Lorenz transformations and creation of logarithmic singularities to the solutions of some nonstrictly hyperbolic semilinear systems with two space variables [1] [2]

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1. This paper deals with some model examples of first order semilinear non-strictly hyperbolic systems and with the singularities of the solutions of the corresponding generalized Cauchy problem for them. The initial data are assumed to have finite jump type discontinuities along two characteristic surfaces Σ_1, Σ_2 which cross transversally along $\Gamma_0 = \Sigma_1 \cap \Sigma_2$. We are interested in the production of logarithmic singularities from the interaction of these piecewise smooth waves at the surface Γ_0 which is not contained in a spacelike manifold. The result proposed here was influenced by the considerations in [M-R1] and [L]. For the sake of completeness we remind of the reader /see [B]/ that for a regular embedded hypersurface Σ, a distribution u, defined in a neighbourhood of Σ, is said to be conormal of order s iff for any finite set of vector fields $V_1, \dots, V_N$ tangent to Σ we have that $V_1 \dots V_N u \in H^s_{loc}$. Suppose now that Σ_i, $1 \le i \le \mu$ are regular characteristic surfaces for the strictly hyperbolic semilinear system $P_m(D)u = F(x, D^{m-1}u)$, $m \ge 1$ and that Σ_1, Σ_2 cross transversally in $\Gamma_0 = \Sigma_1 \cap \Sigma_2$, while Σ_i, $3 \le i \le \mu$ are passing through Γ_0. We assume that $D^{m-1}u$ is locally bounded on the domain of definition Ω of u and $\Sigma_i \cap \{t > 0\}$ is located in the domain of determinacy of $\Omega \cap \{t < 0\}$ for all i, $1 \le i \le \mu$. The solution u is conormal with respect to Σ_1 and Σ_2 for $t < 0$ and it has singular support disjoint from Γ_0. Then one can prove /see [B]/ that $singsupp\ u \subset \cup_{i=1}^{\mu} \Sigma_i$ and u is conormal at all points of $\Sigma_i \setminus \Gamma_0$. Let us note that the characteristic surfaces Σ_i locally cut space-time into 2μ wedges and by definition u is said to be piecewise smooth in $t < 0$ or $t > 0$ if u is smooth in the closure of each wedge.

Suppose now that we have an interaction of two piecewise smooth in $t < 0$ waves described by the previous system. Then it was shown in [M-R2] that the solution u remains piecewise smooth in $t > 0$ provided that Γ_0 is contained in a spacelike hypersurface. Similarily, if one studies the Cauchy problem with piecewise smooth data singular accross $\Gamma_0 \subset \{t = 0\}$, there is a local existence of a piecewise smooth solution singular along the characteristic hypersurfaces passing through Γ_0.

In contrast with [M-R1, M-R2] and [L] we investigate a non-strictly hyperbolic

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semilinear system (the generalized Cauchy problem) and we find out a necessary and sufficient condition for creation of logarithmic type singularity.

§1. Statement of the problem and formulation of the main results

Consider the following semilinear non-strictly hyperbolic 5×5 system in $\mathbf{R}^1_t \times \mathbf{R}^2_{x_1,x_2}$:

$$
(1)\qquad \begin{cases}
\sqrt{2}\partial_t u_1 + \sqrt{2}\partial_{x_1} u_1 + \partial_t u_2 + \partial_{x_1} u_2 + \partial_{x_2} u_2 = 0\\
\partial_t u_1 + \partial_{x_1} u_1 + \partial_{x_2} u_1 + \sqrt{2}\partial_t u_2 + \sqrt{2}\partial_{x_2} u_2 = 0\\
(\partial_{x_1} + \partial_{x_2} + \partial_t) v = w_1\\
\partial_t w_1 + 2\partial_{x_1} w_1 + 2\partial_{x_2} w_2 = 0\\
2\partial_{x_1} w_1 + \partial_t w_2 - 2\partial_{x_1} w_2 = \psi(4t - x_1 - x_2) u_1 u_2
\end{cases}
$$

equipped by Cauchy data on the non-characteristic hyperplane $\alpha : t = \dfrac{x_1+x_2}{4}$: $w_1, w_2, v|_{t<\frac{x_1+x_2}{4}} = 0$ and such that $u_1 = (t-x_1)^{k_1}\theta(t-x_1)$, $u_2 = (t-x_2)^{k_2}\theta(t-x_2)$ for $t < \dfrac{x_1+x_2}{4}$, $\psi(\tau) = \tau^{k_3}\theta(\tau)$, $\forall \tau \in \mathbf{R}^1$. As usual, $\theta(\tau)$ stands for the Heaviside function and $k_i \in \mathbf{Z}_+$, $i = 1,2,3$. Classical solutions can exist for $k_i \geq 3$, $i = 1,2,3$. Moreover, *supp* $\psi u_1 u_2 \subset \{t \geq x_1, t \geq x_2\}$.

This is our main result.

Theorem 1. *There exists a (k_1+k_2+2) order linear partial differential operator with constant coefficientd $M(D)$, $D = (\partial_{x_1}, \partial_{x_2}, \partial_t)$ and such that $M(D)w_1$ has a square root-logarithmic type singularity across the light cone surface of the future $K_2^+ = \{2t = |x|\}$ and singsupp $Mw_1 = K_2^+ \cup \Gamma^+$, where $\Gamma^+ = \{x_1 = x_2 = t \geq 0\}$. Consider now the stright line collinear with the radial vector field $l = \partial_t + \partial_{x_1} + \partial_{x_2}$, starting from the point $P_3 \in \alpha$ and hitting the cone K_2^+ at the point P_1. Then $M(D)v$ is C^∞ smooth in a neighbourhood of the line segment P_3P_1 located outside K_2^+ and over the plane α and $M(D)v$ has a logarithmic-square root type singularity accross K_2^+, singsupp $Mv = K_2^+ \cup \Gamma^+$.*

Remark 1. At the end of this paper a necessary and sufficient condition for the existence of logarithmic type singularity of $M(D)w_1$ is found. The operator $M(D)$ is given by: $M(D) = (3\partial_{x_1} + \partial_{x_2} + \partial_t)^{k_1+1}(\partial_{x_1} + 3\partial_{x_2} + \partial_t)^{k_2+1}$.

$M(D)w_1$ has a loharithmic-square root type singularity accross K_2^+ if, by definition, $M(D)w_1 = P(x,t)\sqrt{f_1(x,t)} + Q(x,t)\log f_2(x,t)$ near K_2^+, where $P, Q \in C^\infty((1-\varepsilon)|x| < 2t < (1+\varepsilon)|x|)$ for some $0 < \varepsilon \ll 1$ and $f_1, f_2 \in C^\infty(0 < |x| \leq 2t < |x|(1+\varepsilon))$, $f_1|_{K_2^+} = 0$ $f_2|_{K_2^+} = 1$, $f_1 > 0$, $f_2 > 0$ for $0 < |x| < 2t < |x|(1+\varepsilon)$.

We propose in Fig. 1 a physical interpretation of the just formulated theorem. We are studying the propagation of five semilinear waves. Two of them are piecewise smooth travelling waves starting from $-\infty$ and the corresponding characteristics are $\Sigma_1 : t - x_1 = 0$, $\Sigma_2 : t - x_2 = 0$. Certainly, Σ_1, Σ_2 are transversal each to other.

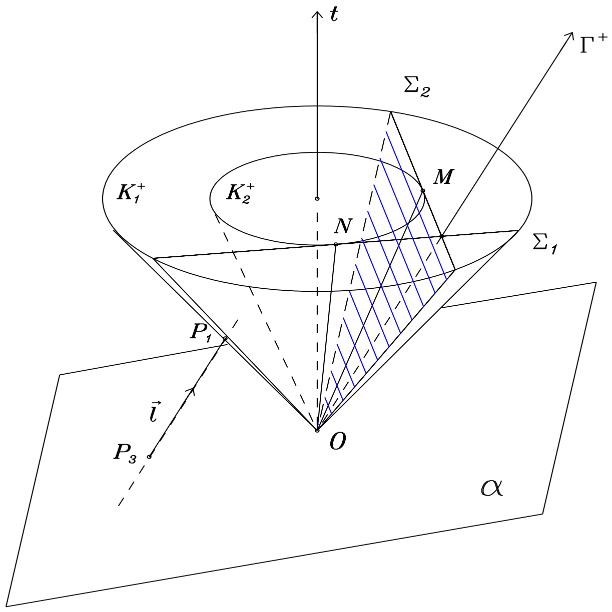

Figure 1:

The other three waves have initial data prescribed on the noncharacteristic plane α. Moreover, $\alpha \cap \Sigma_1 \cap \Sigma_2 = 0$, 0 being the origin in $\mathbf{R}^3$. Our waves have a collision at the ray $\Gamma^+ || l$. The stright line Γ^+ is not contained in a space like manifold. The hyperplanes Σ_1, Σ_2 are tangential to the characteristic cone surface of the future $K_1^+ = \{t = |x|, x \in \mathbf{R}^2\}$ of the system (1) but they are transversal to the second characteristic cone surface of the future K_2^+ and $0l$ is located between K_1^+ and K_2^+. Due to the interaction of the waves at Γ^+ and the tangencity of $\Sigma_{1,2}$ to K_1^+ new singularities of w_1, w_2, v were born. More precisely, *singsupp* $Mw_{1,2} =$ *singsupp* $Mv = K_2^+ \cup \Gamma^+$ and $Mw_{1,2}$, Mv possess logarithmic-square root type singularities accross K_2^+.

Some preliminary notes

At first we shall show that (1) is non-strictly hyperbolic system with respect to t. To do this we shall write (1) in the following form:

$$A_0 \partial_t U + A_1 \partial_{x_1} U + A_2 \partial_{x_2} U = F, \tag{2}$$

where

$$A_0 = \begin{pmatrix} \sqrt{2} & 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \sqrt{2} & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ 0 \\ w_1 \\ 0 \\ \psi u_1 u_2 \end{pmatrix}, \qquad U = \begin{pmatrix} u_1 \\ u_2 \\ v \\ w_1 \\ w_2 \end{pmatrix}.$$

Thus, $\det(\tau A_0+\xi_1 A_1+\xi_2 A_2) = \left[2(\tau+\xi_1)(\tau+\xi_2) - (\tau+\xi_1+\xi_2)^2\right](\tau+\xi_1+\xi_2)(\tau^2 - 4\xi_1^2 - 4\xi_2^2) = (\tau^2-\xi_1^2-\xi_2^2)(\tau+\xi_1+\xi_2)(\tau^2-4\xi_1^2-4\xi_2^2)$. So we find the real smooth roots of the characteristic equation

$$\tau_{1,2} = \pm\sqrt{\xi_1^2+\xi_2^2}, \qquad (\xi_1,\xi_2) \neq 0$$

$$\tau_{3,4} = \pm 2\sqrt{\xi_1^2+\xi_2^2}, \qquad (\xi_1,\xi_2) \neq 0$$

$$\tau_5 = -(\xi_1+\xi_2)$$

Obviously, $\tau_1 \neq \tau_2$, $\tau_3 \neq \tau_4$, $\tau_{1,2} \neq \tau_{3,4}$ and $\tau_{3,4} = \tau_5 \iff \xi_1 = \xi_2 = 0$. On the other hand, $\tau_5 = \tau_{1,2} \iff \xi_1\xi_2 = 0$, $(\xi_1,\xi_2) \neq 0$. So we conclude that (1) is non-strictly hyperbolic system w.r. to t. Geometrically, the line $\tau = -(\xi_1+\xi_2)$, $\xi_1, \xi_2 = 0$ is a generatrix of the characteristic cone $\tau^2 = (\xi_1^2+\xi_2^2)$. We shall see now that the hyperplane $t = \dfrac{x_1+x_2}{4}$ is non-characteristic to (1). More generally, let $\Phi \equiv t + \alpha_1 x_1 + \alpha_2 x_2$. Then $\Phi = 0$ is non-characteristic to (1) iff

$$\det(\Phi_t A_0 + \Phi_{x_1} A_1 + \Phi_{x_2} A_2) \neq 0 \text{ on } \Phi = 0, \quad \text{i. e. iff}$$

$$\det(A_0 + \alpha_1 A_1 + \alpha_2 A_2) \neq 0. \tag{3}$$

But we know that $\det(\tau A_0+\xi_1 A_1+\xi_2 A_2) = (\tau+\xi_1+\xi_2)(\tau^2-(\xi_1^2+\xi_2^2))(\tau^2-4(\xi_1^2+\xi_2^2)) \Rightarrow \det(A_0+\alpha_1 A_1+\alpha_2 A_2) = \det(\tau A_0+\xi_1 A_1+\xi_2 A_2)|_{\tau=1,\xi_1=\alpha_1,\xi_2=\alpha_2}$.

Conclusion. The hyperplane $\Phi = t+\alpha_1 x_1+\alpha_2 x_2 = 0$ is noncharacteristic to our system (1) iff $\begin{cases} \alpha_1^2+\alpha_2^2 \neq 1, 1/4 \\ \alpha_1+\alpha_2 \neq -1 \end{cases}$. As $\alpha_1 = \alpha_2 = 1/4$ for $t = \dfrac{x_1+x_2}{4}$ we conclude that the initial hyperplane is non-characteristic /free surface /or/ space-like one/ to (1).

As each classical solution of the linear PDE $(\partial_t + \partial_{x_1})u_1 = 0$ has the form $u_1 = f_1(t-x_1)$, $f_1 \in C^1$, $(\partial_t + \partial_{x_2})u_2 = 0$, $u_2 = f_2(t-x_2)$, $f_2 \in C^1$, f_1, f_2 being arbitrary functions we see that the 1st and 2nd equations of the system (1) are identically satisfied in $\mathbf{R}^2$ by $u_1 = (t-x_1)^{k_1}\theta(t-x_1)$, $u_2 = (t-x_2)^{k_2}\theta(t-x_2)$. Our next step is to eliminate w_2 from the last two equations of (1). To do this we differentiate the 4th equation w.r. to x_1 and the 5th equation w.r. to x_2. Therefore,

$$(\partial^2_{tx_1} + 2\partial^2_{x_1})w_1 + 2\partial^2_{x_2} w_2 + \partial^2_{tx_2} w_2 = \frac{\partial}{\partial x_2}(\psi u_1 u_2).$$

On the other hand, the 4th equation gives:

$$\partial_t^2 w_1 + 2\partial^2_{x_1 t} w_1 + 2\partial^2_{x_2 t} w_2 = 0.$$

This way,

$$\partial_t^2 w_1 - 4(\partial^2_{x_1} + \partial^2_{x_2}) w_1 = -2\frac{\partial}{\partial x_2}(\psi u_1 u_2), \text{ i. e.} \tag{4}$$

$$\Box_2 w_1 = -2\frac{\partial}{\partial x_2}(\psi u_1 u_2) = -2\psi \frac{\partial}{\partial x_2}(u_1 u_2) + 2\psi' u_1 u_2$$

and $\psi \equiv \psi(4t - x_1 - x_2)$.

§2. Lorenz transformations applied to some hyperbolic equations

A non-degenerate linear change of the variables (x_1, x_2, t) /respectively (x_1, x_2, x_3, t)/ is called a Lorenz change iff it concerves the hyperbolic equation $\Box_c u = u_{tt} - c^2(u_{x_1x_1} + u_{x_2x_2}) = f$ /respectively $\Box_c u = u_{tt} - c^2(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}) = f$/, $c = \text{const} > 0$, up to the constant c /see [N]/.

We are looking for a Lorenz transformation of the following form:

$$\begin{cases} y_1 = \lambda(t - x_1) + (t - x_2) \\ y_2 = (t - x_1) + \lambda(t - x_2) \\ \tau = 4t - x_1 - x_2, \ \lambda = \text{ const } \neq 0. \end{cases} \tag{5}$$

Thus the Cauchy data of (1) are prescribed on the hyperplane $\tau = 0$ and we are looking for a solution in the half space $\tau > 0$ (i.e. $t > \dfrac{x_1 + x_2}{4}$). We point out that *singsupp* $u_1 = \{t = x_1\} = \Sigma_1$, *singsupp* $u_2 = \{t = x_2\} = \Sigma_2$ and that the wedge $W = \{(x_1, x_2, t) : t \geq x_1, t \geq x_2\}$ has the edge $\Gamma : \left| \begin{array}{l} x_1 = t \\ x_2 = t \end{array} \right.$. Assuming the change (5) to be nondegenerate we see that (5) transforms the wedge W into the wedge $\tilde{W}$ whose edge $\tilde{\Gamma}$ is the τ axes: $(\tau = 2t, y_1 = y_2 = 0)$. Equivalently, $\psi(4t - x_1 - x_2) = \psi(\tau) = \tau^{k_3}\theta(\tau) \equiv \tau_+^{k_3}$. Moreover, if $\lambda > 0$ then $\tilde{W}$ turns out to be a wedge /acute cenral angle/ contained in $\mathbf{R}^2_{y_1 y_2}$. So (5) can be rewritten as

$$\begin{cases} y_1 = t(\lambda + 1) - \lambda x_1 - x_2 \\ y_2 = t(\lambda + 1) - x_1 - \lambda x_2 \\ \tau = 4t - x_1 - x_2 \end{cases} \text{ and } 0 \neq \begin{vmatrix} -\lambda & -1 & \lambda + 1 \\ -1 & -\lambda & \lambda + 1 \\ -1 & -1 & 4 \end{vmatrix} = 2(\lambda^2 - 1),$$

i.e. we must take $\lambda \neq 1$. Easy computations give us:

$$\frac{\partial u}{\partial x_1} = -\lambda \frac{\partial u}{\partial y_1} - \frac{\partial u}{\partial y_2} - \frac{\partial u}{\partial \tau}$$

$$\frac{\partial u}{\partial x_2} = -\frac{\partial u}{\partial y_1} - \lambda \frac{\partial u}{\partial y_2} - \frac{\partial u}{\partial \tau}$$

$$\frac{\partial u}{\partial t} = (\lambda+1)\frac{\partial u}{\partial y_1} + (\lambda+1)\frac{\partial u}{\partial y_2} + 4\frac{\partial u}{\partial \tau}.$$

The change (5) is Lorenzian one iff $\lambda \neq \pm 1$ and $\Box_2$ is transformed in $\Box_c$ with some $c > 0$.

Then

$$\Box_2 u = u_{tt} - 4(u_{x_1x_1} + u_{x_2x_2}) = \left[(\lambda+1)\frac{\partial}{\partial y_1} + (\lambda+1)\frac{\partial}{\partial y_2} + 4\frac{\partial}{\partial \tau}\right]^2 u$$

$$-4\left(\lambda\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial \tau}\right)^2 u - 4\left(\frac{\partial}{\partial y_1} + \lambda\frac{\partial}{\partial y_2} + \frac{\partial}{\partial \tau}\right)^2 u$$

$$= \Bigg[\left((\lambda+1)^2 - 4(\lambda^2+1)\right)\frac{\partial^2}{\partial y_1^2} + \left((\lambda+1)^2 - 4(\lambda^2+1)\right)\frac{\partial^2}{\partial y_2^2} + 8\frac{\partial^2}{\partial \tau^2} + (2(\lambda+1)^2$$

$$-16\lambda)\frac{\partial^2}{\partial y_1 \partial y_2} + (8(\lambda+1) - 8(\lambda+1))\frac{\partial^2}{\partial y_1 \partial \tau} + (8(\lambda+1) - 8 - 8\lambda)\frac{\partial^2}{\partial y_2 \partial \tau}\Bigg] u.$$

We put $2(\lambda+1)^2 - 16\lambda = 0 \Rightarrow \lambda_{1,2} = 3 \pm 2\sqrt{2}$ and we take $\lambda = 3 + 2\sqrt{2} = (1+\sqrt{2})^2 \neq 1$.

So the operator $\Box_2$ takes the form

$$\text{(6)}\quad \Box_2 u = 8\frac{\partial^2 u}{\partial \tau^2} - 16(3+2\sqrt{2})\left(\frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2}\right) = 8(u_{\tau\tau} - 2(3+2\sqrt{2})(u_{y_1y_1} + u_{y_2y_2})).$$

This way we conclude that (4) has the following form in the new coordinates (y_1, y_2, τ):

$$\frac{1}{8}\Box_2 w_1 = \frac{\partial^2 w_1}{\partial \tau^2} - 2(3+2\sqrt{2})\left(\frac{\partial^2 w_1}{\partial y_1^2} + \frac{\partial^2 w_1}{\partial y_2^2}\right)$$

$$\text{(7)}\qquad = \frac{1}{4}\psi(\tau)u_1(t-x_1)u'(t-x_2) + \frac{1}{4}\psi'(\tau)u_1(t-x_1)u_2(t-x_2).$$

Thus, $\Box_2 = 8\Box_c$, $c = \sqrt{2(3+2\sqrt{2})} = 2 + \sqrt{2}$.

On the other hand, the first two equations from (5) show that $t - x_1 = \dfrac{\lambda y_1 - y_2}{\lambda^2 - 1}$, $t - x_2 = \dfrac{\lambda y_2 - y_1}{\lambda^2 - 1}$. So

$$\frac{\partial^2 w_1}{\partial \tau^2} - 2(3+2\sqrt{2})\left(\frac{\partial^2 w_1}{\partial y_1^2} + \frac{\partial^2 w_1}{\partial y_2^2}\right) = \frac{1}{4}\psi(\tau)u_1\left(\frac{\lambda y_1 - y_2}{\lambda^2 - 1}\right)u_2'\left(\frac{\lambda y_2 - y_1}{\lambda^2 - 1}\right)$$

$$\text{(8)}\qquad +\frac{1}{4}\psi'(\tau)u_1\left(\frac{\lambda y_1 - y_2}{\lambda^2 - 1}\right)u_2\left(\frac{\lambda y_2 - y_1}{\lambda^2 - 1}\right)$$

$\lambda = 3 + 2\sqrt{2}$, and the support of the right hand side is contained in $\lambda y_1 - y_2 \geq 0$, $\lambda y_2 - y_1 \geq 0$, $\tau \geq 0$, $w_1|_{\tau<0} = 0$.

There are no difficulties to compute the inverse transformation of (5). It is of Lorenz type, certainly. According to the 3rd equation in (5):

$$\tau = 2t + (t - x_1) + (t - x_2) = 2t + \frac{y_1 + y_2}{\lambda + 1} \quad \text{i.e.}$$

$$t = \tau - \frac{y_1 + y_2}{2(\lambda + 1)}.$$

So the inverse transformation of (5) is given by:

$$\text{(9)} \qquad \begin{cases} x_1 = \dfrac{\tau}{2} - \dfrac{(\lambda - 3)y_2 + y_1(3\lambda - 1)}{2(\lambda^2 - 1)} \\ x_2 = \dfrac{\tau}{2} - \dfrac{(\lambda - 3)y_1 + y_2(3\lambda - 1)}{2(\lambda^2 - 1)} \\ t = \dfrac{\tau}{2} - \dfrac{y_1 + y_2}{2(\lambda + 1)} \end{cases}.$$

Suppose now that $u(x, t) \in C^1(\mathbf{R}^3)$. Then the Lorenz change (5) implies: $\frac{\partial u}{\partial \tau} = \frac{1}{2}\left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial t}\right)$. Therefore, the radial vector field $l = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial t}$ is transformed under the diffeomorphism (5) into the vector field $2\frac{\partial}{\partial \tau}$.

Let us consider the smooth nondegenerate change in $\mathbf{R}^3$:

$$\text{(10)} \qquad z_1 = \frac{\lambda y_1 - y_2}{\lambda^2 - 1}, \qquad z_2 = \frac{\lambda y_2 - y_1}{\lambda^2 - 1}, \qquad \tau = \tau, \quad \text{i. e.}$$

the 0τ axes is conserved.

Having in mind that $\frac{\partial u}{\partial y_1} = \frac{\partial u}{\partial z_1}\frac{\lambda}{\lambda^2 - 1} - \frac{\partial u}{\partial z_2}\frac{1}{\lambda^2 - 1} = \frac{1}{\lambda^2 - 1}\left(\lambda\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}\right)u$ and $\frac{\partial u}{\partial y_2} = \frac{1}{\lambda^2 - 1}\left(\lambda\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_1}\right)u$, $\frac{\partial u}{\partial \tau} = \frac{\partial u}{\partial \tau}$ we rewrite (8) as:

$$\text{(11)} \qquad \frac{\partial^2 w_1}{\partial \tau^2} - \frac{2(3 + 2\sqrt{2})}{(\lambda^2 - 1)^2}\left[\left(\lambda\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}\right)^2 + \left(\lambda\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_1}\right)^2\right] w_1 =$$

$$= \frac{1}{4}\psi(\tau)u_1(z_1)u_2'(z_2) + \frac{1}{4}\psi'(\tau)u_1(z_1)u_2(z_2).$$

Thus,

$$\text{(12)} \qquad \frac{\partial^2 w_1}{\partial \tau^2} - \frac{1}{16(1 + \sqrt{2})^2}\left((\lambda^2 + 1)\frac{\partial^2}{\partial z_1^2} + (\lambda^2 + 1)\frac{\partial^2}{\partial z_2^2} - 4\lambda\frac{\partial^2}{\partial z_1 \partial z_2}\right) w_1 =$$

$$= \frac{1}{4}(\partial_\tau + \partial_{z_2})\psi u_1 u_2.$$

Certainly, the second order operator in the brackets is strictly elliptic in the plane $0_{z_1 z_2}$ / $\lambda = (1+\sqrt{2})^2$/.

The inverse change of (10) is given by the formula

(13) $\qquad y_1 = \lambda z_1 + z_2, \qquad y_2 = \lambda z_2 + z_1, \tau = \tau,$ i.e. in matrix form

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = \frac{1}{\lambda^2 - 1}\begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \lambda & 1 \\ 1 & \lambda \end{pmatrix},$$

$\det A = \lambda^2 - 1$. Put $c^2 = \frac{\lambda^2+1}{16(1+\sqrt{2})^2}$, $a = -\frac{4\lambda}{1+\lambda^2} \Rightarrow c^2 = \frac{3(3+2\sqrt{2})}{8(1+\sqrt{2})^2} = \frac{3}{8}$, $a = -\frac{2}{3}$.

This way (12) takes the form:

$$\frac{\partial^2 w_1}{\partial \tau^2} - c^2\left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + a\frac{\partial^2}{\partial z_1 \partial z_2}\right) w_1 = \tag{14}$$

$$= \frac{1}{4}(\partial_\tau + \partial_{z_2})\psi(\tau)u_1(z_1)u_2(z_2),$$

where $c = \frac{1}{2}\sqrt{\frac{3}{2}}$, $a = -\frac{2}{3}$.

Our next step is to find the fundamental solution $E(x,t)$ of the linear operator $Q(D)$ participating in the left hand side of (14). Thus:

$$\frac{\partial^2 E}{\partial \tau^2} - c^2\left(\frac{\partial^2 E}{\partial z_1^2} + \frac{\partial^2 E}{\partial z_2^2} + a\frac{\partial^2 E}{\partial z_1 \partial z_2}\right) = \delta(\tau)\otimes\delta(z), \tag{15}$$

where δ is the standard Dirac delta function supported at the origin. Applying the inverse change $\left| \begin{array}{l} y = A^{-1}z \\ \tau = \tau \end{array}\right.$ to (10) we know that (15) transforms into

$$\frac{\partial^2 E}{\partial \tau^2} - 2(1+\sqrt{2})^2\left(\frac{\partial^2 E}{\partial y_1^2} + \frac{\partial^2 E}{\partial y_2^2}\right) = \delta(\tau)\otimes\delta(Ay), \tag{16}$$

The identity $\delta(Ay) = \frac{\delta(y)}{|\det A|} = \frac{\delta(y)}{\lambda^2-1} = \frac{\delta(y)}{4\sqrt{2}(1+\sqrt{2})^2}$ gives us

$$\square_{2+\sqrt{2}} E = \frac{\delta(\tau)\otimes\delta(y)}{4\sqrt{2}(1+\sqrt{2})^2} \tag{17}$$

and therefore $E.4\sqrt{2}(1+\sqrt{2})^2$ is a fundamental solution of $\square_{2+\sqrt{2}}$.

Conclusion: Put

$$E(y,\tau) = \frac{1}{4\sqrt{2}(1+\sqrt{2})^2}\cdot\frac{\theta((2+\sqrt{2})\tau-|y|)}{2\pi(2+\sqrt{2})\sqrt{(2+\sqrt{2})^2\tau^2-|y|^2}} \tag{18}$$

$$= \frac{1}{16\sqrt{2}(1+\sqrt{2})^3\pi}\frac{\theta((2+\sqrt{2})\tau-|y|)}{\sqrt{2(1+\sqrt{2})^2\tau^2-|y|^2}}.$$

Then $\square_{2+\sqrt{2}}E = \delta(\tau, y)$ /see [V] for example or [H]/.

Going back to the coordinates (z_1, z_2) we obtain:

$$Q(D)E(z,\tau) = \delta(\tau)\otimes\delta(z),\ E(z,\tau) = \frac{1}{16\sqrt{2}(1+\sqrt{2})^3\pi}\cdot\frac{\theta(\sqrt{2}(1+\sqrt{2})\tau-|A^{-1}z|)}{\sqrt{2(1+\sqrt{2})^2\tau^2-|A^{-1}z|^2}}.$$

So according to (14):

$$\begin{cases} Q(D)w_1 = g(z,\tau),\ g(z,\tau) = \frac{1}{4}(\partial_\tau+\partial_{z_2})\psi(\tau)u_1(z_1)u_2(z), \\ w_1|_{\tau<0} = 0. \end{cases} \tag{19}$$

Certainly, the Cauchy problem (19) is satisfied in the sense of Schwartz distributions $D'(\mathbf{R}^3)$. Obviously, *supp* $g \subset \{\tau \geq 0\}$ and more precisely, *supp* $g \subset \{\tau \geq 0, z_1 \geq 0, z_2 \geq 0\}$, *supp* $E \subset \{0 \leq \frac{|A^{-1}z|}{2+\sqrt{2}} \leq \tau\}$.

The theory of the generalized Cauchy problem for strictly hyperbolic constant coefficients differential operators in D' /[V]/ gives that (19) has a unique solution that can be written in a convolutional form:

(20)

$$w_1(z,\tau) = E * g(z,\tau) = \tilde{c}\int\int\int_{\mathbf{R}^3}\frac{\theta(\tau-\mu-|A^{-1}(\frac{z-\nu}{2+\sqrt{2}})|)g(\nu_1,\nu_2,\mu)\,d\nu_1 d\nu_2 d\mu}{\sqrt{2(1+\sqrt{2})^2(\tau-\mu)^2-|A^{-1}(z-\nu)|^2}} =$$

$$= c_1\int\int\int_{\mathbf{R}^3}\frac{\theta(\tau-\mu)-|\frac{A^{-1}}{2+\sqrt{2}}(z-\nu)|)g(\nu_1,\nu_2,\mu)\,d\nu_1 d\nu_2 d\mu}{\sqrt{(\tau-\mu)^2-|\frac{A^{-1}}{2+\sqrt{2}}(z-\nu)|^2}},\quad c_1 = \frac{1}{32(1+\sqrt{2})^4\pi}.$$

We point out that $\hat{K}_{(z,\tau)} = \{(\nu,\mu) : \mu \geq 0, \tau-\mu \geq |\frac{A^{-1}}{2+\sqrt{2}}(z-\nu)|\}$ is the interior of the cone / cone of the past/ with vertex at the point (z,τ), $\tau \geq 0$. Fix some $0 \leq \mu = \mu_0 \leq \tau$. Then $K_{(z,\tau)} = \partial\hat{K}_{(z,\tau)} \cap \{\mu = \mu_0\} = \{(\nu_1,\nu_2) : |\frac{A^{-1}}{2+\sqrt{2}}(z-\nu)| = \tau-\mu_0\}$. Evidently, $|\frac{A^{-1}}{2+\sqrt{2}}(z-\nu)|^2 = (\tau-\mu_0)^2$ is a second order curve contained in the 2 dimensional plane $\mathbf{R}^2_\nu$. On the other hand, $|z-\nu| = |AA^{-1}(z-\nu)| \leq$

$\|A\||A^{-1}(z-\nu)| \leq (2+\sqrt{2})\|A\|(\tau-\mu_0)$ and consequently $K_{(z,\tau)}$ is located inside a circle centered at z and with radius $(2+\sqrt{2})\|A\|(\tau-\mu_0)$. So we conclude that $K_{(z,\tau)}$ is an ellipse /not circle/. Therefore, $\hat{K}_{(z,\tau)}$ is a cone whose basis is an ellipse. The integral (20) exists if $g \in C(\tau \geq 0)$, *supp* $g \subset \{\tau \geq 0\}$. Then

$$w_1(z,\tau) = \tag{21}$$

$$= c_1 \int_0^\tau \int \int_{|A^{-1}(\nu-z)| \leq (2+\sqrt{2})(\tau-\mu)} \frac{g(\nu_1,\nu_2,\mu)\, d\nu_1 d\nu_2 d\mu}{\sqrt{(2+\sqrt{2})^2(\tau-\mu)^2 - |A^{-1}(z-\nu)|^2}}.$$

The standard change $\left| \begin{array}{c} A^{-1}(\nu-z) = \tau p(2+\sqrt{2}),\ p \in \mathbf{R}^2 \\ \tau-\mu = \alpha\tau, \quad \alpha \in \mathbf{R}^1 \end{array} \right.$ in (21) gives us:

$$w_1(z,\tau) = \tag{22}$$

$$= \frac{c_1}{(2+\sqrt{2})\tau} \int_0^1 \int \int_{|p|\leq\alpha} \frac{g(z+\tau(2+\sqrt{2})Ap, \tau(1-\alpha))}{\sqrt{\alpha^2-|p|^2}} \tau^3 (2+\sqrt{2})^2 |A|\, dp_1 dp_2 d\alpha$$

$$= c_2\tau^2 \int_0^1 \int \int_{|p|\leq\alpha} \frac{g(z+\tau(2+\sqrt{2})Ap, \tau(1-\alpha))}{\sqrt{\alpha^2-|p|^2}}\, dp d\alpha,\ c_2 = \text{ const } > 0.$$

Assume that $g \in C(\tau \geq 0)$. Then $w_1(z,\tau) \in C(\tau \geq 0)$. Moreover, if $g \in C^k_\gamma(\tau \geq 0)$, where $\gamma \in \mathbf{R}^1$, $\gamma = z_1, z_2$ or τ then $w_1 \in C^k_\gamma(\tau \geq 0)$, $k \in Z_+$. Suppose now that $g \in C^{0,1}(\tau \geq 0)$ on each compact $D \subset \{\tau \geq 0\}$, i.e. g is Lipschitz continuous with respect to $(z_1, z_2, \tau \geq 0)$ on each compact $D \subset \{\tau \geq 0\}$. Then $w_1 \in C^{0,1}(\tau \geq 0)$ on D.

We point out that if $g \in C(\mathbf{R}^3)$ and *supp* $g \subset \{\tau \geq 0\}$ then $g(z,0) = 0$. The assumption $g \in C^{0,1}(D)$, where D is an arbitrary compact in $\mathbf{R}^3$ implies $w_1 \in C^{0,1}(D)$ and *supp* $w_1 \subset \{\tau \geq 0\}$. Assume that $\alpha_0 = \min(k_1-2, k_2-2, k_3-2) \geq 0$. Then $g \in C^{\alpha_0}$ and therefore $w_1 \in C^{\alpha_0}(\tau \geq 0)$.

Our next step is to estimate from above *supp* $w_1(x,t)$. As we know, *supp* $w_1 \subset \overline{\textit{supp } E + \textit{supp } g}$ in the (z,τ) coordinates and the symbol $+$ stands for the arithmetical sum of the sets *supp* E and *supp* g. Thus

$$\textit{supp } w_1(z,\tau) \subset \{(z,\tau) : |A^{-1}z| \leq (2+\sqrt{2})\tau\} + \{(z,\tau) : z_1 \geq 0, z_2 \geq 0, \tau \geq 0\},$$

as *supp* $g = \{(z,\tau) : z_1 \geq 0, z_2 \geq 0, \tau \geq 0\}$.

Consequently,

$$\textit{supp}\, w_1 \subset \{(z,\tau) : \tau-\mu \geq \frac{|A^{-1}(z-\nu)|}{2+\sqrt{2}} \text{ for some } (\nu,\mu) \in \mathbf{R}^3, \nu_1 \geq 0, \nu_2 \geq 0, \mu \geq 0\}$$

$$= \bigcup_{\nu_1\geq 0, \nu_2 \geq 0, \mu \geq 0} \hat{\hat{K}}_{(\nu,\mu)},$$

$\hat{\hat{K}}_{(\nu,\mu)}$ being the interior of the cone of the future with vertex at (ν, μ).

From geometric reasons it is clear that $\cup \hat{\hat{K}}_{(\nu,\mu)}$ will be contained in the union of the following sets: I octant, the "solid" cone of the future $\hat{\hat{K}}_{(0,0)}$ and the located outside of the first octant envelopes of two 1-parameter families of characteristic conical surface of the future, namely $\{K_{(p,0,0)}\}_{p\geq 0}$ and $\{K_{(0,q,0)}\}_{q\geq 0}$ with vertexes at $(p, 0, 0)$, $(0, q, 0)$. We shall find the envelope of the first family of characteristics only. Thus $\tau^2(2+\sqrt{2})^2 = |A^{-1}(z-p)|^2 = (\lambda^2+1)(z_1-p)^2 + (\lambda^2+1)z_2^2 + 4\lambda(z_1-p)z_2$, i.e. $\tau^2 = 3(z_1-p)^2 + 3z_2^2 + 2(z_1-p)z_2$. We differentiate the last equality with respect to p and we get: $6(z_1-p)+2z_2 = 0 \Rightarrow z_1 - p = -\frac{1}{3}z_2$. So the equation of the envelopes takes the form $\tau^2 = \frac{8}{3}z_2^2$. One can easily see that we are interested in the plane $\Gamma_1 : \tau = -\sqrt{\frac{8}{3}}z_2, \;\; z_2 \leq 0$. There are no difficulties to verify that the characteristic hyperplane Γ_1 is tangential to the cone $K_{(0,0)}$ surface along the cone generatrix $l_0 : \left| \begin{array}{l} z_1 = -\frac{z_2}{3} \\ \tau = -\sqrt{\frac{8}{3}}z_2 \end{array} \right.$.

In a similar way we find the envelope $\Gamma_2 : \tau = -\sqrt{\frac{8}{3}}z_1, \;\; z_1 \leq 0$, of the characteristic cones $\{K_{(0,q,0)}\}_{q\geq 0}$. The characteristic hyperplane Γ_2 is tangential to the cone surface $K_{(0,0)}$ along the cone generatrix $m_0 : \left| \begin{array}{l} z_2 = -\frac{z_1}{3} \\ \tau = -\sqrt{\frac{8}{3}}z_1 \end{array} \right.$. Having in mind the fact that the characteristics are invariant under smooth nondegenerate changes we can go back to the old coordinates (x_1, x_2, t) and conclude that *supp* $w_1(x,t)$ is contained in a domain located over α, i.e. $4t - x_1 - x_2 \geq 0$ and bounded by the characteristic cone surface $\tilde{K}_{(0,0)} : 2t = \sqrt{x_1^2 + x_2^2}$ and the characteristic hyperplanes /surfaces/ $\tilde{\Gamma}_1 : t = \frac{x_1}{4+\sqrt{8/3}} + \frac{1+\sqrt{8/3}}{4+\sqrt{8/3}}x_2$, $\tilde{\Gamma}_2 : t = \frac{x_2}{4+\sqrt{8/3}} + \frac{1+\sqrt{8/3}}{4+\sqrt{8/3}}x_1$. Certainly, $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ are tangential to $\tilde{K}_{(0,0)}$ along some generatrixes $\tilde{l}_1$, $\tilde{m}_1$ of $\tilde{K}_{(0,0)}$. The details are left to the reader.

Put $L = \frac{1}{4}(\partial_\tau + \partial_{z_2})$. Then (19) is rewritten as:

$$\left\{ \begin{array}{l} Q(D)w_1 = L(\psi u_1 u_2) \equiv g, \;\; supp\; g \subset \{\tau \geq 0\} \\ w_1|_{\tau<0} = 0 \end{array} \right. . \tag{23}$$

Consider now the generalized Cauchy problem in $D'(\mathbf{R}^3)$:

$$(24) \qquad \begin{cases} Q(D)w_2 = \psi u_1 u_2, \ \mathit{supp}\ \psi \subset \{\tau \geq 0\} \\ w_2|_{\tau<0} = 0 \end{cases}.$$

According to the theory of generalized Cauchy problem /[H], [V]/ there exists a unique solution of (24) which is given by $w_2(z,\tau) = E * \psi u_1 u_2$. In fact we have in $D'(\mathbf{R}^3) : Q(D)w_2 = Q(D)(E * \psi u_1 u_2) = Q(D)E * \psi u_1 u_2 = \delta * \psi u_1 u_2 = \psi u_1 u_2$ which implies $L(Qw_2) = L(\psi u_1 u_2)$ in $D'(\mathbf{R}^3)$, i.e. $Q(Lw_2) = g(z,\tau)$. Moreover, $\mathit{supp}\ w_2 \subset \{\tau \geq 0\} \Rightarrow \mathit{supp}\ Lw_2 \subset \{\tau \geq 0\}$. Thus,

$$(25) \qquad \begin{cases} Q(Lw_2) = g, \mathit{supp}\ g \subset \{\tau \geq 0\} \\ Lw_2|_{\tau<0} = 0 \end{cases}$$

According to the uniqueness result of the generalized Cauchy problem applied to (23), (25) we get: $w_1 = Lw_2$ and therefore $w_1 = L(E * \psi u_1 u_2)$. On the other hand, $u_1(\lambda) \in C^{k_1-1}(\mathbf{R}^1)$, $u_1(\lambda) \in C^{k_1-1,1}(\mathbf{R}^1)$, $\partial_{z_1}^{k_1+1} u_1 = k_1!\delta(z_1)$, $\partial_{z_2}^{k_2+1} u_2 = k_2!\delta(z_2)$, $u_2 \in C^{k_2-1,1}(\mathbf{R}^1)$. So $M(\partial_{z_1}, \partial_{z_2})w_1 \equiv \partial_{z_1}^{k_1+1}\partial_{z_2}^{k_2+1} L(E * \psi u_1 u_2) = L(E * \psi(\tau)\delta(z_1)\delta(z_2))k_1!k_2!$ and therefore $Q(D)(M(\partial_{z_1}, \partial_{z_2})w_1) = Q(E * L(\psi\delta(z_1)\delta(z_2)))\times \times k_1!k_2! = k_1!k_2! L(\psi\delta(z_1)\delta(z_2)) \Rightarrow \{z_1 = z_2 = 0, \tau \geq 0\} = \mathit{singsupp}\ L(\psi\delta(z_1)\delta(z_2)) \subset \mathit{singsupp}\ Mw_1$. A simple modification of formulas (20), (21) enables us to conclude that /see [C] or [V]/:

$$(26) \qquad Mw_1 = \partial_{z_1}^{k_1+1}\partial_{z_2}^{k_2+1} w_1(z,\tau)$$

$$= \frac{c}{4}\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial z_2}\right)\left[\theta\left(\tau - \frac{|A^{-1}z|}{2+\sqrt{2}}\right)\int_0^{\tau - \frac{|A^{-1}z|}{2+\sqrt{2}}} \frac{\psi(\mu)\, d\mu}{\sqrt{(\tau-\mu)^2 - \left(\frac{|A^{-1}z|}{2+\sqrt{2}}\right)^2}}\right],$$

$c =$ const $\neq 0$, $z \neq 0$.

Further on we shall carefully investigate the properties of

$$(27) \qquad I(z,\tau) = \theta\left(\tau - \frac{|A^{-1}z|}{2+\sqrt{2}}\right)\int_0^{\tau - \frac{|A^{-1}z|}{2+\sqrt{2}}} \frac{\psi(\mu)\, d\mu}{\sqrt{(\tau-\mu)^2 - \left(\frac{|A^{-1}z|}{2+\sqrt{2}}\right)^2}}.$$

Remark 2. $\mathit{supp}\ L(E * \psi(\tau)\delta(z_1)\delta(z_2))) \subset \mathit{supp}\ (E * \psi\delta(z_1)\delta(z_2)) \subseteq \mathit{supp}\ E + \{(z,t) : z = 0, \tau \geq 0\} = \mathit{supp}\ E = \{(z,\tau) : \tau \geq \frac{|A^{-1}z|}{2+\sqrt{2}}\} = \hat{\hat{K}}_{(0,0)}$ and $\mathit{singsupp}\ E = \{(z,\tau) : \tau = \frac{|A^{-1}z|}{2+\sqrt{2}}\} = K_{(0,0)}$, $\mathit{singsupp}\ Mw_1 = \{z_1 = z_2 = 0, \tau \geq 0\} \cup \{\tau = \frac{|A^{-1}z|}{2+\sqrt{2}}, z \neq 0\}$.

Proof of the main Theorem 1

Consider the third equation of our system (1):

$$\begin{cases} (\partial_t + \partial_{x_1} + \partial_{x_2})v = w_1(z,t) \\ v|_{t<\frac{x_1+x_2}{4}} = 0 \end{cases} \tag{28}$$

As we know, in the (z,τ) coordinates it takes the form

$$\begin{cases} 2\dfrac{\partial v}{\partial \tau} = w_1(z,\tau) \\ v|_{\tau<0} = 0, \end{cases} \tag{29}$$

$v \in D'(\mathbf{R}^3), w_1 \in D'(\mathbf{R}^3), w_1|_{\tau<0} = 0$.

Differentiating the generalized Cauchy problem (29) with respect to z_1 and z_2 we get for $z \neq 0$:

$$\begin{cases} \dfrac{\partial}{\partial \tau}(\partial_{z_1}^{k_1+1}\partial_{z_2}^{k_2+1}v) = \dfrac{1}{2}\partial_{z_1}^{k_1+1}\partial_{z_2}^{k_2+1}w_1 = \\ = c_1\left(\dfrac{\partial}{\partial \tau} + \dfrac{\partial}{\partial z_2}\right)\left[\theta\left(\tau - \dfrac{|A^{-1}z|}{2+\sqrt{2}}\right)\displaystyle\int_0^{\tau-\frac{|A^{-1}z|}{2+\sqrt{2}}} \dfrac{\psi(\mu)\,d\mu}{\sqrt{(\tau-\mu)^2 - \left(\frac{|A^{-1}z|}{2+\sqrt{2}}\right)^2}}\right] \\ \partial_{z_1}^{k_1+1}\partial_{z_2}^{k_2+1}v|_{\tau<0} = 0, \end{cases} \tag{30}$$

$c_1 =$ const $\neq 0$, where *supp* $\partial_{z_1}^{k_1+1}\partial_{z_2}^{k_2+1}w_1 \subset \hat{\hat{K}}_{(0,0)}$ and $Mw_1 = \partial_{z_1}^{k_1+1}\partial_{z_2}^{k_2+1}w_1$.

Thus,

$$\begin{cases} \dfrac{\partial}{\partial \tau}Mv = \dfrac{1}{2}Mw_1 \\ Mv|_{\tau<0} = 0, \end{cases} \tag{31}$$

supp $Mw_1 \subset \hat{\hat{K}}_{(0,0)}$ and therefore $Mv = 0$ outside the cone of the future $\hat{\hat{K}}_{(0,0)}$.

Under the additional assumption that Mw_1 is continuous we have:

$$Mv(z,\tau) = \begin{cases} 0, \quad \tau \leq \dfrac{|A^{-1}z|}{2+\sqrt{2}} \\ \dfrac{1}{2}\displaystyle\int_{\frac{|A^{-1}z|}{2+\sqrt{2}}}^{\tau} Mw_1(z,s)\,ds, \quad \tau \geq \dfrac{|A^{-1}z|}{2+\sqrt{2}},\ z \neq 0 \end{cases} \tag{32}$$

i.e.

$$
(33)\quad Mv(z,\tau)=\begin{cases} 0, \quad \tau \le \dfrac{|A^{-1}z|}{2+\sqrt{2}} \\ \dfrac{c}{8}\left[\displaystyle\int_0^{\tau-\frac{|A^{-1}z|}{2+\sqrt{2}}} \frac{\psi(\mu)\,d\mu}{\sqrt{(\tau-\mu)^2-\left(\frac{|A^{-1}z|}{2+\sqrt{2}}\right)^2}} + \int_{\frac{|A^{-1}z|}{2+\sqrt{2}}}^{\tau} \frac{\partial}{\partial z_2}\right. \\ \left.\left(\displaystyle\int_0^{s-\frac{|A^{-1}z|}{2+\sqrt{2}}} \frac{\psi(\mu)\,d\mu}{\sqrt{(\tau-\mu)^2-\left(\frac{|A^{-1}z|}{2+\sqrt{2}}\right)^2}}\right) ds\right], \quad \tau > \dfrac{|A^{-1}z|}{2+\sqrt{2}}. \end{cases}
$$

So for $\tau > \dfrac{|A^{-1}z|}{2+\sqrt{2}} > 0$: $Mv = \dfrac{c}{8}\left(I(z,\tau)+\displaystyle\int_{\frac{|A^{-1}z|}{2+\sqrt{2}}}^{\tau}\frac{\partial}{\partial z_2}I(z,s)\,ds\right)$. After the standard change $\tau-\mu\to\mu$ we can rewrite (33) as;

$$
(34)\qquad Mv(z,\tau)=\begin{cases} 0, \quad \tau \le \dfrac{|A^{-1}z|}{2+\sqrt{2}} \\ \dfrac{c}{8}\left[-\displaystyle\int_{\tau}^{\frac{|A^{-1}z|}{2+\sqrt{2}}} \frac{\psi(\tau-\mu)d\mu}{\sqrt{\mu^2-\left(\frac{|A^{-1}z|}{2+\sqrt{2}}\right)^2}} + \int_{\frac{|A^{-1}z|}{2+\sqrt{2}}}^{\tau} \frac{\partial}{\partial z_2}\right. \\ \left.\left(\displaystyle\int_{\frac{|A^{-1}z|}{2+\sqrt{2}}}^{s} \frac{\psi(s-\mu)\,d\mu}{\sqrt{\mu^2-\left(\frac{|A^{-1}z|}{2+\sqrt{2}}\right)^2}}\right) ds\right], \quad \tau > \dfrac{|A^{-1}z|}{2+\sqrt{2}}. \end{cases}
$$

Certainly, $z \neq 0$ in (34).

Conclusion: We have to compute $\displaystyle\int_{\frac{|A^{-1}z|}{2+\sqrt{2}}}^{\tau} \frac{\psi(\tau-\mu)\,d\mu}{\sqrt{\mu^2-\left(\frac{|A^{-1}z|}{2+\sqrt{2}}\right)^2}} = I(z,\tau)$.

Remark 3. Consider the equation $\left(\dfrac{\partial}{\partial x_1}+\dfrac{\partial}{\partial x_2}+\dfrac{\partial}{\partial t}\right)v_1 = w_1$, $v_1|_{t-\frac{x_1+x_2}{4}<0} = 0$. One can easily see that the straight line Γ passing through each point $0 \neq \tilde{A} \in \alpha$ and colinear with the vector $l=(1,1,1)$ is hitting the cone surfaces of the future and the past $K^+_{(0,0)}, K^-_{(0,0)} = \{(x,t) : \pm 2t = \sqrt{x_1^2+x_2^2}\}$ at one point only.

In fact, $\Gamma : \begin{array}{l} x_1 = s+a_1 \\ x_2 = s+a_2 \\ x_3 = s+a_3 \end{array}$, $\tilde{A} = (a_1,a_2,a_3) \Rightarrow 4a_3 = a_1+a_2$. In order to find $K_{(0,0)}\cap\Gamma$ we have to solve the system $4t^2 = x_1^2+x_2^2$, $x_i = s+a_i$, $i=1,2$, $t = s+a_3$. So

$s_{1,2}^2 = s^2 = \dfrac{a_1^2 + a_2^2 - 4a_3^2}{2} = \dfrac{3a_1^2 + 3a_2^2 - 2a_1a_2}{4}$ and the quadratic form in the right hand side is positively definite. Therefore $s_{1,2}$ are real roots and $s_1 \neq s_2 \iff \tilde{A} \neq 0$.

Lemma 1. *Consider the integral* $V_m = \int \frac{P_m(x)\,dx}{\sqrt{x^2+c}}$, $c =$ *const* < 0, $x > \sqrt{-c}$, $P_m = x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_{l-2}x^{m-l+2} + a_{l-1}x^{m-l} + \ldots + a_{m-1}x + a_m$. *Then there exists a uniquely determined polynomial of order* $m-1$, $Q_{m-1} = b_0x^{m-1} + b_1x^{m-2} + b_2x^{m-3} + \ldots + b_{l-2}x^{m-l+1} + b_{l-1}x^{m-l} + \ldots + b_{m-3}x^2 + b_{m-2}x + b_{m-1}$, *and a constant* λ_m *such that*

$$V_m = Q_{m-1}\sqrt{x^2+c} + \lambda_m V_0,\ V_0 = \int \frac{dx}{\sqrt{x^2+c}} = \ln(x + \sqrt{x^2+c}), \tag{35}$$

$$V_1 = \sqrt{x^2+c} + a_1V_0.$$

This is the elementary and well known proof, $m \geq 2$. Differentiating (35) we have

$$\frac{P_m}{\sqrt{x^2+c}} = Q'_{m-1}\sqrt{x^2+c} + \frac{x}{\sqrt{x^2+c}}Q_{m-1} + \frac{\lambda_m}{\sqrt{x^2+c}}. \text{ i.e.}$$

$P_m = (x^2+c)Q'_{m-1} + Q_{m-1} + \lambda_m$ /in the case $m = 0 : Q_{-1} \equiv 0$, $\lambda_0 = 1$; $m = 1 \rightarrow Q_0 \equiv 1, \lambda_1 = a_1$/. For the unknown coefficients of the polynomial Q_{m-1} and for λ_m /i.e. $m+1$ unknown coefficients/ we get the following linear system:

$$\begin{aligned} mb_0 &= 1 \\ (m-1)b_1 &= a_1 \\ c(m-1)b_0 + (m-2)b_1 &= a_2 \\ &\cdots \\ c(m-l+1)b_{l-2} + (m-l)b_l &= a_l \\ &\cdots \\ 2cb_{m-3} + b_{m-1} &= a_{m-1} \\ cb_{m-2} + \lambda_m &= a_m \end{aligned}; \qquad a_{m+1}\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \\ \lambda_m \end{pmatrix} = \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_{m-1} \\ a_m \end{pmatrix},$$

where the matrix A_{m+1} has the following structure. We have on the main diagonal the elements $m, (m-1), \ldots, 1, 1$; on the first line paralel to the main diagonal and located below it stands 0 and on the second line paralel to the main diagonal and located below it we have: $c(m-1), c(m-2), \ldots, c$. All the other elements of A_{m+1} are 0. So we conclude that $\det A_{m+1} = m!$ and that $\lambda_m = \frac{\det B_{m+1}}{\det A_{m+1}}$, where the matrices A_{m+1}, B_{m+1} coincide up to the last column. The last column of B_{m+1} is $(1, a_1, \ldots, a_{m-1}, a_m)^t$.

Corollary: $\lambda_m \neq 0$ in (35) iff $\det B_{m+1} \neq 0$ and note that $V_m \in C^\infty$ for $x > \sqrt{-c}$.

Let us compute now $I(z, \tau)$.

Certainly, $\psi(\tau-\mu)=\theta(\tau-\mu)(\tau-\mu)^{k_3}=\theta(\tau-\mu)\sum_{k=0}^{k_3}\binom{k_3}{k}\tau^{k_3-k}(-1)^k\mu^k$.

To simplify the notations in computing $I(z,\tau)$ we put $\varphi(z)=\dfrac{|A^{-1}z|}{2+\sqrt{2}}\in C^\infty(z\neq 0)$ and we shall write k instead of k_3. So if $z\neq 0$ and $\tau\geq\varphi(z)$:

$$I(z,\tau)=\int_{\varphi(z)}^{\tau}\frac{\varphi(\tau-\mu)\,d\mu}{\sqrt{\mu^2-\varphi^2(z)}}=\sum_{l=0}^{k}\binom{k}{l}\tau^{k-l}(-1)^l\int_{\varphi(z)}^{\tau}\frac{\mu^l\,d\mu}{\sqrt{\mu^2-\varphi^2(z)}} \tag{36}$$

The change $\mu=\lambda\varphi(z)$, $\varphi(z)\geq 0$, $\varphi(z)=0\iff z=0$ trandforms (36) into

$$I(z,\tau)=\sum_{l=0}^{k}\binom{k}{l}\tau^{k-l}\varphi^l(z)(-1)^l\int_1^{\frac{\tau}{\varphi(z)}}\frac{\lambda^l\,dl}{\sqrt{\lambda^2-1}},\quad \lambda\geq 1.$$

According to (35) there exist a polynomial $Q_{l-1}(\lambda)$ of order $(l-1)$ and a constant λ_l such that

$$I(z,\tau)=\sum_{l=0}^{k}\binom{k}{l}(-1)^l\tau^{k-l}\varphi^l(z)\Big[Q_{l-1}(\lambda)\sqrt{\lambda^2-1}+ \tag{37}$$

$$+\lambda_l\ln(\lambda+\sqrt{\lambda^2-1})\Big]\Big|_{\lambda=1}^{\lambda=\frac{\tau}{\varphi(z)}}\Longrightarrow I(z,\tau)=$$

$$=\sum_{l=0}^{k}\binom{k}{l}(-1)^l\tau^{k-l}\varphi^l(z)\left[Q_{l-1}\left(\frac{\tau}{\varphi(z)}\right)\sqrt{\frac{\tau^2}{\varphi^2(z)}-1}+\lambda_l\ln\left(\frac{\tau}{\varphi(z)}+\sqrt{\frac{\tau^2}{\varphi^2(z)}-1}\right)\right].$$

Evidently, $l=0\Rightarrow Q_{-1}\equiv 0$, $\lambda_0=1$; $l=1\Rightarrow\lambda_1=0$, $Q_0=1$ and $I(z,\tau)\in C^\infty(\tau>\varphi(z)>0)$. Logarithmic terms participate in $I(z,\tau)$ if $\lambda_{l_0}\neq 0$ $/0\leq l_0\leq k/$.

Consider now (38) $\displaystyle\int_{\varphi(z)}^{\tau}\frac{\partial}{\partial z_2}(I(z,s))\,ds$, where $I(z,s)=\sum_{l=0}^{k}\binom{k}{l}(-1)^l s^{k-l}\varphi^l(z)\times$

$$\times\left[Q_{l-1}\left(\frac{s}{\varphi(z)}\right)\sqrt{\frac{s^2}{\varphi^2(z)}-1}+\lambda_l\ln\left(\frac{s}{\varphi(z)}+\sqrt{\frac{s^2}{\varphi^2(z)}-1}\right)\right],\quad s\geq\varphi(z)>0.$$

One can easily see that (38) contains the following four different types of integrals:

1. $\displaystyle\int_{\varphi(z)}^{\tau}\left(\frac{s}{\varphi(z)}\right)^p Q'_{l-1}\left(\frac{s}{\varphi(z)}\right)\cdot\sqrt{\frac{s^2}{\varphi^2(z)}-1}\,ds=\varphi(z)\int_1^{\frac{\tau}{\varphi(z)}}\lambda^p Q'_{l-1}\frac{(\lambda^2-1)}{\sqrt{\lambda^2-1}}\,d\lambda\in$ $C^\infty(\tau>\varphi(z)>0)$, $p\geq 1$, and the last integral is of the type (35).

2. $\displaystyle\int_{\varphi(z)}^{\tau}\frac{s^p}{\varphi^p(z)}Q_{l-1}\left(\frac{s}{\varphi(z)}\right)\cdot\frac{1}{\sqrt{\frac{s^2}{\varphi^2(z)}-1}}\,ds=\varphi(z)\int_1^{\frac{\tau}{\varphi(z)}}\lambda^p Q_{l-1}(\lambda)\frac{d\lambda}{\sqrt{\lambda^2-1}}\in$ $C^\infty(\tau>\varphi(z)>0)$, $p\geq 2$.

3. $\int_{\varphi(z)}^{\tau} \frac{s^{k-l}}{\varphi^{k-l}(z)} Q_{l-1}\left(\frac{s}{\varphi(z)}\right) \sqrt{\frac{s^2}{\varphi^2(z)} - 1}\, ds = \varphi(z) \int_1^{\frac{\tau}{\varphi(z)}} \lambda^{k-l} Q_{l-1}(\lambda) \sqrt{\lambda^2 - 1}\, d\lambda =$ $\varphi(z) \int_1^{\frac{\tau}{\varphi(z)}} \lambda^{k-l} Q_{l-1}(\lambda)(\lambda^2 - 1) \frac{d\lambda}{\sqrt{\lambda^2 - 1}} \in C^{\infty}(\tau > \varphi(z) > 0)$.

4. $\int_{\varphi(z)}^{\tau} \left(\frac{s}{\varphi(z)}\right)^p \frac{1 + \frac{s}{\varphi(z)}/\sqrt{s^2/\varphi^2 - 1}}{s/\varphi(z) + \sqrt{s^2/\varphi^2(z) - 1}}\, ds = \varphi(z) \int_1^{\frac{\tau}{\varphi(z)}} \lambda^p \frac{1 + \lambda/\sqrt{\lambda^2 - 1}}{\lambda + \sqrt{\lambda^2 - 1}}\, d\lambda,$ $p \geq 1$. Thus, $\int_{\varphi(z)}^{\tau} \left(\frac{s}{\varphi(z)}\right)^p \frac{1 + \frac{s}{\varphi(z)}/\sqrt{s^2/\varphi^2 - 1}}{s/\varphi(z) + \sqrt{s^2/\varphi^2(z) - 1}}\, ds = \varphi(z) \int_1^{\frac{\tau}{\varphi(z)}} \frac{\lambda^p}{\sqrt{\lambda^2 - 1}}\, d\lambda \in$ $C^{\infty}(\tau > \varphi(z) > 0)$, $p \geq 1$.

Combining (30)–(34), (37), (38) – p. 1, 2, 3, 4 and using the fact that under the inverse change $(z, \tau) \to (x, t)$ the characteristic cone surface $\{(z, \tau) : \tau = \frac{|A^{-1}z|}{2 + \sqrt{2}}, z \neq 0\}$ is mapped onto the characteristic cone $\{4t^2 = x_1^2 + x_2^2,\ t > 0\}$ and the ray $\{z_1 = z_2 = 0,\ \tau \geq 0\}$ is mapped onto the ray $\{t = x_1 = x_2,\ t \geq 0\}$ we complete the proof of our Theorem 1.

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