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A Gibbs point process of diffusions: existence and uniqueness

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Abstract. *In this work we consider a system of infinitely many interacting diffusions as a marked Gibbs point process. With this perspective, we show, for a large class of stable and regular interactions, existence and (conjecture) uniqueness of an infinite-volume Gibbs process. In order to prove existence we use the specific entropy as a tightness tool. For the uniqueness problem, we use cluster expansion to prove a Ruelle bound, and conjecture how this would lead to the uniqueness of the Gibbs process as solution of the Kirkwood-Salsburg equation.*

1 Introduction and set-up

Consider a Langevin dynamics on $\mathbb{R}^d$ of the form

$$dX_s = dB_s - \frac{1}{2}\nabla V(X_s)ds, \quad s \in [0, 2\beta],\ \beta > 0, \tag{1.1}$$

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The author wishes to warmly thank S. Rœlly and H. Zessin for the many discussions and insights into the topic. The research of the author has been partially funded by Deutsche Forschungsgemeinschaft (DFG) — SFB1294/1 - 318763901 and Deutsch-Französische Hochschule (DFH) — DFDK 01-18.

where B is an $\mathbb{R}^d$-valued Brownian motion, and $V : \mathbb{R}^d \to \mathbb{R}$ is an ultracontractive potential, i.e. outside of some compact subset of $\mathbb{R}^d$,

$$\exists \delta', \mathfrak{a}_1, \mathfrak{a}_2 > 0, \quad V(x) \geq \mathfrak{a}_1 |x|^{d+\delta'} \text{ and } \Delta V(x) - \frac{1}{2}|\nabla V(x)|^2 \leq -\mathfrak{a}_2 |x|^{2+2\delta'}. \tag{1.2}$$

Under these conditions there exists a unique strong solution to (1.1) (see e.g. [12]), which generates an ultracontractive semigroup (see [6],[2]). Moreover, the law of X starting at $X_0 = 0$ is a measure R such that, for any $\delta < \delta'/2$,

$$\int e^{\|m\|_\infty^{d+2\delta}} R(dm) < +\infty. \tag{1.3}$$

For the rest of this work, let $\delta > 0$ as above be fixed.

The question we wish to explore in this work is how to construct a physically meaningful Gibbsian interaction between infinitely many such diffusions starting at random locations. More precisely, we model such a system as a marked Gibbs point process: locations and marks will describe, respectively, starting points and paths of these diffusions. We will then solve the non-trivial questions of existence and uniqueness of the infinite-volume measure for a large class of stable and regular path interactions.

After introducing the Gibbsian framework, we present an existence result via the entropy method of [11]: we use the specific entropy as a tightness tool to prove convergence of a sequence of finite-volume Gibbs measures and show that this limit satisfies the Gibbsian property (that is, the DLR equations). In section 4 we then use the method of cluster expansion – introduced by S. Poghosyan, D. Ueltschi, and H. Zessin in [8], [10] – and the Kirkwood-Salsburg equation to show a Ruelle bound for a regime of small activity, and conjecture that uniqueness of the constructed infinite-volume Gibbs process associated to path interactions follows.

2 Gibbsian formalism for marked point processes

The state space we consider in this work is $\mathscr{E} = \mathbb{R}^d \times C_0$, where $C_0 := C_0\big([0,2\beta];\mathbb{R}^d\big)$, $\beta > 0$, is the set of continuous paths $m : [0,2\beta] \to \mathbb{R}^d$ with initial value $m(0) = 0$. An element $\mathbf{x} = (x,m) \in \mathscr{E}$ is identified with the path $\big(x + m(t)\big)_{t\in[0,2\beta]}$ of starting point $x \in \mathbb{R}^d$ and trajectory $m \in C_0$.

Denote by $\mathscr{M}$ the set of locally-finite point measures (or *configurations*) on $\mathscr{E}$, which are of the form $\gamma = \sum_i \delta_{(x_i,m_i)} \in \mathscr{M}$; we often identify a configuration γ with its support $\{(x_i,m_i)\}_i \subset \mathscr{E}$.

Let $\mathscr{B}_b(\mathbb{R}^d)$ be the subset of bounded Borel sets of $\mathbb{R}^d$. Let $\mathscr{M}_f$ denote the subset of finite configurations, and for any $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, let $\mathscr{M}_\Lambda \subset \mathscr{M}_f$ denote the restriction to starting points inside Λ, and for any configuration $\gamma \in \mathscr{M}$, let $\gamma_\Lambda := \gamma \cap \big(\Lambda \times C_0\big) \in \mathscr{M}_\Lambda$.

Let $\mathscr{P}(\mathscr{M})$ denote the set of probability measures on $\mathscr{M}$: these are called *marked point processes*. As reference process we consider, for any $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, the marked Poisson point process π^z_Λ on $\mathscr{E}$ with intensity measure $z\, dx_\Lambda \otimes R(dm)$. The coefficient z is a positive real number, dx_Λ is the Lebesgue measure on Λ, and the probability measure R is the path measure of the solution of (1.1) starting at 0. In other words, the starting points are drawn in Λ according to a Poisson process, and the marks are diffusion paths starting at these Poisson points.

We add interaction between the points of a configuration by considering an energy functional that takes into account both the locations and the marks.

Assumption 1.1 For any finite marked point configuration $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \in \mathscr{M}_f$, $N \geq 1$, its *energy* is given by the following functional

$$H(\gamma) = \sum_{i=1}^{N} \Psi(\mathbf{x}_i) + \sum_{i=1}^{N} \sum_{j<i} \Phi(\mathbf{x}_i, \mathbf{x}_j) \in \mathbb{R} \cup \{+\infty\}, \tag{1.4}$$

where

- ◇ The *self-potential* term Ψ satisfies $\inf_{x \in \mathbb{R}^d} \Psi(x,m) \geq -k_\Psi \|m\|_\infty^{d+\delta}$ for some constant $k_\Psi > 0$;
- ◇ The *two-body potential* Φ is defined by

$$\Phi(\mathbf{x}_i, \mathbf{x}_j) = \left(\phi(x_i - x_j) + \int_0^{2\beta} \tilde{\phi}(m_i(s) - m_j(s))ds \right) \mathbb{1}_{\{|x_i - x_j| \leq a_0 + \|m_i\|_\infty + \|m_j\|_\infty\}}, \tag{1.5}$$

  where ϕ (acting on on the initial location of the diffusions) is a *radial* (i.e. $\phi(x) = \phi(|x|)$) and *stable* $\mathbb{R}$-valued pair potential in the sense of [13], with stability constant $\mathfrak{c}_\phi \geq 0$, bounded from below, with $\phi(u) \leq 0$ for $u \geq a_0$ (see Figure 1.1); $\tilde{\phi}$ (acting on the dynamics of the diffusions) is a non-negative pair potential.

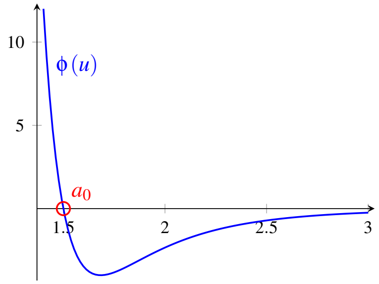

Figure 1.1: An example of radial and stable pair potential ϕ is the *Lennard-Jones* potential $\phi(u) = 16((\frac{3/2}{u})^{12} - (\frac{3/2}{u})^6)$; its zero is at $a_0 = \frac{3}{2}$.

Remark 1.2 *(i)* The stability of the point-interaction potential ϕ and the non-negativity of the mark-interaction potential $\tilde{\phi}$ guarantee stability (in the sense introduced in Lemma 1.5) of the energy H of a marked-point configuration; the fact that ϕ is bounded from below is used to prove the stability of the conditional energy (see Lemma 1.7).

(ii) The indicator function in (1.5) can be interpreted as follows: when the starting points are far enough from each other, the two diffusions do not interact; if their paths do not intersect, they may interact only if $|x_1 - x_2| \leq a_0 + \|m_1\|_\infty + \|m_2\|_\infty$. See Figure 1.2. Notice that the range of interaction is finite but not uniformly bounded.

Definition 1.3 For any $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, the free-boundary-condition *finite-volume Gibbs measure* on Λ with energy H and activity $z > 0$ is the probability measure P^z_Λ on $\mathscr{M}_\Lambda$

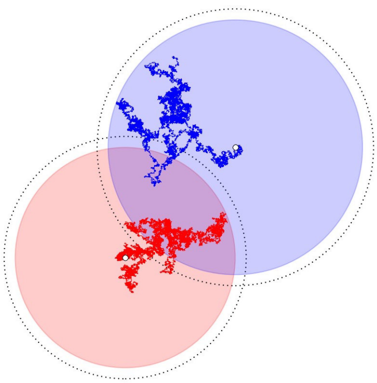

Figure 1.2: The paths of two Langevin diffusions in $\mathbb{R}^2$ which interact. Each circle is centred in the starting point, and its radius corresponds to their maximum displacement in the time interval $[0,1]$. The dotted circle represent the "security" distance $a_0/2$.

defined by

$$P^z_\Lambda(d\gamma) := \frac{1}{Z^z_\Lambda} e^{-H(\gamma_\Lambda)} \pi^z_\Lambda(d\gamma). \tag{1.6}$$

In this work we investigate the existence and uniqueness, as Λ increases to cover the whole space $\mathbb{R}^d$, of an infinite-volume Gibbs measure, in the following sense:

Definition 1.4 A probability measure P on $\mathscr{M}$ is said to be an *infinite-volume Gibbs measure* with energy H and activity $z > 0$, denoted by $P \in \mathscr{G}(H,z)$, if it satisfies, for any $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$ and any positive, bounded, and measurable functional $F : \mathscr{M} \to \mathbb{R}$, the following *DLR equation* (for Dobrushin-Landford-Ruelle)

$$\int_{\mathscr{M}} F(\gamma) P(d\gamma) = \int_{\mathscr{M}} \frac{1}{Z^z_\Lambda(\xi)} \int_{\mathscr{M}_\Lambda} F(\gamma_\Lambda \xi_{\Lambda^c}) e^{-H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})} \pi^z_\Lambda(d\gamma)\, P(d\xi), \tag{DLR}$$

where $H_\Lambda(\gamma)$ is the *conditional energy* of the configuration γ in Λ given its exterior:

$$H_\Lambda(\gamma) := \lim_{r \to +\infty} H\big(\gamma_{\Lambda \oplus B(0,r)}\big) - H\big(\gamma_{\Lambda \oplus B(0,r) \setminus \Lambda}\big), \tag{1.7}$$

with $\Lambda \oplus B(0,r) := \big\{x \in \mathbb{R}^d : \exists y \in \Lambda,\ |y - x| \le r\big\}$.

3 Existence of an infinite-volume Gibbs point process via the entropy method

Under Assumption 1.1 on the energy functional H, the following three lemmas provide the groundwork for the existence theorem.

Lemma 1.5 The following stability condition holds: setting $\mathfrak{c}_H := k_\Psi \vee \mathfrak{c}_\phi$,

$$H(\gamma) \ge -\mathfrak{c}_H \sum_{(x,m) \in \gamma} \big(1 + \|m\|_\infty^{d+\delta}\big), \quad \gamma \in \mathscr{M}_f. \tag{1.8}$$

In order to control the support of the Gibbs point process, we define the subset of *tempered configurations* as the union $\mathscr{M}^{\mathrm{temp}} := \bigcup_{\mathbf{t} \in \mathbb{N}} \mathscr{M}^{\mathbf{t}}$, where $\mathscr{M}^{\mathbf{t}}$ is the set of all configurations $\gamma \in \mathscr{M}$ such that, for all $l \in \mathbb{N}^*$, $\sum_{(x,m) \in \gamma_{B(0,l)}} (1 + \|m\|_\infty^{d+\delta}) \le \mathbf{t}\, l^d$.

Lemma 1.6 For any bounded $\Lambda \subset \mathbb{R}^d$ and $\mathbf{t} \geq 1$, there exists a random variable $\mathfrak{r} = \mathfrak{r}(\gamma_\Lambda, \mathbf{t}) < +\infty$ such that the limit in (1.7) stabilises, i.e.

$$H_\Lambda(\gamma) = H\big(\gamma_{\Lambda \oplus B(0,\mathfrak{r})}\big) - H\big(\gamma_{\Lambda \oplus B(0,\mathfrak{r}) \setminus \Lambda}\big).$$

We say that $\mathfrak{r}(\gamma_\Lambda, \mathbf{t})$ is the *finite but random range* of the interaction $H_\Lambda(\gamma)$.

Lemma 1.7 Fix $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$. For any $\mathbf{t} \geq 1$, there exists a constant $\mathfrak{c}'(\Lambda, \mathbf{t}) \geq 0$ such that the following stability of the conditional energy holds: uniformly for all $\xi \in \mathscr{M}^{\mathbf{t}}$,

$$H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c}) \geq -\mathfrak{c}'(\Lambda, \mathbf{t}) \sum_{(x,m) \in \gamma_\Lambda} \big(1 + \|m\|_\infty^{d+\delta}\big), \quad \gamma_\Lambda \in \mathscr{M}_\Lambda. \tag{1.9}$$

We endow the set $\mathscr{P}(\mathscr{M})$ of probability measures on $\mathscr{M}$ with the topology of local convergence (see [4], [5]). More precisely,

Definition 1.8 A functional F on $\mathscr{M}$ is called *local* and *tame* if there exist a set $\Delta \in \mathscr{B}_b(\mathbb{R}^d)$ and a constant $a > 0$ such that, for all $\gamma \in \mathscr{M}$, $F(\gamma) = F(\gamma_\Delta)$ and $|F(\gamma)| \leq a\,\big(1 + \sum_{(x,m) \in \gamma_\Delta} (1 + \|m\|_\infty^{d+\delta})\big)$.

We denote by $\mathscr{L}$ the set of all local and tame functionals. The topology $\tau_{\mathscr{L}}$ of *local convergence* on $\mathscr{P}(\mathscr{M})$ is defined as the weak* topology induced by $\mathscr{L}$, i.e. the smallest topology on $\mathscr{P}(\mathscr{M})$ under which all the mappings $P \mapsto \int F\,dP$, $F \in \mathscr{L}$, are continuous.

Let us now recall the concept of specific entropy of a probability measure on $\mathscr{M}$.

Definition 1.9 Given two probability measures Q and Q' on $\mathscr{M}$, the *specific entropy* of Q with respect to Q' is defined by

$$\mathcal{I}(Q|Q') = \lim_{\Lambda_n \nearrow \mathbb{R}^d} \frac{1}{|\Lambda_n|} I_{\Lambda_n}(Q|Q'),$$

where $\Lambda_n = [-n, n)^d$, and the *relative entropy* of Q with respect to Q' on Λ is defined as

$$I_\Lambda(Q|Q') := \begin{cases} \displaystyle\int \log f \, dQ_\Lambda & \text{if } Q_\Lambda \preccurlyeq Q'_\Lambda \text{ with } f := \frac{dQ_\Lambda}{dQ'_\Lambda}, \\ \qquad +\infty & \text{otherwise,} \end{cases}$$

where Q_Λ (resp. Q'_Λ) is the image of Q (resp. Q') under the mapping $\gamma \mapsto \gamma_\Lambda$.

The specific entropy with respect to π^z is well defined as soon as Q is invariant under translations on the lattice. Moreover, we underline that for any $a > 0$, the a-entropy level set

$$\mathscr{P}(\mathscr{M})_{\leq a} := \left\{ Q \in \mathscr{P}(\mathscr{M}) : \mathcal{I}(Q|\pi^z) \leq a \right\}$$

is relatively compact for the local convergence topology $\tau_{\mathscr{L}}$, as proved in [5].

Putting together the technical conditions described in this section yields the existence of an infinite-volume Gibbs measure P^z, for *any* activity $z > 0$.

Theorem 1.10 For any energy functional H as in Assumption 1.1 and any activity $z > 0$, there exists at least one infinite-volume Gibbs measure $P^z \in \mathscr{G}(H, z)$.

Sketch of proof. (i) For $\Lambda_n = [-n,n)^d$, consider the sequence $(P^z_{\Lambda_n})_{n\geq 1}$ of finite-volume Gibbs measures, and build the empirical field $(\bar{P}^z_n)_{n\geq 1}$ by stationarising it w.r.t. lattice translations.

(ii) Use uniform bounds on the specific entropy to show the convergence, up to a subsequence, to an infinite-volume measure P^z.

(iii) Prove, using an ergodic property, that P^z carries only the space of tempered configurations.

(iv) Noticing that $\bar{P}^z_n$ does not satisfy the (DLR) equations, introduce a new sequence $(\hat{P}^z_n)_n$ asymptotically equivalent to $(\bar{P}^z_n)_n$ but satisfying (DLR).

(v) Use appropriate approximation technique to show that P^z satisfies (DLR) too.

For details, see [11]. □

Example 1.11 Let $d = 2$. A concrete example of functions satisfying the above assumptions is as follows:

Consider as reference diffusion a Langevin dynamics with $V(x) = |x|^4$; the diffusion is ultracontractive with $\delta' = 2$. The invariant measure $\mu(dx) = e^{|x|^4} dx$ is a Subbotin measure (see [15]).

Consider as self interaction $\Psi(\mathbf{x}) = -\|m\|_\infty^{5/2}$; as interaction between the initial locations a *Lennard-Jones* pair potential $\phi(u) = au^{-12} - bu^{-6}$, $a, b > 0$; as interaction between the marks any non-negative pair potential $\tilde{\phi}$.

4 Uniqueness of Gibbs measure via cluster expansion

The method of cluster expansion relies in finding a regime of small activity $0 < z \le \bar{z}$ in which the partition function Z^z_Λ can be written as the exponential of an absolutely converging series of *cluster* terms. It should then be possible to write an equation (the so-called *Kirkwood-Salsburg equation*, see e.g. [14]) for the correlation functions of the infinite-volume Gibbs measure P^z constructed above. We conjecture that under some assumptions, such an equation has a unique solution, which would lead to the uniqueness of the infinite-volume Gibbs measure. Here we use a strategy developed in [9]. For this section, we make the following additional

Assumption 1.12 The potential ϕ (on initial locations of the diffusions) is *integrable* in $\mathbb{R}^d$: $\|\phi\|_1 < +\infty$; the potential $\tilde{\phi}$ (on the dynamics of the diffusions) is *bounded*: $\|\tilde{\phi}\|_\infty < +\infty$.

The partition function is given, for any $\Lambda \subset \mathbb{R}^d$, by

$$Z^z_\Lambda = 1 + \sum_{N\ge 1} \frac{z^N}{N!} \int_{(\Lambda\times C_0)^N} \exp\Big\{ - \sum_{1\le i\le N} \Psi(x_i, m_i) - \sum_{1\le i<j\le N} \Big(\phi\left(|x_i - x_j|\right) + \int_0^{2\beta} \tilde{\phi}\left(|m_i(s) - m_j(s)|\right) ds \Big) \mathbb{1}_{\{|x_i - x_j| \le a_0 + \|m_i\|_\infty + \|m_j\|_\infty\}} \Big\} dx_1 \cdots dx_N \, R(dm_1) \cdots R(dm_N). \tag{1.10}$$

Theorem 1.13 Consider an energy functional H satisfying Assumption 1.1 and Assumption 1.12. Then the two-body potential Φ satisfies a *modified regularity* condition. Therefore, there exists $\bar{z} > 0$ such that, for any activity $z \le \bar{z}$, the partition function above converges absolutely and a Ruelle bound holds.

Proof. In order to guarantee the absolute convergence of (1.10), we check whether the pair potential Φ satisfies a *modified* $\mathfrak{c}$-*regularity for the functional* $\mathfrak{a}$ (terminology from [10]; introduced in [8]), i.e. that for any $\mathbf{x}_1 = (x_1, m_1)$, the following inequality holds

$$z e^{\mathfrak{c}} \int e^{\mathfrak{a}(\mathbf{x}_2)} |\Phi(\mathbf{x}_1, \mathbf{x}_2)| e^{-\Psi(\mathbf{x}_2)} dx_2 \, R(dm_2) \le \mathfrak{a}(\mathbf{x}_1). \tag{1.11}$$

We consider here $\mathfrak{c} = \mathfrak{c}_\phi$, and a function of the form $\mathfrak{a}(x,m) = \mathfrak{a}(m) = a_1 \|m\|^d_\infty$, where

$$a_1 = \|\phi\|_1 + \left(2\beta \|\tilde{\phi}\|_\infty k_d b_d (a_0^d + 1) \right),$$

with k_d such that $(x+y+z)^d \leq k_d(x^d+y^d+z^d)$, and b_d the volume of the unit ball in $\mathbb{R}^d$. Recalling that the self potential Ψ is such that $\Psi(\mathbf{x}) \geq -k_\Psi \|m\|_\infty^{d+\delta}$. Set $\rho := \int e^{(a_1+k_\Psi)\|m\|_\infty^{d+2\delta}} R(dm) \overset{(1.3)}{<} +\infty$; the modified regularity condition reads

$$ze^{\mathfrak{c}_\phi} \int_{C_0} e^{a_1\|m_2\|_\infty^d} \int_{\mathbb{R}^d} \left|\phi(x_2-x_1)\right| + \left(\int_0^{2\beta} \tilde{\phi}(m_2(s)-m_1(s))ds\right) \mathbb{1}_{\{|x_1-x_2|\leq a_0+\|m_1\|_\infty+\|m_2\|_\infty\}}$$
$$dx_2\ \mathrm{e}^{k_\Psi\|m_2\|_\infty^{d+\delta}} R(dm_2) \leq a_1\|m_1\|_\infty^d.$$

Estimating the l.h.s. leads to the following condition:

$$z \leq \frac{\|m_1\|_\infty^d}{\rho e^{\mathfrak{c}_\phi}(\|m_1\|_\infty^d+1)},$$

which holds as soon as $z \leq (2\rho e^{\mathfrak{c}_\phi})^{-1} =: \bar{z} = \inf_{m_1} \frac{\|m_1\|_\infty^d}{\rho e^{\mathfrak{c}_\phi}(\|m_1\|_\infty^d+1)}$. Applying results in [8], this implies the absolute convergence of (1.10). Moreover, in [9] S. Poghosyan and H. Zessin prove that a Ruelle bound also holds. □

The unique step towards uniqueness which is now missing is the proof that the Kirkwood-Salsburg equation has a unique solution. We state the following conjecture:

Conjecture 1.14 For any activity $z \leq \bar{z}$, the Kirkwood-Salsburg equation has a unique solution.

Assuming the above conjecture to hold true, we obtain the following

Corollary 1.15 For any activity $z \leq \bar{z}$, the infinite-volume measure P^z constructed in Theorem 1.10 is the unique Gibbs measure in $\mathscr{G}(H,z)$.

Conclusions and outlook. In [3], D. Dereudre showed the equivalence between the law of an infinite-dimensional interacting SDE with Gibbsian initial law, and a Gibbs point process on the path space, with a certain energy functional.

It is a natural question to ask whether a Gibbs point process with energy functional H as in Assumption 1.1 is the law of infinite dimensional interacting SDE. Using Malliavin derivatives, D. Dereudre proved that Gibbs point processes with regular H are the law of SDEs with a certain non-markovian drift. See [1] and [7] in the lattice case.

The existence and uniqueness results presented here could therefore be useful to obtain a criterium for the solution of infinite-dimensional SDEs. This is a work in progress.

Bibliography

[1] Dai Pra, P., Rœlly, S.: *An existence result for infinite-dimensional Brownian diffusions with non-regular and non-Markovian drift*, Markov Proc. Rel. Fields **10**, 113–136 (2004).

[2] Davies, E. W.: *Heat kernels and spectral theory*, Cambridge University Press (1989).

[3] Dereudre, D.: *Interacting Brownian particles on pathspace*, ESAIM: Probability and Statistics **7**, 251–277 (2003).

[4] Georgii, H.-O.: *Gibbs Measures and Phase Transitions*, De Gruyter studies in Mathematics 9, 2nd ed. (2011).

[5] Georgii, H.-O., Zessin, H.: *Large deviations and the maximum entropy principle for marked point random fields*, Probab. Theory Relat. Fields **96**(2), 177–204 (1993).

[6] Kavian, O., Kerkyacharian, G., Roynette, B.: *Quelques remarques sur l'ultracontractivité*, J. Funct. Anal. **111**, 155–196 (1993).

[7] Minlos, R.A., Rœlly, S., Zessin, H.: *Gibbs states on space-time*, Pot. Anal. **13**, 367–408 (2000).

[8] Poghosyan, S., Ueltschi, D.: *Abstract cluster expansion with applications to statistical mechanical systems*. J. Math. Ph. **50**(5), 053509 (2009).

[9] Poghosyan, S., Zessin, H.: *Penrose-stable Interactions in Classical Statistical Mechanics*, manuscript in preparation (2020).

[10] Poghosyan S., Zessin, H.: *Construction of limiting Gibbs processes and the uniqueness of Gibbs processes*, In: Lectures in Pure and Applied Mathematics **6**, Potsdam University Press (2020).

[11] Rœlly, S., Zass, A.: *Marked Gibbs processes with unbounded interaction: an existence result*, arXiv:1911.12800 (2019).

[12] Royer, G.: *An initiation to logarithmic Sobolev inequalities*. AMS (2007).

[13] Ruelle, D.: *Superstable interactions in classical statistical mechanics*, Comm. Math. Phys., **18**(2), 127–159 (1970).

[14] Ruelle, D.: *Statistical mechanics: rigorous results*. Imperial college Press (1999).

[15] Subbotin, M.T.: *On the law of frequency of error*. Mat. Sb., **31** 296–301 (1923).