbf{u}, \mathbf{e}, y)(\cdot, t)\|_{H^3(\mathbb{R}^N)} + \|(V_t, \mathbf{u}_t, \mathbf{e}_t, y_t)(\cdot, t)\|_{H^2(\mathbb{R}^N)} + \|\nabla b\|_{H^3(\mathbb{R}^N)}) \leqslant \delta_1,$$

then

$$\begin{aligned} \|(V, \mathbf{u}, \mathbf{e}, y)(\cdot, t)\|^2_{H^3(\mathbb{R}^N)} &+ \|(V_t, \mathbf{u}_t, \mathbf{e}_t, y_t)(\cdot, t)\|^2_{H^2(\mathbb{R}^N)} \\ &\leqslant C_1(\|(V, \mathbf{u}, \mathbf{e}, y)(\cdot, 0)\|^2_{H^3(\mathbb{R}^N)} + \|(V_t, \mathbf{u}_t, \mathbf{e}_t, y_t)(\cdot, 0)\|^2_{H^2(\mathbb{R}^N)}) \exp(-\alpha_1 t) \end{aligned}$$

for any $t \in [0, S]$ and some positive constants α_1 and C_1.

Remark 3.4

Using Green's formulation, the system (3.1) can be reduced to the pure hyperbolic system and the proof of the local solution can be established by a standard contraction mapping principle, see *e.g.* [12],[13]. As for the proof of Lemma 3.3 is similar to the proof of Lemma 2.2, the most difference is to establish the third derivatives estimate on y, we need take ∂_x^3 on the both sides of $(2.4)_3$ and multiply the resulting equation by $\partial_x^3 y$, then integrate it over $\mathbb{R}^N$. Combining all estimates, we conclude the proof of Lemma 3.3. Theorem 3.1 follows from the continuous argument by using Lemma 3.2 and Lemma 3.3.

Corollary 3.5

Suppose that $b(x) = \mathcal{N} > 0$ (*positive constant*) and $n(\cdot, 0) - \mathcal{N} \in H^3(\mathbb{R}^N), \mathbf{u}(\cdot, 0) \in H^3(\mathbb{R}^N), \nabla\Phi(\cdot, 0) \in H^3(\mathbb{R}^N)$ and $T(\cdot, 0) - T^0 \in H^3(\mathbb{R}^N)$. Then there exists sufficiently small constant $\delta_0 > 0$, depending only on $\mathcal{N}$, such that if

$$\|(n(\cdot, 0) - \mathcal{N}, \mathbf{u}(\cdot, 0), \nabla\Phi(\cdot, 0), T(\cdot, 0) - T^0)\|_{H^3(\mathbb{R}^N)}$$

$$+\|(n_t, \mathbf{u}_t, \nabla\Phi_t, T_t)(\cdot, 0)\|_{H^2(\mathbb{R}^N)} \leqslant \delta_0$$

Then the Cauchy problem (3.1)-(3.2) exists an unique global smooth solution $(n(x,t), \mathbf{u}(x,t), \Phi(x,t), T(x,t))$ for all $t \geqslant 0$. Moreover,

$$\|(n(\cdot, t) - \mathcal{N}, \mathbf{u}(\cdot, t), \nabla\Phi(\cdot, t), T(\cdot, t) - T^0)\|^2_{H^3(\mathbb{R}^N)} + \|(n_t, \mathbf{u}_t, \nabla\Phi_t, T_t)(\cdot, t)\|^2_{H^2(\mathbb{R}^N)}$$

$$\leqslant C_0[\|(n(\cdot, 0) - \mathcal{N}, \mathbf{u}(\cdot, 0), \nabla\Phi(\cdot, 0), T(\cdot, 0) - T^0)\|^2_{H^3(\mathbb{R}^N)}$$

$$+\|(n_t, \mathbf{u}_t, \nabla\Phi_t, T_t)(\cdot, 0)\|^2_{H^2(\mathbb{R}^N)}] \exp(-\alpha_0 t)$$

for some positive constants α_0 and C_0.

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