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Propagation of Gibbsianness for Infinite-Dimensional Diffusions with Space-Time Interaction

S. Rœlly[1] **and W.M. Ruszel**[2]

[1] Universität Potsdam, Institut für Mathematik, Am Neuen Palais 10, D-14469 Potsdam, Germany
[2] Technical University Delft, Delft Institute of Applied Sciences, Mekelweg 4, 2628 CD Delft, The Netherlands

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Abstract. We consider infinite-dimensional diffusions where the interaction between the coordinates has a finite extent both in space and time. In particular, it is not supposed to be smooth or Markov. The initial state of the system is Gibbs, given by a strong summable interaction. If the strongness of this initial interaction is lower than a suitable level, and if the dynamical interaction is bounded from above in a right way, we prove that the law of the diffusion at any time t is a Gibbs measure with absolutely summable interaction. The main tool is a cluster expansion in space uniformly in time of the Girsanov factor coming from the dynamics and exponential ergodicity of the free dynamics to an equilibrium product measure.

KEYWORDS: infinite-dimensional diffusion, cluster expansion, non-Markov drift, Girsanov formula, ultracontractivity, planar rotors

AMS SUBJECT CLASSIFICATION: 60K35, 60H10

1. Introduction

In this paper we study propagation of Gibbsianness for a class of infinite-dimensional diffusions with general space-time interaction. The diffusion $X = (X_i(t))_{t\geq 0, i\in\mathbb{Z}^d}$ solves the Stochastic Differential Equation (2.5) where the dynamical interaction splits into a suitable self-interaction and a bounded (possibly) non-regular space interaction with time memory.

Diffusions with memory, such as stochastic delay equations, are indeed very useful for stochastic modeling e.g. in biomathematics, mathematical finance or physics, where delays in the dynamics can represent memory, inertia in financial systems or time-delayed response of physical systems (see e.g. [1,6,10,26] or [19]).

Recall that simple transformations of Gibbs measures may not preserve the Gibbsianness property. The phenomenon was identified by van Enter, Fernandéz and Sokal in [23] and, since then, an extensive effort has been made to find various situations where such pathologies may arise. An example of a transformation which could yield non-Gibbs measures is time-evolution. More precisely, consider a system of interacting particles (or spins) living on a certain space S and distributed at time $t = 0$ according to some Gibbs measure ν. It may happen that, although the system converges, as time goes to infinity, towards another Gibbs measure μ, under certain conditions on ν and μ, there exists a period of time where the time-evolved measure is not Gibbs any more, since an associated (absolutely summable) interaction does not exist. Such unexpected behavior was pointed out in the following cases: for discrete state space S and spin-flip dynamics in [20,21], in the mean-field set-up, see [7,13], for Markovian diffusions on circles, called planar rotors, in [24,25] and for continuous unbounded spins following independent Ornstein – Uhlenbeck dynamics, see [16]. Note that dynamical Gibbs-non-Gibbs transitions have also been investigated from a large-deviation point of view in [22].

Here, on the contrary, we are interested in the *conservation of Gibbsianness* regime for particles living in continuous state spaces. We search for conditions which assure that the time-evolved measure of a system of interacting particles starting from a Gibbs distribution stays Gibbsian during its *whole time-evolution*.

It turns out that for short-time evolutions, conservation of Gibbsianness is robust, as was proved in [5] for Markovian $\mathbb{R}^{\mathbb{Z}^d}$-valued diffusions and in [18] for a particular class of non-Markovian $\mathbb{R}^{\mathbb{Z}^d}$-valued diffusions. Earlier propagation of Gibbsianness results during the whole time-evolution could already be obtained in [5] in the following particular case: the $\mathbb{R}^{\mathbb{Z}^d}$-valued diffusion is prescribed through a *Markov* interaction function b, itself defined as the gradient of a Hamiltonian.

Consider a diffusion $X = (X_i(t))_{t\geq 0, i\in\mathbb{Z}^d}$ where the dynamical interaction term consists of an ultracontractive self-interaction U (which will constrain the free system to converge fast towards a reference product measure) and a bounded non-regular and non-Markov space-time interaction b regulated by a multiplicative scalar factor β. (The parameter β could be considered as a *dynamical inverse temperature*.) We prove that, for any initial Gibbs measure with inverse temperature β_0 bounded above ($\beta_0 < \bar{\beta}_0$), and for a dynamical interaction below a certain intensity ($\beta < \bar{\beta}$), the law of the diffusion at any time t is a Gibbs measure on $\mathbb{R}^{\mathbb{Z}^d}$, described by an absolutely summable interaction. In that sense, Gibbsianness propagates for a very large class of $\mathbb{R}^{\mathbb{Z}^d}$-valued diffusion dynamics which include time-delayed terms. As a corollary, our method leads to a constructive existence result for a class of infinite-dimensional SDE with small (possibly) non-Markovian drift. Finally in Section 3.6 we state the corresponding propagation of Gibbsianness result for a system of planar rotors.

There are in the present paper two main differences and improvements with respect to the paper [5]. First, the Girsanov density of the approximating finite-dimensional diffusions contains stochastic integrals, which cannot be turned into ordinary (bounded) integrals as was done for gradient diffusions. In particular the local interaction functionals Ψ introduced in (3.5) are highly unbounded, even not everywhere defined and their control should be done via (exponential) moments. Secondly, since the interaction b between the coordinates contains a time component, one cannot make use any more of the decoupling method as in [5], which was a simple way to compare the infinite-dimensional dynamics with another, much simpler. To bypass these difficulties, the main tool is a cluster expansion in space — uniform in time — of Girsanov factors coming from the dynamics .

The rest of the paper is divided into the following sections. 2. Framework and main result. 3. Proof of the main theorem with, in particular, the cluster expansion and the estimates of the cluster weights. In Section 3.6 we come back to examples and applications.

2. Framework and main result

In this section we define the necessary framework for our study and state our main result.

2.1. Interaction and Gibbs measures

The main mathematical concept considered in this paper is that of a Gibbs measure on the configuration space $\mathbb{R}^{\mathbb{Z}^d}$. It is based on a so-called interaction function, of which we now recall the definition.

Definition 2.1. An **interaction** ϕ on $\mathbb{R}^{\mathbb{Z}^d}$ is a collection of functions ϕ_Λ from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}$, where Λ is any finite subset of $\mathbb{Z}^d$, satisfying the following properties.

1. ϕ is $\mathcal{F}_\Lambda$-measurable, where $\mathcal{F}_\Lambda$ denotes the sigma-field generated by the canonical projections on $\mathbb{R}^\Lambda$.
2. ϕ is absolutely summable, which means that $\sum_{\Lambda \ni i} ||\phi_\Lambda||_\infty < +\infty$ for all $i \in \mathbb{Z}^d$.

We also recall some other summability assumptions which can be satisfied by an interaction.

(A1) (strong summability) $\sup_{i\in\mathbb{Z}^d}\sum_{\Lambda\ni i}(|\Lambda|-1)||\phi_\Lambda||_\infty < +\infty$, where $|\Lambda|$ denotes the cardinality of Λ.

(A2) (finite-body interaction) $\phi_\Lambda \equiv 0$ as soon as $|\Lambda|$ is large enough.

(A3) (finite-range interaction) $\phi_\Lambda \equiv 0$ as soon as the diameter of Λ is large enough.

Remark that, for bounded interactions, (A3)⇒ (A2)⇒ (A1).

Given an interaction ϕ we define the associated Hamiltonian function $h = (h_\Lambda)_{\Lambda\subset\mathbb{Z}^d}$ by

$$h_\Lambda : \mathbb{R}^\Lambda \times \mathbb{R}^{\Lambda^c} \to \mathbb{R}, \quad h_\Lambda(x_\Lambda, z_{\Lambda^c}) = \sum_{\Lambda':\Lambda'\cap\Lambda\neq\emptyset} \phi_{\Lambda'}(x_\Lambda z_{\Lambda^c}), \tag{2.1}$$

where z is called the boundary condition. We write as usual $x_\Lambda z_{\Lambda^c}$ as shorthand for the concatenation of the configuration x restricted to Λ and the configuration z restricted to Λ^c.

The *finite-volume Gibbs measure* with interaction ϕ at inverse temperature β_0 with boundary condition z w.r.t. an a-priori measure m on $\mathbb{R}$ is the probability measure given by

$$\nu_{\Lambda,z}(dx_\Lambda) = \frac{1}{Z_\Lambda^z}\exp(-\beta_0 h_\Lambda(x_\Lambda, z_{\Lambda^c}))\, m^{\otimes\Lambda}(dx_\Lambda) \tag{2.2}$$

where Z_Λ^z is the renormalizing factor. If the measure m is finite, the scalar Z_Λ^z, also called partition function, is finite too.

As usual the finite-volume measure with *free boundary conditions* is defined by

$$\nu_\Lambda(dx_\Lambda) = \frac{1}{Z_\Lambda}\exp\Big(-\beta_0\sum_{A\subset\Lambda}\phi_A(x_\Lambda)\Big)\, m^{\otimes\Lambda}(dx_\Lambda). \tag{2.3}$$

We can now define the concept of (infinite-volume) Gibbs measure.

Definition 2.2. The measure ν is a Gibbs measure with interaction ϕ at inverse temperature β_0 if for all finite $\Lambda \subset \mathbb{Z}^d$ and smooth $\mathcal{F}_\Lambda$-measurable test functions f, the so-called DLR equations are satisfied

$$\int f(x_\Lambda)\,\nu(dx) = \int\int f(x_\Lambda)\,\nu_{\Lambda,z}(dx_\Lambda)\,\nu(dz), \tag{2.4}$$

which means that the measure $\nu_{\Lambda,z}$ is a regular version of the conditional probability $\nu(dx_\Lambda \mid x_{\Lambda^c} = z_{\Lambda^c})$. One denotes by $\mathcal{G}_{\beta_0}(\phi)$ the set of such Gibbs measures.

2.2. Infinite-volume dynamics

On the path space $\Omega = C(\mathbb{R}_+, \mathbb{R})^{\mathbb{Z}^d}$, endowed by the canonical sigma-field $\mathcal{F}$, we consider the infinite-dimensional diffusion defined as solution of the Stochastic Differential Equation:

$$\begin{cases} dX_i(t) = dB_i(t) + \big(-\frac{1}{2}U'(X_i(t)) + \beta\, b_i([0,t], X)\big)\, dt, \quad i \in \mathbb{Z}^d, \\ X(0) \sim \nu, \end{cases} \tag{2.5}$$

where $(B_i)_{i\in\mathbb{Z}^d}$ is a sequence of real-valued independent Brownian motions, U is a self-potential function, and the drift term of the i^{th} coordinate at time t, $b_i([0,t], \cdot)$, may possibly depend on the values of the other coordinates of the process on the whole time interval $[0,t]$. Thus the process X could be non-Markov.

We denote by Q^ν the law of the solution of the SDE (2.5) (resp. Q^x if the initial condition is deterministic, i.e. $\nu = \delta_x$).

We now state the precise assumptions satisfied by the drift term.

(B1) The self-potential $U : \mathbb{R} \to \mathbb{R}$ is smooth and *ultracontractive*, in such a way that the one-dimensional free dynamics

$$dx(t) = dB(t) - \frac{1}{2}U'(x(t))dt \tag{2.6}$$

generates a semi-group which maps $L^2(m)$ into $L^\infty(m)$, where m is its unique stationary probability measure: $m(dx) = (1/Z)\, \exp\{-U(x)\}dx$.

(B2) The space-time interaction is the product of a scalar intensity parameter β with a functional $b = (b_i)_i$ on Ω which is adapted and local in space and time: There exists a finite neighborhood $\mathcal{N} \subset \mathbb{Z}^d$ around 0 and a finite memory-time $t_0 > 0$ such that for all $i \in \mathbb{Z}^d$, $\omega \in \Omega$, $b_i([0,t],\omega) = b_i(t, (\omega_{i+\mathcal{N}}(s) : t - t_0 \leq s \leq t))$.

(B3) The drift functional b is bounded, i.e. there exists $\bar{b} > 0$ such that $\sup_{i\in\mathbb{Z}^d} \sup_{\omega\in\Omega} \sup_{t\geq 0} |b_i([0,t],\omega)| \leq \bar{b}$.

The following theorem is the main result of our paper.

Theorem 2.1. *Consider Q^ν, the law of the infinite-dimensional SDE* (2.5) *with a drift satisfying assumptions* (B1)–(B3) *and suppose that the initial distribution ν is a Gibbs measure in $\mathcal{G}_{\beta_0}(\phi)$ where ϕ satisfies the strong summability assumption* (A1)*. There exists a bound $\bar{\beta}_0 > 0$ for the initial inverse temperature and a bound $\bar{\beta} > 0$ for the intensity of the space-time interaction such that, if $0 \leq \beta \leq \bar{\beta}$ and $0 \leq \beta_0 \leq \bar{\beta}_0$, for all $t \geq 0$ the time-evolved measure $Q^\nu \circ X(t)^{-1}$ is a Gibbs measure w.r.t. some interaction ϕ^t, which is then absolutely summable.*

Corollary 2.1. *The proof of the above Theorem* 2.1 *provides a constructive way to obtain a solution of the SDE* (2.5) *at any time t for small β as limit* (*in terms of cluster expansions*) *of finite-dimensional approximations, whose existence* (*and uniqueness*) *is ensured by the assumption* (B3).

3. Proof

The dynamics we deal with are obtained by perturbing through the interaction $\beta\, b$ a system of independently evolving components. The law on Ω of the non-interacting system, also called the infinite-dimensional *free system*, corresponding to $\beta = 0$ and the deterministic initial value $x \in \mathbb{R}^{\mathbb{Z}^d}$, is denoted by P^x and is the product law

$$P^x = \otimes_{i\in\mathbb{Z}^d} P_i^{x_i}$$

where $P_i^{x_i}$ is the law on $C(\mathbb{R}_+, \mathbb{R})$ of the one-dimensional SDE (2.6) with initial condition $x_i \in \mathbb{R}$. We denote by $p_t(x_i, \cdot)$ its density function at time t with respect to m:

$$P_i^{x_i} \circ X(t)^{-1}(dy_i) = p_t(x_i, y_i)\, m(dy_i). \tag{3.1}$$

3.1. A finite-dimensional approximation

As usual, we approximate the infinite-volume dynamics by a sequence of finite-volume dynamics. Let Λ be a finite subset of $\mathbb{Z}^d$, and define

$$\Lambda^- = \{i \in \Lambda : \{i + \mathcal{N}\} \subset \Lambda\} \tag{3.2}$$

its $\mathcal{N}$-interior.

Let Q_Λ^x denote the law of the finite-volume dynamics

$$\begin{cases} dX_i(t) = dB_i(t) + \Big(-\dfrac{1}{2}U'(X_i(t)) + \beta\, b_i([0,t], X)\Big)\, dt, \quad i \in \Lambda^-, \\ dX_i(t) = dB_i(t) - \dfrac{1}{2}U'(X_i(t))\, dt, \quad i \in \Lambda \setminus \Lambda^-, \\ X_\Lambda(0) = x_\Lambda. \end{cases} \tag{3.3}$$

It is a perturbation of the finite-volume free dynamics $P_\Lambda^x = \otimes_{i\in\Lambda} P_i^{x_i}$.

3.2. Cluster expansion of the finite-dimensional density

First we expand the finite-volume density of the perturbed system w.r.t. the free system.

Lemma 3.1. *At any time t, $Q_\Lambda^x \circ X(t)^{-1}$ is absolutely continuous with respect to $P_\Lambda^x \circ X(t)^{-1}$ on $\mathbb{R}^\Lambda$ and its density is given by*

$$f_\Lambda^t(x, y) := \frac{dQ_\Lambda^x \circ X(t)^{-1}}{dP_\Lambda^x \circ X(t)^{-1}}(y_\Lambda) = \mathbb{E}_{P^{xy}_{\Lambda,[0,t]}}\Big[\exp\Big(-\sum_{A\subset\Lambda} \Psi_{A,[0,t]}(X)\Big)\Big]. \tag{3.4}$$

$P^{xy}_{[0,t]}$ denotes the law of the bridge on $[0,t]$ obtained by conditioning P_Λ to be at time 0 in x_Λ and at time t in y_Λ, and the functional $\Psi_{A,[0,t]}$ satisfies

$$\Psi_{A,[0,t]}(X) = \begin{cases} -\beta\int_0^t b_i([0,s],X)d\overline{B}_i(s) + \frac{\beta^2}{2}\int_0^t b_i^2([0,s],X)ds & \text{if } \exists\, i : A = \mathcal{N} + i \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

where the process $\overline{B}$ is defined as

$$\overline{B}_i(t)(\omega) = \omega_i(t) + \frac{1}{2}\int_0^t U'(\omega_i(s))ds.$$

Proof. By Girsanov's Theorem,

$$dQ^x_\Lambda(X) = \exp\Big(\sum_{i\in\Lambda^-}\Big(\beta\int_0^t b_i([0,s],X)d\overline{B}_i(s) - \frac{\beta^2}{2}\int_0^t b_i^2([0,s],X)ds\Big)\Big)dP^x_\Lambda(X)$$

$$=: M_{\Lambda,t}(X)\, dP^x_\Lambda(X).$$

Let f be a bounded local function on $\mathbb{R}^{\mathbb{Z}^d}$. Then

$$\begin{aligned}\mathbb{E}_{Q^x_\Lambda}(f(X(t))) &= \mathbb{E}_{P^x_\Lambda}\Big(M_{\Lambda,t}(X)f(X(t))\Big)\\ &= \int \mathbb{E}_{P^{xy}_{\Lambda,[0,t]}}\Big(M_{\Lambda,t}(X)f(X(t))\Big)p_t(x_\Lambda,y_\Lambda)\, m(dy_\Lambda)\\ &= \int f(y_\Lambda)\mathbb{E}_{P^{xy}_{\Lambda,[0,t]}}(M_{\Lambda,t}(X))p_t(x_\Lambda,y_\Lambda)\, m(dy_\Lambda)\end{aligned}$$

which leads to the desired result. □

Remark 3.1. The functional Ψ is not defined a priori on the whole path space Ω, but only for $\omega \in \Omega' \subset \Omega$ for which the stochastic integral $\int_0^t b_i([0,s],\omega)d\omega_i(s)$ makes sense.

If we would assume the initial inverse temperature to be very small (i.e. β_0 vanishing), we could use the usual cluster expansion techniques with respect to both β_0 and β in space to obtain a perturbative result around the free stationary case ($\beta_0 = \beta = 0$). As we would like to treat the more general case where β_0 is not necessarily close to 0, we now develop a more involved space-time cluster expansion technique, which allows us to control space and time simultaneously.

In the following let us perform, for a fixed time t, the cluster expansion for $f^t_\Lambda(x,y)$ w.r.t. the intensity β of the dynamical perturbation.

We decompose the time interval $[0,t]$ into M subintervals $I_j := [jT,(j+1)T]$ with length $T = t/M$, where T is a time-step length larger than the range t_0 of the time-memory of the drift b, in such a way that $[jT - t_0, jT + t_0] \subset [(j-1)T,(j+1)T]$. This latter condition is important to control the range of the time interaction.

A *temporal edge* is a unit space-time pair of the form (i, I_j) with $i \in \mathbb{Z}^d$ and $j \in \mathbb{N}$. Its vertices are the points (i, jT) and $(i,(j+1)T)$ in $\mathbb{Z}^d \times \mathbb{R}_+$. A *space cluster* γ^j, $j \in \mathbb{N}$ is a finite collection of pairwise space-connected temporal edges, that is $\gamma^j = \{(i_1, I_j), \ldots, (i_m, I_j)\}, i_1, \ldots, i_m \in \mathbb{Z}^d$, where the sequence of subsets $i_1 + \mathcal{N}, i_2 + \mathcal{N}, \cdots, i_m + \mathcal{N}$ is connected:

$$(i_1 + \mathcal{N}) \cap (i_2 + \mathcal{N}) \neq \emptyset, \ldots, (i_{m-1} + \mathcal{N}) \cap (i_m + \mathcal{N}) \neq \emptyset.$$

Two space clusters γ^j_1 and γ^j_2 are called *compatible* if no temporal edge of the first one is space-connected with any temporal edge of the other one.

A *time cluster* τ^i, $i \in \mathbb{Z}^d$, is a finite collection of temporal edges of the following type $\tau^i = \{(i, I_j), \ldots, (i, I_{j+r})\}, j, r \in \mathbb{N}$. We call *space-time cluster* Γ a non-empty collection of space and time clusters of the form $\Gamma = \{\gamma^{j_1}_1, \ldots, \gamma^{j_s}_s; \tau^{i_1}_1, \ldots, \tau^{i_p}_p\}$. The *spatial support* of Γ is the set denoted by $[\Gamma]$ of all vertices belonging to the temporal edges which compose Γ. We denote by $[\Gamma]_{k,l}$ the set of all vertices belonging to the temporal edges which compose Γ except $j = k$ and $j = l$. Two space-time clusters are called *non-intersecting* if their space clusters are compatible and their time clusters are disjoint.

Proposition 3.1. *There exist cluster weights $K^t_\Gamma(x,y)$ indexed by space-time clusters $\Gamma \subset \Lambda \times [0,t]$, which depend on t, β, x and y such that*

$$f^t_\Lambda(x,y) = 1 + \sum_{v \in \mathbb{N}^*} \sum_{\{\Gamma_1,\ldots,\Gamma_v\}} K^t_{\Gamma_1}(x,y) \cdots K^t_{\Gamma_v}(x,y) \tag{3.6}$$

where the last summation is on all pairwise non-intersecting space-time clusters Γ_l included in $\Lambda \times [0,t]$.

Proof. For simplicity, write $\Psi_{k,j}$ instead of $\Psi_{k+\mathcal{N},I_j}$. We expand (3.4) decomposing the bridge $P^{xy}_{\Lambda,[0,t]}$ on the time interval $[0,T]$ into a concatenation of bridges of the form $P^{x^{(j)}x^{(j+1)}}_{\Lambda,I_j}$:

$$\begin{aligned} f^t_\Lambda(x,y) &= \mathbb{E}_{P^{xy}_{\Lambda,[0,t]}}\Big[\exp\Big(-\sum_{k\in\Lambda^-} \Psi_{k+\mathcal{N},[0,t]}(X)\Big)\Big] \\ &= \int\int \prod_{j=0}^{M-1} \prod_{k\in\Lambda^-} e^{-\Psi_{k,j}(X)} \bigotimes_{0\le j\le M-1} P^{x^{(j)}x^{(j+1)}}_{\Lambda,I_j}(dX) \end{aligned}$$

$$\times \prod_{\substack{i\in\Lambda\\0\le j\le M-2}} p_T(x_i^{(j)},x_i^{(j+1)}) \bigotimes_{\substack{i\in\Lambda\\0\le j\le M-2}} m(dx_i^{(j+1)})$$

$$= \int \prod_{j=0}^{M-1} \int \prod_{k\in\Lambda^-} e^{-\Psi_{k,j}(X)} P_{\Lambda,I_{j-1}}^{x^{(j-1)}x^{(j)}}(dX) P_{\Lambda,I_j}^{x^{(j)}x^{(j+1)}}(dX)$$

$$\times \prod_{\substack{i\in\Lambda\\0\le j\le M-2}} p_T(x_i^{(j)},x_i^{(j+1)}) \bigotimes_{\substack{i\in\Lambda\\0\le j\le M-2}} m(dx_i^{(j+1)})$$

where $x^{(0)} := x \in \mathbb{R}^\Lambda$ and $x^{(M)} := y \in \mathbb{R}^\Lambda$. Now use

$$\prod_{k\in\Lambda^-} e^{-\Psi_{k,j}(X)} = \prod_{k\in\Lambda^-}(1+e^{-\Psi_{k,j}(X)}-1)$$
$$= 1 + \sum_{n\ge1} \sum_{\{\gamma_1^j,\dots,\gamma_n^j\}} \prod_{l=1}^{n} \prod_{(k,I_j)\in\gamma_l^j} (e^{-\Psi_{k,j}(X)}-1)$$

where the last summation is over all pairwise compatible space clusters included in $\Lambda\times[0,t]$.

On the other hand, for $z^{(0)},\dots,z^{(M)}\in\mathbb{R}$,

$$\begin{aligned}\prod_{j=0}^{M-2} p_T(z^{(j)},z^{(j+1)}) &= \prod_{j=0}^{M-2}(1+p_T(z^{(j)},z^{(j+1)})-1)\\ &= 1+\sum_\tau \prod_{I_j\in\tau}(p_T(z^{(j)},z^{(j+1)})-1)\\ &= 1+\sum_{p\ge1}\sum_{\{\tau_1,\dots,\tau_p\}}\prod_{u=1}^{p}\prod_{I_j\in\tau_u}(p_T(z^{(j)},z^{(j+1)})-1)\end{aligned} \tag{3.7}$$

where the summation on the second line is over all collections τ of time intervals of the type $I_j\subset[0,t]$ and the last summation on the third line is over all pairwise disjoint collections of such consecutive time intervals. One obtains

$$f_\Lambda^t(x,y) = \int_{\mathbb{R}^{|\Lambda|(M-1)}} \prod_{j=0}^{M-1} \int_{\mathbb{R}^{|\Lambda|}} \Big(1+\sum_{n\ge1}\sum_{\{\gamma_1^j,\dots,\gamma_n^j\}}\prod_{l=1}^{n}\prod_{(k,I_j)\in\gamma_l^j}(e^{-\Psi_{k,j}(X)}-1)\Big)$$
$$\times P_{\Lambda,I_{j-1}}^{x^{(j-1)}x^{(j)}}(dX) P_{\Lambda,I_j}^{x^{(j)}x^{(j+1)}}(dX)$$
$$\times \prod_{i\in\Lambda}\Big(1+\sum_{p\ge1}\sum_{\{\tau_1^i,\dots,\tau_p^i\}}\prod_{u=1}^{p}\prod_{I_j\in\tau_u^i}(p_T(x_i^{(j)},x_i^{(j+1)})-1)\Big) \bigotimes_{\substack{i\in\Lambda\\0\le j\le M-1}} m(dx_i^{(j)}) \tag{3.8}$$

$$=: 1 + \sum_{v \geq 1} \sum_{\{\Gamma_1, \dots, \Gamma_v\}} K^t_{\Gamma_1}(x, y) \cdot \ldots \cdot K^t_{\Gamma_v}(x, y)$$

where the last summation is over all pairwise non-intersecting space-time clusters Γ_l included in $\Lambda \times [0, t]$. Therefore, for $\Gamma = \{\gamma_1^{j_1}, \dots, \gamma_s^{j_s}; \tau_1^{i_1}, \dots, \tau_p^{i_p}\}$, the cluster weight K^t_Γ is defined by

$$K^t_\Gamma(x, y) = \int \prod_{m=1}^{s} \int \prod_{k \in \gamma_m^{j_m}} \left(e^{-\Psi_{k, j_m}(X)} - 1\right) P_{\Lambda, I_{j_m - 1}}^{x^{(j_m - 1)} x^{(j_m)}}(dX) P_{\Lambda, I_{j_m}}^{x^{(j_m)} x^{(j_m + 1)}}(dX)$$

$$\times \prod_{u=1}^{p} \prod_{I_j \in \tau_u^{i_u}} (p_T(x_{i_u}^{(j)}, x_{i_u}^{(j+1)}) - 1) \bigotimes_{(i,j) \in [\Gamma]_{0,M}} m(dx_i^{(j)}) \tag{3.9}$$

$$= \int \prod_{m=1}^{s} \mathcal{K}(\gamma_m^{j_m}) \prod_{u=1}^{p} \prod_{I_j \in \tau_u^{i_u}} (p_T(x_{i_u}^{(j)}, x_{i_u}^{(j+1)}) - 1) \bigotimes_{(i,j) \in [\Gamma]_{0,M}} m(dx_i^{(j)})$$

with, for any $2 \leq j \leq M - 2$,

$$\mathcal{K}(\gamma^j) := \int \prod_{k \in \gamma^j} (e^{-\Psi_{k,j}(X)} - 1) \bigotimes_{i \in \Lambda} P_{i, I_{j-1}}^{x_i^{(j-1)} x_i^{(j)}}(dX_i) P_{i, I_j}^{x_i^{(j)} x_i^{(j+1)}}(dX_i)$$

and for the space-cluster γ^1, taking into account the fixed boundary condition $x^{(0)} = x$

$$\mathcal{K}(\gamma^1) := \int \prod_{k \in \gamma^1} (e^{-\Psi_{k,1}(X)} - 1) \bigotimes_{i \in \Lambda} P_{i, I_0}^{x_i x_i^{(1)}}(dX_i) P_{i, I_1}^{x_i^{(1)} x_i^{(2)}}(dX_i),$$

resp. for the space-cluster γ^{M-1} on the time interval $I_{M-1} = [t - T, t]$, taking into account the fixed boundary condition $x^{(M)} = y$

$$\mathcal{K}(\gamma^{M-1}) := \int \prod_{k \in \gamma^{M-1}} (e^{-\Psi_{k, M-1}(X)} - 1) \bigotimes_{i \in \Lambda} P_{i, I_{M-2}}^{x_i^{(M-2)} x_i^{(M-1)}}(dX_i) P_{i, I_{M-1}}^{x_i^{(M-1)} y_i}(dX_i).$$

□

3.3. Cluster estimates

The next step is to estimate the cluster weights $K^t_\Gamma(x, y)$, defined by (3.9), as a function of the small parameter β.

Proposition 3.2. *Let $\Gamma = \{\gamma_1^{j_1}, \dots, \gamma_s^{j_s}; \tau_1^{i_1}, \dots, \tau_p^{i_p}\}$ be a space-time cluster. There exists a function $\lambda(\beta) > 0$ vanishing when β tends to 0 such that the cluster weight $K^t_\Gamma(x, y)$ is bounded uniformly in time and space as follows:*

$$\sup_{t \geq 0} \sup_{x,y} |K^t_\Gamma(x, y)| \leq \lambda(\beta)^{|\Gamma|} \tag{3.10}$$

where $|\Gamma|$, *the cardinality of* Γ, *is the total number of unit temporal edges which compose* Γ.

Proof. To bound the cluster weights we need to interchange integration and products in (3.9). Therefore we make use of the following inequalities, generalizing Hölder inequalities, see [17], Lemma 5.2.

Lemma 3.2. *Let* $(\mu_z)_{z\in\chi}$ *be a family of probability measures, each one defined on a measurable space* E_z *where the elements* z *belong to some finite set* χ. *Let* $(g_k)_k$ *be a family of functions on* $E_\chi = \times_{z\in\chi} E_z$ *such that each* g_k *is* χ_k*-local for a certain* $\chi_k \subset \chi$ *in the sense that for all* $e \in E_\chi$, $g_k(e) = g_k(e_{|\chi_k})$, *and let* $(\rho_k)_k$ *be positive numbers such that, for all* $z \in \chi$, $\sum_{\{k:\chi_k \ni z\}} 1/\rho_k \leq 1$. *Then*

$$\Big|\int_{E_\chi} \prod_k g_k \otimes_{z\in\chi} d\mu_z\Big| \leq \prod \Big(\int_{E_{\chi_k}} |g_k|^{\rho_k} \underset{z\in\chi_k}{\otimes} d\mu_z\Big)^{1/\rho_k}. \tag{3.11}$$

We apply Lemma 3.2 with $\chi := \gamma^j + \mathcal{N}$, $\chi_k := k + \mathcal{N}$, $E_z := C(\mathbb{R}_+, \mathbb{R})$, $g_k := e^{-\Psi_{k,j}} - 1$, $\mu_k := P_{k,I_{j-1}}^{x_k^{(j-1)} x_k^{(j)}} \otimes P_{k,I_j}^{x_i^{(j)} x_k^{(j+1)}}$ and $\rho_i = 4|\mathcal{N}|$ for all i. Since for each $i \in \Lambda$, there are at most $|\mathcal{N}|$ factors k such that $\Psi_{k,j}(X)$ depends on X_i, the assumption $\sum_{k+\mathcal{N}\ni i} 1/(4|\mathcal{N}|) \leq 1$ is satisfied. We then obtain the upper bound

$$\begin{aligned} |\mathcal{K}(\gamma^j)| &\leq \prod_{k\in\gamma^j} \Big[\int (e^{-\Psi_{k,j}} - 1)^{4N} \underset{i\in k+\mathcal{N}}{\otimes} P_{i,I_{j-1}}^{x_i^{(j-1)},x_i^{(j)}}(dX_i) P_{i,I_j}^{x_i^{(j)},x_i^{(j+1)}}(dX_i)\Big]^{1/4|\mathcal{N}|} \\ &=: \prod_{k\in\gamma^j} \mathbb{K}_{k,j}(x^{(j-1)}, x^{(j)}, x^{(j+1)}). \end{aligned} \tag{3.12}$$

Remark at this place that we especially used the space-locality of the interaction b (assumption (B2)). Therefore

$$\begin{aligned} |K_\Gamma^t(x,y)| \leq \int & \prod_{m=1}^{s} \prod_{k\in\gamma_m^{j_m}} \mathbb{K}_{k,j}(x^{(j-1)}, x^{(j)}, x^{(j+1)}) \\ & \times \prod_{u=1}^{p} \prod_{I_j\in\tau_u^{i_u}} (p_T(x_{i_u}^{(j)}, x_{i_u}^{(j+1)}) - 1) \underset{(i,j)\in[\Gamma]_{0,M}}{\otimes} m(dx_i^{(j)}). \end{aligned} \tag{3.13}$$

We apply once more Lemma 3.2 to bound the right hand side of (3.13) by

$$\prod_{m=1}^{s} \prod_{k\in\gamma_m^{j_m}} \Big(\int \mathbb{K}_{k,j}^{N_1}(x^{(j-1)}, x^{(j)}, x^{(j+1)}) \underset{i,j}{\otimes} m(dx_i^{(j)})\Big)^{1/N_1}$$

$$\times \prod_{u=1}^{p} \prod_{I_j \in \tau_u^{i_u}} \left(\int (p_T(x_{i_u}^{(j)}, x_{i_u}^{(j+1)}) - 1)^{N_2} \underset{i,j}{\otimes} m(dx_i^{(j)}) \right)^{1/N_2}$$

for any right choice of N_1, N_2 satisfying $2|\mathcal{N}|/N_1 + 2/N_2 \leq 1$. Choose e.g. $N_1 = 4|\mathcal{N}|$ and $N_2 = 4$. In the two next lemmas we will show that the first integral describing the spatial interaction (resp. the second integral describing the time interaction) is bounded uniformly in t, x and y by a function $C_1(\beta)$ (resp. by $C_2(\beta)$), which leads to

$$|K_\Gamma^t(x,y)| \leq C_1(\beta)^{\sum_m |\gamma_m^{jm}|} C_2(\beta)^{\sum_u |\tau_u^{i_u}|} \leq \max(C_1, C_2)(\beta)^{\sum_m |\gamma_m^{jm}| + \sum_u |\tau_u^{i_u}|} \tag{3.14}$$

which yields the claim (3.10) with $\lambda(\beta) := \max(C_1, C_2)(\beta)$. □

In the next lemma we prove appropriate upper bounds for the spatial interaction, that is for the integral of $\mathbb{K}$, treating first the case where the space cluster γ^j does not contain any boundary temporal edge, that is $j \neq 0$ and $j \neq M$.

Lemma 3.3. *Let $j = 1, \dots, M-1$. There exists a positive real number C_1 depending only on β (and uniform in t, x, y, k and j), vanishing when β goes to 0, such that the following upper bound holds*

$$\int \mathbb{K}_{k,j}^{4|\mathcal{N}|}(x^{(j-1)}, x^{(j)}, x^{(j+1)}) \otimes_{i \in k+\mathcal{N}} m(dx_i^{(j)}) \leq C_1(\beta)^{4|\mathcal{N}|}. \tag{3.15}$$

Proof. Let us fix k. Then

$$\begin{aligned}
&\int \mathbb{K}_{k,j}^{4|\mathcal{N}|}(x^{(j-1)}, x^{(j)}, x^{(j+1)}) \otimes_{i \in k+\mathcal{N}} m(dx_i^{(j)}) \\
&\leq \int\int (e^{-\Psi_{k,j}(X)} - 1)^{4|\mathcal{N}|} \underset{i \in k+\mathcal{N}}{\otimes} P_{i,I_{j-1}}^{x_i^{(j-1)} x_i^{(j)}}(dX_i)\, P_{i,I_j}^{x_i^{(j)} x_i^{(j+1)}}(dX_i) \\
&\quad \times m(dx_i^{(j-1)}) m(dx_i^{(j)}) m(dx_i^{(j+1)}) \\
&= \mathbb{E}_{P_\Lambda}\big((e^{-\Psi_{k,j}(X)} - 1)^{4|\mathcal{N}|}\big).
\end{aligned}$$

We remark that, for any $\zeta \in \mathbb{R}$,

$$\begin{aligned}
(e^\zeta - 1)^{4|\mathcal{N}|} &= \zeta^{4|\mathcal{N}|} \left(\int_0^1 e^{u\zeta} du \right)^{4|\mathcal{N}|} \\
&= \zeta^{4|\mathcal{N}|} \int_0^1 \dots \int_0^1 \exp\{(u_1 + \dots + u_{4|\mathcal{N}|})\zeta\} du_1 \dots du_{4|\mathcal{N}|}.
\end{aligned}$$

Hence

$$\mathbb{E}_{P_\Lambda}\big((e^{-\Psi_{k,j}(X)}-1)^{4|\mathcal{N}|}\big)\big) = \int\limits_{[0,1]^{4|\mathcal{N}|}} \mathbb{E}_{P_\Lambda}(\Psi_{k,j}^{4|\mathcal{N}|} e^{-(u_1+\ldots+u_{4|\mathcal{N}|})\Psi_{k,j}})\, du_1 \ldots du_{4|\mathcal{N}|}.$$

The expectation above can be written as

$$\frac{\partial}{\partial z^{4|\mathcal{N}|}} \mathbb{E}_{P_\Lambda}(e^{-z\Psi_{k,j}})\Big|_{z=u_1+\ldots+u_{4|\mathcal{N}|}},$$

the $4|\mathcal{N}|^{th}$-derivative of the Laplace transform L of the functional $\Psi_{k,j}$ at $z = u_1 + \ldots + u_{4|\mathcal{N}|}$. Let us analyse L:

$$\begin{aligned}
L(z) &= \mathbb{E}_{P_\Lambda}(e^{-z\Psi_{k,j}}) \\
&= \mathbb{E}_{P_\Lambda}\bigg(\exp\Big[z\beta \int\limits_{I_j} b_k([0,s],X)d\overline{B}_k(s) - z^2\beta^2 \int\limits_{I_j} b_k^2([0,s],X)ds\Big] \\
&\qquad \times \exp\Big[z\Big(z-\frac{1}{2}\Big)\beta^2 \int\limits_{I_j} b_k^2([0,s],X)ds\Big]\bigg) \\
&\le \mathbb{E}^{1/2}_{P_\Lambda}\bigg(\exp\Big[2z\beta \int\limits_{I_j} b_k(s,X)d\overline{B}_k(s) - \frac{(2z\beta)^2}{2} \int\limits_{I_j} b_k^2(s,X)ds\Big]\bigg) \\
&\qquad \times \mathbb{E}^{1/2}_{P_\Lambda}\bigg(\exp\Big[z(2z-1)\beta^2 \int\limits_{I_j} b_k^2([0,s],X)ds\Big]\bigg) \\
&= \mathbb{E}^{1/2}_{P_\Lambda}\bigg(\exp\Big[z(2z-1)\beta^2 \int\limits_{I_j} b_k^2([0,s],X)ds\Big]\bigg)
\end{aligned}$$

due to the P_Λ-martingale property of

$$t \mapsto \exp\Big[2z\beta \int\limits_{jT}^{t} b_k([0,s],X)d\overline{B}_k(s) - \frac{(2z\beta)^2}{2} \int\limits_{jT}^{t} b_k^2([0,s],X)ds\Big].$$

To bound not only L but its derivatives, we extend it to the complex plane and notice that

$$\Big|\frac{\partial}{\partial z^{4|\mathcal{N}|}} L(z)\Big| \le \frac{4|\mathcal{N}|\,!}{\rho^{4|\mathcal{N}|}} \sup_{\{\zeta\in\mathbf{C}:|\zeta-z|=\rho\}} |L(\zeta)| \tag{3.16}$$

as soon as L is well defined on $B(z,\rho) = \{\zeta \in \mathbf{C} : |\zeta - z| \le \rho\}$. On $B(z,\rho)$ one has

$$\Big|\exp\Big[\zeta(2\zeta-1)\beta^2 \int\limits_{I_j} b_k^2([0,s],X)ds\Big]\Big| \le \exp\Big[\mathcal{R}e(2\zeta^2-\zeta)\beta^2 \int\limits_{I_j} b_k^2([0,s],X)ds\Big]$$

$$\leq \exp\Big[3(\rho\beta)^2 \int_{I_j} b_k^2([0,s],X)ds\Big]$$
$$\leq \exp\big(3(\rho\beta)^2 T\bar{b}^2\big).$$

Therefore (3.16) becomes $\big|(\partial/\partial z^{4|\mathcal{N}|})\, L(z)\big| \leq (4|\mathcal{N}|\,!)/(\rho^{4|\mathcal{N}|}) \exp(3(\rho\beta)^2 T\bar{b}^2)$. We minimize the r.h.s. choosing $\rho^2 = (2|\mathcal{N}|)/(3T\beta^2\bar{b}^2)$. Thus

$$\Big|\frac{\partial}{\partial z^{4|\mathcal{N}|}} L(z)\Big| \leq c\,(\beta^2 T)^{2|\mathcal{N}|}$$

where c is a positive constant depending only on $\bar{b}$ and $|\mathcal{N}|$. Taking the time step T of the order of $1/\beta$, this leads to the desired inequality (3.15) with $C_1(\beta) := \sqrt{c}\sqrt{\beta}$. □

Let us add a short comment how to compute a similar upper bound in the case of $j = 0$ (resp. in the case $j = M$, in a symmetric way). In that case the spatial support of the cluster γ^0 (resp. γ^M) contains the vertex x (resp. y). In that case one space boundary is fixed (equal to x or y) and we have to control integrals of the type

$$\int\int (e^{-\Psi_{k,0}(X)} - 1)^{4|\mathcal{N}|} \bigotimes_{i\in k+\mathcal{N}} P_{i,I_0}^{x_i x_i^{(1)}}(dX_i) m(dx_i^{(1)}) = \mathbb{E}_{P_\Lambda^x}\big((e^{-\Psi_{k,0}(X)} - 1)^{4|\mathcal{N}|}\big).$$

We then can use the same arguments as in Lemma 3.3, that is identify an exponential martingale and make use of the boundedness of the drift b.

To estimate the time-interaction upper bound C_2 appearing in (3.14), i.e. the fourth moment of the transition kernel p_t of the one-dimensional free dynamics, we also have to distinguish between different types of time clusters composing the space-time cluster Γ: those containing a boundary temporal edge I_0 or I_M and the other time clusters. The next lemma provides an upper bound in that latter case.

Lemma 3.4. *There exists positive constants c', c'' depending only on the self potential U such that*

$$\Big(\int (p_{1/\beta}(z,z') - 1)^4 m(dz)m(dz')\Big)^{(1/4)} \leq c' e^{-c''/\beta}. \tag{3.17}$$

Proof. First

$$\int (p_T(z,z') - 1)^4 m(dz)m(dz') \leq \int ||p_T(\cdot,\cdot) - 1||_{L^\infty}^4\, m(dz)m(dz')$$
$$= ||p_T(\cdot,\cdot) - 1||_{L^\infty}^4.$$

Now, under the ultracontractivity assumption (B1) on the self-interaction U, one has a uniform exponential convergence of p_T to 1 (see e.g. the details of the proof in the appendix of [4]). Moreover the rate of convergence is equal to the spectral gap of p_T. Thus

$$\exists\, c', c'' > 0, \forall T > 0, \quad ||p_T(\cdot,\cdot) - 1||^4_{L^\infty} \leq c' e^{-c''T} \tag{3.18}$$

where c'' is the spectral gap of $(p_t)_t$. We obtain the claim (3.17) taking $T = 1/\beta$. □

When the time cluster τ composing Γ contains the boundary temporal edge I_0 (resp. I_{M-1}) one has to estimate the simple integral $\int (p_T(x,z) - 1)^4 m(dz)$ (resp. $\int (p_T(z,y) - 1)^4 m(dz)$) instead of the above double integration (3.17) under $m \otimes m$. It vanishes with an exponential rate uniformly in x and y when T tends to infinity. Therefore one can take in (3.14) the upper bound $C_2(\beta) := c' \exp\{-c''/\beta\}$.

3.4. Cluster expansion and estimates of the logarithm of the finite-dimensional density

To complete Proposition 3.1 we are now computing an expansion of the logarithm of the density at time t of the finite-dimensional SDE (3.3).

Proposition 3.3. *For β small enough, the logarithm of the Radon – Nikodym derivative* (3.4) *expands as* $\log f^t_\Lambda(x,y) = -\sum_{\Delta\subset\Lambda} \Phi^t_\Delta(x,y)$ *with*

$$\Phi^t_\Delta(x,y) = \sum_{n\geq 0} \sum_{\substack{\{\Gamma_1,\dots,\Gamma_n\} \\ Tr(\Gamma_1,\dots,\Gamma_n)=\Delta}} C(\Gamma_1,\dots,\Gamma_n)\, \mathcal{K}^{x,y}(\Gamma_1)\cdots \mathcal{K}^{x,y}(\Gamma_n) \tag{3.19}$$

where the second sum runs over all collections of disjoint space-time clusters such that their union is connected and $C(\Gamma_1,\dots,\Gamma_n)$ are purely combinatorial coefficients independent of x and y.

Proof. We already know that the density function (3.4) decomposes as

$$f^t_\Lambda(x,y) = \mathbb{E}_{P^{x,y}_{\Lambda,[0,t]}} \Big[\exp\Big(-\sum_{A\subset\Lambda} \Psi_{A,[0,t]}(X)\Big)\Big],$$

which expands as in (3.6) with cluster weights of the form $K^t_\Gamma(x,y)$. We now use the Kotecký and Preiss criterion proven in [14] to derive an expansion of its logarithm. Let Γ be a space-time cluster. We say that another space-time cluster Γ' is *incompatible* with Γ if their associated supports intersect, and we denote this property by the symbol $\Gamma \nsim \Gamma'$. Take now $\bar\beta$ small enough such that for $\beta \leq \bar\beta$,

$$\sup_{x,y\in\mathbb{R}} \sup_{t>0} \sum_{\Gamma'\nsim\Gamma} |K^t_{\Gamma'}(x,y)| e^{|\Gamma'|+\log(|\Gamma'|)} \Big(\leq \sum_{\Gamma'\nsim\Gamma} |\Gamma'| (\lambda(\beta) e)^{|\Gamma'|}\Big) \leq |\Gamma|. \tag{3.20}$$

So, following assertion (2) in [14] the logarithm of $f^t_\Lambda(x,y)$ is expandable, and the following holds:

$$\ln(f^t_\Lambda(x,y)) = \sum_{n\geq 0}\sum_{\Gamma_1,\ldots,\Gamma_n} C(\Gamma_1,\ldots,\Gamma_n)\,\mathcal{K}^{x,y}(\Gamma_1)\cdot\ldots\cdot\mathcal{K}^{x,y}(\Gamma_n). \tag{3.21}$$

The second sum runs over collections of compatible space-time clusters such that their union is connected and $C(\Gamma_1,\ldots,\Gamma_n)$ are combinatorial coefficients coming from the Taylor expansion. Let us now order the space-time clusters in terms of their spatial projections, which are subsets of Λ: If Tr denotes the projection on the spatial support we rewrite (3.21) as $-\sum_{\Delta\subset\Lambda}\Phi^t_\Delta(x,y)$ where Φ^t_Δ is an interaction function given by (3.19).

Moreover Φ^t_Δ is $\mathcal{F}_\Delta\times\mathcal{F}_\Delta$-measurable since the cluster weights $\mathcal{K}^{x,y}(\Gamma)$ depend on x on $supp(\Gamma)\cap(\mathbb{Z}^d\times\{0\})$ and on y on $supp(\Gamma)\cap(\mathbb{Z}^d\times\{t\})$ whose traces are included in Δ. □

Moreover, Kotecký and Preiss provide a useful estimate of the convergence rate of the interaction function Φ^t_Δ in terms of Δ, see inequality (4) in [14]:

Lemma 3.5. *The function Φ^t_Δ satisfies*

$$\lim_{\beta\to 0}\sup_{i\in\mathbb{Z}^d}\sup_{t>0}\sum_{\Delta\ni i}(|\Delta|-1)||\Phi^t_\Delta||_\infty = 0. \tag{3.22}$$

Proof. Indeed Kotecký and Preiss proved the following bound for the interaction function:

$$\sup_{i\in\mathbb{Z}^d}\sup_{t>0}\sum_{\Delta\ni i}(|\Delta|-1)||\Phi^t_\Delta||_\infty \leq 1.$$

Therefore, since the sum on Δ converges uniformly in i and t, we can interchange the limit in β and the summation over Δ to obtain the desired result (3.22). □

3.5. Gibbsianness of the double-layer measure and Kozlov's representation theorem

The rest of the proof of Theorem 2.1 follows the same structure as Steps 2 and 3 of [5], Section 4, in which the drift b is Markov and gradient. Nevertheless, to make our paper self-contained, we sketch the main arguments without giving as much detail. The time-evolved measure we are interested in, $Q^\nu\circ X(t)^{-1}$, is indeed a ν-mixture of the measures $Q^x\circ X(t)^{-1}$ whose approximating densities are $f^t_\Lambda(x,\cdot)$. Therefore, in order to prove the Gibbsianness of $Q^\nu\circ X(t)^{-1}$, we will prove as an intermediate step, the Gibbsianness of the so-called double-layer measure (or measure on the bi-space) $\mathbf{Q}^\nu := Q^\nu\circ(X(0),X(t))^{-1}$ defined on the space $\mathbb{R}^{\mathbb{Z}^d\times\{0,t\}}$.

Lemma 3.6. *Let $\nu \in \mathcal{G}_{\beta_0}(\phi)$, where the interaction ϕ satisfies* (A1). *There exist an upper bound $\bar{\beta}_0 > 0$ for the initial inverse temperature and an upper bound $\bar{\beta} > 0$ for the intensity of the dynamical interaction such that, for $\beta_0 \leq \bar{\beta}_0$ and $\beta \leq \bar{\beta}$, the measure $\boldsymbol{Q}^\nu$ is a Gibbs measure on the bi-space $\mathbb{R}^{\mathbb{Z}^d \times \{0,t\}}$ w.r.t. the a priori measure $m \otimes m$ with an interaction associated to the Hamiltonian*

$$\boldsymbol{H}_{(\Delta,\Delta')}(x,y) := h_\Delta - \sum_{i \in \Delta \cup \Delta'} \log(p_t(x_i, y_i)) + \sum_{A \subset \mathbb{Z}^d; A \cap (\Delta \cup \Delta') \neq \emptyset} \Phi_A^t(x,y) \quad (3.23)$$

where h is the Hamiltonian function derived from ϕ and (Δ, Δ') is short for $(\Delta \times \{0\}) \cup (\Delta' \times \{t\})$.

Proof. Since the interaction ϕ of the initial Gibbs measure satisfies (A1), there exists $\bar{\beta}_0 > 0$ such that for $\beta_0 \leq \bar{\beta}_0$,

$$\beta_0 \sup_{i \in \mathbb{Z}^d} \sum_{\Lambda \ni i} (|\Lambda| - 1) ||\phi_\Lambda||_\infty < 1. \quad (3.24)$$

This assumption implies Dobrushin's uniqueness condition, as is proved e.g. in [8], Proposition (8.8). In particular, for $\beta_0 \leq \bar{\beta}_0$, $\mathcal{G}_{\beta_0}(\phi)$ contains as unique element ν, which can be approximated e.g. by the sequence of finite-volume Gibbs measure ν_Λ with free boundary condition.

Since the sequence $Q_\Lambda^{\nu_\Lambda}$ converges towards Q^ν when Λ increases to $\mathbb{Z}^d$, their joint projection at times 0 and t converges towards $\mathbf{Q}^\nu := Q^\nu \circ (X(0), X(t))^{-1}$ on $\mathbb{R}^{\mathbb{Z}^d \times \{0,t\}}$. $\mathbf{Q}^\nu$ is Gibbs with respect to the a priori measure $\mathbf{m}(dx, dy) = p_t(x,y) m(dx) m(dy)$ and with interaction

$$\Psi_\Delta(x,y) := \phi_\Delta(x) + \Phi_\Delta^t(x,y), \quad x, y \in \mathbb{R}^{\mathbb{Z}^d}, \Delta \subset \mathbb{Z}^d. \quad (3.25)$$

It follows now from (3.24) and (3.25) that there exists a bound $\bar{\beta}$ for the intensity of the dynamical interaction such that, for any $\beta \leq \bar{\beta}$, the Dobrushin's uniqueness assumption is satisfied for Ψ on the bi-space. Therefore $\mathbf{Q}^\nu$ is the unique Gibbs measure on the bi-space associated to the interaction (3.25) or, equivalently, the unique Gibbs measure associated to the Hamiltonian (3.23) and the a priori measure $m \otimes m$. □

Now the measure $\mathbf{Q}^\nu$ can be easily desintegrated in a Gibbsian way w.r.t. the finite-dimensional projections at time t, $\mathbf{Q}^\nu(\cdot \mid X_{\Lambda^c}(t) = y_{\Lambda^c})$, which are defined for a.e. y.

Lemma 3.7. *Fix a finite set $\Lambda \subset \mathbb{Z}^d$. Denote by $\boldsymbol{Q}^{\nu, y_{\Lambda^c}}$ the conditional law of $Q^\nu \circ (X(0), X(t))^{-1}$ given $\{X_{\Lambda^c}(t) = y_{\Lambda^c}\}$. $\boldsymbol{Q}^{\nu, y_{\Lambda^c}}$ is a Gibbs measure on $\mathbb{R}^{(\mathbb{Z}^d \times \{0\}) \cup (\Lambda \times \{t\})}$ with reference measure m and Hamiltonian $\boldsymbol{H}^{y_{\Lambda^c}}$ defined by*

$$\boldsymbol{H}^{y_{\Lambda^c}}_{(\Delta,\Delta')}(x, z_\Lambda) = \boldsymbol{H}_{(\Delta,\Delta')}(x, z_\Lambda y_{\Lambda^c}), \quad (\Delta, \Delta') \subset \mathbb{Z}^d \times \Lambda. \quad (3.26)$$

Furthermore $\boldsymbol{Q}^{\nu,y_{\Lambda^c}}$ can be decoupled as follows

$$\boldsymbol{Q}^{\nu,y_{\Lambda^c}}(dx,dz_\Lambda) = \frac{1}{Z_\Lambda^{y_{\Lambda^c}}} \prod_{i\in\Lambda} p_t(x_i,z_i) \exp\Big(-\sum_{A\cap\Lambda\neq\emptyset} \Phi_A^t(x, z_\Lambda y_{\Lambda^c})\Big) \times m^{\otimes\Lambda}(dz_\Lambda)\,\tilde{Q}^{\nu,y_{\Lambda^c}}(dx)$$

where $\tilde{Q}^{\nu,y_{\Lambda^c}}(dx)$ is the unique Gibbs measure on $\mathbb{R}^{\mathbb{Z}^d}$ defined by the interaction $\tilde{\Phi}^{y_{\Lambda^c}}$ given by

$$\begin{cases} \tilde{\Phi}_i^{y_{\Lambda^c}} & = \phi_i(x) - 1\!\!1_{\{i\in\Lambda^c\}} \log(p_t(x_i,y_i)), \quad i\in\mathbb{Z}^d, \\ \tilde{\Phi}_\Delta^{y_{\Lambda^c}} & = \phi_\Delta(x) - 1\!\!1_{\Delta\cap\Lambda=\emptyset}\Phi_\Delta^t(x,y_\Lambda^c), \Delta\subset\mathbb{Z}^d, \quad |\Delta|\geq 2. \end{cases} \tag{3.27}$$

Indeed, due to the estimates already obtained, it is straightforward to show that, for β small enough, the interaction $\tilde{\Phi}^{y_{\Lambda^c}}$ satisfies Dobrushin's uniqueness condition uniformly in y and in Λ, as a perturbation of the initial interaction, see [5], Lemma 10 and Lemma 11.

Lemma 3.8. *The conditional law of $Q^\nu \circ X(t)^{-1}$ given $\{X_{\Lambda^c}(t) = y_{\Lambda^c}\}$ admits a density w.r.t. $m^{\otimes\Lambda}(dz_\Lambda)$ given by*

$$g_\Lambda^{t,y_{\Lambda^c}}(z_\Lambda) = \frac{1}{Z_\Lambda^{y_{\Lambda^c}}} \int_{\mathbb{R}^{\mathbb{Z}^d}} \prod_{i\in\Lambda} p_t(x_i,z_i) \exp\Big(-\sum_{A\cap\Lambda\neq\emptyset} \Phi_A^t(x,z_\Lambda y_{\Lambda^c})\Big)\tilde{Q}^{\nu,y_{\Lambda^c}}(dx). \tag{3.28}$$

Moreover this density is bounded from below and from above uniformly in y and t, and it is quasilocal, i.e. $\lim_{\Delta\to\mathbb{Z}^d} \sup_{z,z':z_\Delta=z'_\Delta} |g_\Lambda^{t,z_{\Lambda^c}}(z_\Lambda) - g_\Lambda^{t,z'_{\Lambda^c}}(z'_\Lambda)| = 0$.

Boundedness and quasilocality of $g_\Lambda^{t,y_{\Lambda^c}}$ allow to apply Kozlov's representation (Theorem 2 in [15]) which insures the existence of an (absolute summable) interaction ϕ^t for $Q^\nu \circ X(t)^{-1}$.

3.6. Additional remarks

3.6.1. Direct applications

In this section we give some concrete examples for which the assumptions (B1)–(B3) on U and b are satisfied, and thus Theorem 2.1 and Corollary 2.1 hold true.

Recall first some sufficient conditions which imply the ultracontractivity of the one-dimensional free dynamics (2.6), assumption (B1):

$$(1)\quad \liminf_{|x|\to\infty} U''(x) > 0, \qquad (2)\quad \exists\, C \text{ s.t. } U'' - \frac{1}{2}(U')^2 \leq C,$$

$$(3) \quad \exists M > 0 \text{ s.t.} \int\limits_{|x|>M} \frac{1}{U'(x)} dx < +\infty.$$

Properties (1) and (2) ensure the existence of a unique strong solution to the SDE (2.6) and the existence of a unique invariant probability measure, whereas property (3) ensures the ultracontractivity of the associated semigroup, see [12].

Example 3.1 (Markovian case). Let U satisfy above assumptions (1)–(3) and b be a Markovian finite-range bounded drift. It thus satisfies (B2) and (B3). This case includes the one treated in [5].

Example 3.2 (Stochastic resonance). One can generalize the free dynamics in such a way that it remains Markovian but is no more time-homogeneous, introducing an external periodic signal in the dynamics (2.6). These models are used to describe the so-called *stochastic resonance* effect, see e.g. [2,11,26]. So, let us consider as concrete example the following dynamics

$$dx(t) = dB(t) - \frac{1}{2}\Big(x^3(t) - x(t) - A\sin(t)\Big)dt, \tag{3.29}$$

where the drift derives from a time-independent potential given by $U(x) := x^4/4 - x^2/2$ together with a bounded time-periodic forcing with amplitude $A > 0$. In that case properties (1)–(3) are satisfied.

Example 3.3 (Free dynamics with delay). One can generalize the free dynamics introducing a delayed feedback. It then becomes non-Markovian. The over-damped particle motion in the double-well quartic potential as introduced in [19] furnishes such an example: $dx(t) = dB(t) - \frac{1}{2}\Big(x^3(t) - x(t) - \alpha x(t-t_0)\Big)dt$, where $\alpha > 0$ is the strength of the feedback.

The following examples are non-Markovian since they include a time memory.

Example 3.4 (Independent dynamics with time memory).
Let U satisfy (1)–(3). We define the drift by

$$b_i([0,t],\omega) := \begin{cases} \int_0^t \varepsilon(s) f(\omega_i(s))ds & \text{if } t < t_0, \\ \int_{t-t_0}^t \varepsilon(s) f(\omega_i(s))ds & \text{if } t \geq t_0, \end{cases} \tag{3.30}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a measurable bounded function and the time-memory function $\varepsilon : [0,\infty) \to \mathbb{R}$ is assumed to be integrable. This kind of drift b is non-Markovian since it depends on a finite time window with length t_0.

Example 3.5 (Interaction with finite extent in space and time).
Let U satisfy (1)–(3). Fix $t_0 > 0$ and define the drift by

$$b_i([0,t],\omega) := \begin{cases} \int_0^t \alpha_i(t-s, \omega(s))dV_s & \text{if } t < t_0, \\ \int_{t-t_0}^t \alpha_i(t-s, \omega(s))dV_s & \text{if } t \geq t_0, \end{cases} \tag{3.31}$$

where the bounded variation integrator V_s can be deterministic or stochastic and adapted. The functions α_i are bounded and spatially local: $\alpha_i(\cdot, x) = \alpha_i(\cdot, x_{\mathcal{N}})$. Therefore b depends on a finite time window with length t_0.

3.6.2. Planar rotors

In this section we would like to discuss how the above result for propagation of Gibbsianness can be adapted to planar rotors diffusions with non-Markovian drift. It leads to a generalization of the conservation results presented in [25], where the authors considered Markovian dynamics.

Let us first introduce the setting. Take now $\mathbb{S}^{\mathbb{Z}^d}$ as configuration space where $\mathbb{S}$ is the unit circle, which we can identify with the space interval $[0, 2\pi)$ where 0 and 2π are considered to be the same points. We consider the solution $X^{\odot} = (X_i^{\odot}(t))_{t\geq 0, i\in\mathbb{Z}^d}$ of the following infinite system of Stochastic Differential Equations

$$\begin{cases} dX_i^{\odot}(t) = dB_i^{\odot}(t) + \beta\ b_i([0,t], X^{\odot})dt, \quad i \in \mathbb{Z}^d, \\ X^{\odot}(0) \sim \nu, \end{cases} \tag{3.32}$$

on the path space $\Omega_S := C(\mathbb{R}_+, \mathbb{S})^{\mathbb{Z}^d}$ endowed by the canonical sigma-field $\mathcal{F}$. $(B_i^{\odot}(t))_{t\geq 0, i\in\mathbb{Z}^d}$ is a sequence of independent Brownian motions living on the circle $\mathbb{S}$ and the drift term of the i^{th} coordinate, again denoted by $b_i(t, \cdot)$, can depend on the values of the other coordinates on the whole time-interval $[0, t]$. Furthermore ν is supposed to be a suitable initial Gibbs measure. Let Q^ν denote the law of the solution of the SDE (3.32) with initial measure ν.

In the following let us present our assumptions.

The interaction defining the initial Gibbs measure is supposed to be strong summable, that is it satisfies (A1). In the framework of planar rotors, since $\mathbb{S}$ is compact, the class of such interactions is indeed much larger than for unbounded spins.

The circle is the simplest compact manifold, hence we get immediately the ultracontractivity of the semigroup associated to the free dynamics, see for example [9] Theorem 3.3 and exercise 3.8.

We assume that the space-time interactions b_i are local in space and time and bounded, that is they satisfy assumptions (B2) and (B3).

Then we can formulate our result in the context of planar rotors. Its proof follows the same steps as in Sections 3.1–3.5, hence we will not repeat it here.

Theorem 3.1. *Consider Q^ν, the law of the infinite-dimensional SDE* (3.32) *with a drift satisfying assumptions* (B2) *and* (B3) *and suppose that the initial distribution ν is a Gibbs measure in $\mathcal{G}_{\beta_0}(\phi)$ where ϕ satisfies the strong summability assumption* (A1). *There exists a bound $\bar{\beta}_0 > 0$ for the initial inverse temperature and a bound $\bar{\beta} > 0$ for the intensity of the space-time interaction* *such that, if $0 \leq \beta \leq \bar{\beta}$ and $0 \leq \beta_0 \leq \bar{\beta}_0$, for all $t \geq 0$ the time-evolved measure $Q^\nu \circ X(t)^{-1}$ is a Gibbs measure w.r.t. some interaction ϕ^t, which is then absolutely summable.*

Corollary 3.1. *The proof of the above Theorem* 3.1 *provides a constructive way to obtain a solution of the SDE* (3.32) *at any time t for small β as limit* (*in terms of cluster expansions*) *of finite-dimensional approximations, whose existence* (*and uniqueness*) *is ensured by the assumption* (B3).

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