Operators on corner manifolds with exit to infinity

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September 19, 2005

Abstract

We study (pseudo-)differential operators on a manifold with edge Z, locally modelled on a wedge with model cone that has itself a base manifold W with smooth edge Y. The typical operators A are corner degenerate in a specific way. They are described (modulo 'lower order terms') by a principal symbolic hierarchy $\sigma(A) = (\sigma_\psi(A), \sigma_\wedge(A), \sigma_{\pmb{\wedge}}(A))$, where σ_ψ is the interior symbol and $\sigma_\wedge(A)(y,\eta)$, $(y,\eta) \in T^*Y \setminus 0$, the (operator-valued) edge symbol of 'first generation', cf. [15]. The novelty here is the edge symbol $\sigma_{\pmb{\wedge}}$ of 'second generation', parametrised by $(z,\zeta) \in T^*Z \setminus 0$, acting on weighted Sobolev spaces on the infinite cone with base W. Since such a cone has edges with exit to infinity, the calculus has the problem to understand the behaviour of operators on a manifold of that kind.
We show the continuity of corner-degenerate operators in weighted edge Sobolev spaces, and we investigate the ellipticity of edge symbols of second generation. Starting from parameter-dependent elliptic families of edge operators of first generation, we obtain the Fredholm property of higher edge symbols on the corresponding singular infinite model cone.

2000 Mathematics Subject Classification: 35J70, 35S05, 58J40

Keywords: Operators on manifolds with edge and conical exit to infinity, Sobolev spaces with double weights on singular cones, parameter-dependent ellipticity, edge and corner symbols.

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Introduction

This paper studies (pseudo-)differential operators on manifolds with conical exit to infinity whose cross section is a (compact) manifold W with smooth edge Y. More precisely, 'at infinity' such a manifold is modelled on a cylinder $\mathbb{R}_+ \times W$, and the metric is assumed to be conical for large $t \in \mathbb{R}_+$, i.e., of the form $dt^2 + t^2 g_W$ for a wedge metric g_W on the cross section W (cf. Definition 1.2 and Section 2.1 below).

The cone $W^\triangle := (\overline{\mathbb{R}}_+ \times W)/(\{0\} \times W)$ itself is interesting as well because of specific corner effects also for $t \to 0$ (near the tip, represented by $\{0\} \times W$, identified with a point v). A calculus for corners of that type is developed in [18]. Operators on a manifold Z with higher edge, modelled on a wedge $W^\triangle \times \Xi$ with edge $\Xi \subseteq \mathbb{R}^p$, have a so called principal edge symbol which consists of a family of operators on $W^\triangle$ parametrised by $(z, \zeta) \in T^*\Xi \setminus 0$, with information both for $t \to 0$ and $t \to \infty$.

The main objective of the present paper is the investigation of such edge symbols for $t \to \infty$.

The general background is as follows. Operators on manifolds with 'higher singularities' (e.g., of edge or corner type) may be studied by an iterative approach, parallel to the process of repeatedly forming cones and wedges, combined with global constructions. The cones and wedges are based on already constructed manifolds of lower singularity order. By order zero we understand the smooth case, by order one the case of cones with smooth cross sections or of wedges with such model cones, etc. The program of the (pseudo-differential) analysis is to iterate suitable symbolic hierarchies, associated with the strata of the configuration and to establish corresponding operator algebras. The symbols should be responsible for the ellipticity (or parabolicity) of operators, parametrices, Fredholm property (or invertibility), and regularity and asymptotics of solutions. The problem with higher singularities is to really manage the iteration and to achieve a transparent formalism. Our paper is devoted to one of the typical elements of this approach, namely, the analysis of edge symbols taking values in operators on an infinite non-smooth cone, here of second generation (which means singularities of second order).

In order to illustrate the idea, we first recall some aspects of the simpler case of a smooth manifold with boundary. The operators are identified with boundary value problems. They have a two-component principal symbolic hierarchy $(\sigma_\psi, \sigma_\partial)$, consisting of the interior and the boundary symbol (indicated by subscripts 'ψ' and '∂', respectively). Boundary value problems are connected with many analytical and topological details (e.g., the transmission or violated transmission property, cf. Boutet de Monvel [4], Vishik and Eskin [25], [9], the Atiyah-Bott obstruction for the existence of Shapiro-Lopatinskij boundary conditions, cf. Atiyah and Bott [1], and APS or global projection conditions, cf. Atiyah, Patodi and Singer [2]). Smooth manifolds M with boundary are a subcategory of manifolds W with smooth edges; in the general case the model cones of local wedges may be non-trivial, i.e., of the form $X^\triangle := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$, for a (smooth compact) base X of non-zero dimension (rather than $\overline{\mathbb{R}}_+$, the inner normal to the boundary of M). As is known from [15] for a manifold W with smooth edge Y, the principal symbolic hierarchy of operators A in the 'edge algebra' consists of two components $(\sigma_\psi, \sigma_\wedge)$, where $\sigma_\wedge$ is the homogeneous principal edge symbol, a generalisation of σ_∂. The edge symbol is a family of operators

$$\sigma_\wedge(A)(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{1}$$

on the infinite open stretched model cone $X^\wedge := \mathbb{R}_+ \times X \ni (r, x)$, parametrised by $(y, \eta) \in T^*Y \setminus 0$; here $\mathcal{K}^{s,\gamma}(X^\wedge)$ is a scale of weighted Sobolev spaces of smoothness $s \in \mathbb{R}$ and weight $\gamma \in \mathbb{R}$, cf. Section 1.2 below. The operators in the edge algebra have a 2×2 block matrix structure which is analogous to that of boundary value problems (without the transmission property, see, e.g., [19]), with all the features such as extra edge conditions of Shapiro-Lopatinskij type if an analogue of the Atiyah-Bott obstruction vanishes, see [8], [15], [16], or global projection conditions in the opposite case, see [20]. The latter effects are governed by the operators (1).

The interior ellipticity of A, i.e., the ellipticity of the operator with respect to σ_ψ (in a suitable edge-degenerate sense) entails the Fredholm property of the operators (1) for every $(y,\eta) \in T^*Y \setminus 0$, and for all weights $\gamma \in \mathbb{R} \setminus D(y)$ for a discrete set $D(y)$ of reals, cf. [16].

The manifold $X^\wedge$ has a conical exit to infinity (for $r \to \infty$). It turns out that certain subordinate exit symbols, that are usually responsible for the compactness of remainders in a parametrix construction up to ∞, are automatically elliptic, see, e.g., [11, Chapter 3]. In other words, ellipticity 'in the finite' of an operator on a manifold with edge is connected with a specific background information on ellipticity on the manifold $X^\wedge$ with conical exit.

A similar result is necessary for operators on manifolds with singularities of second (and higher) order. In the present paper we investigate this phenomenon for the case of second order singularities. This is far from being straightforward, compared with the first order case. Formally, we replace X from the discussion before by a compact manifold W with edge Y. Then $W^\wedge = \mathbb{R}_+ \times W$ has a conical exit to infinity; however, also the edge $Y^\wedge$ of $W^\wedge$ has an exit. This requires corresponding new elements of the edge calculus. Also the analogues of the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ with $W^\wedge$ instead of $X^\wedge$ have to be introduced as a next generation of weighted Sobolev spaces, now with two axial weights, one for the inner cone axis direction $r \in \mathbb{R}_+$ near zero and one for the corner axis direction $t \in \mathbb{R}_+$ near zero, and with an additional control for $t \to \infty$.

Starting from differential operators $\boldsymbol{A}$ on $W^\wedge \times \Xi$, $\Xi \subseteq \mathbb{R}^p$ open, with a corresponding edge-corner degeneracy in stretched coordinates (cf. the formulas (32) and (33) below), we introduce parameter-dependent edge symbols $\sigma_{\boldsymbol{\wedge}}(\boldsymbol{A})(z,\zeta)$ in a pseudo-differential set-up. Under a natural ellipticity condition, in Section 3.3 we show the Fredholm property on $\mathbb{W}^\wedge$ between weighted corner Sobolev spaces $\mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge)$ (cf. Definition 2.2) for every $(z,\zeta) \in T^*\Xi \setminus 0$; here $\mathbb{W}$ denotes the stretched manifold associated with W, and $\boldsymbol{\gamma} = (\gamma,\theta)$ is a pair of weights with $\gamma \in \mathbb{R}$ belonging to the inner cone axis variable $r \in \mathbb{R}_+$ and $\theta \in \mathbb{R}$ to the corner axis variable $t \in \mathbb{R}_+$.

More generally, the ellipticity of $\boldsymbol{A}$ suggests additional edge conditions on $Y \times \Xi$, satisfying an analogue of the Shapiro-Lopatinskij condition, expressed by an operator family $\sigma_{\boldsymbol{\wedge}}(\boldsymbol{\mathcal{A}})(z,\zeta)$ belonging to a 2×2 block matrix $\boldsymbol{\mathcal{A}}$ with $\boldsymbol{A}$ as the upper left corner. In place of $\mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge)$ we then have a direct sum of spaces with $\mathcal{K}^{s-\frac{n+1}{2},\theta-\frac{n+1}{2}}(Y^\wedge)$ as the second component (here $n+1$ is the codimension of Y in W).

As noted before, the new aspect is the control of such an edge ellipticity on $Y^\wedge$ up to ∞, connected with edge-corner degenerate operators on wedges of the form $W^\wedge \times \Xi$. This requires a systematic approach in terms of operator-valued amplitude functions taking values in the edge algebra on W and with a typical degeneracy in t from the corner singularity. The surprising observation is that the corner degeneracy from

$t \to 0$ automatically causes the desired exit effects for $t \to \infty$ when the operators contain a parameter $\zeta \neq 0$ in degenerate form $t\zeta$ and an extra weight factor $t^{-\mu}$.

The paper is organised as follows. In Chapter 1 we introduce the notion of a manifold with edge and conical exit to infinity, including a new variant of weighted edge Sobolev spaces, cf. Definition 1.8. Moreover, we develop the necessary tools on edge amplitude functions and the so called edge algebra, here with parameters, cf. [7].

Chapter 2 is devoted to edge symbols of second generation, acting on weighted Sobolev spaces $\mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge)$ on the infinite stretched cone $\mathbb{W}^\wedge$ with base $\mathbb{W}$. The spaces themselves encode specific information both for $t \to 0$ and $t \to \infty$. The neighbourhood of $t = 0$ corresponds to the corner situation of [18], while $t \to \infty$ is the novelty from the geometry of a manifold with edge and conical exit. One of the main issues is to see the continuity of edge symbols of second generation in those spaces up to infinity. This is checked first for the case of typical 'corner-degenerate' differential operators. After that we pass to the pseudo-differential case, cf. Theorem 2.14 and Corollary 2.15.

In Chapter 3 we consider (t,τ)-depending amplitude functions $\mathfrak{a}(t,\tau,\zeta) = \widetilde{\mathfrak{a}}(t\tau, t\zeta)$ that take values in the edge algebra on a compact (stretched) manifold $\mathbb{W}$ with edge. We show that the parameter-dependent ellipticity of $\tilde{\mathfrak{a}}(\tilde{\tau},\tilde{\zeta})$, $(\tilde{\tau},\tilde{\zeta}) \in \mathbb{R}^{1+p}$, entails the exit ellipticity of $\mathrm{op}_t(t^{-\mu}\mathfrak{a}(t,\tau,\zeta))$ for $\zeta \neq 0$, which is a necessary information for additional elliptic edge conditions, provided that an analogue of the above mentioned topological obstruction vanishes. In the final section we return to the behaviour of corner symbols at infinity and give a proof of Theorem 2.14.

Operators on non-compact manifolds with a control of amplitude functions up to corresponding 'exits to infinity' have been studied by many authors in different contexts before. Let us mention here the papers of Nirenberg and Walker [12], Parenti [13] and Cordes [5]; they emphasize the role of mapping properties in standard Sobolev spaces globally in the Euclidean space, connected with suitable symbolic estimates at infinity. By changing the nature of the spaces and the symbols at infinity we may obtain, of course, calculi with completely different properties. However, this is not the main point of our paper, although, for instance, operators of Fuchs type on manifolds with conical singularities can be seen from such a point of view. The aspect that standard Sobolev spaces on cones $X^\wedge$ at infinity are natural for manifolds with smooth edges is essential also for our calculus.

1 The edge calculus with parameters

1.1 Manifolds with edge and conical exit

A manifold W with edge Y can be described by its associated stretched manifold $\mathbb{W}$ as follows. $\mathbb{W}$ is a C^∞ manifold with boundary, and $\partial\mathbb{W}$ is an X bundle over Y, for a compact C^∞ manifold X and a C^∞ manifold Y. Let $\pi : \partial\mathbb{W} \to Y$ be the bundle

projection. Then W is the image under the map $p : \mathbb{W} \to W$ defined by $p|_{\partial\mathbb{W}} = \pi$ and $p|_{\mathbb{W}\setminus\partial\mathbb{W}} = \mathrm{id}_{\mathbb{W}\setminus\partial\mathbb{W}}$.
An example is $W = X^{\triangle} \times Y$ (cf. the notation in the introduction) with Y and X as before; then we have $\mathbb{W} = \overline{\mathbb{R}}_+ \times X \times Y$, and $\partial\mathbb{W}$ is just the trivial bundle $X \times Y$.
We set $\mathbb{W}_{\mathrm{sing}} := \partial\mathbb{W}$, $\mathbb{W}_{\mathrm{reg}} := \mathbb{W} \setminus \partial\mathbb{W}$.
Let $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(\mathbb{W})$ denote the space of all differential operators on $\mathbb{W} \setminus \partial\mathbb{W}$ of order μ with smooth coefficients, which are locally near Y in the splitting of variables $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$, $\Omega \subseteq \mathbb{R}^q$ open, of the form

$$r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r, y)(-r\partial_r)^j (rD_y)^{\alpha} \qquad (2)$$

with coefficients $a_{j\alpha} \in C^{\infty}\big(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X)\big)$ (with $\mathrm{Diff}^{\nu}(X)$ being the space of all differential operators of order ν on X with smooth coefficients); in the latter notation the C^{∞} manifold X is not necessarily compact. We will call the operators in $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(\mathbb{W})$ edge-degenerate.
The manifolds in this paper are assumed to be countable unions of compact sets. Then $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(\mathbb{W})$ is a Fréchet space in a natural way.
The manifolds with edge form a category with natural morphisms. In particular, isomorphisms $W \to \widetilde{W}$ can be described on the level of the corresponding stretched manifolds, namely as diffeomorphisms $\mathbb{W} \to \widetilde{\mathbb{W}}$ in the category of C^{∞} manifolds with boundary, the restrictions of which to $\partial\mathbb{W}$ induce isomorphisms $\partial\mathbb{W} \to \partial\widetilde{\mathbb{W}}$ as X-bundles over the edges Y and $\widetilde{Y}$, respectively.

Remark 1.1 *Let $\chi : \mathbb{W} \to \widetilde{\mathbb{W}}$ be an isomorphism between the (stretched) manifolds $\mathbb{W}$ and $\widetilde{\mathbb{W}}$ with edges Y and $\widetilde{Y}$, respectively. Then the operator push forward under $\chi_{\mathrm{int}} : \mathbb{W}_{\mathrm{reg}} \to \widetilde{\mathbb{W}}_{\mathrm{reg}}$ induces an isomorphism $\chi_* : \mathrm{Diff}^{\mu}_{\mathrm{deg}}(\mathbb{W}) \longrightarrow \mathrm{Diff}^{\mu}_{\mathrm{deg}}(\widetilde{\mathbb{W}})$.*

Definition 1.2 *A Riemannian metric on* $\mathrm{int}\mathbb{W}$ *is said to be a wedge metric if it has, locally near $\partial\mathbb{W}$ in the splitting of variables $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$, the form $dr^2 + r^2 g_X(r, y) + g_{\Omega}(r, y)$ for Riemannian metrics g_X and g_{Ω} on X and Ω, respectively, smoothly depending on the variables (up to $r = 0$).*

Let $\mathbb{W}$ be a (stretched) compact manifold with edge Y and consider the cylinder $\mathbb{R} \times \mathbb{W}$ in the splitting variables (t, w). We want to identify $\mathbb{R} \times \mathbb{W}$ with a manifold with conical exit.

Let W be a manifold with edge Y and $\mathbb{W}$ its associated stretched manifold. Then $W \times D$, for any C^{∞} manifold D, is a manifold with edge $Y \times D$, and $\mathbb{W} \times D$ is the associated stretched manifold.

Definition 1.3 *A manifold M with edge and conical exit to infinity is defined as a manifold with edge which contains a submanifold $(1, \infty) \times W$ for a compact manifold W with edge such that $M \setminus \{(1, \infty) \times W\}$ is compact, and $(1, \infty) \times \mathbb{W}$ is endowed with a metric of the form $dt^2 + t^2 g_{\mathbb{W}}$ for a wedge metric $g_{\mathbb{W}}$ on $\mathbb{W}$ (cf. Definition 1.2).*

Example 1.4 *If W is a manifold with egde, the cylinder $\mathbb{R} \times W$ can be endowed with the structure of a manifold with edge and conical exit for $t \to \pm\infty$ when we endow $(-\infty, -1) \times \mathbb{W}$ and $(1, \infty) \times \mathbb{W}$ with the metrics $dt^2 + t^2 g_{\mathbb{W}}^{\pm}$ for wedge metrics $g_{\mathbb{W}}^{\pm}$. Let $W_{\asymp}$ denote the corresponding manifold with conical exits; then $\mathbb{W}_{\asymp}$ is the associated stretched manifold.*

Far from the edge a manifold in the sense of Definition 1.3 belongs to the category of C^∞ manifolds with conical exit.
A C^∞ manifold Γ is called a manifold with conical exit to infinity if there exists a C^∞ submanifold N of codimension 1 such that $(1, \infty) \times N$ is a C^∞ submanifold of Γ, endowed with a cone metric $dt^2 + t^2 g_N$ for some Riemannian metric g_N on N.
We then interpret $t \to \infty$ as a conical exit. In future, for simplicity, we will assume that N can be regarded as a submanifold of a closed C^∞ manifold $\widetilde{N}$ and Γ as a submanifold of a manifold $\widetilde{\Gamma}$ such that

$$\widetilde{\Gamma} \setminus \{(1, \infty) \times \widetilde{N}\} \quad \text{is compact}, \tag{3}$$

and $(1, \infty) \times \widetilde{N}$ is endowed with a Riemannian metric of the form $dt^2 + t^2 g_{\widetilde{N}}$ for a Riemannian metric $g_{\widetilde{N}}$ on $\widetilde{N}$ which restricts to g_N on N. Let us say that a manifold $\widetilde{\Gamma}$ with conical exit is closed if it has the property (3).
By a function which is homogeneous of order $\nu \in \mathbb{R}$ in the large, we understand any $\chi \in C^\infty(\widetilde{\Gamma})$ such that $\chi_\infty := \chi|_{(1,\infty)\times\widetilde{N}}$ satisfies the relation $\chi_\infty(\lambda t, n) = \lambda^\nu \chi_\infty(t, n)$ for all $(t, n) \in (1, \infty) \times \widetilde{N}$ and all $\lambda \geq 1$. In a similar manner we define the homogeneity of order ν in the large for functions of $C^\infty(\Gamma)$.
On a C^∞ manifold $\widetilde{\Gamma}$ with conical exit to infinity satisfying the condition (3), we have a natural notion of weighted Sobolev spaces $H^{s;\delta}(\widetilde{\Gamma})$, $s, \delta \in \mathbb{R}$, cf. [17]. In order to define analogous spaces on Γ, we specify a kind of cut-off functions. We say that an element $\chi \in C^\infty(\Gamma)$ is a conical cut-off function on Γ if it is homogeneous of order 0 in the large and if both $\operatorname{supp} \chi \cap (\Gamma \setminus \{(1, \infty) \times N\})$ and $\operatorname{supp} \chi \cap (\{1\} \times N)$ are compact sets.

Definition 1.5 *Let $H^{s;\delta}_{\text{loc}}(\Gamma)$ denote the subspace of all $u \in H^s_{\text{loc}}(\Gamma)$ such that for every conical cut-off function χ on Γ we have $\chi u \in H^{s;\delta}(\widetilde{\Gamma})$ (the latter function is interpreted as the extension by zero of χu from Γ to $\widetilde{\Gamma}$).*

Remark 1.6 *Let W be a manifold with conical exit to infinity in the sense of Definition* 1.3*. Then $W \setminus Y$ and Y are C^∞ manifolds with conical exit.*

The stretched manifold $\mathbb{W}$ associated with a manifold W with edge Y and conical exit to infinity in the sense of Definition 1.3 has similar natural properties as stretched manifolds in general. $\mathbb{W}$ is a C^∞ manifold with boundary $\partial\mathbb{W}$ and conical exit to infinity, $\partial\mathbb{W}$ is an X bundle over Y, and we have a canonical projection $\mathbb{W} \to W$ as explained in Section 1.1. However, also Y itself is a closed C^∞ manifold with exit in the usual sense.

Remark 1.7 *Let X be a closed C^∞ manifold; then similarly as in Example* 1.4 *we have the manifoldf $X_{\asymp}$ modelled on $\mathbb{R} \times X$ and endowed with a metric $dt^2 + t^2 g_X$ for $|t| > 1$, where g_X is a Riemannian metric for every $\mu \in \mathbb{R}$on X. We then have the weighted Sobolev spaces $H^{s;\delta}(X_{\asymp})$. We will apply this to the case $X = 2\mathbb{W}$ for a compact* (*stretched*) *manifold $\mathbb{W}$ with edge.*

1.2 Weighted Sobolev spaces

In the following consideration we fix an $R^\mu(\lambda) \in L^\mu_{\mathrm{cl}}(X;\mathbb{R}^l)$ which is parameter-dependent elliptic of order $\mu \in \mathbb{R}$ that induces isomorphisms $R^\mu(\lambda) : H^s(X) \to H^{s-\mu}(X)$ for all $\lambda \in \mathbb{R}^l$ and all $s \in \mathbb{R}$. As is well known, such operator families for every $\mu \in \mathbb{R}$, exist. In the following definition we employ the Mellin transform $Mf(v) := \int_0^\infty r^{v-1} f(r)\,\mathrm{d}r$, first for $f \in C_0^\infty(\mathbb{R}_+)$ (which gives us an entire function in $v \in \mathbb{C}$) and then for other distributions (also vector-valued ones). Then v will vary on a weight line $\Gamma_\beta = \{v \in \mathbb{C} : \mathrm{Re}\, v = \beta\}$.

In this paper by a cut-off function on the half-axis we understand any $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ that is equal to 1 in a neighbourhood of $r = 0$. Let us now fix some notation on weighted spaces on a (stretched) cone with base X.

The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s, \gamma \in \mathbb{R}$ is defined as the completion of $C_0^\infty(X^\wedge)$ with respect to the norm $\left\{\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(\mathrm{Im}\, v)Mf(v)\|^2_{L^2(X)}\,\mathrm{d}v\right\}^{\frac{1}{2}}$ for $n = \dim X$ and for any choice of an order reducing family $R^s(\lambda) \in L^s_{\mathrm{cl}}(X;\mathbb{R})$ in the above sense.

The space $H^s_{\mathrm{cone}}(X^\wedge)$ for $s \in \mathbb{R}$ is defined as the subspace of all $f \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)|_{\mathbb{R}_+ \times X}$, such that for every cut-off function $\omega(r)$ on $\overline{\mathbb{R}}_+$ and every $\varphi \in C_0^\infty(X)$ supported in a coordinate neighbourhood U in X we have $(1-\omega)\varphi f \in \chi^* H^s(\mathbb{R}^{n+1})$, where $\chi : \mathbb{R}_+ \times U \to \mathbb{R}_+ \times V$ for any open set $V \subset S^n$ has the form $\mathrm{id}_{\mathbb{R}_+} \times \chi_1$ for a diffeomorphism $\chi_1 : U \to V$.

Finally, we set

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega f + (1-\omega)g : f \in \mathcal{H}^{s,\gamma}(X^\wedge), g \in H^s_{\mathrm{cone}}(X^\wedge)\}$$

for any cut-off function $\omega(r)$, and $\mathcal{S}^\gamma(X^\wedge)_\varepsilon := \bigcap_{k\in\mathbb{N}} \langle r\rangle^{-k}\mathcal{K}^{k,\gamma+\varepsilon-\frac{1}{k+1}}(X^\wedge)$ for any $\varepsilon > 0$.

Note that there is another equivalent definition of the spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ in terms of a mixture between the Mellin and the Fourier transform. Namely, $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ is the completion of the space $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} := \left\{\frac{1}{2\pi i}\int\limits_{\Gamma_{\frac{n+1}{2}-\gamma}}\int\limits_{\mathbb{R}^n} \langle v,\xi\rangle^{2s} |(M_{r\to v}F_{x\to\xi}u)(v,\xi)|^2\,\mathrm{d}v\,\mathrm{d}\xi\right\}^{\frac{1}{2}}. \quad (4)$$

Moreover, if X is a closed compact C^∞ manifold, $\{U_1, \dots, U_N\}$ an open covering by coordinate neighourhoods, $\{\varphi_1, \dots, \varphi_N\}$ a subordinate partition of unity, $\chi_j : U_j \to$

$\mathbb{R}^n$ a system of charts, the expression $\Big\{ \sum_{j=1}^{N} \|(\varphi_j f)(r, \chi_j^{-1}(x))\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)} \Big\}^{\frac{1}{2}}$ is a norm equivalent to (4).

Our next objective is to introduce a class of weighted Sobolev spaces on a manifold with edge and conical exit. The notion is based on the so-called abstract edge Sobolev spaces on $\mathbb{R}^q$, modelled by means of a Hilbert space E endowed with a group $\kappa = \{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : E \to E$, $\lambda \in \mathbb{R}_+$, which is strongly continuous in λ and satisfies $\kappa_\lambda \kappa_{\lambda'} = \kappa_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \mathbb{R}_+$ (in this case we simply say that E is endowed with a group action). Then the space $\mathcal{W}^s(\mathbb{R}^q, E)$ for $s \in \mathbb{R}$ is defined as the completion of $\mathcal{S}(\mathbb{R}^q, E)$ (the Schwartz space of E-valued functions in $\mathbb{R}^q$) with respect to the norm

$$\left\{ \int \langle \eta \rangle^{2s} \|\kappa^{-1}_{\langle \eta \rangle} \hat{u}(\eta)\|^2_E \, \mathrm{d}\eta \right\}^{\frac{1}{2}}, \tag{5}$$

with $\hat{u}(\eta)$ being the Fourier transform of u in $\mathbb{R}^q$. Note that when we replace $\langle \eta \rangle$ in the expression (5) by any other strictly positive function $p(\eta)$ which satisfies an estimate $c_1 \langle \eta \rangle \le p(\eta) \le c_2 \langle \eta \rangle$ for certain constants $c_1 \le c_2$, we obtain an equivalent norm. Below we use, for instance, $p(\eta) = [\eta]$ which is definded as any strictly positive C^∞ function satisfying $[\eta] = |\eta|$ for $|\eta| \ge c$ for some constant $c > 0$.
As an example (and for purposes below) observe that, when we set $E = H^s(\mathbb{R}^{n+1}_{\tilde{x}})$ with $(\kappa_\lambda v)(\tilde{x}) = \lambda^{\frac{n+1}{2}} v(\lambda \tilde{x})$, $\lambda \in \mathbb{R}_+$, we have $H^s(\mathbb{R}^{n+1} \times \mathbb{R}^q) = \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^{n+1}_{\tilde{x}}))$ for every $s \in \mathbb{R}$.
Another example are the weighted edge Sobolev spaces based on $E = \mathcal{K}^{s,\gamma}(X^\wedge)$ with the group action $\kappa_\lambda u(r, x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$, $\lambda \in \mathbb{R}_+$. In this case we set

$$\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)). \tag{6}$$

Observe that for $\mathcal{W}^{\infty,\gamma}(X^\wedge \times \mathbb{R}^q) := \bigcap_{s \in \mathbb{R}} \mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ we have

$$\omega(r) r^\beta \mathcal{W}^{\infty,\gamma}(X^\wedge \times \mathbb{R}^q) = \omega(r) \mathcal{W}^{\infty,\gamma+\beta}(X^\wedge \times \mathbb{R}^q)$$

for every cut-off function ω; however, a similar relation for $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ for finite s is not true.

Let $\mathbb{W}$ be a (stretched) manifold with edge Y and assume that $\mathbb{W}$ is compact. According to the definition of $\mathbb{W}$, there is a finite open covering of $\mathbb{W}$ by coordinate neighbourhoods U which do not intersect $\partial\mathbb{W}$ and (relatively) open sets V diffeomorphic to $[0, 1) \times X \times \Omega \ni (r, x, y)$ for some open $\Omega \subset \mathbb{R}^q$, $q = \dim Y$, where X is the base of the local model cone for $\mathbb{W}$ near $\partial\mathbb{W}$. We then define the scale of weighted Sobolev spaces

$$\mathcal{W}^{s,\gamma}(\mathbb{W}) \tag{7}$$

as the set of all $u \in H^s_{\mathrm{loc}}(\mathbb{W}_{\mathrm{reg}})$ such that locally near $\partial\mathbb{W}$ in the splitting of coordinates (r, x, y) the function ωu belongs to $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ for any cut-off function ω which is equal to 1 near $\partial\mathbb{W}$ and zero outside a small neighbourhood of $\partial\mathbb{W}$.

A slight modification of the global definition allows us to define spaces of the kind $\mathcal{W}^{s,\gamma}(\mathbb{R}^p \times \mathbb{W})$ for any $p \in \mathbb{N}$ and a compact (stretched) manifold $\mathbb{W}$ with edge.

We need weighted edge Sobolev spaces also on other non-compact manifolds with edge. In order to avoid lenghtly generalities, we content ourselves with the case $\mathbb{W}^\wedge = \mathbb{R}_+ \times \mathbb{W} \ni (t, \cdot)$ for a compact (stretched) manifold $\mathbb{W}$ with edge. If $(\varphi_\iota)_{\iota\in\mathbb{N}}$ is a (countable) locally finite partition of unity on $\mathbb{R}$, then we form the space

$$\mathcal{W}^{s,\gamma}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{W}) \tag{8}$$

to be the set of all sums $u = \sum_{\iota\in\mathbb{N}} \varphi_\iota u_\iota$ for arbitrary $u_\iota \in \mathcal{W}^{s,\gamma}(\mathbb{R} \times \mathbb{W})$ supported in $\mathbb{R}_+$ with respect to r. Moreover, let

$$\mathcal{W}^{s,\gamma}_{\mathrm{comp}}(\mathbb{R}_+ \times \mathbb{W}) \tag{9}$$

denote the subspace of elements of (8) of compact support.

We now pass to the definition of spaces on a manifold $\mathbb{W}_\asymp$ as in Example 1.4, for the case that $\partial\mathbb{W}$ is a trivial X bundle over the edge Y; the simple generalisation to arbitrary $\mathbb{W}$ is left to the reader. Since $\mathbb{W}$ is a C^∞ manifold with boundary, there is a collar neighbourhood $\mathbb{V}$ of $\partial\mathbb{W}$, $\mathbb{V} \cong \overline{\mathbb{R}}_+ \times \partial\mathbb{W}$. Let us choose a partition of unity on $\mathbb{W}$ of the form $(\omega, (1-\omega))$ for a function $\omega \in C^\infty(\mathbb{W})$ which is equal to 1 near $\partial\mathbb{W}$ and supported in $\partial\mathbb{W} \times \left[0, \frac{2}{3}\right)$. By notation the manifold $\mathbb{W}_\asymp$ is modelled on a cylinder such that there is a diffeomorphism $\vartheta : \mathbb{R} \times \mathbb{W} \to \mathbb{W}_\asymp$ which is compatible with the dilation, i.e. $\vartheta(\lambda t, m) = \lambda\vartheta(t, m)$ for all $|t| \geq C$, $\lambda \geq 1$, where the multiplication on the right means the dilation on the cones $(1, \infty) \times \mathbb{W}$ and $(-\infty, -1) \times \mathbb{W}$ in a natural way. Let $\{G_1, \ldots, G_N\}$ be a covering of Y by coordinate neighbourhoods (as Y is compact) and let $\{\varphi_1, \ldots, \varphi_N\}$ be a subordinate partition of unity.
Let $\mathbb{V}_{j,\mathrm{reg}} \subset \mathbb{V}$ denote the preimages of $\mathbb{R}_+ \times X \times G_j$ under $\mathbb{V} \cong \overline{\mathbb{R}}_+ \times \partial\mathbb{W}$ ($\partial\mathbb{W} = X \times Y$) and choose corresponding 'charts' $\nu_j : \mathbb{V}_{j,\mathrm{reg}} \to \mathbb{R}_+ \times X \times \mathbb{R}^q$ defined as the composition of $\mathbb{V}_{j,\mathrm{reg}} \to \mathbb{R}_+ \times X \times G_j$ with $\mathbb{R}_+ \times X \times G_j \to \mathbb{R}_+ \times X \times \mathbb{R}^q$, $(r, x, \tilde{y}) \mapsto (r, x, \alpha_j(\tilde{y}))$, for a corresponding chart $\alpha_j : G_j \to \mathbb{R}^q$ on Y. Moreover, let us form $\chi_j : \mathbb{R} \times \mathbb{V}_{j,\mathrm{reg}} \to \mathbb{R} \times \mathbb{R}_+ \times X \times \mathbb{R}^q$ by $\chi_j(t, w) := (t, \nu_j(w))$. Finally, consider the map $\beta : \mathbb{R} \times \mathbb{R}_+ \times X \times \mathbb{R}^q \to \mathbb{R} \times \mathbb{R}_+ \times X \times \mathbb{R}^q$ defined by

$$\beta(t, r, x, y) := (t, [t]r, x, [t]y). \tag{10}$$

We now turn to a crucial definition of weighted edge Sobolev spaces on the (stretched) manifold $\mathbb{W}_\asymp$ with edge and exit to infinity. Similarly as in Remark 1.7, the manifold $\mathbb{W}_\asymp$ is diffeomorphic to $\mathbb{R} \times \mathbb{W}$. Let $2\mathbb{W}$ denote the double of $\mathbb{W}$ consisting of two copies $\mathbb{W}_\pm$ of $\mathbb{W}$, glued together along the common boundary $\partial\mathbb{W} = \mathbb{W}_{\mathrm{sing}}$ (usually we identify $\mathbb{W}$ with $\mathbb{W}_+$). Since $2\mathbb{W}$ is a closed compact C^∞ manifold, we have the spaces $H^{s,\delta}((2\mathbb{W})_\asymp)$, cf. Remark 1.7. Distributions u on $(\mathbb{W}_{\mathrm{reg}})_\asymp$ that vanish near $(\mathbb{W}_{\mathrm{sing}})_\asymp$ will tacitly be regarded as distributions on $(2\mathbb{W})_\asymp$ as the zero extension of u to $(\mathbb{W}_-)_\asymp$. In particular, if ω is a cut-off function on $\mathbb{W}$ as before (i.e., $\omega \equiv 1$ near $\mathbb{W}_{\mathrm{sing}}$), then $1-\omega$, defined on $\mathbb{W}$ and vanishing near $\mathbb{W}_{\mathrm{sing}}$, is also extended by zero to $2\mathbb{W}$. The functions $\omega, (1-\omega)$, etc., will also be regarded as functions on $\mathbb{W}_\asymp$ and $(2\mathbb{W})_\asymp$, respectively.

Definition 1.8 *The space* $\mathcal{W}^{s,\gamma;\delta}(\mathbb{W}_{\asymp})$ *for* $s, \gamma, \delta \in \mathbb{R}$ *is defined as the restriction to* $\mathbb{W}_{\asymp}$ *of the completion of* $C_0^\infty(\mathbb{R} \times \mathbb{W}_{\mathrm{reg}})$ *with respect to the norm*

$$\Big\{ \|(1-\omega)u\|^2_{H^{s;\delta}((2\mathbb{W})_{\asymp})} + \sum_{j=1}^N \|\omega\varphi_j u \circ \chi_j^{-1} \circ \beta^{-1}\|^2_{\langle t,\tilde{y}\rangle^{-\delta}\mathcal{W}^s(\mathbb{R}_t \times \mathbb{R}^q_{\tilde{y}}, \mathcal{K}^{s,\gamma}(\mathbb{R}_{+,\tilde{r}} \times X))} \Big\}^{\frac{1}{2}}.$$

1.3 Edge amplitude functions

We now establish some tools on pseudo-differential operators on a compact (stretched) manifold $\mathbb{W}$ with edge Y, here in parameter-dependent form. Concerning the basics we refer to the monograph [17].

The main information is coming from edge-degenerate symbols of the form

$$r^{-\mu}\tilde{b}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta, \tilde{\lambda}=r\lambda} \tag{11}$$

with symbols $\tilde{b}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda}) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q+l}_{\tilde{\rho},\xi,\tilde{\eta},\tilde{\lambda}})$, the Hörmander's space of classical symbols in the covariables $(\tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda})$, where $\Sigma \subset \mathbb{R}^n$, $\Omega \subset \mathbb{R}^q$ are open sets, $n = \mathrm{dim}X$, $q = \mathrm{dim}Y$. The covariable $\lambda \in \mathbb{R}^l$ plays the role of a parameter. Note that the differential operators of the form (2) have local symbols of the form (11) (where $l = 0$); here $x \in \Sigma$ corresponds to local coordinates on X. With symbols (11) we can associate families of pseudo-differential operators globally on X by forming

$$r^{-\mu}\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}, \tilde{\lambda}) = r^{-\mu}\sum_{k=1}^M \delta_k(\kappa_k^{-1})_* \,\mathrm{op}_x(\tilde{b}_k)(r, y, \tilde{\varrho}, \tilde{\eta}, \tilde{\lambda})\vartheta_k \tag{12}$$

where $\{\tilde{b}_k\}_{k=1,\dots,M}$ is a system of symbols of the kind (11) (up to the weight factor $r^{-\mu}$), corresponding to a system of charts $\kappa_k : X_k \to \Sigma$ on X, with $\{X_k\}_{k=1,\dots,M}$ being an open covering, $\{\delta_1, \dots, \delta_M\}$ a subordinate partition of unity, and $\{\vartheta_1, \dots, \vartheta_M\}$ a set of functions $\vartheta_k \in C_0^\infty(X_k)$ such that $\vartheta_k \equiv 1$ on $\mathrm{supp}\,\delta_k$, $k = 1, \dots, M$. Moreover, $\mathrm{op}_x(\tilde{b})$ denotes the pseudo-differential action with respect to the Fourier transform in $\mathbb{R}^n \ni x$.

If X is a C^∞ manifold (not necessarily compact), $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^m)$ will denote the space of all classical pseudo-differential operators $A(\nu)$ on X with parameters $\nu \in \mathbb{R}^m$ (the local amplitude functions $b(x, \xi, \nu)$ with the covariables (ξ, ν)), while $L^{-\infty}(X; \mathbb{R}^m) = \mathcal{S}(\mathbb{R}^m, L^{-\infty}(X))$. Note that $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^m)$ is a Fréchet space in a natural way.

For the operator function $\tilde{p}$ in (12) we then have

$$\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}, \tilde{\lambda}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q+l}_{\tilde{\rho},\tilde{\eta},\tilde{\lambda}})). \tag{13}$$

We call the operator family $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}, \tilde{\lambda})$ parameter-dependent elliptic if the homogeneous principal part of the local amplitude functions b_k do not vanish for $(\tilde{\rho}, \tilde{\eta}, \tilde{\lambda}) \neq 0$ up to $r = 0$ (we tacitly assume the invariance under the symbol push forwards belonging to coordinate diffeomorphisms for the manifold X). An example is the system of local amplitude functions

$$r^{-\mu}\langle r\rho, \xi, r\eta, r\lambda\rangle^{\mu}. \tag{14}$$

An essential aspect of the edge calculus is that pseudo-differential operators with respect to r and y are defined by means of certain adequate quantisations, mainly the Mellin quantisation with respect to r. We define the Mellin pseudo-differential operator of weight $\gamma \in \mathbb{R}$ by

$$\mathrm{op}_M^{\gamma}(f)u(r) = \int\int_0^{\infty} \Big(\frac{r}{r'}\Big)^{-(\frac{1}{2}-\gamma+i\rho)} f\Big(r, r', \frac{1}{2}-\gamma+i\rho\Big)u(r')\frac{\mathrm{d}r'}{r'}đ\rho,$$

where $f(r, r', v)$ is a 'double' Mellin symbol depending on $(r, r') \in \mathbb{R}_+ \times \mathbb{R}_+$ and the covariable $v \in \Gamma_{\frac{1}{2}-\gamma}$. More precisely, the imaginary part of v is regarded as the covariable; in this sense we also use a notation like $S_{\mathrm{cl}}^{\mu}(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma})$ for the corresponding space of scalar symbols; in general, our symbols will be operator-valued. More generally, we will employ Mellin symbols of the kind

$$f(r, r', v, \nu) \in C^{\infty}(\mathbb{R}_+ \times \mathbb{R}_+, L_{\mathrm{cl}}^{\mu}(X; \Gamma_{\frac{1}{2}-\gamma} \times \mathbb{R}^m))$$

with covariables (v, ν). As before $v \in \mathbb{C}$ varies on $\Gamma_{\frac{1}{2}-\gamma}$, i.e., the covariable itself is $(\mathrm{Re}\, v, \nu) \in \mathbb{R}^{1+m}$. In the following, if E is a Fréchet space, $U \subset \mathbb{C}$ an open set, then $\mathcal{A}(U, E)$ denotes the space of all holomorphic functions in U with values in E (note that $\mathcal{A}(U, E) = \mathcal{A}(U)\hat{\otimes}_{\pi} E$ where $\hat{\otimes}_{\pi}$ means the (completed) projective tensor product).
Let $L_{\mathrm{cl}}^{\mu}(X; \mathbb{C}\times\mathbb{R}^m)$ denote the subspace of all $h(v, \nu) \in \mathcal{A}(\mathbb{C}_v, L_{\mathrm{cl}}^{\mu}(X; \mathbb{R}^m))$ such that $h(\beta + i\rho, \nu) \in L_{\mathrm{cl}}^{\mu}(X; \mathbb{R}_{\rho,\nu}^{1+m})$ for every $\beta \in \mathbb{R}$, uniformly in intervals $c \le \beta \le c'$ for arbitrary $c \le c'$. We will employ the following Mellin quantisation result, combined with a so called kernel cut-off procedure.

Theorem 1.9 *Let $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}, \tilde{\lambda})$ be as in the formula* (13)*, with y varying in an open set $\Omega \subset \mathbb{R}^q$, and let*

$$p(r, y, \rho, \eta, \lambda) := \tilde{p}(r, y, r\rho, r\eta, r\lambda).$$

Then there exists $\tilde{h}(r, y, v, \tilde{\eta}, \tilde{\lambda}) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \Omega, L_{\mathrm{cl}}^{\mu}(X; \mathbb{C} \times \mathbb{R}_{\tilde{\eta},\tilde{\lambda}}^{q+l}))$ such that

$$h(r, y, v, \eta, \lambda) := \tilde{h}(r, y, v, r\eta, r\lambda)$$

satisfies

$$\mathrm{op}_r(p)(y, \eta, \lambda) = \mathrm{op}_M^{\beta}(h)(y, \eta, \lambda) \bmod C^{\infty}(\Omega, L^{-\infty}(X^{\wedge}; \mathbb{R}_{\eta,\lambda}^{q+l})) \tag{15}$$

for every $\beta \in \mathbb{R}$, and $\tilde{h}$ is unique mod $C^{\infty}(\overline{\mathbb{R}}_+ \times \Omega, L^{-\infty}(X; \mathbb{C} \times \mathbb{R}_{\tilde{\eta},\tilde{\lambda}}^{q+l}))$ *(both sides of* (15) *are interpreted as pseudo-differential families on X, regarded as maps $C_0^{\infty}(X^{\wedge}) \to C^{\infty}(X^{\wedge})$).*

A proof of this theorem may be found in [8], see also [7] for the case with parameters. From the construction we have the following observation:

Remark 1.10 *For* $p_0(r,y,\rho,\eta,\lambda) := \tilde{p}(0,y,r\rho,r\eta,r\lambda)$, $h_0(r,y,v,\eta,\lambda) := \tilde{h}(0,y,v,r\eta,r\lambda)$, *we have* $\mathrm{op}_r(p_0)(y,\eta,\lambda) = \mathrm{op}_M^{\beta}(h_0)(y,\eta,\lambda) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^{q+l}_{\eta,\lambda}))$ *for every* $\beta \in \mathbb{R}$. *For* $\kappa_\delta u(r,x) := \delta^{\frac{n+1}{2}} u(\delta r, x)$, $\delta \in \mathbb{R}_+$, *we obtain the homogeneity*

$$\begin{aligned} \mathrm{op}_r(p_0)(y,\delta\eta,\delta\lambda) &= \kappa_\delta \mathrm{op}_r(p_0)(y,\eta,\lambda)\kappa_\delta^{-1}, \\ \mathrm{op}_M^{\beta}(h_0)(y,\delta\eta,\delta\lambda) &= \kappa_\delta \mathrm{op}_M^{\beta}(h_0)(y,\eta,\lambda)\kappa_\delta^{-1} \end{aligned}$$

for all $\delta \in \mathbb{R}_+$.

Let us now choose cut-off functions $\omega_1, \omega_2, \omega_3$ on $\overline{\mathbb{R}}_+$ such that

$$\omega_2 \equiv 1 \text{ on } \operatorname{supp} \omega_1, \quad \omega_1 \equiv 1 \text{ on } \operatorname{supp} \omega_3, \tag{16}$$

and cut-off functions σ and $\tilde{\sigma}$. We then form the family of operators

$$\begin{aligned} a(y,\eta,\lambda) := \sigma(r)\{ & r^{-\mu}\omega_1(r[\eta,\lambda])\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta,\lambda)\omega_2(r'[\eta,\lambda]) \\ & + r^{-\mu}(1-\omega_1(r[\eta,\lambda]))\mathrm{op}_r(p)(y,\eta,\lambda)(1-\omega_3(r'[\eta,\lambda]))\}\tilde{\sigma}(r'); \end{aligned} \tag{17}$$

$a(y,\eta,\lambda)$ is an operator-valued symbol in the sense of the following definition.

Definition 1.11 *Let* E *and* $\widetilde{E}$ *be Hilbert spaces with group actions* $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ *and* $\{\tilde{\kappa}_\delta\}_{\delta\in\mathbb{R}_+}$, *respectively. Moreover, let* $\mu \in \mathbb{R}$ *and* $U \subset \mathbb{R}^p$ *an open set. Then* $S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ *is defined as the subset of all* $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E,\widetilde{E}))$ *such that*

$$\sup_{(y,\eta)\in K\times\mathbb{R}^q} \langle\xi\rangle^{-\mu+|\beta|} \|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D_y^\alpha D_\eta^\beta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,\widetilde{E})}$$

is finite for all $(y,\eta) \in K \times \mathbb{R}^q$ *for every* $K \Subset U$ *and all multi-indices* $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$.

The dimensions p and q in the latter definition are independent; so we can replace the covariable by $(\eta,\lambda) \in \mathbb{R}^{q+l}$.

As is known from the local pseudo-differential calculus on manifolds with edge, we have

$$a(y,\eta,\lambda) \in S^\mu(\Omega \times \mathbb{R}^{q+l}_{\eta,\lambda}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)) \tag{18}$$

for every $s \in \mathbb{R}$. Recall that the group action on the spaces $\mathcal{K}^{s,\gamma}(X)$ has been defined in connection with the edge Sobolev spaces (6).

In the following discussion, to make the operators more concrete, we can imagine local symbols (11) to be of the form (14).

Proposition 1.12 *Let* $\tilde{p}(\tilde{\eta},\tilde{\lambda}) \in L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{q+l})$, *and set*

$$a(\eta,\lambda) := r^{-\mu}(1-\omega_1(r[\eta,\lambda]))\,\mathrm{op}_r(p)(\eta,\lambda)(1-\omega_3(r'[\eta,\lambda]))$$

for $p(r,\eta,\lambda) := \tilde{p}(r, r\eta, r\lambda)$*. Then for every fixed* $\lambda \in \mathbb{R}^l \setminus \{0\}$ *we have*

$$a(\eta,\lambda) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)) \tag{19}$$

for every $s,\gamma \in \mathbb{R}$*. Moreover, if* $\tilde{p}_j$*,* $j \in \mathbb{N}$*, is a sequence tending to zero in* $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{q+l})$*, the associated symbols* $a_j(\eta,\lambda), j \in \mathbb{N}$*, tend to zero in* $S^\mu_{\mathrm{cl}}(\mathbb{R}^q_\eta; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ *for every fixed* $\lambda \in \mathbb{R}^l \setminus \{0\}$ *and* $s,\gamma \in \mathbb{R}$*.*

Proof. In [6] it is shown that when $\chi(\eta,\lambda)$ is an excision function in $(\eta,\lambda) \in \mathbb{R}^{q+l}$ (i.e., C^∞ and $\chi(\eta,\lambda) = 0$ for $|\eta,\lambda| < \tilde{c}$, $\chi(\eta,\lambda) = 1$ for $|\eta,\lambda| > c$ for some $0 < \tilde{c} < c$), the function $\chi(\eta,\lambda)a(\eta,\lambda)$ belongs to $S^\mu(\mathbb{R}^{q+l}_{\eta,\lambda}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$. Since $\tilde{p}(\tilde{\eta},\tilde{\lambda})$ is independent of r, it is even classical in (η,λ). The choice of χ is not essential. For every fixed λ it follows that $\chi(\eta,\lambda)a(\eta,\lambda)$ is a classical symbol of η alone. Moreover, for $\lambda \neq 0$ we have $|\lambda| \geq c$ for some $c > 0$ and hence $|\eta,\lambda| \geq c$ for all $\eta \in \mathbb{R}^q$. Thus $\chi(\eta,\lambda)a(\eta,\lambda)$ is equal to $a(\eta,\lambda)$ for our fixed $\lambda \neq 0$. This gives us the relation (19). The second assertion of Proposition 1.12 is also a consequence of [6]. □

For our calculus we need other essential ingredients, namely the so called smoothing Mellin plus Green operators which we first explain on the level of corresponding (operator-valued) amplitude functions. Let us first specify Definition 1.11 to classical operator-valued symbols.

An element $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ is said to be classical, if there are homogeneous components

$$a_{(\mu-j)}(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E,\widetilde{E}))$$

(i.e., $a_{(\mu-j)}(y,\delta\eta) = \delta^{\mu-j}\tilde{\kappa}_\delta a_{(\mu-j)}(y,\eta)\kappa_\delta^{-1}$ for all $\delta \in \mathbb{R}_+$), $j \in \mathbb{N}$, such that for any excision function $\chi(\eta)$ we have $a(y,\eta) - \chi(\eta)\sum_{j=0}^N a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q; E, \widetilde{E})$ for every $N \in \mathbb{N}$. Let $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \widetilde{E})$ denote the space of classical symbols. If we talk about classical or non-classical symbols we also write '(cl)' as subscript. All notions in connection with operator-valued symbols have a straightforward extension to the case when E or $\widetilde{E}$ are Fréchet spaces with group action. That means, e.g., for $\widetilde{E}$ that this space is written as a projective limit of Hilbert spaces $\varprojlim_{k\in\mathbb{N}} \widetilde{E}^k$ with continuous embeddings $\widetilde{E}^{k+1} \hookrightarrow \widetilde{E}^k \hookrightarrow \ldots \hookrightarrow \widetilde{E}^0$, where $\widetilde{E}^0$ is endowed with a group action which restricts to group actions on $\widetilde{E}^k$ for every k. We then say that the Fréchet space $\widetilde{E}$ is endowed with a group action. Now, if E is a Hilbert space, $\widetilde{E}$ a Fréchet space, both equipped with group actions, we have the spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E}^k)$ with continuous embeddings $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E}^{k+1}) \hookrightarrow S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E}^k)$ for all k, and we set $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E}) = \bigcap_{k\in\mathbb{N}} S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E}^k)$.

Concerning the definition when both E and $\widetilde{E}$ are Fréchet spaces with group action, cf. [17].

Let $M^{-\infty}(X;\Gamma_\beta \times \mathbb{R}^q)$ for any $\beta \in \mathbb{R}$ denote the space of all $f(v,\eta) \in \mathcal{S}(\Gamma_\beta \times \mathbb{R}^q, L^{-\infty}(X))$ such that there is an $\varepsilon > 0$ (depending on f) and an $h(v,\eta) \in \mathcal{A}(\{\beta - \varepsilon < \operatorname{Re} v < \beta + \varepsilon\}, \mathcal{S}(\mathbb{R}^q, L^{-\infty}(X))$ such that $h|_{\Gamma_\beta \times \mathbb{R}^q} = f$ and $h(\gamma + i\rho, \eta) \in \mathcal{S}(\Gamma_\beta \times \mathbb{R}^q, L^{-\infty}(X))$ for every $\gamma \in (\beta - \varepsilon, \beta + \varepsilon)$, uniformly in compact subintervals. The subspace $M^{-\infty}(X;\Gamma_\beta \times \mathbb{R}^q)_\varepsilon$ of all $f \in M^{-\infty}(X;\Gamma_\beta \times \mathbb{R}^q)$ with fixed $\varepsilon > 0$ is a Fréchet space, and $M^{-\infty}(X;\Gamma_\beta \times \mathbb{R}^q)$ itself is the union over all ε. This allows us to speak about C^∞ functions with values in $M^{-\infty}(X;\Gamma_\beta \times \mathbb{R}^q)$. The dimension q is arbitrary; so we can apply this for $(\eta,\lambda) \in \mathbb{R}^{q+l}$ in place of $\eta \in \mathbb{R}^q$.
Let

$$f(y,v) \in C^\infty\big(\Omega, M^{-\infty}\big(X;\Gamma_{\frac{n+1}{2}-\gamma}\big)\big), \tag{20}$$

$\Omega \subseteq \mathbb{R}^q$ open, and form the operator function

$$m(y,\eta,\lambda) := r^{-\mu+j}\tilde{\omega}_1(r[\eta,\lambda])\operatorname{op}_M^{\gamma-\frac{n}{2}}(f)(y)(\eta,\lambda)^\alpha \tilde{\omega}_2(r'[\eta,\lambda]), \tag{21}$$

$\alpha \in \mathbb{N}^{q+l}$, where $\tilde{\omega}_1(r)$ and $\tilde{\omega}_2(r)$ are arbitrary cut-off functions. Then we have

$$m(y,\eta,\lambda) \in S_{\mathrm{cl}}^{\mu-j+|\alpha|}(\Omega \times \mathbb{R}^{q+l}; E, \widetilde{E}) \tag{22}$$

for $E = \mathcal{K}^{s,\gamma}(X^\wedge), \widetilde{E} = \mathcal{K}^{\infty,\gamma-\mu+j}(X^\wedge)$, $s \in \mathbb{R}$.

Symbols of the type (21) for $j = 0$ and $\alpha = 0$ will be called smoothing Mellin symbols of the edge calculus, while for $|\alpha| \le j$ and $j > 0$ we obtain examples of so called Green symbols. The definition of the latter kind of symbols is as follows.

Definition 1.13 *A family of continuous operators $g(y,\eta,\lambda) \in C^\infty(\Omega \times \mathbb{R}^{q+l}, \mathcal{L}(E,\widetilde{E}))$ for $\Omega \subset \mathbb{R}^q$ open, $E = \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-}$, $\widetilde{E} = \mathcal{K}^{\infty,\delta}(X^\wedge) \oplus \mathbb{C}^{j_+}$ with certain $j_\pm \in \mathbb{N}$ is called a Green symbol of order μ, if*

$$g_0(y,\eta,\lambda) := \operatorname{diag}(1, \langle\eta,\lambda\rangle^{-\frac{n+1}{2}}) g(y,\eta,\lambda) \operatorname{diag}(1, \langle\eta,\lambda\rangle^{\frac{n+1}{2}})$$

has the properties

$$g_0(y,\eta,\lambda) \in S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^{q+l}; \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-}, \mathcal{S}^\delta(X^\wedge)_\varepsilon \oplus \mathbb{C}^{j_+})$$

and

$$g_0^*(y,\eta,\lambda) \in S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^{q+l}; \mathcal{K}^{s,-\delta}(X^\wedge) \oplus \mathbb{C}^{j_+}, \mathcal{S}^{-\gamma}(X^\wedge)_\varepsilon \oplus \mathbb{C}^{j_-})$$

for some $\varepsilon > 0$ and all $s \in \mathbb{R}$. Here g_0^* means $(g_0 u, v)_{\mathcal{K}^{0,0} \oplus \mathbb{C}^{j_+}} = (u, g_0^* v)_{\mathcal{K}^{0,0} \oplus \mathbb{C}^{j_-}}$ pointwise (in the sense of formal adjoints) for all $u \in C_0^\infty(X^\wedge) \oplus \mathbb{C}^{j_-}$, $v \in C_0^\infty(X^\wedge) \oplus \mathbb{C}^{j_+}$.

Local edge amplitude functions of the calculus with edge Y of dimension q and including parameters $\lambda \in \mathbb{R}^k$ will have the form

$$\operatorname{diag}(a + m, 0)(y,\eta,\lambda) + g(y,\eta,\lambda), \tag{23}$$

where $a(y, \eta, \lambda)$ is of the form (17), moreover

$$m(y, \eta, \lambda) = r^{-\mu}\omega_1(r[\eta, \lambda]) \operatorname{op}_M^{\gamma-\frac{n}{2}}(f)(y)\omega_2(r'[\eta, \lambda]) \tag{24}$$

is given in terms of a smoothing Mellin symbol (20), and $g(y, \eta, \lambda)$ is a 2×2 block matrix Green symbol as in Definition 1.13. The cut-off functions ω_1, ω_2 in (24) are arbitrary; the specific choice does not change (24) up to a Green symbol of the type of an upper left corner. Therefore, instead of $\tilde{\omega}_1, \tilde{\omega_2}$ in (21), we simply take the same cut-off functions as in (17).

Theorem 1.14 *The operator-valued amplitude function $a(y, \eta, \lambda)$ given by* (17) *admits a representation of the form*

$$\begin{aligned} a(y, \eta, \lambda) = \sigma(r)\{&r^{-\mu}\omega_1(r)\operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(y, \eta, \lambda)\omega_2(r') \\ &+ r^{-\mu}(1-\omega_1(r))\operatorname{op}_r(p)(y, \eta, \lambda)(1-\omega_3(r'))\}\tilde{\sigma}(r') + g(y, \eta, \lambda), \end{aligned}$$

where $g(y, \eta, \lambda)$ is a Green symbol in the sense of Definition 1.13 *with $j_- = j_+ = 0$ and $\varepsilon = \infty$.*

A proof of this result may be found in [10].

1.4 The edge algebra

In this section we prepare some necessary material on (pseudo-)differential operators on a (stretched) compact manifolfd $\mathbb{W}$ with edge Y. The calculus consists of 2×2 block matrix operators of the form

$$\mathcal{A} : \mathcal{W}^{s,\gamma}(\mathbb{W}) \oplus H^{s-\frac{n+1}{2}}(Y, J_-) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}) \oplus H^{s-\frac{n+1}{2}-\mu}(Y; J_+) \tag{25}$$

for an order $\mu \in \mathbb{R}$, a weight $\gamma \in \mathbb{R}$ and $J_-, J_+ \in \operatorname{Vect}(Y)$ (here $\operatorname{Vect}(\cdot)$ denotes the set of all smooth complex vector bundles over the space in the brackets). The global Sobolev spaces $\mathcal{W}^{s,\gamma}(\mathbb{W})$ are defined in Section 1.2, cf. the formula (7), and $H^s(Y, J)$ for $J \in \operatorname{Vect}(Y)$ denotes the standard Sobolev space of smoothness s of distributional sections in J.

The operators (25) are connected with weight and bundle data, denoted by

$$\boldsymbol{g} = (\gamma, \gamma - \mu), \quad \boldsymbol{v} = (J_-, J_+). \tag{26}$$

Those are assumed to be known and fixed in any concrete case. We also could consider operators that refer to distributional sections in bundles $E, F \in \operatorname{Vect}(\mathbb{W})$; the corresponding generalisation is straightforward and will not be discussed here.

We are interested in operators (25) depending on an additional parameter $\lambda \in \mathbb{R}^l$ that is formally involved in the local definition as an extra edge covariable. In addition we have to define the class of smoothing parameter-dependent operators.

In the spaces $\mathcal{W}^{0,0}(\mathbb{W}) \oplus H^0(Y,J)$, $J \in \mathrm{Vect}(Y)$, we fix natural scalar products (from corresponding L^2 spaces, with measures belonging to fixed Riemannian metrics on the manifolds and Hermitian metrics in the bundles). We then have non-degenerate sesquilinear pairings

$$(\cdot,\cdot) : \{\mathcal{W}^{s,\gamma}(\mathbb{W}) \oplus H^s(Y,J)\} \times \{\mathcal{W}^{-s,-\gamma}(\mathbb{W}) \oplus H^{-s}(Y,J)\} \longrightarrow \mathbb{C}. \tag{27}$$

Now $\mathcal{Y}^{-\infty}(\mathbb{W})$ (for the fixed data (26)) is defined to be the set of all operators $\mathcal{G}$ which are continuous in the sense $\mathcal{G} : \mathcal{W}^{s,\gamma}(\mathbb{W}) \oplus H^{s'}(Y,J_-) \longrightarrow \mathcal{W}^{\infty,\gamma-\mu+\varepsilon}(\mathbb{W}) \oplus H^{\infty}(Y,J_+)$ for some $\varepsilon = \varepsilon(\mathcal{G}) > 0$, for all $s,s' \in \mathbb{R}$, such that the formal adjoint $\mathcal{G}^*$ with respect to the pairing (27) induces continuous operators $\mathcal{G}^* : \mathcal{W}^{s,-\gamma+\mu}(\mathbb{W}) \oplus H^{s'}(Y,J_+) \longrightarrow \mathcal{W}^{\infty,-\gamma+\varepsilon}(\mathbb{W}) \oplus H^{\infty}(Y,J_-)$ for all $s,s' \in \mathbb{R}$. If we want to indicate ε we write for the moment $\mathcal{Y}^{-\infty}(\mathbb{W})_\varepsilon$ which is a Fréchet space in a natural way. We then set $\mathcal{Y}^{-\infty}(\mathbb{W};\mathbb{R}^l) := \bigcup_{\varepsilon>0} \mathcal{S}(\mathbb{R}^l, \mathcal{Y}^{-\infty}(\mathbb{W})_\varepsilon)$, which is the space of all smoothing parameter-dependent edge operators associated with (26).

Definition 1.15 *The space $\mathcal{Y}^{\mu}(\mathbb{W};\mathbb{R}^l)$ (belonging to weight and bundle data (26)) is defined to be the set of all operator families of the form*

$$\begin{aligned}\mathcal{A}(\lambda) = &\sum_{j=1}^{L} \mathrm{diag}(\sigma\varphi_j,\varphi_j)\mathcal{A}_j(\lambda)\,\mathrm{diag}(\tilde{\sigma}\psi_j,\psi_j)\\ &+ \mathrm{diag}(1-\sigma,0)\mathcal{A}_{\mathrm{int}}(\lambda)\,\mathrm{diag}(1-\tilde{\tilde{\sigma}},0) + \mathcal{C}(\lambda)\end{aligned}$$

with the following ingredients:

(i) *$\sigma,\tilde{\sigma},\tilde{\tilde{\sigma}}$ are functions in $C^\infty(\mathbb{W})$ supported in a collar neighbourhood of $\partial\mathbb{W}$ and equal to 1 near $\partial\mathbb{W}$, $\tilde{\sigma} \equiv 1$ on $\mathrm{supp}\,\sigma$, $\sigma \equiv 1$ on $\mathrm{supp}\,\tilde{\tilde{\sigma}}$; furthermore, φ_j,ψ_j are elements of $C_0^\infty(\mathbb{V}_j)$ where $(\mathbb{V}_j)_{j=1,\dots,L}$ is a system of neighbourhoods $\mathbb{V}_j \cong [0,1) \times X \times \Omega_j$ on $\mathbb{W}$, $\Omega_j \subseteq \mathbb{R}^q$ open, and the sets Ω_j correspond to charts on Y belonging to an open covering of Y by coordinate neighbourhoods; moreover, $\sum_{j=1}^{L}\varphi_j = 1$ in a neighbourhood of $\partial\mathbb{W}$ and $\psi_j \equiv 1$ on $\mathrm{supp}\,\varphi_j$ for all j;*

(ii) *$\mathcal{A}_j(\lambda) = (\chi_j^{-1})_*\,\mathrm{Op}(\mathfrak{a}_j)(\lambda)$, where $(\chi_j^{-1})_*$ is the operator push forward under $\chi_j^{-1} : [0,1) \times X \times \Omega_j \to \mathbb{V}_j$, with the pseudo-differential action $\mathrm{Op}(\cdot) := \mathrm{Op}_y(\cdot)$ in $y \in \Omega_j$, and $\mathfrak{a}_j(y,\eta,\lambda)$ an amplitude function of the form*

$$\mathfrak{a}_j(y,\eta,\lambda) := \mathrm{diag}(a_j(y,\eta,\lambda) + m_j(y,\eta,\lambda),0) + \mathfrak{g}_j(y,\eta,\lambda), \tag{28}$$

where $a_j(y,\eta,\lambda)$ is of the form (17), $m_j(y,\eta,\lambda)$ of the form (21) for $j = 0$, $\alpha = 0$, and $\mathfrak{g}_j(y,\eta,\lambda)$ is a Green symbol in the sense of Definition 1.13 for $\delta = \gamma - \mu$, with the fibre dimensions $j_\pm$ of $J_\pm$ (the operator push forwards also take into account the transition maps of the bundle);

(iii) *$\mathcal{A}_{\mathrm{int}}(\lambda) \in L^{\mu}_{\mathrm{cl}}(\mathrm{int}\,\mathbb{W};\mathbb{R}^l)$ and $\mathcal{C}(\lambda) \in \mathcal{Y}^{-\infty}(\mathbb{W};\mathbb{R}^l)$.*

As a consequence of the definition, the operators $\mathcal{A}(\lambda)$ are continuous in the sense of (25) for every $\lambda \in \mathbb{R}^l$, $s \in \mathbb{R}$.

The principal symbolic structure

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}),\ \sigma_\wedge(\mathcal{A}))$$

of $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2} \in \mathcal{Y}^\mu(\mathbb{W};\mathbb{R}^l)$ consists of the (parameter-dependent) homogeneous principal symbol $\sigma_\psi(\mathcal{A}_{11})$ of $\mathcal{A}_{11}$ of order μ as a function on $T^*(\mathbb{W}_{\mathrm{reg}}) \times \mathbb{R}^l \setminus 0$ (using that $\mathcal{A}_{11} \in L^\mu_{\mathrm{cl}}(\mathbb{W}_{\mathrm{reg}};\mathbb{R}^l)$), $\sigma_\psi(\mathcal{A}) := \sigma_\psi(\mathcal{A}_{11})$ and the parameter-dependent homogeneous principal edge symbol is a family of continuous operators

$$\sigma_\wedge(\mathcal{A})(y,\eta,\lambda) : \mathcal{K}^{s,\gamma}(X^\wedge) \oplus J_{-,y} \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \oplus J_{+,y} \tag{29}$$

for all $s \in \mathbb{R}$, parametrised by $(y,\eta,\lambda) \in T^*Y \times \mathbb{R}^l \setminus 0$.

The homogeneity of $\sigma_\psi(\mathcal{A})$ is as usual, while the homogeneity of $\sigma_\wedge(\mathcal{A})$ means

$$\sigma_\wedge(\mathcal{A})(y,\delta\eta,\delta\lambda) = \delta^\mu \operatorname{diag}(\kappa_\delta,0)\sigma_\wedge(\mathcal{A})(y,\eta,\lambda) \operatorname{diag}(\kappa_\delta^{-1},0)$$

for all $\delta \in \mathbb{R}_+$ and $(\eta,\lambda) \neq 0$.

Theorem 1.16 *Let $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W};\mathbb{R}^l)$, $\mathcal{B} \in \mathcal{Y}^\nu(\mathbb{W};\mathbb{R}^l)$ (with weight and bundle data such that the composition makes sense) implies $\mathcal{AB} \in \mathcal{Y}^{\mu+\nu}(\mathbb{W};\mathbb{R}^l)$, and we have $\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ (in the sense of componentwise composition).*

The proof of this result is similar to that of the corresponding composition property in the case without parameters, see, e.g., [10] which also explains the role of Theorem 1.14.

Let us set $\mathcal{Y}^{\mu;0}(\mathbb{W};\mathbb{R}^l) := \mathcal{Y}^\mu(\mathbb{W};\mathbb{R}^l)$ and $\mathcal{Y}^{\mu;-1}(\mathbb{W};\mathbb{R}^l) := \{\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W};\mathbb{R}^l) : \sigma(\mathcal{A}) = 0\}$. The elements $\mathcal{A}_{-1} \in \mathcal{Y}^{\mu;-1}(\mathbb{W};\mathbb{R}^l)$ have again a pair of principal symbols $\sigma(\mathcal{A}_{-1}) = (\sigma_\psi(\mathcal{A}_{-1}), \sigma_\wedge(\mathcal{A}_{-1}))$, now of order $\mu - 1$. More generally, for every $k \in \mathbb{N}$ we can define $\mathcal{Y}^{\mu;-(k+1)}(\mathbb{W};\mathbb{R}^l)$ to be the set of all $\mathcal{A}_{-(k+1)} \in \mathcal{Y}^{\mu;-k}(\mathbb{W};\mathbb{R}^l)$ such that $\sigma_\psi(\mathcal{A}_{-(k+1)}) = 0$ and $\sigma_\wedge(\mathcal{A}_{-(k+1)}) = 0$.

Remark 1.17 (i) *$\mathcal{A} \in \mathcal{Y}^{\mu;-1}(\mathbb{W};\mathbb{R}^l)$ implies that the operator (25) is compact for every $\lambda \in \mathbb{R}^l$, $s \in \mathbb{R}$;*

(ii) *let $\mathcal{A} \in \mathcal{Y}^{\mu;-k}(\mathbb{W};\mathbb{R}^l)$, $\mathcal{B} \in \mathcal{Y}^{\nu;-m}(\mathbb{W};\mathbb{R}^l)$ satisfy the conditions of Theorem 1.16. Then we have $\mathcal{AB} \in \mathcal{Y}^{\mu+\nu;-(k+m)}(\mathbb{W};\mathbb{R}^l)$ and $\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$;*

(iii) *let $\mathcal{A}_j \in \mathcal{Y}^{\mu;-j}(\mathbb{W};\mathbb{R}^l)$, $j \in \mathbb{N}$, be an arbitrary sequence, where the weight strip with $\varepsilon > 0$ that is involved in the smoothing Mellin symbols and in the Green symbols is independent of j (cf. the formulas (20), (24) and Definition 1.13). Then there is $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W};\mathbb{R}^l)$ such that $\mathcal{A} - \sum_{j=1}^N \mathcal{A}_j \in \mathcal{Y}^{\mu;-(N+1)}(\mathbb{W};\mathbb{R}^l)$ for every $N \in \mathbb{N}$, and $\mathcal{A}$ is unique $\operatorname{mod} \mathcal{Y}^{-\infty}(\mathbb{W};\mathbb{R}^l)$.*

2 Operators near exits to infinity

2.1 Sobolev spaces on a cone with singular cross section

In this section we study a new scale of Sobolev spaces on an infinite stretched cone $\mathbb{R}_+ \times \mathbb{W}$ for a compact (stretched) manifold $\mathbb{W}$ with edge Y, with multiple weights coming from the interior model cone half-axis $\mathbb{R}_+ \ni r$ of $\mathbb{W}$ and the axial variable $t \in \mathbb{R}_+$ on our infinite cone. In this paper a cut-off function on $\overline{\mathbb{R}}_+$ is any real-valued $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ which is equal to 1 for $0 \leq r < \varepsilon$ for some $\varepsilon > 0$.
Let us first formulate a kind of weighted L^2 spaces on the local wedge $\mathbb{R}_+ \times X \times \mathbb{R}^q$ for a closed compact C^∞ manifold X. Let us set $\mathcal{K}^{0,\gamma}(X^\wedge) := r^{-\frac{n}{2}}(\omega r^\gamma + (1-\omega))L^2(\mathbb{R}_+ \times X)$ for some cut-off function $\omega(r)$, $\gamma \in \mathbb{R}$, with $L^2(\mathbb{R}_+ \times X)$ being taken with the measure $\mathrm{d}r\mathrm{d}x$ and $\mathrm{d}x$ is associated with a Riemannian metric on X. On the space $\mathcal{K}^{0,\gamma}(X^\wedge)$ we fix the group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$, given by $\kappa_\lambda u(r,x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$, $\lambda \in \mathbb{R}_+$ and form, according to the general definition of Section 1.3, the edge Sobolev space $\mathcal{W}^0(\mathbb{R}^q, \mathcal{K}^{0,\gamma}(X^\wedge))$.

Now for $\mathbb{W}$ we choose a finite covering of neighbourhoods $\{U_1, \ldots, U_L, U_{L+1}, \ldots, U_N\}$ such that $U_j \cap \partial\mathbb{W} \neq \emptyset$ for $1 \leq j \leq L$, $U_j \cap \partial\mathbb{W} = \emptyset$ for $L+1 \leq j \leq N$. Moreover, let $\{\varphi_1, \ldots, \varphi_N\}$ be a subordinate partition of unity. Let

$$\chi_j : U_j \to \overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q,\ 1 \leq j \leq L, \quad \chi_j : U_j \to \mathbb{R}^{1+n+q},\ L+1 \leq j \leq N$$

be charts on $\mathbb{W}$ (the notation 'chart' for $1 \leq j \leq L$ is used here in a generalised sense). Then we define the space $\mathcal{W}^{0,\gamma}(\mathbb{W})$ as the space of all sums $u = \sum_{j=1}^N \varphi_j u_j$ such that $u_j \in \chi_j^* \mathcal{W}^0(\mathbb{R}^q, \mathcal{K}^{0,\gamma}(X^\wedge))$ for $j = 1, \ldots, L$, and $u_j \in \chi_j^* L^2(\mathbb{R}^{1+n+q})$ for $j = L+1, \ldots, N$. Let us endow the space $\mathcal{W}^{0,\gamma}(\mathbb{W})$ with a scalar product

$$(u,v)_{\mathcal{W}^{0,\gamma}(\mathbb{W})} := \sum_{j=1}^{L} (f_j, g_j)_{\mathcal{W}^0(\mathbb{R}^q, \mathcal{K}^{0,\gamma}(X^\wedge))} + \sum_{j=L+1}^{N} (f_j, g_j)_{L^2(\mathbb{R}^{1+n+q})} \tag{30}$$

for $f_j := (\chi_j^{-1})^* \varphi_j u_j$ and $g_j := (\chi_j^{-1})^* \varphi_j v_j$ belonging to $\mathcal{W}^0(\mathbb{R}^q, \mathcal{K}^{0,\gamma}(X^\wedge))$ for $1 \leq j \leq L$ and to $L^2(\mathbb{R}^{1+n+q})$ for $L+1 \leq j \leq N$. Then $\mathcal{W}^{0,\gamma}(\mathbb{W})$ is a Hilbert space. We will take (30) for $\gamma = 0$ as the reference scalar product for formal adjoints with respect to the non-degenerate sesquilinear pairing $(\cdot,\cdot) : \mathcal{W}^{s,\gamma}(\mathbb{W}) \times \mathcal{W}^{-s,-\gamma}(\mathbb{W}) \to \mathbb{C}$ for every $s, \gamma \in \mathbb{R}$.
For purposes below we set

$$\mathcal{W}^{s,\gamma}(\mathbb{W})_\varepsilon := \bigcap_{k \in \mathbb{N}} \mathcal{W}^{k, \gamma + \varepsilon - \frac{1}{k+1}}(\mathbb{W})$$

with the Fréchet topology of the projective limit. We now form parameter-depending pseudo-differential operators

$$\mathfrak{a}(\lambda) := \mathrm{Op}_y(a + m + g)(\lambda)$$

with amplitude function $a(y, \eta, \lambda) + m(y, \eta, \lambda) + g(y, \eta, \lambda)$, where $a(y, \eta, \lambda)$ is of the form (17), $m(y, \eta, \lambda)$ of the form (21) for $j = 0$ and $\alpha = 0$, and $g(y, \eta, \lambda)$ is a Green symbol in the sense of Definition 1.13 with $j_- = j_+ = 0$. Such operators act continuously between 'comp' and 'loc' versions of edge Sobolev spaces for every $\lambda \in \mathbb{R}^l$, namely

$$\mathfrak{a}(\lambda) : \mathcal{W}^{s,\gamma}_{\mathrm{comp}(y)}(X^\wedge \times \Omega) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathrm{loc}(y)}(X^\wedge \times \Omega) \tag{31}$$

for all $s \in \mathbb{R}$.

Given a compact stretched manifold $\mathbb{W}$ with edge, by $\mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^l)$, for $\mu \in \mathbb{R}$, $\mathbf{g} = (\gamma, \gamma - \mu)$, $\gamma \in \mathbb{R}$, we denote the space of all operator families $\mathfrak{a}(\lambda) + \mathfrak{c}(\lambda) : C_0^\infty(\mathbb{W}_{\mathrm{reg}}) \to C^\infty(\mathbb{W}_{\mathrm{reg}})$ which belong to $L^\mu_{\mathrm{cl}}(\mathbb{W}_{\mathrm{reg}}; \mathbb{R}^l_\lambda)$, where $A(\lambda)$ is locally near $\partial\mathbb{W}$ in a splitting of variables $(r, x, y) \in X^\wedge \times \Omega$ of the form (31), while $\mathfrak{c}(\lambda)$ is a Schwartz function in $\lambda \in \mathbb{R}^l$ with values in the space of smoothing operators. Here an operator $\mathfrak{c}$ is called smoothing if

$$\mathfrak{c} : \mathcal{W}^{s,\gamma}(\mathbb{W}) \longrightarrow \mathcal{W}^{\infty,\gamma-\mu}(\mathbb{W})_\varepsilon, \quad \mathfrak{c}^* : \mathcal{W}^{s,-\gamma+\mu}(\mathbb{W}) \longrightarrow \mathcal{W}^{\infty,-\gamma}(\mathbb{W})_\varepsilon$$

are continuous for a certain $\varepsilon > 0$ and all $s \in \mathbb{R}$, with C^* being the formal adjoint with respect to the scalar product of $\mathcal{W}^{0,0}(\mathbb{W})$.

Theorem 2.1 *For every $\mu, \gamma \in \mathbb{R}$ the space $\mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^l)$, associated with the weight data $\mathbf{g} = (\gamma, \gamma - \mu)$, contains an order reducing family $\mathfrak{r}^\mu_\gamma(\lambda)$ which induces isomorphisms $\mathfrak{r}^\mu_\gamma(\lambda) : \mathcal{W}^{s,\gamma}(\mathbb{W}) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W})$ for all $s \in \mathbb{R}$, $\lambda \in \mathbb{R}^l$.*

A proof of this result is given in [3]. The ideas of the proof may also be found in [21] in the more special case of boundary value problems without the transmission property.

Definition 2.2 *By $\mathcal{H}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge)$ for $s \in \mathbb{R}$, $\boldsymbol{\gamma} = (\gamma, \theta) \in \mathbb{R}^2$, we denote the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{W}_{\mathrm{reg}})$ with respect to the norm*

$$\Big\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{\dim \mathbb{W}}{2} - \theta}} \| \mathfrak{r}^s_\gamma(\mathrm{Im}\, w)(Mu)(w) \|^2_{\mathcal{W}^{0,\gamma-s}(\mathbb{W})} \, dw \Big\}^{\frac{1}{2}}.$$

Here $M = M_{t \to w}$ is the Mellin transform on $\mathbb{R}_+ \ni t$ with covariable $w \in \mathbb{C}$. Moreover, we define

$$\mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge) := \{ \omega u + (1 - \omega) v : u \in \mathcal{H}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge), v \in \mathcal{W}^{s,\gamma;0}(\mathbb{W}_\asymp)|_{\mathbb{R}_+ \times \mathbb{W}} \},$$

cf. Definition 1.8, where $\omega(t)$ is any cut-off function.

Remark 2.3 *The spaces in Definition 2.2 are independent of the specific choice of order reductions or cut-off functions.*

Observe that there is another equivalent definition of the space $\mathcal{H}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge)$.

Let $\{U_1, \dots, U_L, U_{L+1}, \dots, U_N\}$ be a covering of $\mathbb{W}$ by open sets such that $U_j \cap \mathbb{W}_{\text{sing}} \neq \emptyset$ for $1 \leq j \leq L$, $U_j \cap \mathbb{W}_{\text{sing}} = \emptyset$ for $L+1 \leq j \leq N$. Let us choose the sets U_j in such a way that there are stretched wedge 'coordinates' $(r, x, y) \in X^\wedge \times \mathbb{R}^q$ for $1 \leq j \leq L$, and let

$$\chi_j : U_j \to X^\wedge \times \mathbb{R}^q,\ 1 \leq j \leq L, \quad \chi_j : U_j \to \mathbb{R}^m,\ \ L+1 \leq j \leq N,$$

be corresponding 'charts', for $m = \dim \mathbb{W}_{\text{reg}}$. Then $\mathcal{H}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge)$ is the completion of $C_0^\infty(\mathbb{R} \times \mathbb{W}_{\text{reg}})$ with respect to the norm

$$\Big\{ \sum_{j=0}^{L} \|(\varphi_j u)(t, \chi_j^{-1}(r,x,y))\|^2_{\mathcal{V}^{s,\theta}(\mathbb{R}_+ \times \mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))} + \sum_{j=L+1}^{N} \|(\varphi_j u)(t, \chi_j^{-1}(\tilde{x}))\|^2_{\mathcal{H}^{s,\theta}(\mathbb{R}_+ \times \mathbb{R}^m)} \Big\}^{\frac{1}{2}}.$$

The space $\mathcal{V}^{s,\theta}(\mathbb{R}_+ \times \mathbb{R}^q, E)$ for a Hilbert space E with group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is defined to be the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^q, E)$ with respect to the norm

$$\Big\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{e+1}{2}-\theta}} \int_{\mathbb{R}^q} \langle w, \eta \rangle^{2s} \|\kappa^{-1}_{\langle w,\eta \rangle} (M_{t \to w} F_{y \to \eta} u)(w, \eta)\|^2_E \, \mathrm{d}w \, \mathrm{d}\eta \Big\}^{\frac{1}{2}}.$$

Here e is a natural number which is given as an extra information in connection with the specific space E; for $E = \mathcal{K}^{s,\gamma}(X^\wedge)$ we take $e = \dim(X^\wedge \times \mathbb{R}^q) = n + 1 + q$.

Lemma 2.4 *Let $\mathcal{M}_\varphi$, $\varphi \in C_0^\infty(\overline{\mathbb{R}}_+)$, denote the operator of multiplication by φ. Then we have $\mathcal{M}_\varphi \in \mathcal{L}(\mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge), \mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge))$, and the map $\varphi \to \mathcal{M}_\varphi$ is continuous on $C_0^\infty(\overline{\mathbb{R}}_+)$ with values in the corresponding space of operators, for all $s \in \mathbb{R}$, $\boldsymbol{\gamma} = (\gamma, \theta) \in \mathbb{R}^2$.*

2.2 Corner-degenerate differential operators

Let $\mathbb{W}$ be a (stretched) compact manifold with edge Y, and let $\Xi \subset \mathbb{R}^p$ be an open set. An element of $\text{Diff}^\mu(\mathbb{R}_+ \times \mathbb{W}_{\text{reg}} \times \Xi)$ is called corner-degenerate if it has the form $\boldsymbol{A} := \text{Op}_z(\boldsymbol{a})$ for an amplitude

$$\boldsymbol{a}(z, \zeta) = t^{-\mu} \sum_{k+|\beta| \leq \mu} b_{k\beta}(t, z)(-t\partial_t)^k (t\zeta)^\beta \tag{32}$$

with coefficients $b_{k\beta}(t,z) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \text{Diff}^{\mu-(k+|\beta|)}_{\text{deg}}(\mathbb{W}))$, cf. the notation in Section 1.1.

Observe that the operators of the kind (32) are locally near $\mathbb{R}_+ \times \mathbb{W}_{\text{sing}} \times \Xi$ in the splitting of variables (t, r, x, y, z) of the form

$$\boldsymbol{A} = t^{-\mu} r^{-\mu} \sum_{j+|\alpha|+k+|\beta| \leq \mu} c_{j\alpha,k\beta}(t, r, y, z)(-rt\partial_t)^k(-r\partial r)^j(rD_y)^\alpha(rtD_z)^\beta \tag{33}$$

with coefficients $c_{j\alpha,k\beta} \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \Omega \times \Xi, \text{Diff}^{\mu-(j+|\alpha|+k+|\beta|)}(X))$. Analogously as in the calculus of edge-degenerate operators, we have a 'higher' edge symbol of $\boldsymbol{A}$, namely,

$$\sigma_{\boldsymbol{\wedge}}(\boldsymbol{A})(z, \zeta) = t^{-\mu} \sum_{k+|\beta| \leq \mu} b_{k\beta}(0, z)(-t\partial_t)^k(t\zeta)^\beta. \tag{34}$$

This is a family of differential operators on $\mathbb{R}_+ \times \mathbb{W}$ with parameters $(z, \zeta) \in T^*\Xi \setminus 0$. Setting

$$(\kappa_\lambda u)(t, w) := \lambda^{\frac{2+n+q}{2}} u(\lambda t, w), \tag{35}$$

we have

$$\sigma_{\boldsymbol{\wedge}}(\boldsymbol{A})(z, \lambda\zeta) = \lambda^\mu \kappa_\lambda \sigma_{\boldsymbol{\wedge}}(\boldsymbol{A})(z, \zeta)\kappa_\lambda^{-1} \tag{36}$$

for all $\lambda \in \mathbb{R}_+$. The operators (34) for every fixed (z, ζ) can be interpreted first as a mapping on $C_0^\infty(\mathbb{R}_+ \times \mathbb{W}_{\text{reg}})$.

Theorem 2.5 *Assume that the coefficients* $b_{k\beta}$ *in* (32) *are independent of* t *for large* t*. Then the operators* $\boldsymbol{a}(z, \zeta) : C_0^\infty(\mathbb{R}_+ \times \mathbb{W}_{\text{reg}}) \longrightarrow C_0^\infty(\mathbb{R}_+ \times \mathbb{W}_{\text{reg}})$ *extend to a family of continuous operators*

$$\boldsymbol{a}(z, \zeta) : \mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\boldsymbol{\gamma}-\mu}(\mathbb{W}^\wedge)$$

for all $s \in \mathbb{R}$ *and* $\boldsymbol{\gamma} = (\gamma, \theta)$ *where* $\boldsymbol{\gamma} - \mu := (\gamma - \mu, \theta - \mu)$*,* $(z, \zeta) \in T^*\Xi \setminus 0$*.*

Proof. For convenience we consider the z-independent case $\boldsymbol{a}(\zeta)$. Let us choose a cut-off function $\sigma(t)$, and set $\boldsymbol{a}_0(\zeta) := \boldsymbol{a}(\zeta)\sigma, \boldsymbol{a}_\infty(\zeta) := \boldsymbol{a}(\zeta)(1 - \sigma)$. Then it suffices to show that

$$\boldsymbol{a}_0(\zeta) : \mathcal{H}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge) \longrightarrow \mathcal{H}^{s-\mu,\boldsymbol{\gamma}-\mu}(\mathbb{W}^\wedge) \tag{37}$$

and

$$\boldsymbol{a}_\infty(\zeta) : \mathcal{W}^{s,\gamma;0}(\mathbb{W}_\asymp) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu;0}(\mathbb{W}_\asymp) \tag{38}$$

are continuous. In the second relation, the coefficients are extended by zero for $t \leq 0$; because of the locality of the operators, the support in the image is contained in $\mathbb{R}_+ \times \mathbb{W}$. The continuity of (37) is a result of [18]. Thus it remains to consider $\boldsymbol{a}_\infty(\zeta)$. As in Definition 1.8, we choose a cut-off function $\omega(r)$ and functions φ_l, $1 \leq l \leq N$; then the proof reduces to the continuity of

$$\boldsymbol{a}_\infty(\zeta)(1 - \omega) : \mathcal{K}^{s,\theta}((2\mathbb{W})^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\theta-\mu}((2\mathbb{W})^\wedge) \tag{39}$$

and, for β as in the formula (10):

$$(\beta \circ \chi_l)_* \varphi_l \boldsymbol{a}_\infty(\zeta)\omega : \\ \mathcal{W}^s(\mathbb{R}_t \times \mathbb{R}^q_{\tilde{y}}, \mathcal{K}^{s,\gamma}(\mathbb{R}_{+,\tilde{r}} \times X)) \longrightarrow \mathcal{W}^{s-\mu}(\mathbb{R}_t \times \mathbb{R}^q_{\tilde{y}}, \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_{+,\tilde{r}} \times X)). \tag{40}$$

In the local variables $(t, r, x, y) \in \mathbb{R} \times \mathbb{R}_+ \times X \times \mathbb{R}^q$ the operators $(\chi_l)_* \varphi_l \boldsymbol{a}_\infty \omega$ (with $(\chi_l)_*$ being the push forward under χ_l) have the form

$$t^{-\mu} r^{-\mu} \sum_{j+|\alpha|+k+|\beta| \leq \mu} c_{j\alpha,k\beta;l}(t, r, y)(-rt\partial_t)^k (-r\partial_r)^j (rD_y)^\alpha (rt\zeta)^\beta \tag{41}$$

with coefficients $c_{j\alpha,k\beta;l}(t, r, y) = \varphi_l(y)\omega(r)c_{j\alpha,k\beta}(t, r, y)$, where $c_{j\alpha,k\beta}$ as in (33) (with $\mathbb{R}^q$ instead of Ω). The choice of the cut-off function σ is unessential. So we may assume that $[t] = t$ on $\mathrm{supp}(1 - \sigma)$. Thus, applying the push forward under (10) to the operators (41), it follows that

$$(\beta \circ \chi_l)_* \varphi_l \boldsymbol{a}_\infty(\zeta) = \tilde{r}^{-\mu} \sum_{j+|\alpha|+k+|\beta| \leq \mu} c_{j\alpha,k\beta;l}\Big(t, \frac{\tilde{r}}{t}, \frac{\tilde{y}}{t}\Big)(-\tilde{r}\partial_t)^k (-\tilde{r}\partial_{\tilde{r}})^j (\tilde{r}D_{\tilde{y}})^\alpha (\tilde{r}\zeta)^\beta,$$

for $\tilde{r} := tr$, $\tilde{y} := ty$. □

Corollary 2.6 *The operator function* (34) *extends to a family of continuous operators*

$$\sigma_{\boldsymbol{\wedge}}(\boldsymbol{A})(z, \zeta) : \mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge) \to \mathcal{K}^{s-\mu,\boldsymbol{\gamma}-\mu}(\mathbb{W}^\wedge)$$

for all $s \in \mathbb{R}$ and $\boldsymbol{\gamma} = (\gamma, \theta) \in \mathbb{R}^2$, and all $(z, \zeta) \in T^*\Xi \setminus 0$.

Let us discuss the principal symbolic structure of operators (32) from the point of view of ellipticity. First we have σ_ψ, the standard homogeneous principal symbol of order μ which is a function on $T^*(\mathbb{R}_+ \times \mathbb{W}_{\mathrm{reg}} \times \Xi) \setminus 0$. In the splitting of variables (t, r, x, y, z) (locally close to the edge Y of W) and covariables $(\tau, \varrho, \xi, \eta, \zeta)$, we have

$$\sigma_\psi(\boldsymbol{A})(t, r, x, y, z, \tau, \varrho, \xi, \eta, \zeta) \\ = t^{-\mu} r^{-\mu} \sum_{j+|\alpha|+k+|\beta| \leq \mu} c_{j\alpha,k\beta}(t, r, y, z; x, \xi)(-rti\tau)^k (-ri\varrho)^j (r\eta)^\alpha (rt\zeta)^\beta,$$

where $c_{j\alpha,k\beta}(t, r, y, z; x, \xi)$ is the homogeneous principal symbol of $c_{j\alpha,k\beta}(t, r, y, z)$ in $(x, \xi) \in T^*X \setminus 0$ of order $\mu - (j + |\alpha| + k + |\beta|)$.

Let us set

$$\tilde{\sigma}_\psi(\boldsymbol{A})(t, r, x, y, z, \tilde{\tau}, \tilde{\varrho}, \xi, \tilde{\eta}, \tilde{\zeta}) \\ := t^\mu r^\mu \sigma_\psi(\boldsymbol{A})(t, r, x, y, z, (rt)^{-1}\tilde{\tau}, r^{-1}\tilde{\varrho}, \xi, r^{-1}\tilde{\eta}, (rt)^{-1}\tilde{\zeta}). \tag{42}$$

By definition $\tilde{\sigma}_\psi(\boldsymbol{A})$ is smooth up to $(t, r) = (0, 0)$. The operator $\boldsymbol{A}$ is said to be corner degenerate elliptic (of order μ) with respect to $\sigma_\psi(\boldsymbol{A})$ if

$$\tilde{\sigma}_\psi(\boldsymbol{A}) \neq 0 \text{ for } (\tilde{\tau}, \tilde{\varrho}, \xi, \tilde{\eta}, \tilde{\zeta}) \neq 0 \text{ and all } (t, r, x, y, z) \text{ up to } (t, r) = (0, 0).$$

In order to define the next principal symbolic level of $\boldsymbol{A}$, we observe that (32) for every fixed (z, ζ) is an edge-degenerate family of differential operators on the stretched manifold $\mathbb{R}_+ \times \mathbb{W}$ with edge $\mathbb{R}_+ \times Y$. There are the principal symbols in the edge variables $(t, y) \in \mathbb{R}_+ \times Y$ and covariables (τ, η). By virtue of the formalism for our higher corner calculus, we treat ζ as an extra edge covariable. Thus the parameter-dependent principal edge symbol of (32) in the splitting of variables (t, r, x, y, z) takes the form

$$\begin{aligned} &\sigma_\wedge(\boldsymbol{A})(t, y, z, \tau, \eta, \zeta) \\ &\quad = t^{-\mu} r^{-\mu} \sum_{j+|\alpha|+k+|\beta| \le \mu} c_{j\alpha,k\beta}(t, 0, y, z)(-rti\tau)^k(-r\partial_r)^j(r\eta)^\alpha(rt\zeta)^\beta \end{aligned} \tag{43}$$

which is a family

$$\sigma_\wedge(\boldsymbol{A})(t, y, z, \tau, \eta, \zeta) : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

of continuous operators for every $(t, y, \tau, \eta, \zeta) \in T^*(\mathbb{R}_+ \times Y) \times \mathbb{R}^p_\zeta \setminus 0$, for every fixed $z \in \Xi (\setminus 0$ stands for $(\tau, \eta, \zeta) \neq 0)$.

Similarly as (42), from (43) we want to pass to the 'reduced' edge symbol

$$\begin{aligned} \tilde{\sigma}_\wedge(\boldsymbol{A})(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta}) &= t^\mu \sigma_\wedge(A(z, t^{-1}\tilde{\zeta}))(t, y, t^{-1}\tilde{\tau}, \eta) \\ &= r^{-\mu} \sum_{j+|\alpha|+k+|\beta| \le \mu} c_{j\alpha,k\beta}(t, 0, y, z)(-ri\tilde{\tau})^k(-r\partial_r)^j(r\eta)^\alpha(r\tilde{\zeta})^\beta. \end{aligned} \tag{44}$$

Since the main issue of our investigation is the behaviour of the edge symbol of $\boldsymbol{A}$ of second generation, i.e., of operators on $\mathbb{W}^\wedge$ when the coefficients $b_{k\beta}(t, z)$ of (32) are frozen at $t = 0$, we assume that the coefficients $c_{j\alpha,k\beta}$ do not depend on t for $t > T$ for some $T > 0$. Nevertheless, it is natural to admit (non-trivial) dependence on $t \le T$ and smoothness up to $t = 0$, according to the behaviour of the corresponding coefficients of the operator for small t.

Proposition 2.7 *Let $\boldsymbol{A}$ be corner degenerate elliptic. Then for every fixed (t, y, z) there is a discrete set $D(t, y, z) \subset \mathbb{R}$ such that*

$$\tilde{\sigma}_\wedge(\boldsymbol{A})(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta}) : \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{45}$$

is a family of Fredholm operators for all $\gamma \in \mathbb{R} \setminus D(t, y, z)$, all $(\tilde{\tau}, \eta, \tilde{\zeta}) \neq 0$, and all $s \in \mathbb{R}$.

Although this result is well known, we want to briefly recall some more background, since for the infinite (stretched) cone $X^\wedge$ of 'first generation' this is just a phenomenon that also plays a role for the cone $W^\wedge$ of second generation, with W being our compact manifold with edge Y.

The Fredholm property of (45) is governed by the ellipticity of the principal symbolic structure $(\sigma_\psi, \sigma_M, \sigma_E)$ of operators on $X^\wedge$. Here σ_ψ denotes the homogeneous principal symbol as usual, σ_M is the principal conormal symbol and σ_E the tuple of exit symbols. The corner degenerate ellipticity of $\boldsymbol{A}$ has the consequence that $\sigma_\psi(\tilde{\sigma}_\wedge(\boldsymbol{A}))$ (as a Fuchs type symbol in the splitting of variables (r, x) on $X^\wedge = \mathbb{R}_+ \times X$) is elliptic and that also $\sigma_E(\tilde{\sigma}_\wedge(\boldsymbol{A}))$ is elliptic as soon as the covariables $(\tilde{\tau}, \eta, \tilde{\zeta})$ are non-zero.

In the present case we have

$$\sigma_\psi(\tilde{\sigma}_\wedge(\boldsymbol{A}))(t, y, z; r, x, \varrho, \zeta) = r^{-\mu} \sum_{j=0}^{\mu} c_{j0,00}(t, 0, y, z; x, \xi)(-ir\varrho)^j, \tag{46}$$

where $c_{j0,00}(t, 0, y, z; x, \zeta)$ is the homogeneous principal symbol of $c_{j0,00}(t, 0, y, t)$ as an operator on X of order $\mu - j$. Moreover, the exit symbol has three components, locally in every cone $Y := \mathbb{R}_+ \times U_1$ for a coordinate neighbourhood $U_1 \subset X$ as follows: let us choose a diffeomorphism $U_1 \to V_1$ to an open set $V_1 \subset S^n$ $(n = \dim X)$ and form the cone $V := \{\tilde{x} \in \mathbb{R}^{n+1} \setminus \{0\} : \tilde{x}/|\tilde{x}| \in V_1\}$. We then obtain a diffeomorphisms $U \to V$ when we extend $U_1 \to V_1$ by homogeneity of order 1 in the axial variable to U. Now we can push forward $\tilde{\sigma}_\wedge(\boldsymbol{A})(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta})$ as a differential operator in $(r, x) \in U$ to a differential operator $\boldsymbol{A}_1(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta})$ in the Euclidean coordinates $\tilde{x} \in V$. Let

$$a(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta}; \tilde{x}, \tilde{\xi}) \tag{47}$$

denote the complete symbol of $\boldsymbol{A}_1$ in the variables and covariables $(\tilde{x}, \tilde{\xi}) \in V \times \mathbb{R}^{n+1} \setminus \{0\}$ depending on the parameters $(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta})$.

Writing for the moment $a(\tilde{x}, \tilde{\xi})$ in place of (47), we obtain a polynomial in $\tilde{\xi}$ of order μ, namely, $a(\tilde{x}, \tilde{\xi}) = \sum_{|\beta| \le \mu} c_\beta(\tilde{x}) \tilde{\xi}^\beta$ with coefficients $c_\beta \in C^\infty(V)$ such that $\chi(\tilde{x}) c_\beta(\tilde{x}) \in S^0_{\mathrm{cl}}(V_{\tilde{x}})$ for any excision function $\chi(\tilde{x})$ (the coefficients also depend on $(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta})$). Let $c_{\beta,(0)}(\tilde{x})$ denote the homogeneous principal part of c_β of order zero in $\tilde{x}$. We then have $\sigma_\psi(a)(\tilde{x}, \tilde{\xi}) = \sum_{|\beta| = \mu} c_\beta(\tilde{x}) \tilde{\xi}^\beta$, and

$$\sigma_{\mathrm{e}}(a)(\tilde{x}, \tilde{\xi}) = \sum_{|\beta| \le \mu} c_{\beta,(0)}(\tilde{x}) \tilde{\xi}^\beta, \quad \sigma_{\psi,\mathrm{e}}(a)(\tilde{x}, \tilde{\xi}) = \sum_{|\beta| = \mu} c_{\beta,(0)}(\tilde{x}) \tilde{\xi}^\beta.$$

The pair $(\sigma_{\mathrm{e}}(a), \sigma_{\psi,\mathrm{e}}(a))$ is just what we call the exit symbol $\sigma_E(\tilde{\sigma}_\wedge(\boldsymbol{A}))$, locally in the cone V.

Now for $(\tilde{\tau}, \eta, \tilde{\zeta}) \ne 0$ we have the exit ellipticity, i.e., (apart from $\sigma_\psi(a)(\tilde{x}, \tilde{\xi}) \ne 0$ for all $\tilde{\xi} \ne 0$) the properties

$$\sigma_{\mathrm{e}}(a)(\tilde{x}, \tilde{\zeta}) \ne 0 \text{ for all } \tilde{x} \in V \text{ and } \tilde{\xi} \in \mathbb{R}^{n+1}$$

and

$$\sigma_{\psi,\mathrm{e}}(a)(\tilde{x}, \tilde{\xi}) \ne 0 \text{ for all } \tilde{x} \in V \text{ and } \tilde{\xi} \in \mathbb{R}^{n+1} \setminus \{0\}.$$

The principal conormal symbol has the form

$$\sigma_M(\tilde{\sigma}_\wedge(\boldsymbol{A}))(t, y, z; v) = \sum_{j=0}^{\mu} c_{j0,00}(t, 0, y, z) v^j : H^s(X) \longrightarrow H^{s-\mu}(X) \tag{48}$$

that is independent of the covariables $(\tilde{\tau}, \eta, \tilde{\zeta})$, with $v \in \mathbb{C}$ being the Mellin covariable which substitutes $-r\partial_r$, cf. the formula (46). The operator function (48) is a holomorphic Fredholm family belonging to $L^{\mu}_{\mathrm{cl}}(X; \Gamma_{\frac{n+1}{2}-\gamma})$ for every fixed $\gamma \in \mathbb{R}$ (and every (t, y, z)). It is also parameter-dependent elliptic with the parameter $\operatorname{Im} v \in \mathbb{R}$. These properties together show that (45) is a family of isomorphisms for all $v \in \mathbb{C} \setminus D_1(t, y, z)$ for a discrete set $D_1(t, y, z) \subset \mathbb{C}$ that intersects every strip of finite width parallel to the imaginary axis in a finite set. Now the ellipticity of (45) with respect to the weight γ is just the condition that (45) is bijective for all $v \in \Gamma_{\frac{n+1}{2}-\gamma}$ (this is independent of $s \in \mathbb{R}$). Thus, the discrete set has the form $D(t, y, z) = \{\gamma \in \mathbb{R} : \Gamma_{\frac{n+1}{2}-\gamma} \cap D_1(t, y, z) \neq \emptyset\}$.

Let us now assume that there is a $\gamma \in \mathbb{R}$ such that (48) is bijective for all $(t, y, z) \in \overline{\mathbb{R}}_+ \times \Omega \times \Xi$ (this only concerns a compact t-interval because by assumption the coefficients stabilise for large t) and all $v \in \Gamma_{\frac{n+1}{2}-\gamma}$.

We want to fill up the family of Fredholm operators (45) by additional finite-dimensional entries to a 2×2 block matrix family of isomorphisms.

Setting $(\kappa_\delta u)(r, x) = \delta^{\frac{n+1}{2}} u(\delta r, x)$, $\delta \in \mathbb{R}_+$, we first have

$$\tilde{\sigma}_\wedge(\boldsymbol{A})(t, y, z, \delta\tilde{\tau}, \delta\eta, \delta\tilde{\zeta}) = \delta^{\mu} \kappa_\delta \tilde{\sigma}_\wedge(\boldsymbol{A})(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta}) \kappa_\delta^{-1}$$

for all $\delta \in \mathbb{R}_+$. This gives us

$$\tilde{\sigma}_\wedge(\boldsymbol{A})\Big(t, y, z, \frac{\tilde{\tau}, \eta, \tilde{\zeta}}{|\tilde{\tau}, \eta, \tilde{\zeta}|}\Big) = |\tilde{\tau}, \eta, \tilde{\zeta}|^{\mu} \kappa_{|\tilde{\tau}, \eta, \tilde{\zeta}|} \tilde{\sigma}_\wedge(\boldsymbol{A})(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta}) \kappa^{-1}_{|\tilde{\tau}, \eta, \tilde{\zeta}|}. \tag{49}$$

Since $\dim \ker \tilde{\sigma}_\wedge(\boldsymbol{A})$ and $\dim \operatorname{coker} \tilde{\sigma}_\wedge(\boldsymbol{A})$ only depend on $(\tilde{\tau}, \eta, \tilde{\zeta})/|\tilde{\tau}, \eta, \tilde{\zeta}|$, it suffices to generate the entries for $(\tilde{\tau}, \eta, \tilde{\zeta}) \in S^{q+p}$ (the unit sphere in $\mathbb{R}^{1+p+q}$) and then to extend them by homogeneity to all $(\tilde{\tau}, \eta, \tilde{\zeta}) \in \mathbb{R}^{1+p+q} \setminus \{0\}$, in a similar manner as (49). Moreover, we may assume that (t, y, z) only vary over a compact subset of $\overline{\mathbb{R}}_+ \times B \times \Xi$, because both y and z play the role of local coordinates on corresponding compact edges Y and Z, respectively, and since the coefficients are independent of t for $t > T$ for some $T > 0$. Globally on $M := I \times Y \times L$, for $I := [0, T]$ and any compact $L \subset \Xi$ (say, a closed ball), we interpret (45) as a family of Fredholm operators, parametrised by

$$S^*M = S^*(\overline{\mathbb{R}}_+ \times Y \times \Xi)|_M$$

(the cosphere bundle over M). Let $K(\cdot)$ denote the K-group over the space in the brackets. Then, as is well known, cf [1], there is an index

$$\operatorname{ind}_{S^*M} \tilde{\sigma}_\wedge(\boldsymbol{A}) \in K(S^*M) \tag{50}$$

of the Fredholom family (45) in the K-group over S^*M. Let $\pi_1 : S^*M \to M$ be the canonical projection (from the projection of the cotangent bundle to the base), and let $\pi_1^* : K(M) \to K(S^*M)$ be the map induced by the bundle pull back. An assumption on the given operator is now the relation

$$\mathrm{ind}_{S^*M}\, \tilde{\sigma}_\wedge(\boldsymbol{A}) \in \pi_1^*(M). \tag{51}$$

It is satisfied in many cases (e.g., Laplace-Beltrami operators belonging to corner metrics). The relation (51) is an analogue of the Atiyah-Bott condition for the existence of Shapiro-Lopatinskij elliptic boundary conditions for the given operator in the special case of boundary value problems, cf. [1]. For edge singularities the condition (51) plays an analogous role for the existence of edge conditions, cf. [16] and [20]. We want to impose edge conditions along M. They will first refer to the edge covariables $(\tilde{\tau}, \eta, \tilde{\zeta})$; then we insert the original covariables in the 'corner degenerate' combination $(t\tau, \eta, t\zeta)$. At the same time we will have edge conditions for $t \to \infty$. As announced before, we do that in terms of a family of isomorphisms

$$\tilde{\sigma}_\wedge(\boldsymbol{\mathcal{A}}) := \begin{pmatrix} \tilde{\sigma}_\wedge(\boldsymbol{A}) & \tilde{\sigma}_\wedge(\boldsymbol{K}) \\ \tilde{\sigma}_\wedge(\boldsymbol{T}) & \tilde{\sigma}_\wedge(\boldsymbol{Q}) \end{pmatrix} : \begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ J_{-,(t,y,z)} \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \\ \oplus \\ J_{+,(t,y,z)} \end{matrix} \tag{52}$$

parametrised by $(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta}) \in T^*M \setminus 0$, for vector bundles $J_\pm \in \mathrm{Vect}(M)$.

The additional entries are locally with respect to y nothing other than homogeneous principal parts of Green symbols in the sense of Definition 1.13, with (y, η, λ) replaced by $(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta})$. In addition we impose smoothness in t up to $t = 0$ which is possible, since the upper left corner is smooth up to zero. The way of constructing (52) is to first choose a corresponding isomorphism for points on S^*M and then to extend it by homogeneity according to

$$\tilde{\sigma}_\wedge(\boldsymbol{\mathcal{A}})(t, y, z, \delta\tilde{\tau}, \delta\eta, \delta\tilde{\zeta})\delta^\mu\, \mathrm{diag}(\kappa_\delta, \lambda^{\frac{n+1}{2}}\, \mathrm{id})\tilde{\sigma}_\wedge(\boldsymbol{\mathcal{A}})(t, y, z, \tilde{\tau}, \eta, \tilde{\zeta})\, \mathrm{diag}(\kappa_\delta, \lambda^{\frac{n+1}{2}}\, \mathrm{id})^{-1}.$$

2.3 Tools on operators with exit conditions at infinity

By abstract calculus we understand elements of the pseudo-differential machinery with operator-valued symbols (cf. Definition 1.11), here globally in $\mathbb{R}^q_y$ with $|y| \to \infty$ being interpreted as a conical exit of $\mathbb{R}^q$ to infinity. Let E and $\widetilde{E}$ be Hilbert spaces with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. Let $\mu, \nu, \nu' \in \mathbb{R}$, an element $a(y, y', \eta) \in C^\infty(\mathbb{R}^q_y \times \mathbb{R}^q_{y'} \times \mathbb{R}^q_\eta, \mathcal{L}(E, \widetilde{E}))$ is said to be a symbol of the space

$$S^{\mu;\nu,\nu'}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E}) \tag{53}$$

if

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^{\alpha'}_{y'} D^\beta_\eta a(y, y', \eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,\widetilde{E})} \le c\langle\eta\rangle^{\mu-|\beta|}\langle y\rangle^{\nu-|\alpha|}\langle y'\rangle^{\nu'-|\alpha'|} \tag{54}$$

for all multi-indices α, α', $\beta \in \mathbb{N}^q$ and all $(y, y', \eta) \in \mathbb{R}^{3q}$, with constants $c = c(\alpha, \alpha', \beta) > 0$.

Remark 2.8 *If* $a \in S^{\mu;\nu,\nu'}$ *and* $b \in S^{\tilde{\mu};\tilde{\nu},\tilde{\nu}'}$ *then* $ab \in S^{\mu+\tilde{\mu};\nu+\tilde{\nu},\nu'+\tilde{\nu}'}$ *(in the sense of pointwise composition of operators, under the assumption that the involved spaces fit together). In particular, if a is independent of* y, y', *i.e.,* $a \in S^{\mu}(\mathbb{R}^q_\eta; E, \widetilde{E})$, *it follows also that* $a \in S^{\mu;0,0}(\mathbb{R}^{3q}; E, \widetilde{E})$ *and* $ab \in S^{\mu+\tilde{\mu};\tilde{\nu},\tilde{\nu}'}(\mathbb{R}^{3q}; E, \widetilde{E})$.

Let us consider operators with 'double' symbols $a(y, y', \eta)$ in the space (53),

$$\mathrm{Op}(a)u(y) = \iint e^{i(y-y')\eta} a(y, y', \eta) u(y') dy' đ\eta. \tag{55}$$

Operators (55) can be equivalently written in terms of 'left' or 'right' symbols $a_L(y, \eta)$ and $a_R(y', \eta)$, respectively. They satisfy similar estimates as (54) with the exception, for instance, in the case of left symbols, that there is no dependence of y'; this allows us to set $\nu' = 0$ and to omit it. In other words, we have the spaces $S^{\mu;\nu}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ of symbols of that kind. Similarly as for scalar operators, there is a one-to-one correspondence $a_L(y, \eta) \to \mathrm{Op}(a)$ between left symbols and associated operators. The same is true in the case of right symbols. Then for every $a(y, y', \eta)$ in the space (53) there are unique left (or right) symbols $a_L(y, \eta)$ (or $a_R(y', \eta)$) in $S^{\mu;\nu+\nu'}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ such that $\mathrm{Op}(a) = \mathrm{Op}(a_L)$ (or $= \mathrm{Op}(a_R)$).

Let us set

$$\mathcal{W}^{s;\delta}(\mathbb{R}^q, E) := \langle y \rangle^{-\delta} \mathcal{W}^s(\mathbb{R}^q, E),$$

$s, \delta \in \mathbb{R}$. The continuity results below are based on the following theorem, proved in different generality in [16] or [23].

Theorem 2.9 *Let* $a(y, \eta) \in S^{\mu;\nu}(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. *Then* $\mathrm{Op}(a)$ *induces continuous operators* $\mathrm{Op}(a) : \mathcal{W}^{s;\delta}(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu;\delta-\nu}(\mathbb{R}^q, \widetilde{E})$ *for all* $s, \delta \in \mathbb{R}$.

In order to localise our amplitude function close to the diagonal, we employ the following result:

Lemma 2.10 *There exists a double symbol* $\omega(t, t')$ *in the exit calculus on the real* t*-axis (independent of the covariable* τ*),* $\omega(t, t') \in S^{0;0,0}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_\tau)$ *with the property*

$$\omega(t, t') = 1 \quad \text{for} \quad |t - t'| < 1, \ \omega(t, t') = 0 \quad \text{for} \quad |t - t'| > 2.$$

The function ω *exists in the form* $\omega(t, t') = \psi\left(\dfrac{(t - t')^2}{1 + (t - t')^2}\right)$ *for any* $\psi \in C_0^\infty(\overline{\mathbb{R}}_+)$ *such that* $\psi(t) = 1$ *for* $t < \frac{1}{2}$, $\psi(t) = 0$ *for* $t > \frac{2}{3}$.

A proof may be found in [11, Section 3.3.3].

Let us now give some examples. In our applications the variable y is replaced by (t, y), and the space E has the form

$$E = \mathcal{K}^{s,\gamma}(X^\wedge) \quad \text{for any fixed} \quad s, \gamma \in \mathbb{R}. \tag{56}$$

Lemma 2.11 *Let* $\varphi(t,y,r) \in C^\infty(\mathbb{R}^{1+q}_{t,y} \times \overline{\mathbb{R}}_{+,r})$ *have the properties*

$$\varphi(t,y,r) = 0 \text{ for } -\infty < t < c_0, \text{ and } \varphi(t,y,r) \text{ is independent of } t \text{ for } t > c_1 \tag{57}$$

for certain constants $0 < c_0 < c_1$ *independent of* (y,r), *and let*

$$\varphi(t,y,r) = 0 \quad \text{for} \quad |y| \geq c_2 \text{ or } r \geq c_2 \tag{58}$$

for some $c_2 > 0$, *independent of* t. *Set* $\tilde{\varphi}(t,\tilde{y},\tilde{r}) := \varphi\big(t, \frac{\tilde{y}}{t}, \frac{\tilde{r}}{t}\big)$, *and let* $\mathcal{M}_{\tilde{\varphi}}$ *denote the operator of multiplication by* $\tilde{\varphi}$ *in the space* $E = \mathcal{K}^{s,\gamma}(X^\wedge)$, $X^\wedge = \mathbb{R}_+ \times X \ni (\tilde{r}, x)$. *Then we have*

$$\mathcal{M}_{\tilde{\varphi}} \in S^{0;0}(\mathbb{R}^{1+q}_{t,\tilde{y}} \times \mathbb{R}^{1+q}_{\tau,\eta}; E, E). \tag{59}$$

Moreover, let $(\varphi_j)_{j\in\mathbb{N}}$ *be a sequence in the space* $C^\infty(\mathbb{R}^{1+q} \times \overline{\mathbb{R}}_+)$ *for which the constants* c_0, c_1 *and* c_2 *are independent of* j, *and let* $\varphi_j \to 0$ *for* $j \to \infty$. *Then we have* $\mathcal{M}_{\tilde{\varphi}_j} \to 0$ *in the space* $S^{0;0}(\mathbb{R}^{1+q}_{t,\tilde{y}} \times \mathbb{R}^{1+q}_{\tau,\eta}; E, E)$. *This is valid for every* $s, \gamma \in \mathbb{R}$.

Proof. In order to show the relation (59), we have to verify the symbolic estimates

$$\|\kappa^{-1}_{\langle\tau,\eta\rangle} \Big\{ D^\alpha_{t,\tilde{y}} D^\beta_{\tau,\eta} a(t,\tilde{y},\tau,\eta) \Big\} \kappa_{\langle\tau,\eta\rangle}\|_{\mathcal{L}(E,E)} \leq c\langle\tau,\eta\rangle^{-|\beta|}\langle t,\tilde{y}\rangle^{-|\alpha|} \tag{60}$$

for all $\alpha, \beta \in \mathbb{N}^{1+q}$, with constants $c = c(\alpha,\beta) > 0$, for all $(t,\tilde{y},\tau,\eta) \in \mathbb{R}^{1+q} \times \mathbb{R}^{1+q}$. In the present case we have $a = \mathcal{M}_{\tilde{\varphi}}$ which is independent of the covariables. Thus it suffices to look at the case $\beta = 0$. We have

$$\kappa^{-1}_{\langle\tau,\eta\rangle} D^\alpha_{t,\tilde{y}} \mathcal{M}_{\tilde{\varphi}} \kappa_{\langle\tau,\eta\rangle} = D^\alpha_{t,y} \tilde{\varphi}(t,\tilde{y},\langle\tau,\eta\rangle^{-1}\tilde{r}) = D^\alpha_{t,\tilde{y}} \varphi\Big(t, \frac{\tilde{y}}{t}, \frac{\langle\tau,\eta\rangle^{-1}\tilde{r}}{t}\Big). \tag{61}$$

For the case $\alpha = 0$ we have to show the uniform boundedness of $\|\mathcal{M}_\varphi\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge))}$ as an operator of multiplication in $\tilde{r} \in \mathbb{R}_+$, where φ depends on the parameters $t, \tilde{y}$ and $\langle\tau,\eta\rangle$. Concerning the dependence with respect to the first variables t and $\frac{\tilde{y}}{t} =: y$, only a compact set in $(t,y) \in \mathbb{R}^{1+q}$ is important, cf. the assumption (57). Then the estimate (60) for $\alpha = 0$ reduces to the following result, known from the properties of the $\mathcal{K}^{s,\gamma}(X^\wedge)$-spaces: if we choose any $\psi(t,y,\tilde{r}) \in C^\infty_0(\mathbb{R}^{1+q} \times \overline{\mathbb{R}}_+)$, the operator of multiplication by $\psi(t,y,c\tilde{r})$ for c varying in an interval $(0,\tilde{c})$, $\tilde{c} > 0$, is uniformly bounded in $\mathcal{K}^{s,\gamma}(X^\wedge)$ for every fixed $s, \gamma \in \mathbb{R}$. In the present case we have $c = (t\langle\eta\rangle)^{-1}$ for $t > c_1, (\tau,\eta) \in \mathbb{R}^{1+q}$. For the derivatives (61) for $\alpha \neq 0$ it suffices to note that we only produce factors proportional to $t^{-|\alpha|}$ for $t > c_1$ which can be estimated (up to a constant) by $\langle t, y\rangle$ on the support of ψ.

To complete the proof, we employ the known fact that the operators of multiplication in $\mathcal{K}^{s,\gamma}(X^\wedge)$ tend to zero as soon as the function itself tends to zero. It is trivial that this remains true for the case of an extra parameter-dependence as is the case in the present situation. □

By a similar technique we can prove the following assertions.

Lemma 2.12 *Let $\omega(t,t')$ be as in Lemma 2.10, and set $\varphi(\tilde{r},t,t') := \omega(t,t')\tilde{\sigma}(\frac{\tilde{r}'}{t})$. Then for the operator $\mathcal{M}_\varphi$ of multiplication by φ we have $\mathcal{M}_\varphi \in S^{0;0,0}(\mathbb{R}^{1+q} \times \mathbb{R}^{1+q}; E, E)$ for the spaces (56).*

Lemma 2.13 *Let $\psi(r,t,t',\tilde{y},\tilde{y}') \in C^\infty(\overline{\mathbb{R}}_{+,r}, C^\infty(\mathbb{R} \times B \times \mathbb{R} \times B))$ for the open unit ball B in $\mathbb{R}^q$, and assume that the support of ψ in $\tilde{y} \in B$ as well as the support of ψ in $\tilde{y}' \in B$ is compact; moreover, let*

$$\psi = 0 \text{ for } -\infty < t < c_0, \text{ and } \psi \text{ independent of } t \text{ for } t > c_1,$$

and, similarly, for the dependence with respect to t', for constants $0 < c_0 < c_1$. Then, setting $\tilde{\psi} = \psi(\frac{\tilde{r}}{t}, t, t', \frac{\tilde{y}}{t}, \frac{\tilde{y}'}{t})\omega(t,t')$ we have $\mathcal{M}_{\tilde{\psi}} \in S^{0;0,0}(\mathbb{R}^{1+q} \times \mathbb{R}^{1+q}; E, E)$ for the spaces (56). In addition, $\psi_j \to 0$ in the space of the above mentioned functions with c_0, c_1 independent of j entails $\mathcal{M}_{\varphi_j} \to 0$ in the symbol space.

2.4 Connections between cylindrical and conical representations

From the shape of corner degenerate differential operators (33) for r near zero (i.e., near $\partial\mathbb{W}$, where the variables on $\mathbb{W}$ split into (r,x,y)), we see that the operator-valued (also in the pseudo-differential case) have corner-degenerate form, i.e., the covariables $\varrho, \tau, \eta, \zeta$ (with $\zeta \in \mathbb{R}^p$ being the extra parameter) are involved in the combination

$$(r\varrho, rt\tau, r\eta, rt\zeta). \tag{62}$$

We will drop for a while the z-variable, because we are discussing operators on $\mathbb{W}^\wedge$ for fixed (z,ζ), $\zeta \neq 0$. Moreover, let the coefficients also depend on (t,t') and (y,y'). The starting point is an operator-valued amplitude function

$$P(r,t,t',y,y',\varrho,\tau,\eta,\zeta) := \widetilde{P}(r,t,t',y,y',r\varrho,rt\tau,r\eta,rt\zeta) \tag{63}$$

for some

$$\widetilde{P}(r,t,t',y,y',\tilde{\varrho},\tilde{\tau},\tilde{\eta},\tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times B \times B, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{2+p+q}_{\tilde{\varrho},\tilde{\tau},\tilde{\eta},\tilde{\zeta}})). \tag{64}$$

Here $B := \{y \in \mathbb{R}^q : |y| < 1\}$ plays the role of a chart on the edge Y; double symbols appear in localisations in terms of a partition of unity on Y. The double symbols in t are motivated later on by certain localisations also in the corner axis direction.

Then the upper left corners of the local edge symbols (23) are to be replaced by amplitude functions of the form

$$a(t,t',y,y',\tau,\eta,\zeta) = (a_M + a_\psi + m + g)(t,t',y,y',\tau,\eta,\zeta) \tag{65}$$

which are defined as follows. First we fix cut-off functions $\sigma, \tilde{\sigma}, \omega_1, \omega_2, \omega_3$ of the same kind as in the formula (17) and set $\chi_1 := 1 - \omega_1$, $\quad \chi_2 := 1 - \omega_3$.

Then

$$
\begin{aligned}
&a_M(t,t',y,y',\tau,\eta,\zeta)\\
&:=\sigma(r)r^{-\mu}t^{-\mu}\omega_1(r[t\tau,\eta,t\zeta])\operatorname{op}_M^{\gamma-\frac{n}{2}}(H)(t,t',y,y',\tau,\eta,\zeta)\omega_2(r'[t\tau,\eta,t\zeta])\tilde{\sigma}(r'), \quad (66)\\
&a_\psi(t,t',y,y',\tau,\eta,\zeta)\\
&:=\sigma(r)r^{-\mu}t^{-\mu}\chi_1(r[t\tau,\eta,t\zeta])\operatorname{op}_r(P)(t,t',y,y',\tau,\eta,\zeta)\chi_2(r'[t\tau,\eta,t\zeta])\tilde{\sigma}(r'), \quad (67)
\end{aligned}
$$

for an operator-valued Mellin amplitude function

$$H(r,t,t',y,y',v,\tau,\eta,\zeta):=\widetilde{H}(r,t,t',y,y',v,rt\tau,r\eta,rt\zeta), \quad (68)$$

for some

$$\widetilde{H}(r,t,t',y,y',v,\tilde{\tau},\tilde{\eta},\tilde{\zeta})\in C^\infty(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times B\times B,L^\mu_{\mathrm{cl}}(X;\mathbb{C}\times\mathbb{R}^{1+p+q}_{\tilde{\tau},\tilde{\eta},\tilde{\zeta}})), \quad (69)$$

where (68) is associated with (64) in a similar manner as H with P in Theorem 1.9. Moreover, analogously as (24) we set

$$m(t,t',y,y',\tau,\eta,\zeta)=r^{-\mu}t^{-\mu}\omega_1(r[t\tau,\eta,t\zeta])\operatorname{op}_M^{\gamma-\frac{n}{2}}(F)(t,t',y,y')\omega_2(r'[t\tau,\eta,t\zeta]) \quad (70)$$

for an element

$$F(t,t',y,y',v)\in C^\infty(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times B\times B,M^{-\infty}(X;\Gamma_{\frac{n+1}{2}-\gamma})). \quad (71)$$

Finally, according to Green symbols of Definition 1.13, the Green summand in (65) has the form of an upper left corner of

$$
\begin{aligned}
g(t,t',y,y',\tau,\eta,\zeta):=&t^{-\mu}G(t,t',y,y',t\tau,\eta,t\zeta)=\operatorname{diag}(1,\langle t\tau,\eta,t\zeta\rangle^{\frac{n+1}{2}})\\
&t^{-\mu}G_0(t,t',y,y',t\tau,\eta,t\zeta)\operatorname{diag}(1,\langle t\tau,\eta,t\zeta\rangle^{-\frac{n+1}{2}})
\end{aligned} \quad (72)
$$

for

$$G(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta})=\operatorname{diag}(1,\langle\tilde{\tau},\eta,\tilde{\zeta}\rangle^{\frac{n+1}{2}})G_0(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta})\operatorname{diag}(1,\langle\tilde{\tau},\eta,\tilde{\zeta}\rangle^{-\frac{n+1}{2}}),$$

where

$$
\begin{aligned}
&G_0(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta})\\
&\quad\in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times B\times B\times\mathbb{R}^{1+q+p}_{\tilde{\tau},\eta,\tilde{\zeta}};\mathcal{K}^{s,\gamma}(X^\wedge)\oplus\mathbb{C}^{j_-},\mathcal{S}^{\gamma-\mu}(X^\wedge)_{2\varepsilon}\oplus\mathbb{C}^{j_+})
\end{aligned} \quad (73)
$$

satisfies

$$
\begin{aligned}
&G_0^*(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta})\\
&\quad\in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times B\times B\times\mathbb{R}^{1+p+q}_{\tilde{\tau},\eta,\tilde{\zeta}};\mathcal{K}^{s,-\gamma+\mu}(X^\wedge)\oplus\mathbb{C}^{j_+},\mathcal{S}^{-\gamma}(X^\wedge)_{2\varepsilon}\oplus\mathbb{C}^{j_-}),
\end{aligned} \quad (74)
$$

$s \in \mathbb{R}$, for some $\varepsilon(g) > 0$.

From the definition of the operator function (65), we see that a has the form

$$a(t,t',y,y',\tau,\eta,\zeta) = \tilde{a}(t,t',y,y',t\tau,\eta,t\zeta) \tag{75}$$

for an operator-valued symbol

$$\tilde{a}(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta}) \in S^\mu(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times B \times B \times \mathbb{R}^{1+q+p}_{\tilde{\tau},\eta,\tilde{\zeta}}; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)). \tag{76}$$

The property (76) follows from (18), (22) and Definition 1.13. The meaning of symbols in the variables (t,t',y,y') with (t,t') varying on $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$ is as usual, namely that the symbolic estimates hold uniformly in (t,t') up to $(0,0)$.

From (75) we can pass to the corresponding operator families

$$\mathrm{Op}_{t,y}(a)(\zeta) = \mathrm{Op}_{t,y}(\omega a)(\zeta) + \mathrm{Op}_{t,y}((1-\omega)a)(\zeta), \tag{77}$$

where $\omega(t,t')$ is as in Lemma 2.10. From now on we ignore the second summand in (77) because $1-\omega(t,t')$ cuts out a neighbourhood of the diagonal in all variables and so contributes smoothing operators.

In order to show the continuity of operators in weighted Sobolev spaces on the stretched cone $\mathbb{W}^\wedge$ up to infinity, analogously as (39) we look at the localisation for $t \to \infty$ (the behaviour for $t \to 0$ is encoded in Mellin terms later on as a part of the corner calculus of [18]). We also take into account localisations on the edge Y on charts from coordinate neighbourhoods to the ball B combined with a partition of unity in local y-variables. In other words, our amplitude functions (75) occur in combination with localising factors, i.e., we have to consider

$$(1-\sigma_1(t))\varphi(y)a(t,t',y,y',\tau,\eta,\zeta)\psi(y')(1-\sigma_2(t')) \tag{78}$$

with cut-off functions σ_1, σ_2 on the t-half axis, $\varphi, \psi \in C_0^\infty(B)$, $\psi \equiv 1$ on $\operatorname{supp}\varphi$. However, since we admitted from the very beginning smooth dependence of t, t' and y, y', we may subsume the factors under the dependence on those variables and return again to the notation (75) by assuming

$$a \equiv 0 \quad \begin{array}{l}\text{for all } y, y' \text{ in a neighbourhood of } \partial B \\ \text{and for all } t, t' \text{ when } t < T_0 \text{ or } t' < T_0\end{array} \tag{79}$$

for some $T_0 > 0$. In addition, since we are mainly interested in principal edge symbols from the higher corner calculus, we assume that

$$\tilde{a}(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta}) \text{ is independent of } t \text{ for } t > T_1 \text{ and of } t' \text{ for } t' > T_1, \tag{80}$$

for some $T_1 > 0$. As explained before, the main point in this section is to show that

$$\boldsymbol{a}(\zeta) := \mathrm{Op}_{t,y}(\omega a)(\zeta) \quad \text{for} \quad \zeta \neq 0$$

induces continuous operators

$$\boldsymbol{a}(\zeta) : \mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge) \to \mathcal{K}^{s-\mu,\boldsymbol{\gamma}-\mu}(\mathbb{W}^\wedge) \tag{81}$$

for all $s \in \mathbb{R}$. By construction, our amplitude function $a(t, t', y, y', \tau, \eta, \zeta)$ refers to the variables in the cylinder $\mathbb{R}_+ \times B \ni (t, y)$ with the support conditions (79) and (80). The operator function a takes values in pseudo-differential operators on $X^\wedge \ni (r, x)$, with $r \in \mathbb{R}_+$ being the inner cone axis direction. To simplify the explanations, we often suppress the action on X and interpret $\boldsymbol{a}(\zeta)$ as a pseudo-differential operator on $\mathbb{R}_+ \times \mathbb{R}_+ \times B \ni (t, r, y)$ for any fixed $\zeta \neq 0$.

Now the specific aspect for the continuity of (81) is to show, analogously as (40), that

$$\beta_* \boldsymbol{a}(\zeta) : \mathcal{W}^s(\mathbb{R}_t \times \mathbb{R}^q_{\tilde{y}}, \mathcal{K}^{s,\gamma}(\mathbb{R}_{+,\tilde{r}} \times X)) \longrightarrow \mathcal{W}^{s-\mu}(\mathbb{R}_t \times \mathbb{R}^q_{\tilde{y}}, \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}_{+,\tilde{r}} \times X)) \tag{82}$$

is continuous, where β is the diffeomorphism

$$\beta : \mathbb{R}_+ \times \mathbb{R}_+ \times B \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^q, \quad \beta(t, r, y) = (t, tr, ty).$$

Since at the moment we are considering the local situation with respect to the edge variable y, the map χ_l in (40) can be ignored, and the localising functions $\varphi_l(y)$ and $\omega(r)$ may also be subsumed under the amplitude function itself (cf. the explanation after the formula (78)). Let us write $\beta = \alpha_2 \circ \alpha_1$ for $\alpha_1(t, r, y) := (t, tr, y)$, $\alpha_2(t, \tilde{r}, y) := (t, \tilde{r}, ty)$, such that $\beta_* = \alpha_{2*} \circ \alpha_{1*}$. By definition we have $\alpha_1 = \mathrm{id}_{\mathbb{R}_{+,t}} \times \beta_1 \times \mathrm{id}_B$ for

$$\beta_1(r) = tr,$$

and $\alpha_2 = \mathrm{id}_{\mathbb{R}_{+,t}} \times \mathrm{id}_{\mathbb{R}_{+,r}} \times \beta'_2$ for $\beta'_2(y) = ty$. Set

$$\beta_2 := \mathrm{id}_{\mathbb{R}_{+,t}} \times \beta'_2 : \mathbb{R}_{+,t} \times B \to \mathbb{R}_{+,t} \times \mathbb{R}^q.$$

From (77) we have $\beta_* \boldsymbol{a}(\zeta) = \beta_{2*} \operatorname{Op}_{t,y}(\omega \beta_{1*} a)(\zeta)$. It follows that

$$\beta_{2*} \operatorname{Op}_{t,y}(\omega \beta_{1*} a)(\zeta)) = \operatorname{Op}_{t,\tilde{y}}(\omega \beta_{2*}(\beta_{1*} a))(\zeta), \tag{83}$$

where β_{2*} on the right hand side has the meaning of a corresponding push forward of symbols,belonging to $y \mapsto \tilde{y} = ty$, applied to $(\beta_{1*} a)(t, t', y, y', \tau, \eta, \zeta)$.

Theorem 2.14 *Let a be an amplitude function of the form* (65) *with the summands* (66)*,* (67)*,* (70) *and* (72)*. Then for every fixed $\zeta \neq 0$ we have*

$$\omega(t, t') \beta_{2*}(\beta_{1*} a)(t, t', \tilde{y}, \tilde{y}', \tau, \tilde{\eta}, \zeta) \in S^{\mu;0,0}(\mathbb{R}^{1+q}_{t,\tilde{y}} \times \mathbb{R}^{1+q}_{t',\tilde{y}'} \times \mathbb{R}^{1+q}_{\tau,\tilde{\eta}}; E, \widetilde{E}) \tag{84}$$

for the spaces $E := \mathcal{K}^{s,\gamma}(X^\wedge)$, $\widetilde{E} = \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$, $s \in \mathbb{R}$.

Corollary 2.15 *For every $\zeta \neq 0$ the operator* (81) *is continuous for all $s \in \mathbb{R}$.*

In fact, (83) generates continuous operators between the spaces in (82), i.e., (82) holds for $\beta_* \boldsymbol{a}(\zeta)$ itself. This yields immediately the assertion.

Theorem 2.14 will be proved in Section 3.4 below.

3 Ellipticity

3.1 Parametrices in the case of non-compact edges

Let $\tilde{\mathfrak{a}}(t, \tilde{\tau}, \tilde{\zeta}) \in C^\infty(\mathbb{R}, \mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}_{\tilde{\tau}} \times \mathbb{R}^p_{\tilde{\zeta}}))$ be an element that is independent of t for $|t| > T$ for some $T > 0$. The weight and bundle data in $\tilde{a}$ are assumed to be

$$\boldsymbol{g} := (\gamma, \gamma - \mu), \quad \boldsymbol{w} := (J_-, J_+), \tag{85}$$

respectively, cf. the notation of Section 1.4.

Definition 3.1 *An operator function of the form* $\mathfrak{a}(t, \tau, \zeta) := \tilde{\mathfrak{a}}(t, t\tau, t\zeta)$ *is called elliptic if*

(i) $\tilde{\mathfrak{a}}(t, \tilde{\tau}, \tilde{\zeta})$ *is parameter-dependent elliptic for every* t*, with parameters* $\tilde{\tau}, \tilde{\zeta}$*;*

(ii) $\mathfrak{a}(t, \tau, \zeta)$ *is parameter-dependent elliptic for every* t*, with parameters* τ, ζ*.*

Note that the property (i) entails the parameter-dependent ellipticity of $\mathfrak{a}(t, \tau, \zeta)$ for $|t| > \varepsilon$ for every $\varepsilon > 0$. In other words, (ii) is an extra requirement only for t close to zero.

Theorem 3.2 *Let* $\tilde{\mathfrak{a}}(t, \tilde{\tau}, \tilde{\zeta}) \in C^\infty(\mathbb{R}, \mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^{1+p}_{\tilde{\tau}, \tilde{\zeta}}))$ *satisfy the condition* (i) *of Definition* 3.1*. Then there exists* $\tilde{\mathfrak{p}}(t, \tilde{\tau}, \tilde{\zeta}) \in C^\infty(\mathbb{R}, \mathcal{Y}^{-\mu}(\mathbb{W}; \mathbb{R}^{1+q}_{\tilde{\tau}, \tilde{\zeta}}))$ *(associated with* $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma)$ *and* $\boldsymbol{w}^{-1} = (J_+, J_-)$ *when* $\mathfrak{a}$ *belongs to* (85)*), satisfying the analogous condition, i.e.* $\tilde{\mathfrak{p}}(t, \tilde{\tau}, \tilde{\zeta})$ *is independent of* t *for* $|t| > T$ *for some* $T > 0$*, and*

$$1 - \tilde{\mathfrak{p}}(t, \tilde{\tau}, \tilde{\zeta})\tilde{\mathfrak{a}}(t, \tilde{\tau}, \tilde{\zeta}),\ 1 - \tilde{\mathfrak{a}}(t, \tilde{\tau}, \tilde{\zeta})\tilde{\mathfrak{p}}(t, \tilde{\tau}, \tilde{\zeta}) \in C^\infty(\mathbb{R}, \mathcal{Y}^{-\infty}(\mathbb{W}; \mathbb{R}^{1+q}_{\tilde{\tau}, \tilde{\zeta}})).$$

This result is known from the general (parameter-dependent) calculus on a compact (stretched) manifold $\mathbb{W}$ with edges.

We now choose a partition of unity on $\mathbb{R} \ni t$ of the form $\chi_1^- + \varphi_1 + \chi_1^+ = 1$ for a function $\varphi_1 \in C_0^\infty(\mathbb{R})$, which is equal to 1 in an interval $-T - \varepsilon \le t \le T + \varepsilon$ for some $T > 0$ and $\varepsilon > 0$; then we find unique $\chi_1^\pm \in C^\infty(\mathbb{R})$, supported in $(-\infty, -T)$ and $(T, +\infty)$, respectively. Moreover, choose functions $\chi_2^\pm \in C^\infty(\mathbb{R})$ and $\varphi_2 \in C_0^\infty(\mathbb{R}_+)$ such that $\varphi_2 \equiv 1$ on $\operatorname{supp} \varphi_1$, $\chi_2^\pm \equiv 1$ on $\operatorname{supp} \chi_1^\pm$, where $\chi_2^\pm$ are also supported in $(0, +\infty)$ and $(-\infty, 0)$, respectively.

Given an operator function $\mathfrak{a}(t, \tau, \zeta) = \tilde{\mathfrak{a}}(t, t\tau, t\zeta)$ for $\tilde{\mathfrak{a}}(t, \tilde{\tau}, \tilde{\zeta}) \in C^\infty(\mathbb{R}, \mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^{1+q}_{\tilde{\tau}, \tilde{\zeta}}))$, we form the operator functions

$$\mathfrak{a}^\pm(t, t', \tau, \zeta) := \chi_1^\pm(t)\omega(t[\zeta], t'[\zeta])\mathfrak{a}(t, \tau, \zeta)\chi_2^\pm(t')$$

and $\mathfrak{a}_0(t, t', \tau, \zeta) := \varphi_1(t)\mathfrak{a}(t, \tau, \zeta)\varphi_2(t')$, where $\omega(t, t')$ is the cut-off function from Lemma 2.10. We then write

$$\mathbf{op}_t(\mathfrak{a})(\zeta) := \mathrm{op}_t(\mathfrak{a}^-)(\zeta) + \mathrm{op}_t(\mathfrak{a}_0)(\zeta) + \mathrm{op}_t(\mathfrak{a}^+)(\zeta). \tag{86}$$

Remark 3.3 *Let* $\tilde{\chi}_j^{\pm}, \tilde{\varphi}_j$, $j = 1, 2$, $\tilde{\omega}$ *be another choice of functions with the above mentioned properties and write* $\tilde{\mathbf{op}}_t(\mathfrak{a})(\zeta)$ *in analogy with* (86) *when we use the functions with tilde. Then for every fixed* $\zeta \neq 0$ *the operators* $\tilde{\mathbf{op}}_t(\mathfrak{a})(\zeta) - \mathbf{op}_t(\mathfrak{a})(\zeta)$ *have an integral kernel in* $\mathcal{S}(\mathbb{R}_t \times \mathbb{R}_{t'}, \mathcal{Y}^{-\infty}(\mathbb{W}))$.

Theorem 3.4 *Let* $\mathfrak{a}(t, \tau, \zeta) = \tilde{\mathfrak{a}}(t, t\tau, t\zeta)$ *for* $\tilde{\mathfrak{a}}(t, \tilde{\tau}, \tilde{\zeta}) \in C^\infty(\mathbb{R}, \mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^{1+q}_{\tilde{\tau},\tilde{\zeta}}))$ *be elliptic* (*in the sense of Definition* 3.1). *Then for every* $\zeta \neq 0$ *the operator*

$$\mathbf{op}_t([t]^{-\mu}\mathfrak{a})(\zeta) : \begin{array}{c} \mathcal{W}^{s,\gamma;\delta}(\mathbb{W}_{\asymp}) \\ \oplus \\ \mathcal{W}^{s-\frac{n+1}{2},\gamma-\frac{n+1}{2};\delta}(Y_{\asymp}, J_-) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu;\delta}(\mathbb{W}_{\asymp}) \\ \oplus \\ \mathcal{W}^{s-\frac{n+1}{2}-\mu,\gamma-\frac{n+1}{2}-\mu;\delta}(Y_{\asymp}, J_+) \end{array} \tag{87}$$

is Fredholm for every $s, \delta \in \mathbb{R}$. *Furthermore, there exists a* $\tilde{\mathfrak{p}}(t, \tilde{\tau}, \tilde{\zeta}) \in C^\infty(\mathbb{R}, \mathcal{Y}^{-\mu}(\mathbb{W}; \mathbb{R}^{1+q}))$ *such that for* $\mathfrak{p}(t, \tau, \zeta) := \tilde{\mathfrak{p}}(t, t\tau, t\zeta)$ *for every fixed* $\zeta \neq 0$ *both*

$$1 - \mathbf{op}_t([t]^{-\mu}\mathfrak{a})(\zeta)\mathbf{op}_t([t]^{\mu}\mathfrak{p})(\zeta), \quad \text{and} \quad 1 - \mathbf{op}_t([t]^{\mu}\mathfrak{p})(\zeta)\mathbf{op}_t([t]^{-\mu}\mathfrak{a})(\zeta)$$

belong to $\mathcal{S}(\mathbb{R} \times \mathbb{R}, \mathcal{Y}^{-\infty}(\mathbb{W}))$, *i.e.,* $\mathbf{op}_t([t]^{\mu}\mathfrak{p})(\zeta)$ *is a parametrix of* $\mathbf{op}_t([t]^{-\mu}\mathfrak{a})(\zeta)$.

Proof. We cover the space $\mathbb{W}_{\asymp}$ by finitely many open sets of the form $\mathbb{W}_{0,\asymp}$ and $\mathbb{W}_{j,\asymp}$, $j = 1, \ldots, N$, for a submanifold $\mathbb{W}_0 \subset \mathbb{W}_{\text{reg}}$ and sets $\mathbb{W}_j \subset \mathbb{W}$ such that $\mathbb{W}_j \cap \partial\mathbb{W} \neq \emptyset$, where $\mathbb{W}_j$ in the splitting of variables (r, x, y) is of the form $[0, 1) \times X \times Y_j$, for coordinate neighbourhoods Y_j on Y diffeomorphic to the open unit ball $B \subset \mathbb{R}^q$. To show the Fredholm property of (87), it suffices to construct local parametrices of the operators $\mathbf{op}_t([t]^{-\mu}\mathfrak{a}(\zeta))\big|_{\mathbb{W}_{j,\asymp}}$ for all $j = 0, \ldots, N$, such that the remainders are compact operators (after the globalisation process, with a partition of unity, etc.).

The local parametrix construction for $j = 0$ corresponds to the case of manifolds with conical exits to infinity and smooth cross section $\mathbb{W}_0$, where our operators come from edge-degenerate families with smooth model cone, axial variable t and edge covariable $\zeta \neq 0$. This case is known and treated in [11, Chapter 3]. Thus it remains to consider the localised operators over $\mathbb{W}_{j,\asymp}$ for $j > 0$. These can be expressed in the splitting of variables

$$(t, r, x, y) \in \mathbb{R} \times [0, 1) \times X \times B.$$

Because $t \to +\infty$ and $t \to -\infty$ are similar, we may content ourselves with the case $t \to +\infty$. Since local parametrices over finite intervals in t easily yield the desired contributions, it remains to study the localisations over $(T, \infty) \times [0, 1) \times X \times B$ for some $T > 0$. This corresponds exactly to the local situation that was studied in Chapter 2.

□

3.2 Edge symbols of second generation

Let us consider the space of all operator families

$$\tilde{\mathfrak{a}}(t, z, \tilde{\tau}, \tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \ \mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^{1+p}_{\tilde{\tau},\tilde{\zeta}})), \tag{88}$$

where the values for every $(t, z, \tilde{\tau}, \tilde{\zeta})$ as operators in $\mathcal{Y}^\mu(\mathbb{W})$ are connected with chosen weight and bundle data

$$\boldsymbol{g} := (\gamma, \gamma - \mu) \quad \text{and} \quad \boldsymbol{w} := (J_-, J_+), \tag{89}$$

respectively. Let

$$\mathcal{Y}^\mu(\mathbb{W}; \mathbb{C} \times \mathbb{R}^p_{\tilde{\zeta}})$$

denote the space of all operator functions $\tilde{\mathfrak{h}}(w, \tilde{\zeta}) \in \mathcal{A}(\mathbb{C}, \mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^p_{\tilde{\zeta}}))$ such that $\tilde{\mathfrak{h}}(\alpha + i\tau, \tilde{\zeta}) \in \mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^{1+p}_{\tau, \tilde{\zeta}})$ for every $\alpha \in \mathbb{R}$, uniformly in compact α-intervals.

Given elements (88) and

$$\tilde{\mathfrak{h}}(t, z, w, \tilde{\zeta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Xi, \mathcal{Y}^\mu(\mathbb{W}; \mathbb{C} \times \mathbb{R}^p_{\tilde{\zeta}})), \tag{90}$$

we set

$$\mathfrak{a}(t, z, \tau, \zeta) := \tilde{\mathfrak{a}}(t, z, t\tau, t\zeta), \quad \mathfrak{a}_0(t, z, \tau, \zeta) := \tilde{\mathfrak{a}}(0, z, t\tau, t\zeta), \tag{91}$$

and

$$\mathfrak{h}(t, z, w, \zeta) := \tilde{\mathfrak{h}}(t, z, w, t\zeta), \quad \mathfrak{h}_0(t, z, w, \zeta) := \tilde{\mathfrak{h}}(0, z, w, t\zeta). \tag{92}$$

With $\mathfrak{a}(t, z, \tau, \zeta)$ and $\mathfrak{h}(t, z, w, \zeta)$ we can associate families of operators

$$\mathrm{op}_M^\delta(\mathfrak{h})(z, \zeta),\ \mathrm{op}_t(\mathfrak{a})(z, \zeta) : \begin{array}{c} C_0^\infty(\mathbb{R}_+ \times \mathrm{int}\, \mathbb{W}) \\ \oplus \\ C_0^\infty(Y^\wedge, J_-) \end{array} \longrightarrow \begin{array}{c} C^\infty(\mathbb{R}_+ \times \mathrm{int}\, \mathbb{W}) \\ \oplus \\ C^\infty(Y^\wedge, J_+) \end{array}$$

which z-wise belong to the parameter-dependent edge calculus of the class $\mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^p_\zeta)$ on the (stretched) manifold $\mathbb{R}_+ \times \mathbb{W} = \mathbb{W}^\wedge$ with the non-compact edge $\mathbb{R}_+ \times Y = Y^\wedge$; here the bundles over $Y^\wedge$ are tacitly identified with those over Y by pull back with respect to the projection $Y^\wedge \to Y$.

Theorem 3.5 *For every operator of the form* (88)*, there exists an element* (90) *such that*

$$\mathrm{op}_M^\delta(\mathfrak{h})(z, \zeta) = \mathrm{op}_t(\mathfrak{a})(z, \zeta) \bmod C^\infty(\Xi, \mathcal{Y}^{-\infty}(\mathbb{W}^\wedge; \mathbb{R}^p_\zeta))$$

for every $\delta \in \mathbb{R}$.

Remark 3.6 *If p and h are as in Theorem* 3.5*, we also have*

$$\mathrm{op}_M^\delta(\mathfrak{h}_0)(z, \zeta) = \mathrm{op}_t(\mathfrak{a}_0)(z, \zeta) \bmod C^\infty(\Xi, \mathcal{Y}^{-\infty}(\mathbb{W}^\wedge; \mathbb{R}^p_\zeta))$$

for every $\delta \in \mathbb{R}$.

We now choose cut-off functions $\omega_1, \omega_2, \omega_3$ on $\overline{\mathbb{R}}_+$ satisfying the condition (16), and set $\chi_1(t) := 1 - \omega_1(t), \chi_2(t) := 1 - \omega_3(t)$. Moreover, we fix cut-off functions $\sigma(t)$ and $\tilde{\sigma}(t)$. Let us form

$$a(z,\zeta) := \sigma(t)t^{-\mu}\Big\{\omega_1(t[\zeta])\operatorname{op}_M^{\theta-\frac{\dim \mathbb{W}}{2}}(\mathfrak{h})(z,\zeta)\omega_2(t'[\zeta]) \\ + \chi_1(t[\zeta])\omega(t[\zeta],t'[\zeta])\operatorname{op}_t(\mathfrak{a})(z,\zeta)\chi_2(t'[\zeta])\Big\}\tilde{\sigma}(t') \tag{93}$$

and

$$\sigma_{\wedge}(a)(z,\zeta) = t^{-\mu}\Big\{\omega_1(t|\zeta|)\operatorname{op}_M^{\theta-\frac{\dim \mathbb{W}}{2}}(\mathfrak{h}_0)(z,\zeta)\omega_2(t'|\zeta|) \\ + \chi_1(t|\zeta|)\omega(t|\zeta|,t'|\zeta|)\operatorname{op}_t(\mathfrak{a}_0)(z,\zeta)\chi_2(t'|\zeta|)\Big\}. \tag{94}$$

Let us set

$$E := \begin{array}{c}\mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge)\\ \oplus \\ \mathcal{K}^{s-\frac{n+1}{2},\theta-\frac{n+1}{2}}(Y^\wedge,J_-)\end{array}, \quad \widetilde{E} := \begin{array}{c}\mathcal{K}^{s-\mu,\boldsymbol{\gamma}-\mu}(\mathbb{W}^\wedge)\\ \oplus \\ \mathcal{K}^{s-\frac{n+1}{2}-\mu,\theta-\frac{n+1}{2}-\mu}(Y^\wedge,J_+)\end{array}, \tag{95}$$

with $\boldsymbol{\gamma} = (\gamma,\theta)$ and for every $s \in \mathbb{R}$; the group action in these spaces is defined by

$$\kappa_\lambda := \operatorname{diag}(\kappa_\lambda^{\mathbb{W}},\ \kappa_\lambda^{Y})$$

for $(\kappa_\lambda^{\mathbb{W}}u)(t,w) := \lambda^{\frac{1+\dim \mathbb{W}}{2}}u(\lambda t,w)$, $(\kappa_\lambda^{Y}v)(t,y) := \lambda^{\frac{1+\dim Y}{2}}v(\lambda t,y)$, $\lambda \in \mathbb{R}_+$, with u and v being in the respective spaces.

Remark 3.7 *The operators* (94) *define a family of continuous operators*

$$\sigma_{\wedge}(a)(z,\zeta) : E \to \widetilde{E}$$

between the spaces (95)*, for all $s \in \mathbb{R}$, $(z,\zeta) \in T^*\Xi \setminus 0$, and we have*

$$\sigma_{\wedge}(a)(z,\lambda\zeta) = \lambda^{\mu}\kappa_\lambda\sigma_{\wedge}(a)(z,\zeta)\kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+$, $(z,\zeta) \in T^*\Xi \setminus 0$.

Theorem 3.8 *We have*

$$a(z,\zeta) \in S^{\mu}(\Xi \times \mathbb{R}^p; E, \widetilde{E}), \tag{96}$$

for the spaces $E, \widetilde{E}$ given by (95) *for every $s \in \mathbb{R}$.*

Proof. The dependence of $a(z,\zeta)$ on z is not the essential difficulty; so we consider the z-independent case. Set $\delta := \theta - \frac{\dim \mathbb{W}}{2}$ and write $a(\zeta) = a_0(\zeta) + a_1(\zeta)$ for

$$a_0(\zeta) := \sigma(t)t^{-\mu}\omega_1(t[\zeta])\operatorname{op}_M^{\delta}(\mathfrak{h})(\zeta)\omega_2(t'[\zeta])\tilde{\sigma}(t'),$$
$$a_1(\zeta) := \sigma(t)t^{-\mu}\chi_1(t[\zeta])\omega(t[\zeta],t'[\zeta])\operatorname{op}_t(\mathfrak{a})(\zeta)\chi_2(t'[\zeta])\tilde{\sigma}(t').$$

We show that $a_i(\zeta) \in S^\mu(\mathbb{R}^p; E, \widetilde{E})$ for $i = 0, 1$. Let us first consider $i = 1$. Moreover, for convenience, we concentrate on the upper left corner of the block matrices; then $E = \mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge)$, $\widetilde{E} = \mathcal{K}^{s-\mu,\boldsymbol{\gamma}-\mu}(\mathbb{W}^\wedge)$. We verify that

$$a_1(\zeta) \in C^\infty(\mathbb{R}^p_\zeta; \mathcal{L}(E, \widetilde{E})). \tag{97}$$

In fact, let $\mathfrak{a}_1(t, t', \tau, \zeta) := \omega(t[\zeta], t'[\zeta])\mathfrak{a}(t, \tau, \zeta)$. Then we know that

$$\mathrm{op}_t(\mathfrak{a}_1)(\zeta) : \mathcal{W}^{s,\gamma}_{\mathrm{comp}}(\mathbb{W}^\wedge) \to \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathrm{loc}}(\mathbb{W}^\wedge)$$

is a C^∞ family of continuous operators, cf. the formulas (8) and (9). Moreover, the operators of multiplication by C_0^∞ functions in t (or t') $\in \mathbb{R}_+$ define continuous operators

$$\begin{aligned} &\tilde{\sigma}(t')\chi_2(t'[\zeta]) : \mathcal{K}^{s,\boldsymbol{\gamma}}(\mathbb{W}^\wedge) \to \mathcal{W}^{s,\gamma}_{\mathrm{comp}}(\mathbb{W}^\wedge), \\ &\sigma(t)t^\mu\chi_1(t[\zeta]) : \mathcal{W}^{s-\mu,\gamma-\mu}_{\mathrm{loc}}(\mathbb{W}^\wedge) \to \mathcal{K}^{s-\mu,\boldsymbol{\gamma}-\mu}(\mathbb{W}^\wedge) \end{aligned}$$

that smoothly depend on ζ. This holds for all $s, \gamma, \theta \in \mathbb{R}$, i.e., we obtain altogether the relation (97). Next we choose an excision function $\chi(\zeta)$ and write $a_1(\zeta) = \chi(\zeta)a_1(\zeta) + (1 - \chi(\zeta))a_1(\zeta)$. Then, because of $(1 - \chi(\zeta))a_1(\zeta) \in C_0^\infty(\mathbb{R}^p_\zeta, \mathcal{L}(E, \widetilde{E}))$, cf. the relation (97), we get

$$(1 - \chi(\zeta))a_1(\zeta) \in S^{-\infty}(\mathbb{R}^p_\zeta; E, \widetilde{E}).$$

Thus it remains to characterise $\chi(\zeta)a_1(\zeta)$. Setting

$$b_1(\zeta) := \chi(\zeta)t^{-\mu}\chi_1(t[\zeta])\omega(t[\zeta], t'[\zeta])\,\mathrm{op}_t(\mathfrak{a})(\zeta)\chi_2(t'[\zeta])$$

we have $\chi(\zeta)a_1(\zeta) = \sigma(t)b_1(\zeta)\tilde{\sigma}(t')$. Using Lemma 2.4 it suffices to consider $b_1(\zeta)$. Let us assume for the moment that $\tilde{\mathfrak{a}}(\tilde{\tau}, \tilde{\zeta})$ is independent of t. From the results of Section 3.4 below we know that $b_1(\zeta) \in C^\infty(\mathbb{R}^p_\zeta, \mathcal{L}(E, \widetilde{E}))$. In addition we have $b_1(\zeta) \to 0$ in that space when $\tilde{\mathfrak{a}}(\tilde{\tau}, \tilde{\zeta}) \to 0$ in the space $\mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^{1+p}_{\tilde{\tau},\tilde{\zeta}})$. Moreover, there is a $C > 0$ such that $b_1(\lambda\zeta) = \lambda^\mu\kappa^{\mathbb{W}}_\lambda b_1(\zeta)(\kappa^{\mathbb{W}}_\lambda)^{-1}$ for all $\lambda \geq 1$, $|\zeta| \geq C$. This yields $b_1(\zeta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^p; E, \widetilde{E})$, and hence, $a_1(\zeta) \in S^\mu(\mathbb{R}^p; E, \widetilde{E})$. Moreover, $\mathfrak{a}(\tilde{\tau}, \tilde{\zeta}) \to 0$ in $\mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^{1+p}_{\tilde{\tau},\tilde{\zeta}})$ entails $a_1(\zeta) \to 0$ in $S^\mu(\mathbb{R}^p; E, \widetilde{E})$.

Let us now consider the general case. Because of the factor $\sigma(t)$ from the left, we may assume that $\tilde{\mathfrak{a}}(t, \tilde{\tau}, \tilde{\zeta})$ vanishes for $t > c$ for some $c > 0$. Let $C^\infty([0, c]_0)$ denote the Fréchet subspace of elements of $C^\infty(\overline{\mathbb{R}}_+)$ that vanish for $t \geq c$. Then $\tilde{\mathfrak{a}}(t, \tilde{\tau}, \tilde{\varrho})$ can be written as a convergent sum $\tilde{\mathfrak{a}}(t, \tilde{\tau}, \tilde{\zeta}) = \sum_{j=0}^\infty \lambda_j\varphi_j(t)\tilde{\mathfrak{a}}_j(\tilde{\tau}, \tilde{\zeta})$ with $\lambda_j \in \mathbb{C}$, $\sum |\lambda_j| < \infty$ and $\varphi_j \in C^\infty([0, c]_0)$, $\tilde{\mathfrak{a}}_j(\tilde{\tau}, \tilde{\zeta}) \in \mathcal{Y}^\mu(\mathbb{W}; \mathbb{R}^{1+p}_{\tilde{\tau},\tilde{\zeta}})$ tending to zero in the respective spaces for $j \to \infty$.

It follows that

$$b_1(\zeta) = \sum_{j=0}^\infty \lambda_j\varphi_k p_j \tag{98}$$

for $p_j := \chi(\zeta)t^{-\mu}\chi_1(t[\zeta])\omega(t[\zeta], t'[\zeta]) \operatorname{op}_t(\mathfrak{a}_j)(\zeta)\chi_2(t'[\zeta])$, $\mathfrak{a}_j(t, \tau, \zeta) := \tilde{\mathfrak{a}}_j(t\tau, t\zeta)$. As we saw, $p_j \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^p; E, \widetilde{E})$ tends to zero in the symbol space; moreover, by Lemma 2.4 the operators of multiplication by φ_j tend to zero in $S^0(\mathbb{R}^p; \widetilde{E}, \widetilde{E})$. This gives us immediately the convergence of (98) in $S^{\mu}(\mathbb{R}^p; E, E)$.

In order to treat $a_0(\zeta)$ we first observe that

$$a_0(\zeta) \in C^{\infty}(\mathbb{R}^p_{\zeta}, \mathcal{L}(E, \widetilde{E})). \tag{99}$$

The pointwise continuity of the operator $a_0(\zeta) : E \to \widetilde{E}$ for every $s, \gamma, \theta \in \mathbb{R}$ is known from [18, Theorem 3.3.6]. In this case, because of the cut-off factors ω_1 and ω_2, there is no exit effect for $t \to \infty$. The smoothness in ζ is an easy consequence of the continuity of the correspondence between (operator-valued) Mellin symbols and associated operators and of the C^{∞} dependence of the Mellin symbols on ζ. From such arguments we also know that when $\mathfrak{h}(t, w, \tilde{\zeta}) \in C^{\infty}(\overline{\mathbb{R}}_+, \mathcal{Y}^{\mu}(\mathbb{W}; \mathbb{C} \times \mathbb{R}^p_{\tilde{\zeta}}))$ tends to zero in that space, also the associated C^{∞} function of operators (99) tends to zero. Let us now assume that $\tilde{\mathfrak{h}}$ is independent of t. Then we have $a_0(\lambda\zeta) = \lambda^{\mu}\kappa^{\mathbb{W}}_{\lambda} a_0(\zeta)(\kappa^{\mathbb{W}}_{\lambda})^{-1}$ for all $\lambda \geq 1$, $|\zeta| \geq c$ for some $c > 0$. This gives us immediately $a_0(\zeta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^p; E, \widetilde{E})$. In addition we also see that $a_0(\zeta)$ tends to zero in the symbol space as soon as $\widetilde{\mathfrak{h}}(w, \tilde{\zeta})$ tends to zero in $\mathcal{Y}^{\mu}(\mathbb{W}; \mathbb{C} \times \mathbb{R}^p_{\tilde{\zeta}})$. For the case of t-dependent $\tilde{\mathfrak{h}}$ we can apply a tensor product argument of a similar structure as before and then finally obtain $a_0(\zeta) \in S^{\mu}(\mathbb{R}; E, \tilde{E})$. □

3.3 The Fredholm property of edge symbols

Theorem 3.9 *Let $\mathfrak{a}(z, \tilde{\tau}, \tilde{\zeta}) \in C^{\infty}(\Xi, \mathcal{Y}^{\mu}(\mathbb{W}; \mathbb{R}^{1+p}_{\tilde{\tau},\tilde{\zeta}}))$ be an operator family connected with* (89)*, and assume that $\mathfrak{a}$ is parameter-dependent elliptic with the parameters $(\tilde{\tau}, \tilde{\zeta}) \in \mathbb{R}^{1+p}$, for every fixed $z \in \Xi$. Then there exists a discrete set $D(z) \subset \mathbb{R}$ such that the family of operators* (94)

$$\sigma_{\wedge}(a)(z, \zeta) : E \to \widetilde{E}$$

is a Fredholm operator for every $\theta \in \mathbb{R} \setminus D(z)$ and every $z \in \Xi$, $\zeta \in \mathbb{R}^p \setminus \{0\}$, $s \in \mathbb{R}$.

Proof. For convenience, we first consider the case when there are no extra edge conditions, i.e., the bundles $J_{\pm}$ are both of fibre dimension zero. For every fixed $(z, \zeta), \zeta \neq 0$, the operator $\sigma_{\wedge}(a)(z, \zeta)$ is elliptic in the edge calculus on the non-compact (stretched) manifold $\mathbb{R}_+ \times \mathbb{W}$ with edge $\mathbb{R}_+ \times Y$. As such there exists a parametrix in this calculus. In order to obtain the Fredholm property in our spaces, we need a control of the parametrix for $t \to 0$ and $t \to \infty$, such that the corresponding local remainders are compact. For $t \to 0$ this problem is discussed in [18] in a corresponding corner calculus with base $\mathbb{W}$. In this theory the Fredholm property requires the bijectivity of the corner conormal symbol

$$\mathfrak{h}(0, z, w, 0) : \mathcal{W}^{s,\gamma}(\mathbb{W}) \to \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}) \tag{100}$$

for all $w \in \Gamma_{\frac{\dim \mathbb{W}+1}{2}-\theta}$, cf. [18, Definition 3.4.1]. As is known, the operators (100) are bijective for every $w \in D_1(z)$ for a certain discrete set $D_1(z) \subset \mathbb{C}$ which intersects every finite strip $c \leq \operatorname{Re} w \leq c'$ in a finite set. Then we have $D(z) = \{\theta \in \mathbb{R} : \Gamma_{\frac{\dim \mathbb{W}+1}{2}-\theta} \cap D_1(z) = \emptyset\}$.

To finish the proof we have to construct a local parametrix for $t \to \infty$. However, this is contained in Theorem 3.4 as an information on $\mathbb{R} \times \mathbb{W} \cong \mathbb{W}_{\asymp}$ locally for $t \to \infty$. The case with additional edge conditions of trace and potential type along the edge is completely analogous; the local parametrices in $\mathbb{R}_+ \times \mathbb{W}$ as well as those for $t \to 0$ and $t \to \infty$ are given by [18] and Theorem 3.4, respectively. □

3.4 The exit behaviour of corner symbols

Let us consider $b(t, t', y, y', \tau, \eta, \zeta) := (\beta_{1*}a)(t, t', y, y', \tau, \eta, \zeta)$, cf. the formula (84), and compute the symbol $c(t, t', \tilde{y}, \tilde{y}', \tau, \tilde{\eta}, \zeta) := (\beta_{2*}b)(t, t', \tilde{y}, \tilde{y}', \tau, \tilde{\eta}, \zeta)$ which will have the property

$$\beta_{2*} \operatorname{Op}_{t,y}(b)(\zeta) = \operatorname{Op}_{t,\tilde{y}}(\beta_{2*}b)(\zeta)$$

(without any remainders). A straightforward computation gives us

$$\begin{aligned}(\beta_{2*} \operatorname{Op}_{t,y}(b))v(t, \tilde{y}) &= (\beta_2^*)^{-1}[\operatorname{Op}_{t,y}(b)((\beta_2^*)v)(t, y)] \\ &= \iint e^{i(t-t')\tau + i(\tilde{y}-\tilde{y}')\tilde{\eta}} b(t, t', \frac{\tilde{y}}{t}, \frac{\tilde{y}'}{t}, \tau, t\tilde{\eta}, \zeta) v(t', \tilde{y}') dt' d\tilde{y}' đ\tau đ\tilde{\eta}\end{aligned}$$

for any $v(t, \tilde{y}) \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^q, E)$, such that $c(t, t', \tilde{y}, \tilde{y}', \tau, \tilde{\eta}, \zeta) = b(t, t', \frac{\tilde{y}}{t}, \frac{\tilde{y}'}{t}, \tau, t\tilde{\eta}, \zeta)$. In order to show Theorem 2.14, we consider the summands from the representation (65) separately. We have $c(t, t', \tilde{y}, \tilde{y}', \tau, \tilde{\eta}, \zeta) = \beta_{2*}(b_M + b_\psi + m_1 + g_1)(t, t', \tilde{y}, \tilde{y}', \tau, \tilde{\eta}, \zeta)$ for

$$b_M = \beta_{1*}a_M, \quad b_\psi = \beta_{1*}a_\psi, \quad m_1 = \beta_{1*}m, \quad g_1 = \beta_{1*}g. \tag{101}$$

We then set

$$c_M := \beta_{2*}b_M, \quad c_\psi := \beta_{2*}b_\psi, \quad m_2 := \beta_{2*}m_1, \quad g_2 := \beta_{2*}g_1. \tag{102}$$

Let us now compute the symbols (101). By definition we have

$$\begin{aligned}(\beta_{1*}a_\psi)(t, t', y, y', \tau, \eta, \zeta)u(\tilde{r}) &= (\beta_1^*)^{-1}(\operatorname{op}_r(\boldsymbol{a}_\psi)(t, t', y, y', \tau, \eta, \zeta)(\beta_1^* u)(r)) \quad (103)\\ &= \operatorname{op}_{\tilde{r}}(\boldsymbol{b}_\psi)(t, t', y, y', \tau, \eta, \zeta)u(\tilde{r})\end{aligned}$$

for

$$\begin{aligned}&\boldsymbol{a}_\psi(r, r', t, t', y, y', \varrho, \tau, \eta, \zeta) \\ &\quad := \sigma(r)r^{-\mu}t^{-\mu}\chi_1(r[t\tau, \eta, t\zeta])\widetilde{P}(r, t, t', y, y', r\varrho, rt\tau, r\eta, rt\zeta)\chi_2(r'[t\tau, \eta, t\zeta])\tilde{\sigma}(r'),\end{aligned}$$

cf. the formulas (63) and (67). This gives us

$$b_\psi(t, t', y, y', \tau, \eta, \zeta) = \operatorname{op}_{\tilde{r}}(\boldsymbol{b}_\psi)(t, t', y, y', \tau, \eta, \zeta) \tag{104}$$

for

$$\boldsymbol{b}_\psi(\tilde{r},\tilde{r}',t,t',y,y',\varrho,\tau,\eta,\zeta)$$
$$:= \tilde{r}^{-\mu}\sigma\big(\frac{\tilde{r}}{t}\big)\chi_1\big(\frac{\tilde{r}}{t}[t\tau,\eta,t\zeta]\big)\widetilde{P}\big(\frac{\tilde{r}}{t},t,t',y,y',\tilde{r}\varrho,\tilde{r}\tau,\frac{\tilde{r}}{t}\eta,\tilde{r}\zeta\big)\chi_2\big(\frac{\tilde{r}'}{t}[t\tau,\eta,t\zeta]\big)\tilde{\sigma}\big(\frac{\tilde{r}'}{t}\big).$$

Analogously, we obtain

$$\begin{aligned} b_M(t,t',y,y',\tau,\eta,\zeta)u(\tilde{r}) &= (\beta_{1*}a_M)(t,t',y,y',\tau,\eta,\zeta)u(\tilde{r}) \\ &= (\beta_1^*)^{-1}(\mathrm{op}_M^{\gamma-\frac{n}{2}}(\boldsymbol{a}_M))(t,t',y,y',\tau,\eta,\zeta)(\beta_1^* u)(r) \\ &= \mathrm{op}_M^{\gamma-\frac{n}{2}}(\boldsymbol{b}_M)u(\tilde{r}) \end{aligned} \tag{105}$$

for

$$\boldsymbol{a}_M(r,r',t,t',y,y',v,\tau,\eta,\zeta)$$
$$:= \sigma(r)r^{-\mu}t^{-\mu}\omega_1(r[t\tau,\eta,t\zeta])\widetilde{H}(r,t,t',y,y',v,\tau,\eta,\zeta)\omega_2(r'[t\tau,\eta,t\zeta])\tilde{\sigma}(r'),$$

cf. the formulas (66) and (69), and

$$\boldsymbol{b}_M(\tilde{r},\tilde{r}',t,t',y,y',v,\tau,\eta,\zeta)$$
$$:= \tilde{r}^{-\mu}\sigma\big(\frac{\tilde{r}}{t}\big)\omega_1\big(\frac{\tilde{r}}{t}[t\tau,\eta,t\zeta]\big)\widetilde{H}(r,t,t',y,y',v,\tilde{r}\tau,\frac{\tilde{r}}{t}\eta,\tilde{r}\zeta)\omega_2\big(\frac{\tilde{r}'}{t}[t\tau,\eta,t\zeta]\big)\tilde{\sigma}\big(\frac{\tilde{r}'}{t}\big).$$

Moreover, we have

$$\begin{aligned} m_1(t,t',y,y',\tau,\eta,\zeta)u(\tilde{r}) &= (\beta_{1*}m)(t,t',y,y',\tau,\eta,\zeta)u(\tilde{r}) \\ &= (\beta_1^*)^{-1}\,\mathrm{op}_{M,r}^{\gamma-\frac{n}{2}}(m)(t,t',y,y',\tau,\eta,\zeta)(\beta_{1*}u)(r) \\ &= \mathrm{op}_{M,\tilde{r}}^{\gamma-\frac{n}{2}}(\boldsymbol{m}_1)u(\tilde{r}) \end{aligned} \tag{106}$$

for

$$m(r,r',t,t',y,y',\tau,\eta,\zeta) = r^{-\mu}t^{-\mu}\omega_1(r[t\tau,\eta,t\zeta])F(t,t',y,y',v)\omega_2(r'[t\tau,\eta,t\zeta])$$

and

$$\boldsymbol{m}_1(\tilde{r},\tilde{r}',t,t',y,y',v,\tau,\eta,\zeta) = \tilde{r}^{-\mu}\omega_1\big(\frac{\tilde{r}}{t}[t\tau,\eta,t\zeta]\big)F(t,t',y,y',v)\omega_2\big(\frac{\tilde{r}'}{t}[t\tau,\eta,t\zeta]\big).$$

Finally, we consider the push forward of the Green operator family $g(t,t',y,y',\tau,\eta,\zeta)$ under β_1. We employ the following characterisation by kernels.

Proposition 3.10 *The conditions* (73) *and* (74) *are equivalent to the existence of a function*

$$K(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta};r,r') \in S_{\mathrm{cl}}^{\mu+1}(\mathbb{R}^{1+q}_{t,y}\times\mathbb{R}^{1+q}_{t',y'}\times\mathbb{R}^{1+q+p}_{\tilde{\tau},\eta,\tilde{\zeta}})\hat{\otimes}_\pi\mathcal{S}^{\gamma-\mu}(X^\wedge_{r,x})_\varepsilon\hat{\otimes}_\pi\mathcal{S}^{-\gamma}(X^\wedge_{r',x'})_\varepsilon,$$

such that

$$G(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta})v(r) = \int_0^\infty K(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta};r[\tilde{\tau},\eta,\tilde{\zeta}],r'[\tilde{\tau},\eta,\tilde{\zeta}])v(r')\mathrm{d}r' \quad (107)$$

for all $v \in \mathcal{K}^{s,\gamma}(X^\wedge_{r',x'})$.

A pointwise kernel characterisation of such a type for Green operators in a cone is obtained by Seiler [24]; this gives rise to a corresponding characterisation of corresponding edge Green symbols on the lines of [22]. In the relation (107) we suppressed the dependence of functions on (x,x') or x', both in K and in the argument functions $v \in \mathcal{K}^{s,\gamma}(X^\wedge_{r',x'})$; clearly, in formulas of the kind (107) we have to integrate over $x' \in X$. We hope that our shorter notation will not cause confusion.

By definition we have the relation (72), i.e.,

$$g(t,t',y,y',\tau,\eta,\zeta)v(r) = t^{-\mu}\int_0^\infty K(t,t',y,y',t\tau,\eta,t\zeta;r[t\tau,\eta,t\zeta],r'[t\tau,\eta,t\zeta])v(r')\mathrm{d}r'$$

which gives us

$$\begin{aligned} g_1(t,t',y,y',\tau,\eta,\zeta)u(\tilde{r}) &= (\beta_{1*}g)(t,t',y,y',\tau,\eta,\zeta)u(\tilde{r}) \\ &= (\beta_1^*)^{-1}g(t,t',y,y',\tau,\eta,\zeta)(\beta_1^*u)(r) \\ &= t^{-(\mu+1)}\int_0^\infty K(t,t',y,y',t\tau,\eta,t\zeta;\frac{\tilde{r}}{t}[t\tau,\eta,t\zeta],\frac{\tilde{r}'}{t}[t\tau,\eta,t\zeta])u(t\tilde{r}')\mathrm{d}\tilde{r}'. \end{aligned} \quad (108)$$

Proof of Theorem 2.14. Using the form (65) of the symbol $a = a_M + a_\psi + m + g$ in the variables $(t,t',y,y',\tau,\eta,\zeta)$, we have to compute

$$b = \beta_{1*}(a) = b_M + b_\psi + m_1 + g_1$$

in the variables $(t,t',y,y',\tau,\eta,\zeta)$, cf. the expressions (103), (104), (105), (106) and (108).

We first observe that in our calculations the choice of the cut-off functions $\omega_1(r)$, $\omega_2(r)$ and $\chi_1(r)$, $\chi_2(r)$ is not essential. Since $\zeta \neq 0$ is fixed, we may assume $[t\tau,t\tilde{\eta},t\zeta] = |t\tau,t\tilde{\eta},t\zeta|$ for all $t \geq T$. Thus, if $\omega(r)$ is any cut-off function and $\chi(r) = 1-\omega(r)$, we have

$$\chi(\frac{\tilde{r}}{t}[t\tau,\eta,t\zeta]) = \chi(\frac{\tilde{r}}{t}[t\tau,t\tilde{\eta},t\zeta]) = \chi(\tilde{r}|\tau,\tilde{\eta},\zeta|) \quad (109)$$

for $t \geq T$, $\tilde{\eta} := \frac{\eta}{t}$, and, similarly, for ω. Since it is known from [6] (cf. also [14] for the case of boundary value problems) that the operator functions (102) are elements of

$$S^\mu(\mathbb{R}^{1+q}_{t,\tilde{y}} \times \mathbb{R}^{1+q}_{t',\tilde{y}'} \times \mathbb{R}^{1+q}; E, \widetilde{E}),$$

then we may concentrate on the exit effects for large $|(t,\tilde{y})|$ and $|(t',\tilde{y}')|$.

Using (109), it follows that

$$\begin{aligned}\omega(t,t')c_\psi(t,t',\tilde{y},\tilde{y}',\tau,\tilde{\eta},\zeta) &= \omega(t,t')b_\psi\big(t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t},\tau,t\tilde{\eta},\zeta\big)\\ &= \mathrm{op}_{\tilde{r}}\big\{\tilde{r}^{-\mu}\sigma\big(\frac{\tilde{r}}{t}\big)\chi_1(\tilde{r}|\tau,\tilde{\eta},\zeta|)\omega(t,t')\\ &\quad \widetilde{P}\big(\frac{\tilde{r}}{t},t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t},\tilde{r}\varrho,\tilde{r}\tau,\tilde{r}\tilde{\eta},\tilde{r}\zeta\big)\chi_2(\tilde{r}'|\tau,\tilde{\eta},\zeta|)\tilde{\sigma}\big(\frac{\tilde{r}'}{t}\big)\big\}\end{aligned}$$

is equal to $\omega(t,t')c_\psi$ modulo a term with compact support in t which is of simpler structure and can easily be identified as an element of $S^{\mu;-\infty,0}(E,\widetilde{E})$, cf. the condition (80).

We now apply a modification of the method of [14] in order to prove that our operator functions are operator-valued symbols of the desired classes. Our case is even simpler than the one in [14], because we do not consider boundary value problems but operators on a closed compact C^∞ manifold X. On the other hand, in the present case we have to observe the extra exit properties with respect to $|t,y|\to\infty$.

The function (64) can be written as a convergent sum

$$\widetilde{P}(r,t,t',y,y',\tilde{\varrho},\tilde{\tau},\tilde{\eta},\tilde{\zeta}) = \sum_{j=0}^{\infty}\lambda_j\varphi_j(r,t,t',y,y')P_j(\tilde{\varrho},\tilde{\tau},\tilde{\eta},\tilde{\zeta}) \tag{110}$$

with $\lambda_j\in\mathbb{C}$, $\sum_{j=0}^{\infty}|\lambda_j|<\infty$, and $\varphi_j\in C^\infty(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times B\times B)$, $P_j\in L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{2+q+p})$ tending to zero in the respective spaces for $j\to\infty$. Because of the assumption on P, we may choose φ_j in such a way that

$$\varphi_j\equiv 0 \text{ for } y,y' \text{ in a neighbourhood of } \partial B, \tag{111}$$

moreover,

$$\varphi_j\equiv 0 \text{ for } t<T_0 \text{ or } \quad t'<T_0, \text{ for some } T_0>0, \tag{112}$$

and

$$\varphi_j \text{ independent of } t \text{ for } t>T_1 \text{ and of } t' \text{ for } t'>T_1, \text{ for some } T>0, \tag{113}$$

cf. the relations (79) and (80).

Let $\tilde{\omega}(t,t')$ be another function as in Lemma 2.10 such that $\tilde{\omega}\equiv 1$ on $\operatorname{supp}\omega$.

We obtain (first formally, and then, including convergence)

$$c_\psi(t,t',\tilde{y},\tilde{y}',\tau,\tilde{\eta},\zeta) = \sum_{j=0}^{\infty}\lambda_j\sigma\big(\frac{\tilde{r}}{t}\big)\varphi_j\big(r,t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t}\big)b_j(\tau,\tilde{\eta},\zeta)\tilde{\omega}(t,t')\tilde{\sigma}\big(\frac{\tilde{r}'}{t}\big) \tag{114}$$

for

$$b_j(\tau,\tilde\eta,\zeta) := \mathrm{op}_{\tilde r}\big(\tilde r^{-\mu}\chi_1(\tilde r[\tau,\tilde\eta,\zeta])P_j(\tilde r\varrho,\tilde r\tau,\tilde r\tilde\eta,\tilde r\zeta)\chi_2(\tilde r'[\tau,\tilde\eta,\zeta])\big). \tag{115}$$

From Proposition 1.12 we know that $b_j(\tau,\tilde\eta,\zeta)$ belongs to

$$S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{1+q}_{\tau,\tilde\eta};\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

for $\zeta\neq 0$, and tends to zero in that space for $j\to\infty$. In addition, the operator of multiplication by $\sigma(\frac{\tilde r}{t})\varphi_j(\frac{\tilde r}{t},t,t',\frac{\tilde y}{t},\frac{\tilde y'}{t})\omega(t,t')$ belongs to $S^{0;0,0}(\mathbb{R}^{3(1+q)};\cdot,\cdot)$ (where dots stand for $\mathcal{K}^{s,\gamma}(X^\wedge)$ for any s,γ) and tends to zero for $j\to\infty$, cf. Lemma 2.11 below. The operator of multiplication by $\tilde\omega(t,t')\tilde\sigma(\frac{\tilde r'}{t})$ can also be treated as a symbol of the class $S^{0;0,0}(\mathbb{R}^{3(1+q)};\cdot,\cdot)$ by Lemma 2.12 (which includes the variables $\tilde y,\tilde y'$ because of the assumption on the support of the amplitude functions). Thus, by virtue of Remark 2.8 below, the factors at λ_j in the sum (114) belong to $S^{\mu;0,0}(\mathbb{R}^{3(1+q)};E,\widetilde E)$ and tend to zero for $j\to\infty$. Thus (114) converges in this space.

Concerning analogous calculations for scalar symbols, see [11, Section 3.3.8].

Let us now analyse the non-smoothing Mellin contribution of ωc which is of the form

$$\begin{aligned}\omega(t,t')c_M(t,t',\tilde y,\tilde y',\tau,\tilde\eta,\zeta) &= \omega(t,t')b_M\big(t,t',\frac{\tilde y}{t},\frac{\tilde y'}{t},\tau,t\tilde\eta,\zeta\big)\\ &= \mathrm{op}^{\gamma-\frac n2}_{M,\tilde r}\big\{\tilde r^{-\mu}\omega_1(\tilde r|\tau,\tilde\eta,\zeta|)\omega(t,t')\widetilde H\big(\frac{\tilde r}{t},t,t',\frac{\tilde y}{t},\frac{\tilde y'}{t},v,\tilde r\tau,\tilde r\tilde\eta,\tilde r\zeta\big)\omega_2(\tilde r'|\tau,\tilde\eta,\zeta|)\big\}.\end{aligned}$$

Here, for simplicity, we omitted the factors $\sigma\big(\frac{\tilde r}{t}\big)$ and $\tilde\sigma\big(\frac{\tilde r'}{t}\big)$ which occur in (69), using the fact that the specific choice of σ and $\tilde\sigma$ is unessential: so without loss of generality we assume $\sigma\equiv 1$ on $\mathrm{supp}\,\omega_1$, $\tilde\sigma\equiv 1$ on $\mathrm{supp}\,\omega_2$.

The function $\widetilde H$ can be written as a convergent sum

$$\widetilde H(r,t,t',y,y',v,\tilde\tau,\tilde\eta,\tilde\zeta) = \sum_{j=0}^{\infty}\lambda_j\varphi_j(r,t,t',y,y')H_j(v,\tilde\tau,\tilde\eta,\tilde\zeta)$$

with $\lambda_j\in\mathbb{C}$, $\sum_{j=0}^{\infty}|\lambda_j|<\infty$, and $\varphi_j\in C^\infty(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\times B\times B)$, $H_j\in L^{\mu}_{\mathrm{cl}}(X;\mathbb{C}\times\mathbb{R}^{1+q+p}_{\tilde\tau,\tilde\eta,\tilde\zeta})$, tending to zero in the respective spaces for $j\to\infty$. We still have the support properties of the kind (79), (80). Then, analogously as for ωc_ψ before, we obtain in the present case

$$\omega(t,t')c_M(t,t',\tilde y,\tilde y',\tau,\tilde\eta,\zeta) = \sum_{j=0}^{\infty}\lambda_j c_j(t,t',\tilde y,\tilde y',\tau,\tilde\eta,\zeta) \tag{116}$$

for

$$c_j(t,t',\tilde{y},\tilde{y}',\tau,\tilde{\eta},\zeta)$$
$$:= \mathrm{op}_{M,\tilde{r}}^{\gamma-\frac{n}{2}}\Big\{\varphi_j\Big(\frac{\tilde{r}}{t},t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t}\Big)\omega(t,t')\omega_1(\tilde{r}|\tau,\tilde{\eta},\zeta|)\tilde{r}^{-\mu}H_j(v,\tilde{r}\tau,\tilde{r}\tilde{\eta},\tilde{r}\zeta)\omega_2(\tilde{r}'|\tau,\tilde{\eta},\zeta|)\Big)\Big\}.$$

This gives us

$$\omega(t,t')c_M(t,t',\tilde{y},\tilde{y}',\tau,\tilde{\eta},\zeta) = \sum_{j=0}^{\infty}\lambda_j\varphi_j\big(\frac{\tilde{r}}{t},t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t}\big)\omega(t,t')b_j(\tau,\tilde{\eta},\zeta)$$

for $b_j(\tau,\tilde{\eta},\zeta) := \mathrm{op}_{M,\tilde{r}}^{\gamma-\frac{n}{2}}\{\tilde{r}^{-\mu}\omega_1(\tilde{r}|\tau,\tilde{\eta},\zeta|)H_j(v,\tilde{r}\tau,\tilde{r}\tilde{\eta},\tilde{r}\zeta)\omega_2(\tilde{r}'|\tau,\tilde{\eta},\zeta|)\}$.

Similarly as before, the operators of multiplication by $\varphi_j\big(\frac{\tilde{r}}{t},t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t}\big)\omega(t,t')$ belong to the space $S^{0;0,0}(\mathbb{R}^{3(1+q)};E,E)$ and tend to zero in that space for $j\to\infty$. Moreover, from [6] and [14] we know that the operator-valued symbols b_j belong to $S^{\mu;0,0}(\mathbb{R}^{3(1+q)};E,\widetilde{E})$ and also tend to zero as $j\to\infty$. Thus (116) converges in $S^{\mu;0,0}(\mathbb{R}^{3(1+q)};E,\widetilde{E})$, which shows the assertion for ωc_M.

The next expression to be analysed is the smoothing Mellin summand of ωc. We have (cf. the notations (70) and (71))

$$\omega(t,t')m_2(t,t',\tilde{y},\tilde{y}',\tau,\tilde{\eta},\zeta) = \omega(t,t')m_1(t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t},\tau,t\tilde{\eta},\zeta)$$
$$= \mathrm{op}_{M,\tilde{r}}^{\gamma-\frac{n}{2}}\{\tilde{r}^{-\mu}\omega_1(\tilde{r}|\tau,\tilde{\eta},\zeta|)\omega(t,t')F(t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t},v)\omega_2(\tilde{r}'|\tau,\tilde{\eta},\zeta|)\}.$$

The arguments for $m_2\in S^{\mu;0,0}(\mathbb{R}^{3(1+q)};E,\widetilde{E})$ are analogous to those for the non-smoothing Mellin term, using again a tensor product argument for the function $F(t,t',y,y',v)$.

The last summand in $\omega(t,t')c(t,t',\tilde{y},\tilde{y}',\tau,\tilde{\eta},\zeta)$ to be characterised is the Green term defined by

$$\omega(t,t')g_2(t,t',\tilde{y},\tilde{y}',\tau,\tilde{\eta},\zeta)u(\tilde{r})$$
$$= \omega(t,t')t^{-(\mu+1)}\int_0^{\infty}K(t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t},t\tau,t\tilde{\eta},t\zeta;\tilde{r}|\tau,\tilde{\eta},\zeta|,\tilde{r}'|\tau,\tilde{\eta},\zeta|)u(\tilde{r}')\mathrm{d}\tilde{r}',$$

for all $u\in\mathcal{K}^{s,\gamma}(X^{\wedge}_{r',x'})$. Similarly as before, since $\zeta\neq 0$ is fixed and t is sufficiently large on the support of the amplitude function, we could replace $[\cdot]$ by $|\cdot|$. Recall that the integration with respect to the variables $x\in X$ is automatically carried out. Let us represent the kernel function K of Proposition 3.10 as a convergent series

$$K(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta};r,r') = \sum_{i=0}^{\infty}\lambda_j k_j(t,t',y,y',\tilde{\tau},\eta,\tilde{\zeta})K_j(r,r')$$

for sequences $\lambda_j \in \mathbb{C}$, $\sum_{j=0}^{\infty} |\lambda_j| < \infty$, $k_j \in S_{\mathrm{cl}}^{\mu+1}(\mathbb{R}^{1+q}_{t,y} \times \mathbb{R}^{1+q}_{t',y'} \times \mathbb{R}^{1+q+p}_{\tilde{\tau},\eta,\tilde{\zeta}})$, $\quad K_j(r,r') \in \mathcal{S}^{\gamma-\mu}(X^{\wedge}_{r,x})_{\varepsilon} \hat{\otimes}_{\pi} \mathcal{S}^{-\gamma}(X^{\wedge}_{r',x'})_{\varepsilon}$ tending to zero in the respective spaces as $j \to \infty$. Since the conditions (79) and (80) are valid in analogous form for the function K with respect to the variables (t,t',y,y'), we can obviously choose the elements k_j in such a way that they also satisfy these conditions.

We then obtain

$$g_2(t,t',\tilde{y},\tilde{y}',\tau,\tilde{\eta},\zeta)u(\tilde{r}) \tag{117}$$
$$= \sum_{j=0}^{\infty} \lambda_j \omega(t,t') t^{-(\mu+1)} k_j(t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t},t\tau,t\tilde{\eta},t\zeta) \int_0^{\infty} K_j(\tilde{r}|\tau,\tilde{\eta},\zeta|,\tilde{r}'|\tau,\tilde{\eta},\zeta|)u(\tilde{r}')d\tilde{r}'.$$

The factors $\omega(t,t')t^{-(\mu+1)}K_j(t,t',\frac{\tilde{y}}{t},\frac{\tilde{y}'}{t},t\tau,t\tilde{\eta},t\zeta)$ are scalar symbols in $S^{\mu;0,0}(\mathbb{R}^{3(1+q)})$ tending to zero as $j \to \infty$. Moreover,

$$G_j(\tau,\tilde{\eta},\zeta) : u(\tilde{r}) \mapsto \int_0^{\infty} K_j(\tilde{r}|\tau,\tilde{\eta},\zeta|,\tilde{r}'|\tau,\tilde{\eta},\zeta|)u(\tilde{r})d\tilde{r}$$

represent symbols in $S^{\mu}(\mathbb{R}^{1+q};E,\widetilde{E})$ tending to zero as $j \to \infty$. This shows that the series (117) converges in the space $S^{\mu;0,0}(\mathbb{R}^{3(1+q)};E,\widetilde{E})$ which completes the proof. □

We have studied so far the upper left corners (75) of local edge symbols. Additional edge conditions up to infinity are encoded by the 12-, 12-, and 22-entries of the Green contributions in general form, cf. the expressions (72) and (73). Finally, the spaces E and $\widetilde{E}$ are to be replaced by $E \oplus \mathbb{C}^{j_-}$ and $\widetilde{E} \oplus \mathbb{C}^{j_+}$. The details are straightforward after the proof for the case of upper left corners and will be omitted; so we only formulate the corresponding generalisation of Theorem 2.14 (with the same notation for the spaces E and $\widetilde{E}$):

Theorem 3.11 *For every fixed $\zeta \neq 0$ the entries*

$$\omega(t,t')\beta_{2*}(\beta_{1*}g)(t,t',\tilde{y},\tilde{y}',\tau,\tilde{\eta},\zeta) =: (\mathfrak{g}_{ij})_{i,j=1,2}(t,t',\tilde{y},\tilde{y}',\tau,\tilde{\eta},\zeta)$$

are symbols of the kind

$$\mathfrak{g}_{12} \in S^{\mu-\frac{n+1}{2};0,0}(\mathbb{R}^{3(1+q)};\mathbb{C}^{j_-},\widetilde{E}),\ \ \mathfrak{g}_{21} \in S^{\mu+\frac{n+1}{2};0,0}(\mathbb{R}^{3(1+q)};E,\mathbb{C}^{j_+}),$$

and $\mathfrak{g}_{22} \in S^{\mu;0,0}(\mathbb{R}^{3(1+q)};\mathbb{C}^{j_-},\mathbb{C}^{j_+})$ (the upper left corner $\mathfrak{g}_{11}$ was characterised in Theorem 2.14).

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