Gibbs states on space-time

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Abstract

Gibbs states on path spaces of the form $\mathcal{C}(\mathbb{R},\mathbb{R})^{\mathbb{Z}^d}$ are constructed by two different methods : as laws of solutions of infinite stochastic differential equations of gradient type or as thermodynamic limits constructed with help of the method of cluster expansions.

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0 Introduction

We construct in this paper Gibbs states in space-time : This means Gibbs states on the space of trajectoires $\Omega = \mathcal{C}(\mathbb{R}, \mathbb{R})^{\mathbb{Z}^d}$ which are specified locally in space-time windows of the from $V = \Lambda \times]a, b[$, $\Lambda \subseteq \mathbb{Z}^d$ finite, $a, b \in \mathbb{R}$, given the configuration of trajectoiries outside this window.

This construction is done by two different methods and for different (classes of) hamiltonians.

The first method uses the idea that Gibbs states on Ω are solutions of infinite-dimensional stochastic differential equations. To be more precise : Let $h = (h_i)_{i \in \mathbb{Z}^d}$ and $\tilde{h} = (\tilde{h}_i)_{i \in \mathbb{Z}^d}$ be suitable hamiltonians on $\mathbb{R}^{\mathbb{Z}^d}$ and ν a Gibbs state on $\mathcal{X} = \mathbb{R}^{\mathbb{Z}^d}$ specified by $\tilde{h}$ and the Lebesgue measure.

Consider then the law Q^ν of the stochastic process $X = (X_{k,t})_{k \in \mathbb{Z}^d, t \geq 0}$, which is the unique solution of the stochastic gradient system

$$\begin{cases} dX_{k,t} &= dW_{k,t} - \frac{1}{2} \nabla_k h_k(X_{\cdot,t}) dt, \\ X_{\cdot,0} &\overset{(\mathcal{L})}{=} \nu \end{cases} \tag{1}$$

$(k \in \mathbb{Z}^d, t \geq 0)$. Here $(W_k)_{k \in \mathbb{Z}^d}$ is a collection of independant Brownian motions. The law Q^ν on Ω is a candidate for a space-time Gibbs state on Ω. We prove here that this indeed is true. But in which sense ? We find that

$$\begin{aligned} \Pi_V^H(\omega, \cdot) &=: Q^\nu(\cdot \mid \omega_j(t), (j,t) \notin V) \\ &= \frac{1}{Z_V^H(\omega_{\partial V})} \exp - H_V(\cdot \omega_{\partial \Lambda \times \bar{I}}) \otimes_{k \in \Lambda} P_{\bar{I}}^{\omega_{k,o}, \omega_{k,b}} \quad Q^\nu - a.s. (2) \end{aligned}$$

Here $\omega = (\omega_i(t))_{i \in \mathbb{Z}^d, t \geq 0} \in \Omega$, $V = \Lambda \times]0, b[$, $\omega_{\partial V}$ resp. $\omega_{\partial \Lambda \times \bar{I}}$ represent the projection of ω on the boundary of V (which is defined by the range of the space-time interaction) resp. the projection of ω on $\bar{I} = [0, b]$ and the boundary of Λ. $P_{\bar{I}}^{x,y}$ denotes the Brownian bridge on $[0, b]$ which starts in x and ends in y. H_V is an Hamiltonian which is explicitely given in terms of the Hamiltonian h, which defines the dynamics (0.1).

In the second approach we obtain similar results for two classes of interactions. Limiting Gibbs states $\Pi_\infty^H = \lim_{V_n \uparrow \mathbb{Z}^d \times \mathbb{R}} \Pi_{V_n}^H$ are constructed for models which are a small perturbation of free fields as well as for a situation which had been studied already in the first part of the paper. The construction is based on the method of cluster expansions.

The organisation of this paper is as follows : in section 1, after having presented the general structure of modern Gibbsian theory in the spirit of the work of Föllmer [Foe1] and Preston [Pre], we introduce space-time models as Gibbsian states on $C(\mathbb{R}^+; \mathbb{R}^n)^{\mathbb{Z}^d}$. This is done carefully with the idea to introduce the main ingredients of a general theory of space-time stochastic processes which are only locally specified in space and time (and not globally as in the classical theory of Markov processes).

Then, in section 2, we construct, for a special class of Hamiltonians and a particular choice of the reference (i.e. free) measure the associated Gibbs states as the laws of an infinite-dimensional stochastic gradient system with Gibbsian initial distribution. This construction allows us to use existence results of [Sh-Sh] in order to discuss the problem of existence for Gibbs states on the path space $C(\mathbb{R}^+;\mathbb{R})^{\mathbb{Z}^d}$. As a result we obtain an injection of the set of initial Gibbs measures into the set of reversible Gibbs states on the path level associated to some corresponding Hamiltonian.

Finally, in section 3, we show, for a class of Hamiltonians indexed by a small parameter ε, using the powerful method of cluster expansions, existence and regularity of the limiting Gibbs states.

1 Specifications and their Gibbs states

1.1 The general framework

In a very general context we can describe the situation as Foellmer [Foe1] and Preston [Pre] in the following way :

Let $(\Omega, \mathcal{F})$ be a standard Borel space, $\mathfrak{V}$ an index set which is partially ordered by a relation $\subseteq$. We assume that this order is directed from above i.e. for each $V_1, V_2 \in \mathfrak{V}$ there exists $V \in \mathfrak{V}$ with $V_1 \subseteq V$ and $V_2 \subseteq V$, and is countably generated i.e. there is a sequence $\{V_n\}_n$ from $\mathfrak{V}$ such that if $V \in \mathfrak{V}$ then $V \subseteq V_n$ for some n. We call such a sequence $\{V_n\}_n$ a countable base in $\mathfrak{V}$.

There is also given a collection $\mathbb{F}$ of two filtrations indexed by $\mathfrak{V}$, which we call bifiltration in Ω :

$$\mathbb{F} = (\hat{\mathcal{F}}_V, \mathcal{F}_V)_{V\in\mathfrak{V}}$$

where $(\hat{\mathcal{F}}_V)_{V\in\mathfrak{V}}$ is a decreasing family of sub-σ-algebras of $\mathcal{F}$ (a "backward filtration") and $(\mathcal{F}_V)_{V\in\mathfrak{V}}$ a family of increasing sub-σ-algebras of $\mathcal{F}$ (a "forward filtration") such that

$$\mathcal{F} \quad = \quad \bigvee_{V\in\mathfrak{V}} \mathcal{F}_V$$

where the right hand side means the minimal σ-algebra containing every $\mathcal{F}_V$, and

$$\mathcal{F} \quad = \quad \mathcal{F}_V \vee \hat{\mathcal{F}}_V \quad \text{for each} \quad V \in \mathfrak{V}.$$

We denote

$$\begin{aligned} \partial_{V'}\mathcal{F}_V &= \mathcal{F}_{V'} \cap \hat{\mathcal{F}}_V \quad \text{if} \quad V \subseteq V' \quad \text{and} \\ \partial\mathcal{F}_V &= \mathcal{F}_V \cap \hat{\mathcal{F}}_V = \partial_V \mathcal{F}_V \end{aligned}$$

the σ-algebras induced at the border of V.

Definition 1 *A Markov field with respect to* $\mathbb{F}$ *is a probability measure* Q *on* $(\Omega, \mathcal{F})$ *such that for each* $V \in \mathfrak{V}$ *and* $f \in b\mathcal{F}_V$ *(i.e.* f *bounded and measurable with respect to* $\mathcal{F}_V$*),*

$$E_Q(f \mid \hat{\mathcal{F}}_V) = E_Q(f \mid \partial\mathcal{F}_V) \quad a.s. \tag{3}$$

where $E_Q(. \mid \mathcal{G})$ *denotes the conditional expectation of* Q *with respect to the* σ*-algebra* $\mathcal{G}$*.*

We shall consider Markov fields Q which are given by an $\mathbb{F}$-specification, that is the collection of kernels satisfying the properties of the following definition.

Definition 2 *An* $\mathbb{F}$*-specification is a collection* Π *of quasi-Markovian kernels* $(\Pi_V)_{V\in\mathfrak{V}}$ *from* $(\Omega, \mathcal{F}_V)$ *in* $(\Omega, \partial\mathcal{F}_V)$*, which means that :*

α) *for each* $\omega \in \Omega, \Pi_V(\omega, .)$ *is either a probability measure on* $(\Omega, \mathcal{F}_V)$ $(V \in \mathfrak{V})$*, or the measure* 0 *;*

β) *for each* $f \in b\mathcal{F}_V$*, the function* $\Pi_V(., f)$ *is measurable with respect to* $\partial\mathcal{F}_V$*;*

satisfying also the following properties :

γ) *for each* $V \in \mathfrak{V}$

$$\Pi_V(., f'f) = f'\Pi_V(., f), f' \in b\partial\mathcal{F}_V, f \in b\mathcal{F}_V$$

δ) *for each* $V \subseteq V'$, $\Pi_{V'}\Pi_V = \Pi_{V'}$ *on* $\mathcal{F}_V$ *(compatibility condition).*

In the special case where $\mathcal{F}_V = \mathcal{F}$ for each $V \in \mathfrak{V}$ we call $\hat{\mathbb{F}}$ the associated bifiltration and Π an $(\hat{\mathcal{F}}_V)_V$-specification or an $\hat{\mathbb{F}}$-specification.

Remark 1 In α) , usually $\Pi_V(\omega, .)$ is supposed to be always a probability measure. Here we assume only a "quasi-markovianity", i.e. we introduce the possibility for the kernel to vanish, but we assume implicitely that we are not in the trivial case where $\{\omega : \Pi_V(\omega, \Omega) = 1\} = \emptyset$ for some V.

Now we are in the position to define the basic object of our study.

Definition 3 *Let* Π *be an* $\mathbb{F}$*-specification. We say that a probability measure* Q *on* $(\Omega, \mathcal{F})$ *is a Gibbs state with specification* Π *if for each* $V \in \mathfrak{V}$ *and* $f \in b\mathcal{F}_V$

$$E_Q(f \mid \hat{\mathcal{F}}_V) = \Pi_V(f) \quad Q - a.s. \tag{4}$$

Remark 2 It is evident from the definition that each Gibbs state is a Markov field. Furthermore, denoting $R_V = \{\omega \in \Omega : \Pi_V(\omega, \Omega) = 1\}$, we see that $R_V \in \hat{\mathcal{F}}_V$ and that each Gibbs state Q is concentrated on each R_V.

The following simple observation will be very useful.

Lemma 1 *Let Q be a probability measure on $(\Omega, \mathcal{F})$ and Π a $\mathbb{F}$-specification. Then Q is a Gibbs state with specification Π if and only if Q is a Markov field with respect to $\mathbb{F}$ and*

$$Q(\Pi_V(f)) = Q(f) \quad \textit{for each} \quad V \in \mathfrak{V} \textit{ and } f \in b\mathcal{F}_V.$$

The main question thus is the construction of specifications. The general idea for such a construction is given by the Gibbsian paradigm : First one constructs a specification Π^0 on the given bi-filtration $(\Omega, \mathbb{F})$, which serves as a reference (the so-called free specification) and then, by means of a given $\mathbb{F}$-Hamiltonian H, one constructs a specification Π^H, which takes into account also the interaction.

To make this precise we formalise the concept of a Hamiltonian :

Definition 4 *A collection $H = (H_V)_{V \in \mathfrak{V}}$ of functions*

$$H_V : \Omega \to \mathbb{R} \cup \{+\infty\},$$

is called an $\mathbb{F}$-Hamiltonian if

- α) *H_V is $\mathcal{F}_V$-measurable for each V ;*
- β) *H is $\mathbb{F}$-additive, i.e.$\forall V, V' \in \mathfrak{V}, V \subseteq V'$, there exists*

  $$H_{(V',V)} : \Omega \to \mathbb{R} \cup \{+\infty\},$$

  measurable with respect to $\partial_{V'}\mathcal{F}_V$, such that

  $$H_{V'} = H_V + H_{(V',V)} .$$

In the special case where $\mathcal{F}_V = \mathcal{F}$ for each $V \in \mathfrak{V}$, we call H an $\hat{\mathbb{F}}$-Hamiltonian.

The following fundamental theorem (cf. [Pre]) allows to construct specifications based on the reference specification Π^0 and a Hamiltonian H :

Theorem 1 *If we define for $V \in \mathfrak{V}, \omega \in \Omega$*

$$\Pi_V^H(\omega, d\omega') = \begin{cases} \frac{1}{Z_V^H(\omega)} \exp(-H_V(\omega'))\Pi_V^0(\omega, d\omega'), & \textit{if } 0 < Z_V^H(\omega) < +\infty \\ 0 & \textit{otherwise} \end{cases} \tag{5}$$

where

$$Z_V^H(\omega) = \int_\Omega \exp(-H_V(\omega'))\Pi_V^0(\omega, d\omega'),$$

then the family $\Pi^H = (\Pi_V^H)_{V\in\mathfrak{V}}$ *defines an* $\mathbb{F}$*-specification.*

We denote by $\mathcal{G}(\Pi^H)$ or also $\mathcal{G}(H, \Pi^0)$ the set of Gibbs states with specification Π^H. They are also called Gibbsian modifications of Π^0 by the Hamiltonian H.

1.2 Finite range Gibbs states

Föllmer's concept of a Gibbs state, as developed above is that of a finite range Gibbs state. We remark here that the usual theory is obtained if the filtration $(\mathcal{F}_V)_{V\in\mathfrak{V}}$ is given by $\mathcal{F}_V = \mathcal{F}$ for each $V \in \mathfrak{V}$. The corresponding bifiltration is denoted by $\hat{\mathbb{F}}$. The following simple observation shows that a Gibbs state with respect to a finite range $\hat{\mathbb{F}}$-specification Π is also a Gibbs state for Π, if Π is considered as an $\mathbb{F}$-specification. To be more precise we have the

Lemma 2 *Let Q be a probability on $(\Omega, \mathcal{F})$, $\mathbb{F}$ a bifiltration in Ω and Π and $\hat{\mathbb{F}}$-specification. If Π has also finite range with respect to $\mathbb{F}$ in the sense that each $\Pi_V(\cdot, f), f \in b\mathcal{F}_V$, is $\partial\mathcal{F}_V$-measurable, and if $Q \in \mathcal{G}(\Pi)$, then Q is an $\mathbb{F}$-Markov field, specified by the $\mathbb{F}$-specification induced by Π .*

For future reference we state here the following obvious consequence. Let $b\mathcal{F}_{\text{loc}}$ be the space of all bounded, measurable functions f on Ω which are local in the sense that $f \in b\mathcal{F}_V$ for some $V \in \mathfrak{V}$.

Proposition 1 *Let $\mathbb{F}$ be a bifiltration in $(\Omega, \mathcal{F})$ and $\Pi = (\Pi_V)_{V\in\mathfrak{V}}$ an $\hat{\mathbb{F}}$-specification having finite range with respect to $\mathbb{F}$. For a countable base $(V_n)_n$ in $\mathfrak{V}$ and a sequence $(q_n)_n$ of probabilities on $(\Omega, \partial\mathcal{F}_{V_n})$ we consider the sequence of probabilities on $(\Omega, \mathcal{F})$, defined by*

$$\Pi_n := \int_\Omega \Pi_{V_n} dq_n \ ,$$

together with a probability Q on $(\Omega, \mathcal{F})$. Then the condition

$$\lim_{n\to\infty} \Pi_n(f) = Q(f), f \in b\mathcal{F}_{\text{loc}} \tag{6}$$

implies that $Q \in \mathcal{G}(\Pi)$ and possesses the property of an $\mathbb{F}$-Markov field.

Proof : Let $V \in \mathfrak{V}$ and $f = f_1 \cdot f_2$, where $f_1 \in b\hat{\mathcal{F}}_V$ is local and $f_2 \in b\mathcal{F}_V$. Then $\Pi_V(f) = f_1\Pi_V(f_2)$ is a local bounded, measurable function, because

Π has finite range, and thus

$$\begin{aligned} Q(f_1 \cdot f_2) &= \lim_n \int_\Omega \Pi^\cdot_{V_n}(f_1 f_2) dq_n \\ &= \lim_n \int_\Omega \Pi^\cdot_{V_n}(\Pi^\cdot_V(f_1 f_2)) dq_n \\ &= \lim_n \int_\Omega \Pi^\cdot_{V_n}(f_1 \Pi^\cdot_V(f_2)) dq_n \\ &= Q(f_1 \Pi_V(f_2)) \end{aligned}$$

Therefore Q is Gibbs for the $(\hat{\mathcal{F}}_V)_V$-specification Π. Since Π is of finite range, lemma 2 implies that Q is an $\mathbb{F}$-Markov field. ■

In the following two paragraphs, we give examples of reference specifications and Hamiltonians H in space as well as in space-time.

1.3 Specifications in space

We recall here the construction of specifications which had been introduced for the first time by Dobrushin, Lanford and Ruelle in the case when the index set $\mathfrak{V}$ modelizes volumes in $\mathbb{Z}^d$ (so called DLR-construction).

Let $\mathcal{X} = \mathcal{N}^{\mathbb{Z}^d}, d \in \mathbb{N}^*$, where $\mathcal{N}$ is some measurable space with some fixed σ-field $\mathcal{A}$. The index-set $\mathfrak{V}$ is the collection of finite subsets Λ of $\mathbb{Z}^d$. The canonical projections are denoted by $X_k, k \in \mathbb{Z}^d$ and X_Λ is the canonical projection on $\mathcal{N}^\Lambda$.

The spatial bifiltration $\mathbb{F}$ in $\mathcal{X}$ is defined in the following way :

$$\begin{aligned} \hat{\mathcal{F}}_\Lambda &= \sigma(X_k; k \notin \Lambda), \\ \mathcal{F}_\Lambda &= \sigma(X_k; k \in \overline{\overline{\Lambda}}) \end{aligned}$$

Here $\overline{\overline{\Lambda}} = \Lambda \cup \partial\Lambda$, where $\partial\Lambda$ is some well-defined frontier of Λ, e.g.

$$\partial\Lambda = \{k \notin \Lambda \colon d(k, \Lambda) < R\}$$

where $0 < R \leq +\infty$ describes the range of the interaction (which can be infinite) and $d(k, \Lambda) = min\{\rho(k,j), j \in \Lambda\}$ where $\rho(k,j) = \sum_{i=1}^d |k_i - j_i|$.

We use also the σ-fields

$$\mathcal{A}_\Lambda = \sigma(X_\Lambda) \; .$$

We now indicate the construction of a free $\mathbb{F}$-specification $(\pi^0_\Lambda)_{\Lambda \subset \mathbb{Z}^d}$ which serves as a reference. Let $(\lambda_k)_{k \in \mathbb{Z}^d}$ be a family of probability measures on $\mathcal{N}$:
for each $x \in \mathcal{X}$, $A' \in \partial\mathcal{F}_\Lambda, A \in \mathcal{A}_\Lambda$

$$\pi^0_\Lambda(x, A' \cap A) = 1_{A'}(x)(\bigotimes_{k \in \Lambda} \lambda_k)(A).$$

To say it in another way, the measure $\pi^0_\Lambda(x,.)$ is equal to

$$\pi^0_\Lambda(x,.) = (\bigotimes_{k\in\Lambda} \lambda_k) \otimes \delta_{x_{\partial\Lambda}}.$$

Let $\phi = (\phi_\Lambda)_{\Lambda\subset\mathbb{Z}^d}$ be a spatial potential on $\mathcal{X}$, i.e. a collection of functions $\phi_\Lambda\colon \mathcal{X} \to \mathbb{R}\cup\{+\infty\}$ which are $\mathcal{A}_\Lambda$-measurable and which satisfy : $\phi_\Lambda \equiv 0$ if the diameter of Λ is larger than $R \leq \infty$. Then, the space $\mathbb{F}$-Hamiltonian $h = (h_\Lambda)_{\Lambda\subset\mathbb{Z}^d}$ on $\mathcal{X}$ generated by ϕ is given by

$$h_\Lambda = \sum_{\Lambda':\Lambda'\cap\Lambda\neq\varnothing} \phi_{\Lambda'}.$$

We thus know from Theorem 1 that the kernels

$$\pi^h_\Lambda(x,dx') = \begin{cases} \frac{1}{Z^h_\Lambda(x)}\exp(-h_\Lambda(x'))\pi^0_\Lambda(x,dx'), & \text{if } 0 < Z^h_\Lambda(x) < +\infty \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

define an $\mathbb{F}$-specification. So it makes sense to consider $\mathcal{G}(h,\pi^0)$, which for obvious reasons is denoted also by $\mathcal{G}(h,(\lambda_k)_k)$.

Remark 3 The fact that π^h is an $\mathbb{F}$-specification remains true if we replace the probability measures λ_k by σ-finite measures.

Example 1 : The DLR-construction above can be done in the special case (which will be fundamental in the following) where $\mathcal{N}$ is the separable Banach space $\mathcal{C}([a,b],\mathbb{R}^n), 0 \leq a < b < +\infty$, and the measures λ_k are Wiener measures $P^{\mu_k}_{[a,b]}$ on $\mathcal{N}$ with an initial condition given by a σ-finite measure μ_k on $\mathbb{R}^n$.

1.4 Specifications in space-time

In this paragraph we introduce the main - and new- objects we need to work in a space-time context. The space is discrete (a lattice model) and time is continuous.

1.4.1 Bifiltrations and local specifications.

Let $\mathcal{C}(\mathbb{R}^+,\mathbb{R}^n)$ denote the space of all continuous functions x on $\mathbb{R}^+$ with values in $\mathbb{R}^n$. Define a sequence of semi norms on it by $p_N(x) = \sup_{0\leq t\leq N} |x(t)|, N = 1,2,\ldots,$. Then define a metric d on $\mathcal{C}(\mathbb{R}^+,\mathbb{R}^n)$ by

$$d(x,y) = \sum_{N=1}^{\infty} \frac{1}{2^N}\cdot\frac{p_N(x-y)}{1+p_N(x-y)}.$$

With this metric $\mathcal{C}(\mathbb{R}^+,\mathbb{R}^n)$ is a standard Borel space (see [Wh]).

We now construct specifications in space-time.

By this we mean specifications on the standard Borel space $\Omega = (\mathcal{C}(\mathbb{R}^+, \mathbb{R}^n))^{\mathbb{Z}^d}$ with respect to a natural bi-filtration $\mathbb{F}$ which is indexed by regions in $\mathbb{Z}^d \times \mathbb{R}^+$. To be more precise :

Let $\mathcal{F}$ be the σ-algebra in Ω generated by the space-time projections $X_{k,t}, k \in \mathbb{Z}^d, t \in \mathbb{R}^+$:

$$X_{k,t} \colon \Omega \to \mathbb{R}^n, X_{k,t}(\omega) = \omega_{k,t} \ .$$

The index set $\mathfrak{V}$ consists of all sets of the form $V = \Lambda \times I$, where $\Lambda \in \mathcal{S}$, i.e. Λ is a finite subset of $\mathbb{Z}^d$, and I is a time interval of the form $I =]a, b[$, $0 \le a < b < +\infty$. With each $V \in \mathfrak{V}$ we associate its region of influence, defined by

$$\bar{\bar{V}} = V \cup \partial V \ ,$$

where $\partial V = (\Lambda \times \partial I) \cup (\partial\Lambda \times \bar{I})$, $\bar{I} = [a, b]$ and $\partial I = \{a, b\} = \bar{I} \setminus I$.

The bifiltration $\mathbb{F}$ in Ω is now defined by

$$\hat{\mathcal{F}}_V = \sigma(X_{k,t}; (k, t) \notin V) \quad (8)$$
$$\mathcal{F}_V = \sigma(X_{k,t}; (k, t) \in \bar{\bar{V}}) \ . \quad (9)$$

The general idea for the construction of $\mathbb{F}$-specifications on Ω is to start the construction locally in space and time and then to extend it globally.

To do this we introduce some additional notations :

Given I and $\Lambda \in \mathcal{S}$ we define the spaces

$$\begin{aligned} \Omega_{\Lambda,\bar{I}} &= (\mathcal{C}(\bar{I}, \mathbb{R}^n))^{\Lambda} \ , \\ \Omega_{\bar{I}} &= (\mathcal{C}(\bar{I}, \mathbb{R}^n))^{\mathbb{Z}^d}, \\ \Omega_{\Lambda} &= (\mathcal{C}(\mathbb{R}^+, \mathbb{R}^n))^{\Lambda}; \end{aligned}$$

We consider in $\Omega_{\bar{I}}$ the spatial bi-filtration $\mathbb{F}_{\bar{I}}$ defined by the traces of $\mathbb{F}$ in $\Omega_{\bar{I}}$. Thus $\mathbb{F}_{\bar{I}} = ((\hat{\mathcal{F}}_{\bar{I}})_\Lambda, (\mathcal{F}_{\bar{I}})_\Lambda)_{\Lambda \in \mathcal{S}}$, where

$$(\hat{\mathcal{F}}_{\bar{I}})_\Lambda = \Omega_{\bar{I}} \cap \hat{\mathcal{F}}_{\Lambda \times I} \quad (10)$$
$$(\mathcal{F}_{\bar{I}})_\Lambda = \Omega_{\bar{I}} \cap \mathcal{F}_{\Lambda \times I} \quad (11)$$

We'll use also the following projections in space-time (resp. space, resp. time) defined on Ω with values in $\Omega_{\Lambda,\bar{I}}$ (resp. in $\Omega_{\bar{I}}$, resp. in Ω_Λ) :

$$\begin{aligned} X_{\Lambda,\bar{I}} &= (X_{k,t})_{k \in \Lambda, t \in \bar{I}}, X_{\Lambda,\bar{I}}(\omega) = \omega_{\Lambda,\bar{I}} \in \Omega_{\Lambda,\bar{I}}, \\ X_{\bar{I}} &= X_{\mathbb{Z}^d,\bar{I}}, X_{\bar{I}}(\omega) = \omega_{\bar{I}} \in \Omega_{\bar{I}} \quad , \\ X_\Lambda &= (X_k)_{k \in \Lambda}, X_\Lambda(\omega) = \omega_\Lambda \in \Omega_\Lambda. \end{aligned}$$

We also introduce the notation

$$Q_{\Lambda,\bar{I}} = Q o X^{-1}_{\Lambda,\bar{I}}, Q_{\bar{I}} = Q o X^{-1}_{\bar{I}}, Q_\Lambda = Q o X^{-1}_\Lambda.$$

Here $\omega \in \Omega$ and Q is a probability measure on $(\Omega, \mathcal{F})$.

We start with the following simple observation :
If Π is a $\mathbb{F}$-specification then, if $V = \Lambda \times I \in \mathfrak{V}$,

$$\rho_V(\omega', f') = \Pi_V(\omega, f' o X_{\bar{I}}), \omega' \in \Omega_{\bar{I}}, f' \in b(\mathcal{F}_{\bar{I}})_\Lambda \tag{12}$$

with $\omega \in \Omega$ chosen in such a way that $\omega_{\partial V} = \omega'_{\partial V}$, defines a local specification ρ on the collection $(\Omega_{\bar{I}}, \mathbb{F}_{\bar{I}})_I$ in the following sense :
$\rho = (\rho_V)_{V \in \mathfrak{V}}$ is a collection of quasi-Markovian kernels ρ_V from $(\Omega_{\bar{I}}, (\mathcal{F}_{\bar{I}})_\Lambda)$ in $(\Omega_{\bar{I}}, \partial(\mathcal{F}_{\bar{I}})_\Lambda)$ having the following two properties (cf. γ) and δ) in Definition 2):

i) $\forall f' \in b\partial(\mathcal{F}_{\bar{I}})_\Lambda, \forall f \in b(\mathcal{F}_{\bar{I}})_\Lambda$

$$\rho_{\Lambda \times I}(f' f) = f' \rho_{\Lambda \times I}(f) \ ;$$

ii) $\forall V \subseteq V', V = \Lambda \times I, V' = \Lambda' \times I'$

$$\rho_{V'} \rho_V = \rho_{V'} \text{ on } (\mathcal{F}_{\bar{I}})_\Lambda. \tag{13}$$

We observe also that for each time interval I the collection $\Pi_{\bar{I}} = (\Pi_{\bar{I},\Lambda})_{\Lambda \in \mathcal{S}}$, defined by

$$\Pi_{\bar{I},\Lambda}(\omega', f') = \rho_{\Lambda \times I}(\omega', f'), \omega' \in \Omega_{\bar{I}}, f' \in b(\mathcal{F}_{\bar{I}})_\Lambda,$$

is a specification on $(\Omega_{\bar{I}}, \mathbb{F}_{\bar{I}})$. For a proper understanding of the situation we remark here that $\Pi_{\bar{I}}$ is a specification indexed by the spatial parameter $\Lambda \in \mathcal{S}$. But the dependance of $\Pi_{\bar{I},\Lambda}(\omega', f')$ of the variable ω' is more complex than that of a specification in space (see section 1.3). It depends also on the boundary values in time $\omega'_{\Lambda,\partial I}$ and not only on ω'_{Λ^c}! Moreover we have

Lemma 3 *If Π is an $\mathbb{F}$-specification and $Q \in \mathcal{G}(\Pi)$, then for each time interval I $Q_{\bar{I}} \in \mathcal{G}(\Pi_{\bar{I}})$, that is the time restriction of a space-time Gibbs state remains a Gibbs state.*

The next proposition gives a condition under which the converse statement is also true.

Proposition 2 *Let $\rho = (\rho_V)_{V \in \mathfrak{V}}$ be a local specification on $(\Omega_{\bar{I}}, \mathbb{F}_{\bar{I}})_I$ and Q be a probability measure on $(\Omega, \mathcal{F})$ such that, for each time interval I, $Q_{\bar{I}} \in \mathcal{G}(\Pi_{\bar{I}})$. Then*

$$\Pi_V(\omega, f' o X_{\bar{I}}) = \rho_V(\omega_{\bar{I}}, f'), \omega \in \Omega, f' \in b(\mathcal{F}_{\bar{I}})_\Lambda \,, \tag{14}$$

defines a specification Π on $(\Omega, \mathbb{F})$. Moreover $Q \in \mathcal{G}(\Pi)$, if additionally Q satisfies the following (two-sided time) Markov property :

$$\forall V \in \mathfrak{V}, \forall f \in b\mathcal{F}_V, \, E_Q(f \mid \hat{\mathcal{F}}_V) = E_Q(f \mid \tilde{\partial}\mathcal{F}_V) \ a.s. \tag{15}$$

Here $\tilde{\partial}\mathcal{F}_{\Lambda \times I} = \bigvee_{\Lambda' \supseteq \Lambda} \partial_{\Lambda' \times I} \mathcal{F}_{\Lambda \times I} = X_{\bar{I}}^{-1}((\hat{\mathcal{F}}_{\bar{I}})_\Lambda)$.

Proof :

It is obvious that Π is a specification on $(\Omega, \mathbb{F})$. If now Q is a probability on $(\Omega, \mathcal{F})$ such that $Q_{\bar{I}} \in \mathcal{G}(\Pi_{\bar{I}})$ for each interval I, then

$$\forall V \in \mathfrak{V}, \forall f \in b\mathcal{F}_V,\ Q(f \mid \tilde{\partial}\mathcal{F}_V) = \Pi_V(., f) \ \ Q - \text{a.s.}$$

Therefore the two-sided Markov property (1.11) of Q immediately implies $Q \in \mathcal{G}(\Pi)$. ∎

Proposition 2 suggests to construct $\mathbb{F}$-specifications by means of local specifications ρ on $(\Omega_{\bar{I}}, \mathbb{F}_{\bar{I}})_I$, which in turn should be constructed using the Gibbsian prescription. Thus we are now looking for a suitable local reference specification and Hamiltonians.

1.4.2 Local reference specifications.

We now discuss the basic example of a local specification on $(\Omega_{\bar{I}}, \mathbb{F}_{\bar{I}})_I$, which will serve as a reference in the following.

Example 2 : Let $V = \Lambda \times I \in \mathfrak{V}$ where $I =]a, b[$ and consider

$$\rho^0_V(\omega, f) = \int_{\Omega_{\Lambda,\bar{I}}} f(\eta\omega_{\Lambda^c}) \bigotimes_{k\in\Lambda} P_{\bar{I}}^{\omega_{k,a},\omega_{k,b}}(d\eta),$$

where $\omega \in \Omega_{\bar{I}}$, $f \in b(\mathcal{F}_{\bar{I}})_\Lambda$. Here $P_{\bar{I}}^{x,y}$ denotes the Brownian bridge between x and y on $\bar{I}$.
It is clear that each ρ^0_V is a Markovian kernel from $(\Omega_{\bar{I}}, \mathcal{F}_{\bar{I},\Lambda})$ in $(\Omega_{\bar{I}}, \partial\mathcal{F}_{\bar{I},\Lambda})$ having property i) of a local specification. We'll see now that the compatibility property (equality (1.11)) holds true too :

Let $V \subseteq V', V = \Lambda \times]a, b[\ V' = \Lambda' \times]a', b'[, f \in b\mathcal{F}_{\bar{I},\Lambda}$ and $\omega \in \Omega_{\bar{I}}$. Then

$$\begin{aligned}
&\textstyle\int_{\Omega_{\bar{I}'}} \rho^0_V(\omega', f)\rho^0_{V'}(\omega, d\omega') \\
&\quad = \textstyle\int_{\Omega_{\Lambda',\bar{I}'}} \rho^0_V(\eta\omega_{\Lambda'^c}, f) \bigotimes_{k\in\Lambda'} P_{\bar{I}'}^{\omega_{k,a'},\omega_{k,b'}}(d\eta) \\
&\quad = \textstyle\int_{\Omega_{\Lambda',\bar{I}'}} \int_{\Omega_{\Lambda,I}} f(\xi\eta_{\Lambda^c}\omega_{\Lambda'^c}) \bigotimes_{\ell\in\Lambda} P_{\bar{I}}^{\eta_{\ell,a},\eta_{\ell,b}}(d\xi) \bigotimes_{k\in\Lambda'} P_{\bar{I}'}^{\omega_{k,a'},\omega_{k,b'}}(d\eta)
\end{aligned}$$

It is sufficient to consider only functions f of the form

$$f = (f_1 \circ X_\Lambda)(f_2 \circ X_{\Lambda'\setminus\Lambda})(f_3 \circ X_{\Lambda'^c})$$

Then, using obvious properties of the underlying Brownian motion, the last integral equals

$$
\begin{aligned}
&\int_{\Omega_{\Lambda',\bar I'}} \bigotimes_{\ell\in\Lambda} P_{\bar I}^{\eta_{\ell,a},\eta_{\ell,b}}(f_1\circ X_\Lambda) f_2(\eta_{\Lambda^c}) f_3(\omega_{\Lambda'^c}) \bigotimes_{k\in\Lambda'} P_{\bar I'}^{\omega_{k,a'},\omega_{k,b'}}(d\eta)\\
&= f_3(\omega_{\Lambda'^c}) \int_{\Omega_{\Lambda',\bar I'}} \Big(\bigotimes_{\ell\in\Lambda} P_{\bar I}^{\eta_{\ell,a},\eta_{\ell,b}}\Big)(f_1\circ X_\Lambda) \bigotimes_{k\in\Lambda} P_{\bar I'}^{\omega_{k,a'},\omega_{k,b'}}(d\eta)\\
&\qquad \int_{\Omega_{\Lambda',\bar I'}} f_2(\eta_{\Lambda^c}) \bigotimes_{k\in\Lambda'\setminus\Lambda} P_{\bar I'}^{\omega_{k,a'},\omega_{k,b'}}(d\eta)\\
&= f_3(\omega_{\Lambda'^c}) \int_{\Omega_{\Lambda',\bar I'}} f_1(\eta_\Lambda) f_2(\eta_{\Lambda^c}) \bigotimes_{k\in\Lambda'} P_{\bar I'}^{\omega_{k,a'},\omega_{k,b'}}(d\eta)\\
&= \rho^0_{V'}(\omega, f),
\end{aligned}
$$

so equality (1.11) is verified.

Remark 4 *Given $\mathbb{F}$ denote by $\hat{\mathbb{F}}_{\bar I}$ the bifiltration in $\Omega_{\bar I}$ which is defined via (1.8) and (1.9) by the bifiltration $\hat{\mathbb{F}}$. We know from Proposition 2 that, if $\rho = (\rho_V)_{V\in\mathfrak{V}}$ is a local $\hat{\mathbb{F}}_{\bar I}$-specification, then $(\Pi_V)_V$, as defined by (1.12), is a $\hat{\mathbb{F}}$-specification.*

We want to remark here that an inspection of the reasoning in example 2 shows that ρ^0 can also be considered as an $\hat{\mathbb{F}}_{\bar I}$-specification and thus induces via (1.12) an $\hat{\mathbb{F}}$-reference specification Π^0 which has finite range with respect to $\mathbb{F}$. Thus if we have also an $\mathbb{F}$-Hamiltonian we can construct the $\hat{\mathbb{F}}$- specification Π^H having finite range with respect to $\mathbb{F}$. Any $Q \in \mathcal{G}(\Pi)$ then is a Markov field with respect to $\mathbb{F}$.

This remark will be used in connection with Proposition 1 in the third part of the paper.

1.4.3 Potentials and Hamiltonians in space-time.

Let $\mathbb{F}$ be the bi-filtration in Ω defined by (1.5), (1.6). Hamiltonians on $(\Omega, \mathbb{F})$ are given by space-time potentials in the sense of

Definition 5 *Let $\Phi = (\Phi_V)_{V\in\mathfrak{V}}$ be a collection of functions $\Phi_V\colon \Omega \to \mathbb{R}\cup\{+\infty\}$.*
Φ is called a potential on Ω, if

- *α) Φ_V is $\mathcal{B}_V$-mesurable for each $V \in \mathfrak{V}$, where $\mathcal{B}_V\colon = \sigma(X_{k,t}\,;\,(k,t)\in V)$.*
- *β) Φ is additive in time, i.e.*

$$\forall\Lambda,\, \forall I \subseteq I',\, \exists\Phi_{(\Lambda,I',I)}\colon \Omega \to \mathbb{R}\cup\{+\infty\}\,,$$

measurable with respect to $\mathcal{B}_{\Lambda\times I'\setminus I}$ such that

$$\Phi_{\Lambda\times I'} = \Phi_{\Lambda\times I} + \Phi_{(\Lambda,I',I)}\;.$$

We say that <u>Φ has range R</u>, if

$$\forall V = \Lambda \times I \,, \forall \Lambda' \colon \Lambda' \cap \Lambda \neq \emptyset, \Lambda' \cap \overline{\overline{\Lambda}}_R^{\,c} \neq \emptyset, \Phi_{\Lambda' \times I} \equiv 0$$

Here $0 < R \leq +\infty$ is given and $\overline{\overline{\Lambda}}_R = \Lambda \cup \partial\Lambda$ where the frontier of Λ is defined like in section 1.2 by

$$\partial\Lambda = \{k \notin \Lambda \colon d(k, \Lambda) < R\}.$$

The space-time potential Φ generates the following space-time Hamiltonian:

$$H_{\Lambda \times I} = \sum_{\Lambda' : \Lambda' \cap \Lambda \neq \varnothing} \Phi_{\Lambda' \times I}.$$

Example 3 Let $\varphi = (\varphi_\Lambda)_{\Lambda \in \mathcal{S}}$ be a spatial potential on $(\mathbb{R}^n)^{\mathbb{Z}^d}$ as defined in section 1.2. Then

$$\Phi_{\Lambda \times I}(\omega) = \begin{cases} \int_I \varphi_\Lambda(\omega_s) ds, & \text{if } \int_I \left| \varphi_\Lambda(\omega_s) \right| ds < +\infty \\ +\infty & \text{otherwise.} \end{cases} \tag{16}$$

defines a space-time-potential on Ω. If φ has range R then also Φ. The associated space-time Hamiltonian is equal to :

$$H_{\Lambda \times I}(\omega) = \int_I \sum_{\Lambda' : \Lambda' \cap \Lambda \neq \varnothing} \varphi_{\Lambda'}(\omega_s) ds.$$

1.4.4 Gibbs states in space-time.

Let $\rho^0 = (\rho^0_V)_{V \in \mathfrak{V}}$ be the local specification on $(\Omega_{\bar{I}}, \mathbb{F}_{\bar{I}})_I$ of example 2 in section 1.4.2 and consider the corresponding specification on $(\Omega, \mathbb{F})$, defined like in Proposition 2 by

$$\Pi^0_V(\omega, f' \circ X_{\bar{I}}) = \rho^0_V(\omega_{\bar{I}}, f'), \omega \in \Omega, f' \in b(\mathcal{F}_{\bar{I}})_\Lambda \,.$$

Moreover, let $H = (H_V)_{V \in \mathfrak{V}}$ be a Hamiltonian on $(\Omega, \mathbb{F})$. We know from Proposition 2 and Theorem 1 that Π^H is a $\mathbb{F}$-specification.

In the following sections we consider, for H chosen in a particular way, the set $\mathcal{G}(\Pi^H)$ of Gibbs states Q on $(\Omega, \mathbb{F})$. Our aim will be to give in this case a description of $\mathcal{G}(\Pi^H)$ in terms of stochastic processes indexed by continuous time in $\mathbb{R}^+$ and state space $\mathbb{R}^{\mathbb{Z}^d}$. In particular we are interested in the <u>existence problem</u> (i.e. $\mathcal{G}(\Pi^H) \neq \emptyset$) and the <u>problem of phase transition</u> (i.e. $\operatorname{card} \mathcal{G}(\Pi^H) > 1$). We do not touch in this paper the uniqueness problem (i.e. $\operatorname{card} \mathcal{G}(\Pi^H) = 1$).

2 Gibbs states in space-time and infinite dimensional diffusions

In this section we consider the case where $n = 1$ and replace $\mathfrak{V}$ by the sub-class $\mathfrak{V}_0$ of all $\Lambda \times I \in \mathfrak{V}$ with $I =]0, b[, b > 0$.

Our aim is now to construct a large class of Gibbs states in $\mathcal{G}(H, \Pi^0)$, where Π^0 is the reference specification built on Brownian bridges (see example 2) and a special Hamiltonian H to be specified a bit later. Candidates for such Gibbs states are laws of infinite dimensional stochastic gradient systems in view of the Girsanov theorem. They are defined in the following way :

Let ϕ be a pair potential on $\mathbb{R}^{\mathbb{Z}^d}$ (i.e. satisfying $\phi_\Lambda \equiv 0$ if $\operatorname{card}\Lambda > 2$, $\Lambda \subset \mathbb{Z}^d$) ; we use in the following the notations $\phi_k =: \phi_{\{k\}}$ and $\phi_{k\ell} =: \phi_{\{k,\ell\}}$, $k, \ell \in \mathbb{Z}^d, k \neq \ell$. We suppose that the associated hamiltonian

$$h_\Lambda(y) = \sum_{k \in \Lambda} \phi_k(y_k) + \sum_{\{k,\ell\} \cap \Lambda \neq \varnothing} \phi_{k\ell}(y_k, y_\ell) \tag{17}$$

is well defined for each Λ finite subset of $\mathbb{Z}^d$ and $y \in \mathbb{R}^{\mathbb{Z}^d}$.

We define now an infinite dimensional gradient system by the following stochastic dynamics :

$$\begin{cases} dX_{k,t} & = dW_{k,t} - \frac{1}{2} \nabla_k h_k(X_{\cdot,t}) dt, \ k \in \mathbb{Z}^d \\ X_{\cdot,0} & \overset{(\mathcal{L})}{=} \nu \end{cases} \tag{18}$$

where $(W_k)_k$ is a family of independent Brownian motions with initial value 0, and is also independent from $(X_{\cdot,0})$. To solve (18) it is necessary to assume that h and the initial law ν satisfy some regularity properties. We adopt the general situation presented in [CRZ], where ϕ is an unbounded finite-range interaction and ν is a tempered measure with polynomial moments. More precisely, under the following assumptions :

$$
\begin{cases}
\alpha) & \exists K>0,\ \underset{k}{sup}\ \left|\phi'_k(0)\right| \le K, \phi''_i \ge -K \\
\beta) & \forall k \in \mathbb{Z}^d,\ \phi'_k \text{ has at most polynomial increasing of degree } p \\
\gamma) & \forall \ell \in \mathbb{Z}^d,\ \underset{k\in\mathbb{Z}^d}{sup}\ \left|\frac{\partial}{\partial x_k}\phi_{k\ell}(0,0)\right| < +\infty \\
\delta) & \underset{k\neq\ell}{sup}\ \left|\frac{\partial^2}{\partial x_\ell \partial x_k}\phi_{k\ell}(0,0)\right| < +\infty,\ \frac{\partial^2}{\partial x_k^2}\phi_{k\ell} \ge -q_{k\ell} \ \text{ where } q_{k\ell}\ge 0, \\
& \qquad\qquad \text{and } \sup_k \sum_\ell q_{k\ell} < +\infty \\
\epsilon) & \exists p>0, (\int x_k^{2p}\nu(dx))_{k\in\mathbb{Z}^d} \in \mathcal{S}'(\mathbb{Z}^d) \ \text{ where} \\
& \qquad\qquad \mathcal{S}'(\mathbb{Z}^d) = \{(y_k)_k \in \mathbb{R}^{\mathbb{Z}^d}, \exists q \in \mathbb{N} \sum_k (|k|+1)^{2q}|y_k|^2 < +\infty\}, \\
& \qquad\qquad \text{(tempered sequences on } \mathbb{Z}^d\text{)}
\end{cases}
\tag{19}
$$

there exists a unique strong solution for the system (18). Furthermore the process $(X_{k,t})_{k,t}$ takes value in $\mathcal{S}'(\mathbb{Z}^d)$ and has bounded polynomial moments (cf.[Sh-Sh] Theorem 4.2).

We denote by Q^ν the law of the solution of the system (2.2). Our question now is the following : Does Q^ν have a Gibbsian structure?

We first recall (section 2.1) the main result of [CRZ], that, when ν is Gibbsian, Q^ν, restricted to any bounded time interval $\bar{I} = [0,b]$ has a spatial Gibbsian structure in the DLR-sense of example 1 section 1.3 with respect to a reference specification built on the Wiener measure on $\bar{I}$ with initial distribution μ_k denoted by $(P_{\bar{I}}^{\mu_k})_k$, the pure spatial bifiltration $\mathbb{F}_{\bar{I}}$ defined in section 1.3, and a Hamiltonian $\tilde{H}$ which is explicitely defined with help of the dynamical potential ϕ and the potential $\tilde{\varphi}$ associated to ν. In a second step (section 2.2) we prove our main result which says that even globally in time, i.e. on the full time interval $\mathbb{R}^+$ ($\mathbb{R}$ in the stationnary case), Q^ν has a Gibbsian structure, but now in the space-time sense of section 1.4, where the Hamiltonian consists of the space-time dynamical part of $\tilde{H}$ and the reference specification is built on Brownian bridges.

2.1 The case of a fixed compact time interval $\bar{I} = [0,b]$.

Let $\tilde{\varphi}$ be a potential on $\mathbb{R}^{\mathbb{Z}^d}$ with associated hamiltonian $\tilde{h}$ such that the set $\mathcal{G}(\tilde{h}, (\mu_k)_{k\in\mathbb{Z}^d})$ of (space)-Gibbs measures on $\mathbb{R}^{\mathbb{Z}^d}$ built on $(\mu_k)_{k\in\mathbb{Z}^d}$ and $\tilde{h}$ (see section 1.3) contains at least one element.

Theorem 2 *([CRZ] Theorem 4.15) Under the assumptions (19) made on ϕ and ν the following assertions are equivalent for a law Q on $\Omega_{\bar{I}}$:*
(i) $Q = Q^{\nu}_{\bar{I}}$, i.e. Q is the law of X, solution of (2.2) in the interval $\bar{I}$ with initial law $\nu \in \mathcal{G}(\tilde{h}, (\mu_k)_k)$.
(ii) $Q \in \mathcal{G}(\tilde{H}, (P^{\mu_k}_{\bar{I}})_k)$, where $\tilde{H} = (\tilde{H}_{\Lambda\times I})_{\Lambda\subset\mathbb{Z}^d}$ is a space $\mathbb{F}$-Hamiltonian, which has the following structure : for $\Lambda \subset \mathbb{Z}^d, \omega \in \Omega_{\bar{I}}$

$$\tilde{H}_{\Lambda\times I}(\omega) = H_{\Lambda\times I}(\omega) + H^{\mathrm{bd}}_{\Lambda\times I}(\omega)$$

where

$$H_{\Lambda\times I}(\omega) = \int_I \sum_{\Lambda':\Lambda'\cap\Lambda\neq\varnothing, card\Lambda'=1,2,3} \varphi_{\Lambda'}(\omega_s)ds$$

and

$$H^{\mathrm{bd}}_{\Lambda\times I}(\omega) = \sum_{\Lambda':\Lambda'\cap\Lambda\neq\varnothing} \left(\frac{1}{2}\phi_{\Lambda'}(\omega_b) - \frac{1}{2}\phi_{\Lambda'}(\omega_0) + \tilde{\varphi}_{\Lambda'}(\omega_0)\right).$$

φ is the following three-body interaction :

- $\varphi_{\{k\}}(y_k) = -\frac{1}{4}\Delta_k\phi_k(y_k) + \frac{1}{8}(\nabla_k\phi_k)^2(y_k)$
- $\varphi_{\{k,\ell\}}(y_k, y_\ell) = -\frac{1}{4}(\Delta_k + \Delta_\ell)\phi_{k,\ell}(y_k, y_\ell) + \frac{1}{4}(\nabla_k\phi_k(y_k)\nabla_k\phi_{k,\ell}(y_k, y_\ell)$
  $+\nabla_\ell\phi_\ell(y_\ell)\nabla_\ell\phi_{k,\ell}(y_k, y_\ell)) - \frac{1}{8}((\nabla_k\phi_{k,\ell})^2 + (\nabla_\ell\phi_{k,\ell})^2)(y_k, y_\ell)$
- $\varphi_{k,\ell,m}(y_k, y_\ell, y_m) = -\frac{1}{4}(\nabla_k\phi_{k,\ell}(y_k, y_\ell)\nabla_k\phi_{k,m}(y_k, y_m) + \nabla_\ell\phi_{k,\ell}(y_k, y_\ell)$
  $\nabla_\ell\phi_{m,\ell}(y_m, y_\ell) + \nabla_m\phi_{k,m}(y_k, y_m)\nabla_m\phi_{m,\ell}(y_m, y_\ell))$
- $\varphi_\Lambda \equiv 0$ *otherwise*

(20)

We compute a concrete example in section 2.3.

2.2 The case of the unbounded time interval $\mathbb{R}^+$.

We are now able to state our main result.

Theorem 3 *i) Under the assumptions (19), for any initial condition $\nu \in \mathcal{G}(\tilde{h}, (\mu_k)_k)$ the full path measure Q^ν is a space-time Gibbs measure element of $\mathcal{G}(H, \Pi^0)$, where $H = (H_{\Lambda\times I})_{\Lambda\subset\mathbb{Z}^d, I=]0,b[}, 0 < b < +\infty$ is the space-time Hamiltonian given above and Π^0 is the reference specification built on Brownian bridges (introduced in section 1.4.2) .*
ii) If the initial condition ν belongs to $\mathcal{G}(h, (dx_k)_k)$, the path measure is stationnary (even reversible), and hence the process can be prolonged to negative time. Then, denoting its distribution on $\mathbb{R}$ by Q^ν as before, we have that $Q^\nu \in \mathcal{G}(H, \Pi^0)$, where $H = (H_{\Lambda\times I})_{\Lambda\subset\mathbb{Z}^d, I=]a,b[}, -\infty < a < b < +\infty$.

Proof : i) 1) For each time interval $I =]0, b[$, $Q^\nu_{\bar{I}}$ is the law of the solution of (18) on the interval $\bar{I}$. Theorem 2 thus implies that $Q^\nu_{\bar{I}} \in \mathcal{G}(\tilde{H}, (P^{\mu_k}_{\bar{I}})_k)$.

Thus $Q^{\nu}_{\bar{I}}$ is a Gibbs state in space in the usual DLR-sense. We now want to use Proposition 2. Thus the question is wether $Q^{\nu}_{\bar{I}}$ is also a Gibbs state specified by some $\Pi_{\bar{I}}$, where $\Pi_{\bar{I}}$ is derived from some local specification ρ on $(\Omega_{\bar{I}}, \mathbb{F}_{\bar{I}})_{\bar{I}}$.

2) The construction of ρ is as follows. We start with the local specification ρ_0 of example 2, consider the corresponding $\mathbb{F}$-specification Π^0, defined by (1.10), and construct with help of H via Theorem 1 the corresponding $\mathbb{F}$-specification Π^H. ρ is finally defined by Π^H via (1.12). We consider now $\Pi^H_{\bar{I}}$. To use Proposition 2 we must show that $Q^{\nu}_{\bar{I}} \in \mathcal{G}(\Pi^H_{\bar{I}})$.

This will be done by desintegrating the Gibbs state $Q^{\nu}_{\bar{I}} \in \mathcal{G}(\tilde{H}, (P^{\mu_k}_{\bar{I}})_k)$ with respect to the projection on $\partial I = \{0, b\}$, denoted by $X_{\partial I}(\omega) = \omega_{\partial I}$. To write this proof we use the following simplified notations :

$$Q = Q^{\nu}_{\bar{I}}, \nu' = Q o X^{-1}_{\partial I}, \mu'_k = P^{\mu_k}_{\bar{I}} o X^{-1}_{\partial I}, \mu'_\Lambda = \underset{k \in \Lambda}{\otimes} \mu'_k \,.$$

Let $g \in b(\hat{\mathcal{F}}_{\bar{I}})_\Lambda$ be of the form $g = g_1 g_2$, where $g_1(\omega) = g_1(\omega_{\Lambda^c})$, $g_2(\omega) = g_2(\omega_{\Lambda, \partial I})$ and let $f \in b(\mathcal{F}_{\bar{I}})_\Lambda$.

Then

$$\begin{aligned} E_Q(gf) \;&=\; E_Q\Big[g_1(\omega_{\Lambda^c}) \frac{1}{Z^{\tilde{H}}_\Lambda(\omega_{\Lambda^c})} \int f(\xi\omega_{\Lambda^c}) g_2(\xi_{\partial I}) \\ &\qquad \exp -\tilde{H}_\Lambda(\xi\omega_{\Lambda^c}) \big(\underset{k\in\Lambda}{\otimes} P^{\mu_k}_{\bar{I}} \big)(d\xi) \Big] \end{aligned} \tag{21}$$

Now desintegrate $P^{\mu_k}_{\bar{I}}$ with respect to the projection $X_{\partial I}$ and Q with respect to $X_{\Lambda^c, \partial I}$:

$$\begin{aligned} P^{\mu_k}_{\bar{I}} &= \int_{\mathbb{R}^2} P^{y_k}_{\bar{I}} \mu'_k(dy_k) \\ Q &= \int Q^{y_{\Lambda^c}} \nu'_{\Lambda^c}(dy_{\Lambda^c}). \end{aligned}$$

Using this we get equality of (21) with

$$\begin{aligned} &\iint g_1(\omega_{\Lambda^c}) \frac{1}{Z^{\tilde{H}}_\Lambda(\omega_{\Lambda^c})} \iint f(\xi\omega_{\Lambda^c}) g_2(y_\Lambda) \\ &\exp -\tilde{H}_\Lambda(\xi\omega_{\Lambda^c}) \big(\underset{k\in\Lambda}{\otimes} P^{y_k}_{\bar{I}} \big)(d\xi) \mu'_\Lambda(dy_\Lambda) Q^{y_{\Lambda^c}}(d\omega_{\Lambda^c}) \nu'_{\Lambda^c}(dy_{\Lambda^c}) \end{aligned} \tag{22}$$

We observe here that equality of (21) and (22) implies, choosing $f \equiv 1$ and g_1, g_2 depending only on $\omega_{\partial I}$, the following quasi Gibbsian structure of ν' :

$$\nu'_\Lambda(dy_\Lambda / y_{\Lambda^c}) = \rho_\Lambda(y) \cdot \mu'_\Lambda(dy_\Lambda) \quad \nu'_{\Lambda^c} - \text{a.s.}[y_{\Lambda^c}], \tag{23}$$

where

$$\rho_\Lambda(y) = \int \frac{1}{Z^{\tilde{H}}_\Lambda(\omega_{\Lambda^c})} \int \exp -\tilde{H}_\Lambda(\xi\omega_{\Lambda^c}) \big(\underset{k\in\Lambda}{\otimes} P^{\mu_k}_{\bar{I}} \big)(d\xi) Q^{y_{\Lambda^c}}(d\omega_{\Lambda^c}).$$

Observe here that $\rho_\Lambda(y)$ is ($\nu' -$ a.s.) strictly positive, therefore $\nu'_\Lambda(\cdot/y_{\Lambda^c})$ and μ'_Λ are ($\nu' -$ a.s.) equivalent.

We thus can replace in (22) $\mu'_\Lambda(dy_\Lambda)$ by $\frac{1}{\rho_\Lambda(y)}\nu'(dy_\Lambda/y_{\Lambda^c})$ and obtain :

$$\iiint g_1(\omega_{\Lambda^c})g_2(y_\Lambda)\frac{1}{Z_\Lambda^{\tilde{H}}(\omega_{\Lambda^c})}\frac{1}{\rho_\Lambda(y)}\int f(\xi\omega_{\Lambda^c})$$
$$\exp -\tilde{H}_\Lambda(\xi\omega_{\Lambda^c})\big(\bigotimes_{k\in\Lambda} P_{\bar{I}}^{y_k}\big)(d\xi)Q^{y_{\Lambda^c}}(d\omega_{\Lambda^c})\nu'_\Lambda(dy_\Lambda/y_{\Lambda^c})\nu'_{\Lambda^c}(dy_{\Lambda^c})$$

which in turn equals

$$E_Q\Big[g_1(\omega_{\Lambda^c})g_2(\omega_{\Lambda,\partial I})\frac{1}{Z_\Lambda^{\tilde{H}}(\omega_{\Lambda^c})}\frac{1}{\rho_\Lambda(\omega_{\partial I})}$$
$$\int f(\xi\omega_{\Lambda^c})\exp -\tilde{H}_\Lambda(\xi\omega_{\Lambda^c})\big(\bigotimes_{k\in\Lambda} P_{\bar{I}}^{\omega_{k,\partial I}}\big)(d\xi)\Big] \tag{24}$$

Finally, we observe that $\tilde{H}_\Lambda(\omega) = H_\Lambda(\omega_I) + H_\Lambda^{bd}(\omega_{\partial I})$ which implies that (24) equals

$$E_Q\Big[g(\omega)\frac{1}{Z_\Lambda^{\tilde{H}}(\omega_{\Lambda^c})}\frac{1}{\rho_\Lambda(\omega_{\partial I})}\exp -H_\Lambda^{bd}(\omega_{\partial I})$$
$$\int f(\xi\omega_{\Lambda^c})\exp -H_\Lambda(\xi\omega_{\Lambda^c})\big(\bigotimes_{k\in\Lambda} P_{\bar{I}}^{\omega_{k,\partial I}}\big)(d\xi)\Big] \tag{25}$$

The equality of (21) and (25) for each g thus gives, for Q a.s. ω,

$$E_Q(f/(\hat{\mathcal{F}}_{\bar{I}})_\Lambda)(\omega) = \mathrm{cte}(\omega_{\Lambda^c},\omega_{\partial I})\int f(\xi\omega_{\Lambda^c})\exp -H_\Lambda(\xi\omega_{\Lambda^c})\big(\bigotimes_{k\in\Lambda} P_{\bar{I}}^{\omega_{k,\partial I}}\big)(d\xi)$$

i.e. $Q \in \mathcal{G}(\Pi_{\bar{I}}^H)$.

3) Next we observe that $Q_{\bar{I}}^\nu$ satisfies the 1-sided Markov property (see theorem 3.6 [Do-Ro]) i.e. $(X_{\cdot,t})_{t\geq 0}$ is a (time) Markov process in the usual sense. Then $Q_{\bar{I}}^\nu$ satisfies also the 2-sided Markov property, as shown in the following Lemma 4. Thus we can apply Proposition 2 to conclude that $Q \in \mathcal{G}(H,\Pi^0)$.

■

Lemma 4 *If a probability measure Q on $(\Omega,\mathcal{F})$ satisfies the one-sided Markov property then also the two-sided Markov property.*

Proof : Let $f \in b\mathcal{F}_V$ and $g \in b\hat{\mathcal{F}}_V$ of the form $g = h_1 g_1 k g_2 h_2$ where $g_1 \in b\mathcal{B}_{\mathbb{Z}^d\times\{a\}}, g_2 \in b\mathcal{B}_{\mathbb{Z}^d\times\{b\}}, h_1 \in b\mathcal{B}_{\mathbb{Z}^d\times]0,a]}, h_2 \in b\mathcal{B}_{\mathbb{Z}^d\times]b,\infty[}, k \in b\mathcal{B}_{\Lambda^c\times\bar{I}}$. The σ-algebras $\mathcal{B}_V$ had been defined in Definition 5.

Then using the equivalence of the left-sided and the right-sided Markov property several times we get

$$
\begin{aligned}
Q(h_1 g_1 k f g_2 h_2) &= Q(h_1 g_1 k f g_2 Q(h_2 \mid \mathcal{B}_{\mathbb{Z}^d \times \{b\}})) \\
&= Q(Q(h_1 \mid \mathcal{B}_{\mathbb{Z}^d \times \{a\}}) g_1 k f g_2 Q(h_2 \mid \mathcal{B}_{\mathbb{Z}^d \times \{b\}})) \\
&= Q(Q(h_1 \mid \mathcal{B}_{\mathbb{Z}^d \times \{a\}}) g_1 k Q(f \mid \tilde{\partial}\mathcal{F}_V) g_2 Q(h_2 \mid \mathcal{B}_{\mathbb{Z}^d \times \{b\}})) \\
&= Q(h_1 g_1 k Q(f \mid \tilde{\partial}\mathcal{F}_V) g_2 h_2)
\end{aligned}
$$

This shows that the two-sided Markov property (equality (1.11)) is true. The proof of ii) is obvious.

■

Let us remark that in the above result of Theorem 3 *i)* the initial parameters ($\tilde{h}$ and μ_k) disappear in the space-time Gibbsian description of Q^ν. It means that, for ϕ fixed, all the laws Q^ν when ν describes the (large!) set of Gibbsian measures on $\mathbb{R}^{\mathbb{Z}^d}$ belong to the same set $\mathcal{G}(H, \Pi^0)$. It is then reasonable to restrict our attention to the case *ii)*, i.e. to stationnary, or even reversible space-time Gibbsian fields.

Then Theorem 3 exhibits an injection from the set $\mathcal{G}(h, (dx_k)_k)$ of Gibbsian measures on the configuration spaces $\mathbb{R}^{\mathbb{Z}^d}$, which correspond to the time projection of the states on Ω, into the set $\mathcal{G}_r(H, \Pi^0)$ of reversible Gibbsian states on Ω associated to the Hamiltonian functional H. Thus, $\mathcal{G}_r(H, \Pi^0)$ is non empty if $\mathcal{G}(h, (dx_k)_k)$ is non empty. Moreover, in situations when phase transition of Gibbsian measures associated to h is known, we derive precise informations on the level of path space. This idea is developped in the next paragraphs 2.3 and 2.4.

In order to obtain a bijection between the sets $\mathcal{G}(h, (dx_k)_k)$ and $\mathcal{G}_r(H, \Pi^0)$ we should prove some converse statement to Theorem 3 *ii)*. This is one of the topics presented in [DP-R-Z]. In this paper, a Gibbs variational principle is proved - in the context of space-time invariant processes - to show that space-time Gibbs state in $\mathcal{G}_r(H, \Pi^0)$ are weak solutions of stochastic differential equations like (2.2), under the asumption that the hamiltonian h is a bounded perturbation of a suitable self potential. Unfortunately, in the example 2.3 below, we can not use these results since the pair interaction $\phi^{(\varepsilon)}_{k,\ell}$ is unbounded on $\mathbb{R}^2$.

2.3 An example of a reversible Gibbs state.

Let $\phi^{(\varepsilon)}$ be the following pair potential on $\mathbb{R}^{\mathbb{Z}^d}$:

$$
\begin{cases}
\phi^{(\varepsilon)}_k(y_k) & = \mathcal{P}(y_k), \\
\phi^{(\varepsilon)}_{k,\ell}(y_k, y_\ell) & = \varepsilon c_{|\ell-k|} y_k y_\ell, k, \ell \in \mathbb{Z}^d, k \neq \ell
\end{cases}
\tag{26}
$$

where $\mathcal{P}$ is a convex polynomial function, and $c \in \mathcal{S}'(\mathbb{Z}^d)$.
It is well known that, for ε small enough, the set $\mathcal{G}(h^{(\varepsilon)}, (dx_k)_k))$ contains

a unique space Gibbs measure ($h^{(\varepsilon)}$ is the Hamiltonian associated to the potential $\phi^{(\varepsilon)}$).

We then immediately derive the following example.

Proposition 3 *Let $H^{(\varepsilon)}$ be the reversible Hamiltonian on the path space Ω defined by taking $\phi = \phi^{(\varepsilon)}$ in* (2.4)*. Then, for ε small enough, $\mathcal{G}_r(H^{(\varepsilon)}, \Pi^0)$ contains at least one Gibbsian field, the law of the unique reversible process solution of* (18)*.*

In this example, since $\phi^{(\varepsilon)}$ is a small perturbation of the self interaction $\mathcal{P}$, $H^{(\varepsilon)}$ comes from a small pair interaction perturbation $\Phi^{(\varepsilon)}$ with the following decomposition as in the formula (1.14) :

$$\Phi^{(\varepsilon)}_{\Lambda\times I}(X) = \int_I \varphi^{(\varepsilon)}_\Lambda(X_{\Lambda,s})ds \tag{27}$$

where

- $\varphi^{(\varepsilon)}_{\{k\}}(y_k) = \varphi^{(0)}_{\{k\}}(y_k) = -\frac{1}{4}\mathcal{P}''(y_k) + \frac{1}{8}\mathcal{P}'(y_k)^2 + \frac{\varepsilon^2}{8}\left(\sum_{j\neq 0} c^2_{|j|}\right) y_k^2$
- $\varphi^{(\varepsilon)}_{\{k,\ell\}}(y_k, y_\ell) = \frac{\varepsilon}{2}c_{|\ell-k|}(\mathcal{P}'(y_k)y_\ell + \mathcal{P}'(y_\ell)y_k) + \frac{\varepsilon^2}{4}\left(\sum_{j\neq \ell,k} c_{|\ell-j|}c_{|k-j|}\right) y_k y_\ell \quad \text{for } k\neq\ell$
- $\varphi^{(\varepsilon)}_\Lambda \equiv 0 \quad$ otherwise

2.4 An example of a phase transition

In [Ne], Nelson affirms that the set $\mathcal{G}(h^\alpha, (dx_k)_k)$ contains more than one element if α is large enough. Here h^α is the Hamiltonian derived from the following pair potential :

$$\begin{cases} \phi^{(\alpha)}_k(y_k) &= y_k^4 - \alpha y_k^2 \\ \phi^{(\alpha)}_{k,\ell}(y_k, y_\ell) &= \frac{1}{2}(y_\ell - y_k)^2 \quad \text{if} \quad |k-\ell| = 1 \quad \text{and} \quad 0 \quad \text{otherwise} \end{cases} \tag{28}$$

It means that the convexity of y^4 is too much perturbated by the term $-\alpha y^2$.

It induces the following phase transition behavior on the path level :

For α large enough, the set $\mathcal{G}_r(H^\alpha, \Pi^0)$ contains more than one reversible Gibbsian field, where H^α is the time reversible Hamiltonian on Ω defined by equation (2.4) *with $\phi = \phi^{(\alpha)}$ given in* (2.12)*.*

2.5 A remark on the Martin boundary

We just gave two examples of $\mathbb{F}$-specifications Π^H on $\Omega = \mathcal{C}(\mathbb{R}, \mathbb{R})^{\mathbb{Z}^d}$ for which the set of Gibbs state $\mathcal{G}(\Pi^H)$ is non void. Taking into account that

Ω is a standard Borel space we thus can apply the general Martin-Dynkin boundary theory (see [Foe2] and [Pre]). This theory implies the existence of a probability kernel $\Pi^H_\infty : \Omega \times \mathcal{F} \to \mathbb{R}^+$ with the following properties :

i) each $\Pi^H_\infty(\cdot, F), F \in \mathcal{F}$ is measurable with respect to the tail field $\hat{\mathcal{F}}_\infty = \bigcap_{V \in \mathfrak{V}} \hat{\mathcal{F}}_V$

ii) each $\Pi^H_\infty(\omega, \cdot), \omega \in \Omega$, is an extremal element of $\mathcal{G}(\Pi^H)$.

Thus each $\Pi^H_\infty(\omega, \cdot)$ is trivial on the tail field, or equivalently has short range correlations (with respect to $(\hat{\mathcal{F}}_V)_{V \in \mathfrak{V}}$). This means

Given any $F \in \mathcal{F}$ and $\varepsilon > 0$, then there exists $V \in \mathfrak{V}$ with

$$\left| \Pi^H_\infty(\omega, F \cap F') - \Pi^H_\infty(\omega, F) \cdot \Pi^H_\infty(\omega, F') \right| < \varepsilon \quad \text{for all} \quad F' \in \hat{\mathcal{F}}_V. \tag{29}$$

Π^H_∞ can be expressed by means of the specification $(\Pi^H_V)_{V \in \mathfrak{V}}$ as follows : if $(V_n)_n$ is a countable base in V then there exist $E \in \hat{\mathcal{F}}_\infty$ such that $Q(E) = 1$ for all $Q \in \mathcal{G}(\Pi^H)$ and

$$\lim_n \Pi^H_{V_n}(\omega, \cdot) = \Pi^H_\infty(\omega, \cdot), \omega \in E. \tag{30}$$

To say it in another way, for all $\omega \notin N$, where N is $\mathcal{G}$ (Π^H)-negligeable in $\hat{\mathcal{F}}_\infty$, (30) is true.

3 Gibbs states in space-time and cluster expansion

Another approach to the problem of the construction of space-time Gibbs states makes use of cluster expansion techniques. It is a powerful theory, for example when the considered model is a small modification of independent fields. In this section we study two systems which modelize infinitely many anharmonical oscillators with a (small) interaction on the lattice $\mathbb{Z}^d$. We present cluster representations of the partition functions and obtain existence and regularity properties of the limiting fields. Let us also quote the work of Jona-Lasinio and Sénéor [JL-S] in which the authors use space-time cluster expansion to compute asymptotic properties of diffusion processes.

3.1 The first model : a small quadratic perturbation of a free field

3.1.1 Its space-time specifications

Consider the polynom $\mathcal{P}_1$ on $\mathbb{R}$:

$$\mathcal{P}_1(z) = z^{2r} + \sum_{j=1}^{2r-2} \alpha_j z^j \ , \ \alpha_j \in \mathbb{R}, r \in \mathbb{N}, r > 1 \tag{31}$$

and the associated Schrödinger operator on $L^2(\mathbb{R}, dz)$

$$Af(z) = -\frac{1}{2}\frac{d^2}{dz^2} f(z) + \mathcal{P}_1(z) f(z) \ .$$

It is a self adjoint operator which admits a smallest eigenvalue λ_0. Let $e^{-\frac{1}{2}\varphi_0}$ the unique associated normalized strictly positive eigenvector. Denote by μ_0 the probability measure on $\mathbb{R}$:

$$\mu_0(dz) = e^{-\varphi_0(z)} dz. \tag{32}$$

It is the unique invariant (and also time reversible) probability measure for the one-dimensional diffusion, solution of the following stochastic differential equation :

$$dX_t = dW_t - \frac{1}{2}\nabla\varphi_0(X_t)dt, t \in \mathbb{R} \tag{33}$$

We denote by $\tilde{P}$ the law on $\mathcal{C}(\mathbb{R}, \mathbb{R})$ of this reversible diffusion process X. With this time stationary measure, we can construct a free field in the following way. As a local reference specification on $\Omega = \mathcal{C}(\mathbb{R}, \mathbb{R})^{\mathbb{Z}^d}$, generalizing example 2 of section 1.4.2, we consider for $V = \Lambda \times I, \Lambda \subset \mathbb{Z}^d, I =]a, b[, -\infty < a < b < +\infty, \omega \in \Omega$,

$$\tilde{\Pi}^0_V(\omega, foX_{\bar{I}}) = \int_{\Omega_{\Lambda,\bar{I}}} f(\eta\omega_{\Lambda^c})(\otimes_{k\in\Lambda}\tilde{P}_{\bar{I}}^{\omega_{k,a},\omega_{k,b}})(d\eta) \tag{34}$$

where $\tilde{P}_{\bar{I}}^{x,y}$ denotes the bridge of the diffusion solution of (3.3) between x and y on $\bar{I}$.

As a natural Gibbsian modification of this free field we introduce as interaction between the coordinates a small nearest neighboor quadratic interaction (cf. example 3 in section 1.4.3); the associated space-time Hamiltonian then satisfies, for $\Lambda \subset \mathbb{Z}^d, I =]a, b[, -\infty < a < b < +\infty$,

$$H^{(\varepsilon)}_{\Lambda\times I}(\omega) = \varepsilon \int_I \sum_{\substack{k,l\in\Lambda \\ |k-l|=1}} \omega_{k,s}\omega_{l,s} ds + \varepsilon \int_I \sum_{\substack{k\in\Lambda, l\in\Lambda^c \\ |k-l|=1}} \omega_{k,s}\omega_{l,s} ds. \tag{35}$$

We are now able to state our main result in this section :

Theorem 4 *For ε small enough, there exists a space-time Gibbs measure Q in $\mathcal{G}(H^{(\varepsilon)}, \tilde{\Pi}^0)$. This measure admits a cluster expansion and is invariant with respect to any space-time translation. Moreover it satisfies the property of short range correlations.*

The proof of this theorem is based on the following convergence result.

Let $V_n = \Lambda_n \times I_n$ be an increasing sequence of sets in $\mathbb{Z}^d \times \mathbb{R}$ which tends to $\mathbb{Z}^d \times \mathbb{R}$ when n tends to infinity. Let ∂V_n be the space-time boundary of V_n as defined in section 1.4.1.

Lemma 5 *For a fixed $\omega \in \Omega$, let $Q_n^{\omega_{\partial V_n}}$ be the probability on Ω_{Λ_n, I_n} indexed by the boundary conditions $\omega_{\partial V_n}$ defined by :*

$$\begin{aligned} Q_n^{\omega_{\partial V_n}}(d\eta) &= \frac{1}{Z^{\varepsilon}_{V_n}(\omega_{\partial V_n})} \exp - H^{(\varepsilon)}_{V_n}(\eta\omega_{\partial V_n})(\otimes_{k\in\Lambda_n} \tilde{P}_{\bar{I}_n})(d\eta) \\ &= \frac{1}{Z^{\varepsilon}_{V_n}(\omega_{\partial V_n})} \int \exp - H^{(\varepsilon)}_{V_n}(\eta\omega_{\partial V_n})(\otimes_{k\in\Lambda_n} \tilde{P}^{x_k,y_k}_{\bar{I}_n})(d\eta) \otimes_{k\in\Lambda_n} \mu^n(dx_k, dy_k) \end{aligned}$$

where $\mu^n = \tilde{P} o X^{-1}_{\partial I_n}$. If

$$\lim_n Q_n^{\omega_{\partial V_n}}(f) =: Q(f), f \in b\mathcal{F}_{loc}$$

then the weak limit Q belongs to $\mathcal{G}(H^{(\varepsilon)}, \tilde{\Pi}^0)$.

Proof : The proof is a direct application of Proposition 1 in connection with Remark 4, which implies that $\Pi^{H^{(\varepsilon)}}$ is not only an $\mathbb{F}$-specification but even an $\hat{\mathbb{F}}$-specification. ■

So, we will now compute the limit of $(Q_n^{\omega_{\partial V_n}})_n$ for the special boundary conditions $\omega_{\partial V_n} \equiv 0$. For simplicity we note

$$Q_n =: Q_n^0.$$

The next paragraphs are devoted to the proof of the convergence of Q_n which is based on the method of cluster expansion. Technical details are contained in [Mi-Ve-Za] where the limit Gibbs state for quantum crystal is constructed. Here, we sketch the important steps of the proof since we will use them in section 3.2 as a basis for the study of the more complicated Gibbsian modification of the free field $\tilde{\Pi}^0$ by the interaction given in (2.11).

3.1.2 The cluster representation of statistical sums

Let $a > 0$ be a real number which we will choose later and $\mathbb{Z}_a \subset \mathbb{R}$ the one-dimensional lattice with step length a.

Let us denote by $I_j = [ja, (j+1)a], j \in \mathbb{Z}$, time intervals of length a. Let

$$\mathbb{Z}_a^{d+1} = \mathbb{Z}^d \times \mathbb{Z}_a$$

be a space-time lattice with scale a for the time. We call temporal edge in $\mathbb{Z}_a^{d+1}$ edges of the following type :

$$b^{\mathrm{temp}} = [(k, ja), (k, (j+1)a)] \equiv (k, I_j), k \in \mathbb{Z}^d \ .$$

We call by plaquette on the interval I_j the following pair of neighbours of temporal edges

$$\square_j^{k,\ell} \equiv \{(k, I_j), (\ell, I_j)\}$$

if $k, \ell \in \mathbb{Z}^d$ are neighbours, i.e. $|k - \ell| = 1$

For every set $B = \{b^{\mathrm{temp}}\}$ of temporal edges we denote by $[B] \subset \mathbb{Z}_a^{d+1}$ the full set of vertices of $b^{\mathrm{temp}} \in B$. We assume that the time interval I is of the form

$$I = [-Na, Na] = \bigcup_{j=-N}^{N-1} I_j$$

For any collection of values $\{y_j, j \in \{-N, \dots, N\}\} \in \mathbb{R}^{2N+1}$ we introduce the conditional distribution

$$\tilde{P}_{\bar{I}}(\ /X_{ja} = y_j, j = -N, \dots, N)$$

which is obtained by fixing the values of the process $(X_t)_{t \in I}$ (solution of (33)) at each time $\{ja, j = -N, \dots, N\}$.

Since the process X_t is Markovian, we have :

$$\tilde{P}_{\bar{I}}(\ /X_{ja} = y_j, j = -N, \dots, N) = \otimes_{j=-N}^{N-1} \tilde{P}_{\bar{I}_j}^{y_j, y_{j+1}}. \tag{36}$$

Let us denote by $p^0(\{y_j\}_{j=-N,\dots,N})$ the density with respect to $\otimes_{j=-N}^{N} \mu_0(dy_j)$ of the joint distribution of $(X_{ja}, j = -N, \dots, N)$. By the Markov property,

$$p^0(\{y_j\}_{j=-N,\dots,N}) = \prod_{j=-N}^{N-1} p_a^0(y_{j+1}/y_j) \tag{37}$$

where $p_a^0(y/x)$ is the density of the transition probability from x to y during a time length a.

With these notations, one can write the partition functions Z_V^ε as follows, for $V = \Lambda \times I$,

$$\begin{aligned} Z_V^\varepsilon \ &=: \int_\Omega \exp\big(-H_V^{(\varepsilon)}(\omega_V 0_{\partial V}\big) \underset{k \in \Lambda}{\otimes} \tilde{P}_{\bar{I}}(d\omega_k) \\ &= \int_{\mathbb{R}^{|\Lambda|(2N+1)}} Z_V(y_V) \prod_{k \in \Lambda} p^0(\{y_{k,j}\}) \otimes_{j=-N}^{N} \mu_0(dy_{k,j}) \end{aligned} \tag{38}$$

where $y_V = \{y_{k,j}, (k,j) \in \Lambda \times \{-N, \dots, N\}\}$ are the possible values of the paths $X_{\Lambda,\bar I}$ at the points $(k, ja) \in \mathbb{Z}_a^{d+1}$, $k \in \Lambda$, $j \in \{-N, \dots, N\}$.

By the Markov property,

$$Z_V(y_V) = \prod_{j=-N}^{N-1} \int \prod_{\substack{k,\ell\in\Lambda \\ |k-\ell|=1}} \exp\bigl(-\varepsilon \int_{I_j} \omega_{k,s}\omega_{\ell,s} ds\bigr) \tilde P_{\bar I_j}^{y_{k,j},y_{k,j+1}}(d\omega_k).$$

Furthermore,

$$\begin{aligned} \prod_{\substack{k,\ell\in\Lambda \\ |k-\ell|=1}} \exp -\varepsilon \int_{I_j} \omega_{k,s}\omega_{\ell,s} ds &= \prod_{\substack{k,\ell\in\Lambda \\ |k-\ell|=1}} \Bigl(1 + \exp\bigl(-\varepsilon \int_{I_j} \omega_{k,s}\omega_{\ell,s} ds\bigr) - 1\Bigr) \\ &= 1 + \sum_{\Gamma^j} \prod_{\Box_j^{k,\ell}\in\Gamma^j} \Bigl(\exp\bigl(-\varepsilon \int_{I_j} \omega_{k,s}\omega_{\ell,s} ds\bigr) - 1\Bigr) \qquad (39) \\ &= 1 + \sum_{s\geq 1} \sum_{\gamma_1^j,\dots,\gamma_s^j} \prod_{m=1}^{s} \prod_{\Box_j^{k,\ell}\in\gamma_m^j} \Bigl(\exp\bigl(-\varepsilon \int_{I_j} \omega_{k,s}\omega_{\ell,s} ds\bigr) - 1\Bigr) \end{aligned}$$

where the summation $\sum_{\Gamma^j}$ takes into account all non empty, non ordered collections of plaquettes $\Gamma^j = \{\Box_j^{k,\ell}\}$ on the interval I_j such that $\{k,\ell\} \subset \Lambda$, and the summation $\sum_{\gamma_1^j,\dots,\gamma_s^j}$ takes into account all non empty collections of pairwise non intersecting, connected sets γ_m^j of such plaquettes. γ is called a <u>contour</u>. Thus, from (39), $Z_V(y_V)$ becomes :

$$Z_V(y_V) = \prod_{j=-N}^{N-1} \Bigl(1 + \sum_{s\geq 1} \sum_{\gamma_1^j,\dots,\gamma_s^j} \prod_{m=1}^{s} \int \prod_{\Box_j^{k,\ell}\in\gamma_m^j} \qquad (40)$$
$$\Bigl(\exp\bigl(-\varepsilon \int_{I_j} \omega_{k,s}\omega_{\ell,s} ds\bigr) - 1\Bigr) \tilde P_{\bar I_j}^{y_{k,j},y_{k,j+1}}(d\omega_k)\Bigr)$$

On the other side, from (37) and (38) :

$$Z_V^{\varepsilon} = \int \prod_{k\in\Lambda} \prod_{j=-N}^{N-1} p_a^0(y_{k,j+1}/y_{k,j}) Z_V(y_V) \bigotimes_{\substack{k\in\Lambda \\ j=-N,\dots,N}} \mu_0(dy_{k,j})$$

Here also we can rewrite, for $k \in \Lambda$ fixed, the product :

$$\begin{aligned}\prod_{j=-N}^{N-1} p_a^0(y_{k,j+1}/y_{k,j}) &= \prod_{j=-N}^{N-1} \big(1 + p_a^0(y_{k,j+1}/y_{k,j}) - 1\big) \\ &= 1 + \sum_{\tau^k} \prod_{I_j \in \tau_n^k} \big(p_a^0(y_{k,j+1}/y_{k,j}) - 1\big) \qquad (41) \\ &= 1 + \sum_{p \geq 1} \sum_{\tau_1^k \ldots, \tau_p^k} \prod_{n=1}^{p} \prod_{I_j \in \tau^k} \big(p_a^0(y_{k,j+1}/y_{k,j}) - 1\big)\end{aligned}$$

Here the summation $\sum_{\tau^k}$ is over all non ordered collections $\{I_j\}$ of different intervals I_j, and the summation $\sum_{\tau_1^k,\ldots,\tau_p^k}$ is over all pairwise non intersecting collections of $\tau_n^k = (I_{j_n}, I_{j_n+1}, \ldots, I_{j_n+r})$ of juxtaposition of I_j.

We can represent each sequence τ_n^k as a collection of temporal edges $\tau_n^k = \{(k, I_{j_n}), \ldots, (k, I_{j_n+r})\}$.

Let us now call an <u>aggregate</u> Γ a connected non empty collection $\Gamma = \{\gamma_1^{j_1}, \ldots, \gamma_s^{j_s}, \tau_1^{k_1}, \ldots, \tau_p^{k_p}\}$ consisting of a collection of contours $\{\gamma_1^{j_1}, \ldots, \gamma_s^{j_s}\}$ and a collection of sequence $\{\tau_1^{k_1}, \ldots, \tau_p^{k_p}\}$ (one of them can be empty).

For each aggregate Γ we introduce a function $\mathfrak{X}_\Gamma(\omega)$ which depends on the values of $\omega \in \Omega$ on the edges $(k, I_j) \in \bar{\Gamma}$ where $\bar{\Gamma}$ is the set of all temporal edges which compose Γ (contours or sequences) :

$$\begin{aligned}\mathfrak{X}_\Gamma(\omega) = \quad & \prod_{m=1}^{s} \prod_{\square_{j_m}^{k,\ell} \in \gamma_m^{j_m}} \big(\exp(-\varepsilon \int_{I_{j_m}} \omega_{k,s}\omega_{\ell,s} ds) - 1\big) \\ & \prod_{n=1}^{p} \prod_{(k_n, I_j) \in \tau_n^{k_n}} \big(p_a^0(y_{k_n,j+1}/y_{k_n,j}) - 1\big) \qquad (42)\end{aligned}$$

with $y_{k,j} = \omega_{k,ja}$.

Now, for every collection $\eta = \{(k, I_j)\}$ of temporal edges in $\mathbb{Z}_a^{d+1}$, we introduce the measure $\tilde{P}^\eta$ on the space $\bigotimes_{(k,I_j)\in\eta} \Omega_{\{k\},\bar{I}_j}$ of pieces of trajectories on the edges of η, by :

$$\tilde{P}^\eta(d\omega_{k,I_j}, (k, I_j) \in \eta) = \otimes_{(k,I_j)\in\eta} \tilde{P}_{\bar{I}_j}^{y_{k,j}, y_{k,j+1}}(d\omega_k) \otimes_{(k,j)\in[\eta]} \mu_0(dy_{k,j}) \qquad (43)$$

Here $[\eta]$ denotes the vertices of temporal edges from η. With these notations we can write from (42) and (43)

$$\begin{aligned} K_\Gamma &=: \int \mathfrak{X}_\Gamma(\omega) d\tilde{P}^{\bar{\Gamma}}(\omega) \qquad (44) \\ &= \int \prod_{m=1}^{s} \int \prod_{\square_{j_m}^{k,\ell} \in \gamma_m^{j_m}} \big(\exp(-\varepsilon \int_{I_{j_m}} \omega_{k,s}\omega_{\ell,s} ds) - 1\big) \\ & \otimes_{(k,aj_m)\in[\bar{\gamma}_m^{j_m}]} \tilde{P}_{\bar{I}_j}^{y_{k,j},y_{k,j+1}}(d\omega_k) \\ & \prod_{n=1}^{p} \prod_{(k_n,I_j)\in\tau_n^{k_n}} \big(p_a^o(y_{k_n,j+1}/y_{k_n,j}) - 1\big) \otimes_{(k,ja)\in[\bar{\Gamma}]} \mu_0(dy_{k,j}) \end{aligned}$$

From (38), (40), (44) we obtain the following representation of Z_V^ε :

$$Z_V^\varepsilon = 1 + \sum_{\{\Gamma_1,\ldots,\Gamma_s\}} \prod_{i=1}^{s} K_{\Gamma_i}\,, \qquad (45)$$

where the summation is taken over all non ordered, non empty collections $\{\Gamma_1,\ldots,\Gamma_s\}$ of pairwise non intersecting aggregates Γ_i such that

$$[\bar{\Gamma}_i] \subset V = \Lambda \times I \subset \mathbb{Z}_a^{d+1}\ .$$

The representation (45) is called a cluster representation for Z_V^ε.

We need now to estimate the values of K_Γ.

3.1.3 Cluster estimates

Proposition 4 *Under suitable choice of the time scale a there exists some constant $\lambda(\varepsilon)$ (small if ε is small) such that the weight K_Γ of the aggregate $\Gamma = \{\gamma_1^{j_1},\ldots,\gamma_s^{j_s},\tau_1^{k_1},\ldots,\tau_p^{k_p}\}$ satisfies the estimate :*

$$\big|K_\Gamma\big| < \lambda(\varepsilon)^{|\Gamma|} \qquad (46)$$

where $|\Gamma| = \operatorname{card}\bar{\Gamma}$ is the number of temporal edges which compose Γ.

To prove the above proposition we need the following abstract integration lemma, which generalizes Hölder inequalities :

Lemma 6 *Let $(\mathcal{E}_x,\mu_x)_{x\in\mathbf{X}}$ be a family of spaces $\mathcal{E}_x$ with probability measures μ_x, indexed by the elements x of some finite set $\mathbf{X}$. Let also $\{f_{Y_i}, Y_i \subset X\}$ be a family of functions f_{Y_i} on $\mathcal{E}_\mathbf{X} = \underset{x\in\mathbf{X}}{\times} \mathcal{E}_x$, indexed by subsets Y_i of $\mathbf{X}$ in such a way that, for any Y_i,*

$$f_{Y_i}(\mathcal{E}) = f_{Y_i}(\mathcal{E}_{|Y_i}), \mathcal{E} = \{\mathcal{E}_x \in \mathcal{E}_\mathbf{X}\}\ .$$

Let $n_{Y_i} > 1$ be numbers satisfying the following conditions :

$$\forall x \in \mathbf{X}, \sum_{Y_i \ni x} \frac{1}{n_{Y_i}} \leq 1. \tag{47}$$

Then

$$\left| \int_{\mathcal{E}_\mathbf{X}} \prod_{Y_i} f_{Y_i} d\mu_\mathbf{X} \right| \leq \prod_{Y_i} \left(\int_{\mathcal{E}_{Y_i}} |f_{Y_i}|^{n_{Y_i}} d\mu_{Y_i} \right)^{1/n_{Y_i}}$$

where $\mu_B = \underset{x \in B}{\otimes} \mu_x$ for any subset $B \subset \mathbf{X}$.

Proof : cf [Mi-Ve-Za], Appendix A.

We use this lemma to estimate the average

$$\int \prod_{\square_j^{k,\ell} \in \gamma^j} \left(e^{-\varepsilon \int_{I_j} \omega_{k,s} \omega_{\ell,s} ds} - 1 \right) \otimes_{(k,I_j) \in \bar{\gamma}^j} \tilde{P}_{\bar{I}_j}^{y_{k,j}, y_{k,j+1}} (d\omega_k)$$

which appears in (44).

Here γ^j is some contour of plaquettes on the interval I_j . So we can apply Lemma 5 with $\mathbf{X} = \bar{\gamma}^j$, $\mathcal{E}_{(k,I_j)} = \Omega_{\{k\},\bar{I}_j}$, $\mu_{(k,\bar{I}_j)} = \tilde{P}_{\bar{I}_j}^{y_{k,j}, y_{k,j+1}} (d\omega_k)$, $Y_i = \square_j^{k,\ell} \in \gamma^j$, the plaquettes of the contour γ^j and $f_{\square_j^{k,\ell}} = \exp(-\varepsilon \int_{I_j} \omega_{k,s} \omega_{\ell,s} ds) - 1$.

Every temporal edge $(k, I_j) \in \bar{\gamma}^j$ appears in no more than 2d plaquettes $\square_j^{k,\ell} \in \gamma^j$ (d is the lattice dimension). Thus, if we attribute to every plaquette a number n_1 such that

$$\frac{2d}{n_1} \leq 1 \tag{48}$$

we obtain (47). Hence, we can write

$$\begin{aligned} &\left| \int \prod_{\square_j^{k,\ell} \in \gamma^j} \left(e^{-\varepsilon \int_{I_j} \omega_{k,s} \omega_{\ell,s} ds} - 1 \right) \otimes_{(k,I_j) \in \bar{\gamma}^j} \tilde{P}_{\bar{I}_j}^{y_{k,j}, y_{k,j+1}} (d\omega_k) \right| \\ &\leq \prod_{\square_j^{k,\ell} \in \gamma^j} \left(\int \left| e^{-\varepsilon \int_{I_j} \omega_{k,s} \omega_{\ell,s} ds} - 1 \right|^{n_1} \otimes_{k,\ell} \tilde{P}_{\bar{I}_j}^{y_{k,j}, y_{k,j+1}} (d\omega_k) \tilde{P}_{\bar{I}_j}^{y_{l,j}, y_{l,j+1}} (d\omega_l) \right)^{1/n_1} \\ &\equiv \prod_{\square_j^{k,\ell} \in \gamma^j} F_{\square_j^{k,\ell}} (y_{k,j}, y_{k,j+1}, y_{\ell,j}, y_{\ell,j+1}) \end{aligned} \tag{49}$$

From (44), (49) we get the following estimate for K_Γ :

$$\begin{aligned} |K_\Gamma| \leq & \int \prod_{m=1}^{s} \prod_{\square_{jm}^{k,\ell} \in \gamma_m^{jm}} F_{\square_{jm}^{k,\ell}} \prod_{n=1}^{p} \prod_{(k_n, I_j) \in \tau_n^{k_n}} \\ & |p_a^o (y_{k_n,j+1} / y_{k_n,j}) - 1| \otimes_{(k,ja) \in \bar{\Gamma}} \mu_0 (dy_{k,j}) \end{aligned} \tag{50}$$

In order to apply another time Lemma 6 we take $\mathbf{X} = [\bar{\Gamma}]$, $\mathcal{E}_{(k,ja)} = \mathbb{R}$, $\mu_{(k,ja)} = \mu_0$, Y_m are either vertices of plaquettes $\Box_j^{k,\ell}$ from contours $\gamma \in \Gamma$ or vertices of interval I_j from sequences $\tau \in \Gamma$.

Every vertex $(k, ja) \in [\bar{\Gamma}]$ appears in no more than 4d plaquettes from Γ and in no more than 2 temporal edges from sequences τ of Γ. Assign to plaquettes a number n_1 and to temporal edges a number n_2 such that

$$\frac{4d}{n_1} + \frac{2}{n_2} \leq 1$$

then assumption (47) is satisfied. Then, by Lemma 6,

$$\begin{aligned}|K_\Gamma| \leq & \prod_{m=1}^{s} \prod_{\Box_{j_m}^{k,\ell} \in \gamma_m^{j_m}} \Big[\int_{\mathbb{R}^4} F^{n_1}_{\Box_{j_m}^{k,\ell}} \mu_0(dy_{k,j_m})\mu_0(dy_{k,j_m+1})\mu_0(dy_{\ell,j_m})\mu_0(dy_{\ell,j_m+1})\Big]^{1/n_1} \\ & \prod_{n=1}^{p} \prod_{I_j \in \tau_n^{k_n}} \Big[\int_{\mathbb{R}^2} \Big|p_a^0(y_{k_n,j+1}/y_{k_n,j}) - 1\Big|^{n_2} \mu_0(dy_{k_n,j})\mu_0(dy_{k_n,j+1})\Big]^{1/n_2} \quad (51)\end{aligned}$$

Taking $n_1 = 8d$, $n_2 = 4$, and denoting by

$$\begin{aligned}&M_1 = M_1(a, \varepsilon) = \\ &\Big[\int_{\mathbb{R}^4} F^{8d}_{\Box_j^{k,\ell}} \mu_0(dy_{k,j})\mu_0(dy_{k,j+1})\mu_0(dy_{\ell,j})\mu_0(dy_{\ell,j+1})\Big]^{1/8d} \quad (52)\end{aligned}$$

for some plaquette $\Box_j^{k,\ell}$ and

$$M_2 = M_2(a) = \Big[\int_{\mathbb{R}^2} |p_a^o(y_2/y_1) - 1|^4 \mu_0(dy_1)\mu_0(dy_2)\Big]^{1/4} \quad (53)$$

it follows from (51)-(53), that

$$|K_\Gamma| < M_1^{(\sum_{m=1}^{s} |\gamma_m^{j_m}|)} \; M_2^{(\sum_{n=1}^{p} |\tau_n^{k_n}|)} \quad (54)$$

where $|\gamma|$ is the cardinal of plaquettes in the contour γ, which is also equal to the half of the cardinal of temporal edges in $\bar{\gamma}$, and $|\tau|$ is the cardinal of intervals in the sequence τ.

We now estimate M_1 and M_2 :

α) Estimation of M_1

We have

$$\begin{aligned}&\int_{\mathbb{R}^4} F^{8d}_{\Box_j^{k,\ell}} \mu_0(dy_{k,j})\mu_0(dy_{k,j+1})\mu_0(dy_{\ell,j})\mu_0(dy_{\ell,j+1}) \\ = & \int_{\mathbb{R}^4} \int \Big|e^{-\varepsilon \int_{I_j} \omega_{k,s}\omega_{\ell,s} ds} - 1\Big|^{8d} \otimes \tilde{P}_{\bar{I}_j}^{y_{k,j},y_{k,j+1}}(d\omega_k)\tilde{P}_{\bar{I}_j}^{y_{l,j},y_{l,j+1}}(d\omega_l) \\ & p_a^0(y_{k,j}, y_{k,j+1})p_a^0(y_{\ell,j}, y_{\ell,j+1}) \frac{\mu_0(dy_{k,j})\mu_0(dy_{k,j+1})\mu_0(dy_{\ell,j})\mu_0(dy_{\ell,j+1})}{p_a^0(y_{k,j}, y_{k,j+1})p_a^0(y_{\ell,j}, y_{\ell,j+1})}\end{aligned}$$

where $p_a^0(y_1, y_2) = \tilde{P}(X_0 = dy_1, X_a = dy_2)/\mu_0(dy_1)\mu_0(dy_2)$ is the density of the joint distribution at times 0 and a of the process X, as introduced in (37).

Lemma 7 *There is an absolute constant $b > 0$ such that*

$$\forall a > a_0, \ \ p_a^o(y_1, y_2) > b \tag{55}$$

The proof can be found in [Mi-Ve-Za] Lemma 5.3 or in [Fa-Mi].

From (55) it follows that the right hand side of (55) is bounded above by

$$\frac{1}{b^2} \iint \Big| e^{-\varepsilon \int_{I_j} \omega_{k,s}\omega_{\ell,s} ds} - 1 \Big|^{8d} \tilde{P}(d\omega_k)\tilde{P}(d\omega_l) \tag{56}$$

To estimate the above integral, remark that

$$\Big| e^{-\varepsilon \int_0^a \omega_{k,s}\omega_{\ell,s} ds} - 1 \Big| = |\varepsilon| \Big| \int_0^a \omega_{k,s}\omega_{\ell,s} ds \Big| \quad \int_0^1 e^{-\varepsilon\tau \int_0^a \omega_{k,s}\omega_{\ell,s} ds} d\tau$$

Then

$$\begin{aligned} &\Big| e^{-\varepsilon \int_0^a \omega_{k,s}\omega_{\ell,s} ds} - 1 \Big|^{8d} = \varepsilon^{8d} \Big| \int_0^a \omega_{k,s}\omega_{\ell,s} ds \Big|^{8d} \\ &\int_{[0,1]^{8d}} e^{-\varepsilon(\tau_1 + \cdots + \tau_{8d}) \int_0^a \omega_{k,s}\omega_{\ell,s} ds} d\tau_1 \ldots d\tau_{8d} \\ &\leq \varepsilon^{8d} \big(\int_0^a \omega_{k,s}^2 ds \big)^{4d} \big(\int_0^a \omega_{\ell,s}^2 ds \big)^{4d} e^{4d|\varepsilon| \int_0^a (\omega_{k,s}^2 + \omega_{\ell,s}^2) ds} \end{aligned}$$

After introducing this inequality in (56) we obtain

$$M_1 < b^{-1/4d} |\varepsilon| \Big[\int (\int_0^a \omega_s^2 ds)^{4d} e^{4d|\varepsilon| \int_0^a \omega_s^2 ds} \tilde{P}(d\omega) \Big]^{1/4d} \tag{57}$$

To control the above exponential moment, let us introduce the function on $\mathbb{C}$:

$$\mathcal{S}(z) = \int e^{z \int_0^a \omega_s^2 ds} \tilde{P}(d\omega).$$

Obviously we have

$$\frac{d^{4d}}{dz^{4d}} \mathcal{S}(z)_{|z = 4d|\varepsilon|} = \int (\int_0^a \omega_s^2 ds)^{4d} e^{4d|\varepsilon| \int_0^a \omega_s^2 ds} \tilde{P}(d\omega). \tag{58}$$

But, decomposing $\mathcal{S}$ in Taylor series, and using Hölder's inequality,

$$\begin{aligned} \mathcal{S}(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \int (\int_0^a \omega_s^2 ds)^n \tilde{P}(d\omega) \\ |\mathcal{S}(z)| &\leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \int (\int_0^a \omega_s^{2n} ds.\, a^{n-1}) \tilde{P}(d\omega) \\ &\leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} a^n \int_{\mathbb{R}} y^{2n} \mu_0(dy). \end{aligned}$$

By (32) μ_0 admits the density function $e^{-\varphi_0}$ for which holds :

Lemma 8 *There exists an absolute constant* $C_1 > 0$ *such that*

$$|e^{-\varphi_0(y)}| \leq C_1 e^{-2|y|^{r+1}/r+1}, y \in \mathbb{R}$$

The parameter $2r$ was introduced in (31) as the degree of the polynom which induces the drift of the process X with invariant measure $e^{-\varphi_0}dx$.

The proof can be found in [Mi-Ve-Za] Lemma 5.4 or in [Fa-Mi].

Lemma 8 implies that

$$\begin{aligned} \int y^{2n}\mu_0(dy) &\leq C_1 \int_0^\infty y^{2n} e^{-2y^{r+1}/r+1} dy \\ &\leq C_2 \int_0^\infty z^{(2n-r)/(r+1)} e^{-z} dz \\ &= C_2 \Gamma\big(\frac{2n+1}{r+1}\big) \end{aligned}$$

The Gamma function Γ can be estimated with help of Stirling's formula ; introduced in $\mathcal{S}(z)$, it gives

$$|\mathcal{S}(z)| \leq C_3 \sum_{n=0}^{\infty} a^n |z|^n n^{-n\frac{r-1}{r+1}} e^{-n\frac{(r-1)}{r+1}} =: C_3 \psi(|z|).$$

The function ψ is an entire function on $\mathbb{C}$ with order $\rho = \frac{r+1}{r-1}$ and finite type $\sigma = \frac{r-1}{r+1}$ (see [Ma]). Thus

$$\psi(|z|) \leq C_4 e^{\bar{\sigma}|za|^{(r+1)/(r-1)}} \tag{59}$$

where $\bar{\sigma} = \frac{r}{r+1}$ and C_4 is an absolute constant.

Using (57), (58), and Cauchy's formula for the derivation of an holomorphic function, we find that $\forall \delta > 0$,

$$M_1 < C_5 |\varepsilon| \frac{1}{\delta} \exp\big[\frac{\bar{\sigma}}{4d}(a(4d|\varepsilon| + \delta))^{(r+1)/(r-1)}\big] \tag{60}$$

where C_5 is an absolute constant.

β) Estimate of M_2

Let us return to the description of the model in 3.1.

To the Schrödinger operator A we can associate the semi-group e^{-tA} on $L^2(\mathbb{R}, dz)$ and denote by $q_t(x, y)$ the associated kernel. There is the following relation between q_t and p_t^0, the transition kernel of the process X introduced in (37):

$$p_t^0(y/x) = \frac{q_t(x, y)}{e^{-\lambda_0 t} e^{-\frac{1}{2}(\varphi_0(x) + \varphi_0(y))}} .$$

On the other side, q_t can be desintegrated as follows : if $\lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n < \cdots$ are eigenvalues of A and ψ_n are its normalized eigenfunctions ($\psi_0 = e^{-\varphi_0/2}$),

$$q_t(x,y) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\psi_n(x)\psi_n(y) \ . \tag{61}$$

Thus we get

$$p_t^0(y/x) - 1 = \sum_{n=1}^{\infty} e^{-(\lambda_n-\lambda_0)t}\frac{\psi_n}{\psi_0}(x)\frac{\psi_n}{\psi_0}(y) \tag{62}$$

Lemma 9 *The following estimate holds*

$$\left|\frac{\psi_n}{\psi_0}(x)\right| \leq C_6 e^{K|\lambda_n|^{\frac{r+1}{2r}}} \tag{63}$$

where $C_6 > 0$ and $K > 0$ are constants which do not depend on n and x.

The proof can be found in [Mi-Ve-Za].

From (62), (63) we get

$$|p_a^0(y/x) - 1| \leq e^{\lambda_0 a} C_1 \sum_{n\geq 1} e^{-(\lambda_n a - K|\lambda_n|^{\frac{r+1}{2r}})}$$

Let $n_0 > 1$ be the smallest number such that $\lambda_n > 0$ and $\lambda_n \geq 3\lambda_0$ for $n \geq n_0$.

Then for

$$a > \frac{3K}{\lambda_{n_0}^{(s-1)/(s+1)}} =: a_0$$

$\forall n \geq n_0,\ \lambda_n a - K\lambda_n^{\frac{r+1}{2r}} \geq \frac{2}{3}a\lambda_n$, which implies :

$$e^{\lambda_0 a} C_6 \sum_{n\geq 1} e^{-(\lambda_n a - K\lambda_n^{\frac{r+1}{2r}})} \leq e^{\lambda_0 a} C_6 \sum_{n\geq 1}^{n_0-1} e^{-(\lambda_n a - K\lambda_n^{\frac{r+1}{2r}})} + C_6 \sum_{n\geq n_0} e^{-\frac{1}{3}a\lambda_n}$$

Since $\lambda_n > C_7 n^{2r/r+1}$ (see [Ti] formula (7.3.8)) the rest of the above serie is bounded by

$$\int_{n_0-1}^{+\infty} e^{-C_8 a y^{2r/r+1}} dy < C_9 e^{-C_8 a(n_0-1)}$$

Finally,

$$e^{\lambda_0 a} C_6 \sum_{1\leq n<n_0} e^{-\lambda_n a + K\lambda_n^{(r+1/2r)}} \leq e^{-(\lambda_1-\lambda_0)a} C_6 \sum_{1\leq n<n_0} e^{K\lambda_n^{\frac{r+1}{2r}}} \leq C_{10} e^{-(\lambda_1-\lambda_0)a}$$

From these estimates we obtain that

$$M_2 \leq C_{11} e^{-C_7 a} \tag{64}$$

with $C_7 = \inf(\lambda_1 - \lambda_0, C_8(n_0 - 1))$ and all $(C_i)_{i\geq 0}$ are absolute constants.

If we take now $a > a_0 = -\frac{1}{3C_7} ln|\varepsilon|$ and choose $\delta = |\varepsilon|^{1/3}$ in (60) we find from (60) and (64) that

$$\begin{cases} M_1 & \leq C_{13}|\varepsilon|^{2/3} \\ M_2 & \leq C_{14}|\varepsilon|^{1/3} . \end{cases}$$

From this and (54), we get the estimate (46) with $\lambda = C|\varepsilon|^{1/3}$ and C is an absolute constante. ∎

3.1.4 The cluster expansion of the measures Q_n.

For any finite set of temporal edges $B = \{(k, I_j)\} \subset \mathbb{Z}_a^{d+1}$ and configuration ω we denote by ω_B its restriction on B.

We shall get now a representation for the average

$$\int \mathcal{A}_B dQ_n \tag{65}$$

where $\mathcal{A}_B$ is a local bounded function on Ω localized on B, and B is included in the set B_n of temporal edges of $V_n = \Lambda_n \times I_n$.

First we formulate some important consequence of the cluster representation (45) obtained for the partition function Z_V^ε.

Let τ be a finite set of temporal edges, included in T the set of all temporal edges in $\mathbb{Z}_a^{d+1}$.

Let us introduce the partition function

$$Z_\tau^\varepsilon = 1 + \sum_{m=1}^{\infty} \sum_{\{\Gamma_1,\ldots,\Gamma_m\}} \prod_{i=1}^{m} K_{\Gamma_i}$$

where the summation is taken over all non ordered non empty collections $\{\Gamma_1, \ldots, \Gamma_m\}$ of pairwise non intersecting aggregates Γ_i such that $\bar{\Gamma}_i \subset \tau$, and K_{Γ_i} defined by (44).

For any subset $\tau \subset \tau'$ we define

$$f_\tau^{\tau'} = \frac{Z_{\tau\setminus\bar{\tau}}^\varepsilon}{Z_{\tau'}^\varepsilon} \tag{66}$$

where $\bar{\tau}$ is the set of edges which have commun points with edges from τ.

The following lemma, which can be found in [Mi-Ma], holds :

Lemma 10 *For ε small enough,*

i) $\exists C_{15} > 0$, independent on τ and τ' such that

$$\left| f_\tau^{\tau'} \right| < C_{15}.2^{|\tau|} \tag{67}$$

ii) The following expansion holds :

$$f_\tau^{\tau'} = 1 + \sum_{\eta=\{\Gamma\},\bar{\Gamma}\subset\tau'}^{\tau} D_\tau(\eta) \prod_{\Gamma\in\eta} K_\Gamma \tag{68}$$

where the summation is over collections η of aggregates Γ such that η is connected, $\tau \cap \cup_{\Gamma\in\eta}\bar{\Gamma} \neq \emptyset$ and $\cup_{\Gamma\in\eta}\bar{\Gamma} \subset \tau'$. The coefficients $D_\tau(\eta)$ do not depend on τ' and the serie is absolutely convergent.

iii) There exist a limit of the expansion (68) when τ' tends to T :

$$f_\tau = \lim_{\tau'\uparrow T} f_\tau^{\tau'} = 1 + \sum_{\eta=\{\Gamma\}}^{\tau} D_\tau(\eta) \prod_{\Gamma\in\eta} K_\Gamma \tag{69}$$

iv) We have the following estimate :

$$\left| f_\tau^{\tau'} - f_\tau \right| < C_{16} 2^{|\bar{\tau}|} (1/2)^{d(\tau,{\tau'}^c)} \tag{70}$$

where $d\left(\tau,{\tau'}^c\right)$ is the length of the smallest path in T which goes from τ to the complement of τ' in T.

v) The following estimate holds : for τ_1, $\tau_2 \subset \tau'$

$$\begin{aligned} &\left| f_{\tau_1\cup\tau_2}^{\tau'} - f_{\tau_1}^{\tau'} \cdot f_{\tau_2}^{\tau'} \right| < C_{17} 3^{|\tau_1|+|\tau_2|} (C\lambda(\varepsilon))^{d(\tau_1,\tau_2)} \\ \text{and} \quad & \\ &\left| f_{\tau_1\cup\tau_2} - f_{\tau_1} \cdot f_{\tau_2} \right| < C_{17} 3^{|\tau_1|+|\tau_2|} (C\lambda(\varepsilon))^{d(\tau_1,\tau_2)} \end{aligned} \tag{71}$$

We now return to the expansion of the mean of the functional $\mathcal{A}_B$ defined in (65). We have

$$\begin{aligned} \int \mathcal{A}_B dQ_n &= \frac{1}{Z_{V_n}^\varepsilon} \int \mathcal{A}_B \, e^{-H_{V_n}^{(\varepsilon)}} \otimes_{\Lambda_n} d\tilde{P}_{I_n} \\ &=: \frac{Z_{V_n}^\varepsilon(\mathcal{A}_B)}{Z_{V_n}^\varepsilon} \end{aligned}$$

where the numerator has the following representation :

$$Z_{V_n}^\varepsilon(\mathcal{A}_B) = \sum_{\substack{\zeta=\{\bar{\Gamma}_j\} \\ \bar{\Gamma}_j\in B_{V_n}}} K_\zeta(\mathcal{A}_B) \left(1 + \sum_{\substack{\eta=\{\Gamma_i\} \\ \bar{\Gamma}_i\in B_{V_n}\setminus(B\cup\bar{\zeta})}} \prod_{\Gamma_i\in\eta} K_{\Gamma_i} \right) \tag{72}$$

and

$$K_\zeta(\mathcal{A}_B) = \int \mathcal{A}_B(\omega) \prod_{j=1}^{m} \mathfrak{X}_{\Gamma_j}(\omega) d\tilde{P}^{\bar{\Gamma}_j}(\omega)$$

(cf. (44)).

¿From (72) and (68) we find

$$\int \mathcal{A}_B dQ_n = \sum_\zeta K_\zeta(\mathcal{A}_B)(Z^\varepsilon_{V_n\setminus(\widehat{B\cup\bar\zeta})})/Z^\varepsilon_{V_n} = \sum_\zeta K_\zeta(\mathcal{A}_B) f^{B_{V_n}}_{B\cup\bar\zeta}$$
$$= \sum_{\zeta,\eta} K_\zeta(\mathcal{A}_B) D_{B\cup\bar\zeta}(\eta) \prod_{\Gamma\in\eta} K_\Gamma$$

Using estimates (67), (69), we can conclude that for ε small enough (which implies $\lambda(\varepsilon)$ small enough) the above serie converges absolutely and uniformly in n, so that

$$\lim_{V_n\uparrow\mathbb{Z}^d\times\mathbb{R}} \int \mathcal{A}_B dQ_n = \sum_\zeta K_\zeta(\mathcal{A}_B)\cdot f_{B\cup\bar\zeta}$$
$$=: \int \mathcal{A}_B dQ$$

The functional $\mathcal{A}_B \longmapsto \int \mathcal{A}_B dQ$ is linear bounded and positive on the algebra of bounded local functions. Then there exists a unique probability measure Q such that

$$Q = \lim_{V_n\uparrow\mathbb{Z}^d\times\mathbb{R}} Q_n\,.$$

The fact that Q satisfies the property of short range correlations is a consequence of a cluster representation for

$$\int \mathcal{A}_{B_1}\mathcal{A}_{B_2} dQ_n - \int \mathcal{A}_{B_1} dQ_n \int \mathcal{A}_{B_2} dQ_n$$

and (71), but also of the general Martin-boundary theory mentioned above.

This completes the proof of theorem 4. ■

3.2 The model associated to a Stochastic Differential Equation

In this section, using the method of cluster expansion, we give an alternative construction of a Gibbsian field associated to the space-time interaction of the type given in the section 2.3.

As in the last section 3.1., we will define a sequence of measure $Q_{2,n}$ with zero boundary configuration and prove its convergence to the desired Gibbsian field. (The fact that the limit is a Gibbs state derives from Lemma 5).

More precisely, let us take $\phi^{(\varepsilon)}$ a pair potential on $\mathbb{R}^{\mathbb{Z}^d}$ defined as follows (particular case of (2.10)) :

$$\begin{cases} \phi^{(\varepsilon)}_{\{k\}}(y_k) & = y_k^{2r}, r\in\mathbb{N}^*, r>1 \\ \phi^{(\varepsilon)}_{\{k,\ell\}}(y_k,y_\ell) & = \varepsilon y_k y_\ell \quad \text{if} \quad |k-\ell|=1 \quad \text{and} \quad 0 \quad \text{otherwise} \end{cases} \tag{73}$$

This induces on the space-time level an interaction $\Phi^{(\varepsilon)}$ as in (27) :

$$\Phi^{(\varepsilon)}_{\Lambda\times I}(X) = \int_I \varphi^{(\varepsilon)}_\Lambda (X_{\Lambda,s})ds$$

where $\varphi^{(\varepsilon)}$ satisfies :

- $\varphi^{(\varepsilon)}_{\{k\}}(y_k) = (r^2/2)y_k^{4r-2} - r(r-1/2)y_k^{2r-2} + (\varepsilon^2 d/4)y_k^2$
- $\varphi^{(\varepsilon)}_{\{k,\ell\}}(y_k, y_\ell) = \begin{cases} (\varepsilon r/2)(y_k^{2r-1}y_\ell + y_\ell^{2r-1}y_k) & \text{if } |k-\ell| = 1 \\ (\varepsilon^2/2)y_k y_\ell & \text{if } |k-\ell| = \sqrt{2} \\ (\varepsilon^2/4)y_k y_\ell & \text{if } |k-\ell| = 2 \\ 0 & \text{otherwise} \end{cases}$

This is a pair interaction with finite range equal to 2.

3.2.1 The free measure and the Hamiltonian

Consider the polynom

$$\mathcal{P}_2(y) = \frac{r^2}{2}y^{4r-2} - \frac{r(2r-1)}{2}y^{2r-2} + \frac{\varepsilon^2 d}{4}y^2 \,.$$

It plays the role of the polynom $\mathcal{P}_1$ in the section 3.1. The new reference process X on $C(\mathbb{R},\mathbb{R})$ is the reversible solution of the stochastic differential equation (33) where $\exp(-\varphi_0/2)$ is the unique normalized strictly positive eigenvector associated to the smallest eigenvalue of the Schrödinger operator :

$$-\frac{1}{2}\frac{d^2}{dz^2} + \mathcal{P}_2(z) \,.$$

We denote the law of X by $\tilde{P}_2$ and construct a local reference specification $\tilde{\Pi}^0_2$ as in (34), where $\tilde{P}$ is replaced by $\tilde{P}_2$.

The Gibbsian modification of this free field we consider now is induced by the Hamiltonian

$$H^{(\varepsilon)}_{2,\Lambda\times I}(\omega) = \int_I \Big(\sum_{k,l\in\Lambda} \varphi^{(\varepsilon)}_{\{k,\ell\}}(\omega_{k,s},\omega_{l,s}) + \sum_{k\in\Lambda,\ell\in\Lambda^c} \varphi^{(\varepsilon)}_{\{k,\ell\}}(\omega_{k,s},\omega_{\ell,s}) \Big) ds \,. \quad (74)$$

Theorem 5 *For ε small enough, there exists a space-time Gibbs measure Q_2 in $\mathcal{G}(H^{(\varepsilon)}_2, \tilde{\Pi}^0_2)$, obtained as limit of the following sequence of probabilities on $\Omega_{\Lambda_n,I_n}, V_n = \Lambda_n \times I_n$:*

$$\begin{aligned} Q_{2,n}(d\eta) &= \frac{1}{Z^\varepsilon_{V_n}} \exp - H^{(\varepsilon)}_{2,V_n}(\eta 0)(\otimes_{k\in\Lambda_n}\tilde{P}_{2,\bar{I}_n})(d\eta) \\ &= \frac{1}{Z^\varepsilon_{V_n}} \Big(\exp - \int_{I_n} \sum_{k,\ell\in\Lambda_n} \varphi^{(\varepsilon)}_{\{k,\ell\}}(\eta_{k,s},\eta_{\ell,s}) \Big) (\otimes_{k\in\Lambda_n}\tilde{P}_{2,\bar{I}_n})(d\eta). \end{aligned}$$

Q_2 admits a cluster expansion and satisfies the property of short range correlations.

3.2.2 The cluster expansion of Z_V^ε

As in section 3.1, the time interval I is divided into intervals I_j with length a. Temporal edges are defined like in section 3.1.2. Plaquettes are defined as pairs of temporal edges in interaction. Thus there are three types of plaquettes.

$$
\begin{array}{lll}
{}_1\square_j^{k,\ell} & \text{are plaquettes with} & |k-\ell| = 1 \\
{}_2\square_j^{k,\ell} & \text{are plaquettes with} & |k-\ell| = \sqrt{2} \\
{}_3\square_j^{k,\ell} & \text{are plaquettes with} & |k-\ell| = 2
\end{array}
$$

Then the partition function Z_V^ε admits the representation (45) where K_{Γ_i} is defined as in (44) except that the term

$$\prod_{\square_{jm}^{k,\ell}} \big(\exp(-\varepsilon \int_{I_{jm}} \omega_{k,s}\omega_{\ell,s} ds) - 1\big)$$

is replaced by

$$\prod_{\alpha=1,2,3} \prod_{{}_\alpha\square_{jm}^{k,\ell}} \big(\exp(-\int_{I_{jm}} \varphi_{k,\ell}^{(\varepsilon)}(\omega_{k,s},\omega_{\ell,s}) ds) - 1\big)$$

3.2.3 Cluster estimates

To estimate K_Γ we apply lemma 6 by choosing numbers n_α, $\alpha = 1,2,3$, corresponding to plaquettes of type α such that inequality (47) holds. More precisely, every temporal edge appears in no more then $2d$ plaquettes of type 1, $2d(d-1)$ plaquettes of type 2 and $2d$ plaquettes of type 3. So we choose n_α such that

$$\frac{2d}{n_1} + \frac{2d(d-1)}{n_2} + \frac{2d}{n_3} \le 1\,. \tag{75}$$

Then the similar estimate for K_Γ as (50) holds where

$$\prod_{\square_{jm}^{k,\ell}} F_{\square_{jm}^{k,\ell}} \text{ is replaced by } \prod_{\alpha=1,2,3} \prod_{{}_\alpha\square_{jm}^{k,\ell}} F_{{}_\alpha\square_{jm}^{k,\ell}}\,.$$

Every vertex (k, ja) appears in no more than $4d$ plaquettes of type 1, $4d(d-1)$ plaquettes of type 2 and $4d$ plaquettes of type 3, and in no more than two series. So, by choosing n_α such that

$$\frac{2}{4} + \frac{4d}{n_1} + \frac{4d(d-1)}{n_2} + \frac{4d}{n_3} \le 1\,,$$

we can apply lemma 6 and get the following estimate (similar to (3.21))

$$
\begin{aligned}
|K_\Gamma| \;\leq\; & \prod_{m=1}^{s}\prod_{\alpha=1}^{3}\prod_{{}_\alpha\square^{k,\ell}_{j_m}\in\gamma_m^{j_m}} \Big[\int_{\mathbb{R}^4} F^{n_\alpha}_{{}_\alpha\square^{k,\ell}_{j_m}}\mu_0(dy_{k,j_m}) \\
& \qquad \mu_0(dy_{k,j_m+1})\mu_0(dy_{\ell,j_m})\mu_0(dy_{\ell,j_m+1})\Big]^{1/n_\alpha} \\
& \prod_{n=1}^{p}\prod_{I_j\in\tau_n^{k_n}} \Big[\int_{\mathbb{R}^2}\Big|p_a^0(y_{k_n,j+1}/y_{k_n,j})-1\Big|^4 \mu_0(dy_{k_n,j})\mu_0(dy_{k_n,j+1})\Big]^{1/4}
\end{aligned}
$$

We then take $n_1 = 24d$, $n_2 = 24d(d-1)$, $n_3 = 24d$ and denote by

$$
M_\alpha^{p\ell} = \Big[\int_{\mathbb{R}^4} F^{n_\alpha}_{{}_\alpha\square^{k,\ell}_{j}}(y_1,y_2,y_3,y_4)\mu_0(dy_1)\mu_0(dy_2)\mu_0(dy_3)\mu_0(dy_4)\Big]^{1/n_\alpha}
$$

and $M^{\text{edge}} = M_2$ like in (53).

As in (54)

$$
|K_\Gamma| < \left(M_\alpha^{p\ell}\right)^{\sum_{m=1}^{s}|\gamma_m^{j_m}|_\alpha} \left(M^{\text{edge}}\right)^{\sum_{n=1}^{p}|\tau_n^{k_n}|}
$$

where $|\gamma|_\alpha$ is the number of plaquettes of type α in the contour γ.

Now we have to estimate the quantities $M_\alpha^{p\ell}$ and M^{edge}.

For $M_2^{p\ell}, M_3^{p\ell}$ and M^{edge} the computations are similar to those of the first model (section 3.1). This implies that, by choosing $a \sim -ln\varepsilon$, we get

$$
\begin{cases} M_\alpha^{p\ell} & < C_{18}\varepsilon^{2/3} \quad \alpha = 2,3 \\ M^{\text{edge}} & < C_{19}\varepsilon^{1/3}\ . \end{cases} \tag{76}
$$

Estimation of $M_1^{p\ell}$:

$$
M_1^{p\ell} < \Big[1/b^2 \iint \Big|\exp(-(\varepsilon r/2)\int_0^a(\omega_{1,s}^{2r-1}\omega_{2,s}+\omega_{1,s}\omega_{2,s}^{2r-1})ds)-1\Big|^{24d}\tilde{P}_2(d\omega_1)\tilde{P}_2(d\omega_2)\Big]^{1/24d}.
$$

But

$$
\begin{aligned}
& \exp(-(\varepsilon r/2)\int_0^a(\omega_{1,s}^{2r-1}\omega_{2,s}+\omega_{1,s}\omega_{2,s}^{2r-1})ds)-1 \\
& = (\varepsilon r/2)\int_0^a(\omega_{1,s}^{2r-1}\omega_{2,s}+\omega_{1,s}\omega_{2,s}^{2r-1})ds \cdot \int_0^1 e^{-(r\varepsilon/2)\tau\int_0^a(\omega_{1,s}^{2r-1}\omega_{2,s}+\omega_{1,s}\omega_{2,s}^{2r-1})}d\tau
\end{aligned}
$$

Together with Young inequality

$$
\left|\omega_1^{2r-1}\omega_2+\omega_2^{2r-1}\omega_1\right| \leq \left|\omega_1\right|^{2r}+\left|\omega_2\right|^{2r}
$$

we get

$$\iint \left|\exp(-(\varepsilon r/2)\int_0^a (\omega_{1,s}^{2r-1}\omega_{2,s}+\omega_{1,s}\omega_{2,s}^{2r-1})ds)-1\right|^{24d}\tilde{P}_2(d\omega_1)\tilde{P}_2(d\omega_2)$$
$$\leq (\varepsilon r/2)^{24d}\iint \left(\int_0^a \left|\omega_{1,s}\right|^{2r}+\left|\omega_{2,s}\right|^{2r})ds\right)^{24d} \tag{77}$$
$$\exp(12r\varepsilon d\int_0^a \left|\omega_{1,s}\right|^{2r}+\left|\omega_{2,s}\right|^{2r})ds)\tilde{P}_2(d\omega_1)\tilde{P}_2(d\omega_2)$$

Let us introduce the function on $\mathbb{C}$:

$$\mathcal{R}(z)=\big(\int \exp(z\int_0^a |\omega_s|^{2r}ds)\tilde{P}_2(d\omega)\big)^2 .$$

Obviously, the right hand side of (77) equals

$$(\varepsilon r/2)^{24d}\frac{d^{24d}}{dz^{24d}}\mathcal{R}(z)\mid_{z=12r\varepsilon d}$$

In a similar way than for the first model, using lemma 6, we can estimate $\mathcal{R}$ by

$$|\mathcal{R}(z)|<\left(C_{20}\sum_{r=1}^{\infty}\frac{a^n|z|^n}{\gamma^n}\right)^2=C_{20}^2\left(\frac{1}{1-\frac{a|z|}{\gamma}}\right)^2$$

for $|z|<\gamma/a$, where γ is a positive constant.

Then $\mathcal{R}$ is analytical on the disc $|z|<\gamma/a$ and for ε,δ small enough.

$$\frac{d^{24d}}{dz^{24d}}\mathcal{R}(z)\mid_{z=12r\varepsilon d}<\frac{C_{21}}{\delta^{24d}}\left(\frac{1}{1-\frac{a(\varepsilon+d)12rd}{\gamma}}\right)^2$$

We choose $\delta=\varepsilon^{1/3}$, and it implies that

$$M_1^{p\ell}<C_{22}\varepsilon^{2/3} .$$

Together with (76) and (77), we find the following cluster estimate :

$$|K_\Gamma|<(C\varepsilon^{2/3})^{|\Gamma|} .$$

The cluster expansion of the measures $Q_{2,n}$ is omitted because it is similar to the computations done in 3.1.4. for Q_n.

This completes the proof of theorem 5.

■

Remark : Since we are not able to assure the uniqueness of Gibbs states in $\mathcal{G}_r(H_2^{(\varepsilon)},\tilde{\Pi}_2^0)$, - see the remarks at the end of paragraph 2.2 - we can not identify Q_2 with the reversible process which appeared in the section 2.3 and which belongs to $\mathcal{G}_r(H^{(\varepsilon)}, \tilde{\Pi}^0)$.

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