Path-dependent infinite-dimensional SDE with non-regular drift: an existence result.

David Dereude *
Sylvie Rœlly †

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Abstract: We establish in this paper the existence of weak solutions of infinite-dimensional shift invariant stochastic differential equations driven by a Brownian term. The drift function is very general, in the sense that it is supposed to be neither small or continuous, nor Markov. On the initial law we only assume that it admits a finite specific entropy.
Our result strongly improves the previous ones obtained for free dynamics with a small perturbative drift. The originality of our method leads in the use of the specific entropy as a tightness tool and on a description of such stochastic differential equation as solution of a variational problem on the path space.

Key-words: Infinite-dimensional SDE, non-Markov drift, non-regular drift, variational principle, specific entropy.

*Laboratoire Paul Painlevé, UMR CNRS 8524, Université Lille1, 59655 Villeneuve d'Ascq Cedex, France, *david.dereudre@univ-lille1.fr*

†Institut für Mathematik der Universität Potsdam, Am Neuen Palais 10, 14469 Potsdam, Germany, *roelly@math.uni-potsdam.de*

1 Introduction

The main object of this paper is the infinite-dimensional stochastic differential equation (SDE)

$$dX_i(t) = \mathbf{b}_t(\theta_i X)\, dt + dB_i(t) \ , \quad i \in \mathbb{Z}^d, \tag{1}$$

on the configuration space $\Omega = \mathcal{C}([0,T],\mathbb{R})^{\mathbb{Z}^d}$, where the drift $\mathbf{b} : [0,T] \times \Omega$ is an adapted functional, θ_i denotes the space-shift on Ω by vector $-i$ and $(B_i)_{i\in\mathbb{Z}^d}$ is a sequence of independent real-valued Brownian motions.

Our aim is to prove the existence of a weak solution of the SDE (1) on the finite time-interval $[0,T]$, where the drift $\mathbf{b}$ is supposed to be as general as possible, in particular *non-Markov* and *non-regular*. Indeed, in Theorem 2.1, we solve the SDE (1) for a path-dependent drift which is supposed to be only uniformly bounded and local, that is

$$\|\mathbf{b}\|_\infty \ := \sup_{t\in[0,T],\omega\in\Omega} |\mathbf{b}_t(\omega)| < +\infty \tag{2}$$

$$\text{and } \mathbf{b}_t(\omega) \ = \ \mathbf{b}_t(\omega_\Delta(s), s \in [0,t]), \text{ for } t \in [0,T], \tag{3}$$

where Δ is a fixed finite subset of $\mathbb{Z}^d$ and $\omega_\Delta = (\omega_i)_{i\in\Delta}$ denotes the coordinates of the path ω indexed by Δ. The initial condition is assumed to be shift-invariant with finite specific entropy. In Section 5 we extend our existence result to drifts $\mathbf{b}$ containing also a Lipschitz unbounded part.

Let us illustrate our main result by a typical example. Let $\beta^+ \neq \beta^-$ be two real numbers and $\Delta \subset \mathbb{Z}^d$ be a set with cardinality N. Define first the function b on $\mathbb{R}^{\mathbb{Z}^d}$ by

$$b(x) := \beta^+ \, \mathbb{1}_{\left\{x_0 \geq \frac{1}{N}\sum_{i\in\Delta} x_i\right\}} + \beta^- \, \mathbb{1}_{\left\{x_0 < \frac{1}{N}\sum_{i\in\Delta} x_i\right\}}. \tag{4}$$

It takes the value β^+ (respectively β^-) if the 0-coordinate x_0 is larger (respectively smaller) than the barycentre of the Δ-coordinates x_Δ. Introducing a δ-delay (with $0 < \delta < T$) consider now the drift $\mathbf{b}_t(\omega) := b(\omega(0 \vee (t-\delta))$. It leads to a stochastic differential delay equation (1) whose discontinuous drift satisfies assumptions (2) and (3). In the above example the time memory of the drift is bounded (by δ), but our approach also allows to deal with path-dependent drift with long-term memory like $\mathbf{b}_t(\omega) := \int_0^t b(s,\omega(s))ds$.

Note that SDE with non-Markov and non-regular drifts are relevant in many fields of applications like mathematical finance, biomathematics or physics, see e.g. [M97], [AHMP07] or [TP01].

Let us briefly recall some results concerning infinite-dimensional SDEs. In the very special Markovian case, when the drift only depends on the present time $\mathbf{b}_t(\omega) = \mathbf{b}_t(\omega(t))$, and the functions $x \mapsto \mathbf{b}_t(x)$ satisfy certain growth condition at infinity, (strong) solutions of (1) with values in a weighted ℓ^2-space were constructed in [SS80] and [F82]; the particular case of gradient drift (i.e. the function b is the gradient of a smooth Hamilton function) was treated earlier in detail in [DR78] and [R99]. For the existence of weak solutions of a Markov SDE with unbounded linear term the theory of Dirichlet forms can also be used fruitfully, see e.g. [AR91]. Very recently, for SDEs with values in Hilbert spaces with non-regular Markovian drift, strong uniqueness results were obtained in several frameworks, see [DPFPR13] and [DPFRV14].
If the drift is non-Markov but satisfies a Lipschitz assumption (see Section 5 for precise definitions), extending straightforwardly the results in [SS80] would provide the existence and uniqueness of a strong solution of (1). For general non-Markov and non-regular drifts $\mathbf{b}$, to our knowledge, till now only particular *perturbative* cases were treated, see [DPR06] and [RR14]. They correspond to the perturbation of a *free* dynamics (involving only a self-interaction term) by a sufficiently small drift. Thus, for example, the existence and uniqueness of solutions of (1) for the drift (4) is known when parameters $\beta^{\pm}$ are small enough.

A fruitful approach to construct solutions of infinite-dimensional SDEs is to describe them as Gibbs measures on a path space. This point of view was initiated for gradient diffusions on a finite time interval in [D87] and developed later in [CRZ96]. The procedure includes here two steps:
i) the construction of Gibbs measures on the path space associated to a suitable Hamiltonian H (depending on the drift $\mathbf{b}$ and on the initial law)
ii) the identification of (some of) them as weak solutions of SDE (1).
When the uniform norm of the drift $\mathbf{b}$ is small enough, step *i*) can be done via the perturbative techniques of cluster expansion, as in [DPR06] and [RR14]. But recently a more general approach, first appeared in [GH96] and based on the compactness of the level sets of the specific entropy density, allowed to construct directly infinite-volume Gibbs measures associated to strong interaction [D09, DDG12]. This entropic method will be our first major tool. When the drift is Markov and regular (i.e. Malliavin-differentiable), step *ii*) can be done via an integration by parts formula on the path space, as in

[CRZ96]. In the general case, a variational principle, which characterizes the shift invariant Gibbs measures as the minimizers of a so-called *free energy* functional, is more suitable. So, we will here identify the Gibbs measure using the variational approach, as in [DPRZ02]: It will be our second major tool.

Our approach underlines to what extent tools from statistical mechanics can be powerful in the framework of stochastic analysis. Let us mention that this strategy has just been applied fruitfully in the framework of stochastic geometry to construct infinite branching tessellations with interaction, see [GST14].

The paper is divided into the following sections. Section 2 contains the framework and first results. In section 3, the proof of the main theorem is given, consisting in the construction of a weak solution of (1) for a bounded drift **b**. In section 4, we will point out some structural properties satisfied by this solution. We present in the last section the extension of the existence result in the setting of unbounded drifts **b** including a bounded non-regular term and an unbounded Lipschitz continuous one.

2 Framework and main result

2.1 State spaces

From now on, without loss of generality, we fix $T = 1$, i.e. the time interval is equal to $[0, 1]$. So the configuration space of the SDE (1) is the canonical space $\Omega = \mathcal{C}([0, 1], \mathbb{R})^{\mathbb{Z}^d}$ equipped with the uniform norm, endowed with the canonical Borel σ-field $\mathcal{F}$ generated by the cylinders. The canonical process on Ω is denoted by $X = (X_i(t))_{i\in\mathbb{Z}^d, t\in[0,1]}$.

For any $i \in \mathbb{Z}^d$, we denote by θ_i the space shift by vector $-i$ which acts on $\mathbb{R}^{\mathbb{Z}^d}$ or on Ω. With $\mathcal{P}(E)$ we denote the space of probability measures on any measurable space $(E, \mathcal{E})$. Moreover,

$$\mathcal{P}_s(\Omega) := \{P \in \mathcal{P}(\Omega),\ P \circ \theta_i^{-1} = P \quad \forall i \in \mathbb{Z}^d\}$$

is the set of probability measures on Ω which are space-shift invariant.

Similarly,

$$\mathcal{P}_s(\mathbb{R}^{\mathbb{Z}^d}) := \{P \in \mathcal{P}(\mathbb{R}^{\mathbb{Z}^d}),\ P \circ \theta_i^{-1} = P \quad \forall i \in \mathbb{Z}^d\}.$$

In a natural way, we take as reference measure on Ω the law $\mathbf{W}$ of the non-interacting infinite system corresponding to $\mathbf{b} = 0$ with a product measure as initial law, i.e.

$$\mathbf{W} = \Big(\int_{\mathbb{R}} W^z \, m(dz) \Big)^{\otimes \mathbb{Z}^d} \in \mathcal{P}_s(\Omega).$$

Here W^z denotes the Wiener measure on $\mathcal{C}([0,1],\mathbb{R})$ with fixed initial condition z and $m \in \mathcal{P}(\mathbb{R})$ is a given probability measure on $\mathbb{R}$.

For any subset $\Lambda \subset \mathbb{Z}^d$ we denote by $X_\Lambda = (X_i)_{i\in\Lambda}$ the projection from Ω on $\mathcal{C}([0,1],\mathbb{R})^\Lambda$. We also define the σ-field

$$\mathcal{F}_\Lambda = \sigma(X_\Lambda(t), t \in [0,1]), \tag{5}$$

and the projection by X_Λ of a probability measure $P \in \mathcal{P}(\Omega)$:

$$P_\Lambda := P \circ X_\Lambda^{-1} \in \mathcal{P}(\mathcal{C}([0,1],\mathbb{R})^\Lambda).$$

Similarly, for any $\mu \in \mathcal{P}(\mathbb{R}^{\mathbb{Z}^d})$, its Λ-marginal law is denoted by $\mu_\Lambda \in \mathcal{P}(\mathbb{R}^\Lambda)$.

2.2 Specific entropy

For μ, ν probability measures on a measurable space $(E, \mathcal{E})$, we denote by $\mathcal{I}(\mu;\nu)$ their *relative entropy* defined as usual by:

$$\mathcal{I}(\mu;\nu) = \begin{cases} \int_E \ln(f)\, d\mu & \text{if } \mu \ll \nu \text{ with density } f \\ +\infty & \text{otherwise} \end{cases}.$$

When the underlying space has a product structure, one localises the entropy in the following way: for any subset $\Lambda \subset \mathbb{Z}^d$ and $\mu, \nu \in \mathcal{P}(\mathbb{R}^{\mathbb{Z}^d})$, $\mathcal{I}_\Lambda(\mu;\nu) := \mathcal{I}(\mu_\Lambda;\nu_\Lambda)$. Now, we recall the definition of the *specific entropy* of a shift invariant probability measure μ on $\mathbb{R}^{\mathbb{Z}^d}$ (with respect to $m^{\otimes\mathbb{Z}^d}$):

$$\mathfrak{I}(\mu) := \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathcal{I}_\Lambda(\mu; m^{\otimes \mathbb{Z}^d}), \tag{6}$$

where the limit above is taken for any increasing sequence $(\Lambda_n)_n$ of finite sets converging to $\mathbb{Z}^d$ and $|\Lambda|$ denotes the cardinal of Λ. Similarly, at the path level, the specific entropy of any shift invariant probability measure $Q \in \mathcal{P}_s(\Omega)$ with respect to $\mathbf{W}$ is given by:

$$\mathfrak{I}(Q) := \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathcal{I}_\Lambda(Q; \mathbf{W}). \tag{7}$$

The concept of *specific* entropy appeared first in [RR67] and we advice for instance Chapter 15, [G11] for a general presentation.

2.3 Results

Our main result is the following theorem.

Theorem 2.1 *Fix an initial probability measure $\mu \in \mathcal{P}_s(\mathbb{R}^{\mathbb{Z}^d})$ with finite specific entropy $\mathfrak{I}(\mu)$ and assume that the drift $\mathbf{b}$ is uniformly bounded and local, that is satisfies (2) and (3). Then the infinite-dimensional SDE (1) admits, at least, one shift-invariant weak solution P with marginal law at initial time μ. Moreover its specific entropy $\mathfrak{I}(P)$ is finite .*

In other words, there exists a probability measure $P \in \mathcal{P}_s(\Omega)$ with μ as marginal at time 0 such that the process $\Big(X_i(t) - X_i(0) - \int_0^t \mathbf{b}_s(\theta_i X) ds\Big)_{i\in\mathbb{Z}^d, t\in[0,1]}$ is a family of P-independent Brownian motions. Moreover the finiteness of the specific entropy of μ propagates at the path level:

$$\mathfrak{I}(\mu) < +\infty \Rightarrow \mathfrak{I}(P) < +\infty.$$

In section 5, an extension of Theorem 2.1 is given in the setting of unbounded drifts.

We now give a more precise description of the set $\mathbf{Sol}$ of solutions of the SDE (1) without prescribing the initial condition.

$$\mathbf{Sol} := \{P \in \mathcal{P}_s(\Omega) \text{ weak solution of (1) with } \mathfrak{I}(P) < +\infty\}.$$

Theorem 2.2 *The set $\mathbf{Sol}$ is convex and its extremal points are ergodic solutions. In particular, for any ergodic probability measure $\mu \in \mathcal{P}_s(\mathbb{R}^{\mathbb{Z}^d})$ with $\mathfrak{I}(\mu) < +\infty$ there exists an ergodic weak solution P of the SDE (1) which admits μ as marginal law at time 0.*

More precisely, each probability measure P in **Sol** admits a unique representation in the following way:

$$P = \int_{\Theta} \pi(u,.)\vartheta(du),$$

where $(\Theta, \mathcal{T}, \vartheta)$ is an auxiliary probability space and π is a kernel on $(\Theta, \mathcal{F})$ such that
(i) for each $F \in \mathcal{F}$, $\pi(.,F)$ is $\mathcal{T}$-measurable and
(ii) for each $u \in \Theta$, $\pi(u,.)$ is an ergodic solution in **Sol**.
This theorem is proved in Section 4 which is devoted to the Gibbs structure of the solutions of (1). The proof involves the representation of Gibbs measures by extremal ones.

Let us note that our approach leads to the explicit construction of a particular solution but do not allow to obtain a uniqueness result. For sake of completeness, let us recall a recent result answering this question, obtained via the cluster expansion method, see [RR14] Corollary 2.4. It only concerns the *perturbative regime*, since the dynamics has to be close to a free dynamics.

Proposition 2.1 *Consider the infinite-dimensional SDE* (1) *with a drift of the form*

$$\mathbf{b}_t(\omega) := -\frac{1}{2}\varphi'(\omega_0(t)) + \tilde{\mathbf{b}}_t(\omega_\Delta(s), s \in [0,t]))$$

where φ *is a smooth ultracontractive self-potential* (*i.e. the semi group of the associated one-dimensional gradient diffusion maps* $L^2(m)$ *into* $L^\infty(m)$)*. Take as initial condition the stationary measure of the free dynamics:* $\mu(dx) = \otimes_{i\in\mathbb{Z}^d} e^{-\varphi(x_i)}dx_i$*. If the interaction term* $\tilde{\mathbf{b}}$ *admits a uniform norm which is* sufficiently small*, then* (1) *admits a* unique *weak solution.*

3 Proof of the main Theorem 2.1

In this section, we present the proof of Theorem 2.1 divided in several steps. The approximate solution of (1) is defined in section 3.1 as a finite volume solution with vanishing fixed external configuration. In section 3.2, we show that a well chosen sequence of approximate solutions is tight for the topology of local convergence on Ω since their specific entropies are uniformly bounded. Then, the identification of any limit point as a Brownian semimartingale with appropriate kernels as local specifications is done in Section 3.3. In Section 3.4, using the preceding sections, we prove that any limit point is a zero of the free energy functional, which is computed as the difference between the specific entropy and the specific energy. Thus, in Section 3.5, we complete the proof by identifying the zeros of the free energy as solutions of (1).

3.1 A sequence of approximate solution.

We define the finite volume approximation of the SDE (1) on Λ, finite subset of $\mathbb{Z}^d$, by

$$\begin{cases} dX_i(t) &= \mathbf{b}_t(\theta_i(X_\Lambda 0_{\Lambda^c}))\,dt + dB_i(t)\;, \quad i \in \Lambda,\; t \in [0,1] \\ X_\Lambda(0) &\sim \mu_\Lambda, \end{cases} \tag{8}$$

where the configuration $X_\Lambda 0_{\Lambda^c}$ is a concatenation of the configuration X on Λ and the constant function 0 outside Λ. With other words, we freeze the external configuration outside Λ to be equal to 0.

Take the increasing sequence of finite cubic volume $\Lambda_n = \{-n, \dots, n-1\}^d \subset \mathbb{Z}^d$. By Girsanov Theorem, for any n, there exists a unique probability measure called $P_n \in \mathcal{P}(\mathcal{C}([0,1],\mathbb{R})^{\Lambda_n})$, weak solution of the SDE (8) on Λ_n. Since μ admits a finite specific entropy, μ_{Λ_n} is absolutely continuous with respect to $m^{\otimes\Lambda_n}$ (with density denoted by f_{Λ_n}) and so

$$\begin{aligned} \frac{dP_n}{dW^{\otimes\Lambda_n}}(X_{\Lambda_n}) &= f_{\Lambda_n}(X_{\Lambda_n}(0))\,\exp -H_{\Lambda_n}(X_{\Lambda_n}0_{\Lambda_n^c}) \\ \text{where } H_\Lambda(X) &= -\sum_{i\in\Lambda}\Big(\int_0^1 \mathbf{b}_t(\theta_i X)\,dX_i(t) - \frac{1}{2}\int_0^1 \mathbf{b}_t^2(\theta_i X)\,dt\Big). \end{aligned} \tag{9}$$

Note that, due to the boundedness of $\mathbf{b}$, the functional H_{Λ_n} is well-defined $W^{\otimes\Lambda_n}$-a.s..

Since we aim at constructing a shift invariant solution of (1), we first introduce a space-periodisation of P_n. Let $P_n^{\text{per}} \in \mathcal{P}(\Omega)$ be the probability measure under which the restrictions of the configurations on disjoint blocks $((\theta_{2kn}X)_{\Lambda_n})_{k\in\mathbb{Z}^d}$ are independent and identically distributed like P_n. Thus we consider the space-averaged probability measure on Ω

$$\bar{P}_n := \frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n} P_n^{\text{per}} \circ \theta_i^{-1} \quad \in \mathcal{P}_s(\Omega). \tag{10}$$

$\bar{P}_n$ is shift invariant by construction. It can be interpreted as the shift invariant extension of the solution of (8) on Λ_n.

3.2 Tightness

We now show that the sequence $(\bar{P}_n)_n$ has an accumulation point for the $\mathcal{L}$-topology of local convergence on $\mathcal{P}(\Omega)$. This topology is defined as the coarsest one such that the maps $P \mapsto P(A)$, from $\mathcal{P}(\Omega)$ to $\mathbb{R}$, are continuous for any local event $A \in \mathcal{F}$. The key argument is the following tightness criterium based on the specific entropy $\mathfrak{I}$ and proved in [G11], Proposition 15.14.

Proposition 3.1 *For any constant $M > 0$, the level set*

$$\{P \in \mathcal{P}_s(\Omega),\ \mathfrak{I}(P) \leq M\}$$

is sequentially compact for the $\mathcal{L}$-topology.

Therefore, we have to prove such a uniform upper bound for the sequence $(\bar{P}_n)_n$.

Proposition 3.2 *The specific entropy of the sequence $(\bar{P}_n)_n$ is uniformly bounded:*

$$\sup_{n\geq 1} \mathfrak{I}(\bar{P}_n) < +\infty.$$

Proof. First, it is straightforward that

$$\mathfrak{I}(\bar{P}_n) = \frac{1}{|\Lambda_n|}\mathcal{I}(P_n; W^{\otimes\Lambda_n}). \tag{11}$$

(for details, see e.g. the arguments of Proposition 15.52 in [G11]). From (9)

$$\begin{aligned}
\mathcal{I}(P_n; W^{\otimes\Lambda_n}) &= \int \ln(f_{\Lambda_n}) d\mu_{\Lambda_n} - E_{P_n}\Big(H_{\Lambda_n}(X_{\Lambda_n} 0_{\Lambda_n^c})\Big) \\
&= \mathcal{I}(\mu_{\Lambda_n}; m^{\otimes\Lambda_n}) \\
&\quad + \sum_{i\in\Lambda_n} E_{P_n}\left(\int_0^1 \mathbf{b}_t(\theta_i(X_{\Lambda_n} 0_{\Lambda_n^c}))\Big(dX_i(t) - \mathbf{b}_t(\theta_i(X_{\Lambda_n} 0_{\Lambda_n^c}))dt\Big)\right) \\
&\quad + \frac{1}{2}\sum_{i\in\Lambda_n} E_{P_n}\left(\int_0^1 \mathbf{b}_t^2(\theta_i(X_{\Lambda_n} 0_{\Lambda_n^c}))dt\right).
\end{aligned} \tag{12}$$

Since P_n is a weak solution of (8), the process $\Big(X_i(t) - \int_0^t \mathbf{b}_t(\theta_i(X_{\Lambda_n} 0_{\Lambda_n^c}))dt\Big)_{i\in\Lambda_n,\ t\in[0,1]}$ is a random vector of independent P_n-Brownian motions. Therefore the second term in the right hand side of (12) vanishes. Due to the finiteness of the specific entropy of μ, we obtain

$$\frac{1}{|\Lambda_n|}\mathcal{I}(P_n; W^{\otimes \Lambda_n}) \leq \sup_{n\geq 1} \frac{1}{|\Lambda_n|}\mathcal{I}(\mu_{\Lambda_n}; m^{\otimes \Lambda_n}) + \frac{1}{2}\|\mathbf{b}\|_\infty^2 < +\infty. \tag{13}$$

With (11), this completes the proof of Proposition 3.2. ■

As corollary we get the

Proposition 3.3 *There exists a subsequence $(\bar{P}_{n_k})_k$ of the sequence $(\bar{P}_n)_n$ which converges for the $\mathcal{L}$-topology to some $\bar{P} \in \mathcal{P}_s(\Omega)$.*

From now on we write for simplicity $\bar{P} = \lim_n \bar{P}_n$ instead of $\bar{P} = \lim_k \bar{P}_{n_k}$. The rest of Section 3 is devoted to the analysis of this limit point $\bar{P}$.

3.3 Structure of the limit point $\bar{P}$

The class of Brownian semimartingales with bounded specific entropy is closed by $\mathcal{L}$-limits, as we will see in what follows.

3.3.1 $\bar{P}$ is a Brownian semimartingale

Recall first the following important structural result for which we give the main lines of the proof.

Lemma 3.1 *Let $Q \in \mathcal{P}_s(\Omega)$ be a probability measure with finite specific entropy $\mathfrak{I}(Q)$. Then there exists an adapted process $(\tilde{\beta}_t)_{t\in[0,1]}$ on Ω such that the family of processes*

$$M_i(t) = X_i(t) - X_i(0) - \int_0^t \tilde{\beta}_t(\theta_i X)ds, \qquad i \in \mathbb{Z}^d, t \in [0,1],$$

are independent Brownian motions under Q. Moreover, the map $(t,\omega) \mapsto \tilde{\beta}_t(\omega)$ is $L^2(dt \otimes dQ)$-integrable and

$$\mathfrak{I}(Q \circ X(0)^{-1}) + \frac{1}{2}E_Q\Big(\int_0^1 \tilde{\beta}_t^2 dt\Big) \leq \mathfrak{I}(Q). \tag{14}$$

Proof.

First let us notice that the specific entropy $\mathfrak{I}(Q)$ admits the following representation as mean of the relative entropy of a conditional probability:

$$\mathfrak{I}(Q) = E_Q\Big(\mathcal{I}_{\{0\}}\big(Q(\cdot|\mathcal{F}^-)|\mathbf{W}\big)\Big),$$

where $\mathcal{F}^- := \sigma(X_i, i < 0)$ (here $<$ denotes the lexicographic order). This result is a version of McMillan theorem, which goes back to the work of Robinson and Ruelle [RR67] and can be proved as in [DP93], Proposition 4.1. Define now $\mathcal{F}^0 := \sigma(X_i, i \neq 0)$. Since $\mathcal{F}^- \subset \mathcal{F}^0$, by Jensen inequality,

$$E_Q\Big(\mathcal{I}_{\{0\}}\big(Q(\cdot|\mathcal{F}^0)|\mathbf{W}\big)\Big) \leq E_Q\Big(\mathcal{I}_{\{0\}}\big(Q(\cdot|\mathcal{F}^-)|\mathbf{W}\big)\Big) < +\infty. \qquad (15)$$

The left hand side in (15), also called local entropy in [FW86] , is then finite. Thus, by [FW86] Theorem 2.4, there exists an adapted process $\tilde{\beta}$ in $L^2(dt \otimes dQ)$ such that

$$M_i(t) = X_i(t) - X_i(0) - \int_0^t \tilde{\beta}_t(\theta_i X)ds, \quad i \in \mathbb{Z}^d, t \in [0,1],$$

are independent Q-Brownian motions.
It remains to show (14) in following essentially the proof of Lemma 8 in [DPRZ02]. ■

Since $\bar{P}$ has finite specific entropy, applying Lemma 3.1 we deduce that it is a Brownian semimartingale characterized by its drift β. The proof of Theorem 2.1 is complete provided we show that $\beta_t(\omega) = \mathbf{b}_t(\omega)$ for $dt \otimes \bar{P}$-almost all t and ω, and that $\bar{P} \circ X(0)^{-1}$ is equal to μ. These identifications will be completed in Section 3.5. The identification of the drift requires sophisticated tools, which we now develop.

3.3.2 Local structure of $\bar{P}$

Define, for $\xi \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$, a reference probability kernel on Ω,

$$\Pi^0_\Lambda(\xi, d\omega) := \otimes_{i\in\Lambda} W^{\xi_i(0)}(d\omega_i) \otimes \delta_{\xi_{\Lambda^c}}(d\omega_{\Lambda^c}). \qquad (16)$$

It corresponds to a Brownian dynamics with fixed initial position inside Λ and frozen path outside Λ. Next we perturb it via the functional defined in (9):

$$\Pi^H_\Lambda(\xi, d\omega) := e^{-H_\Lambda(\omega)}\, \Pi^0_\Lambda(\xi, d\omega). \qquad (17)$$

Note that Π_Λ^H is a probability kernel since $e^{-H_\Lambda(\omega)}$ is a Π_Λ^0-martingale. It corresponds to a solution of (1) on Λ with fixed initial condition $\xi_\Lambda(0)$ and frozen path outside ξ_{Λ^c}. We also define a probability kernel with a wider interaction range, which will be useful in the sequel:

$$\Pi_\Lambda^{H,+}(\xi, d\omega) := \frac{1}{Z_\Lambda(\xi)}\, e^{-H_{\Lambda^+}(\omega)}\, \Pi_\Lambda^0(\xi, d\omega), \tag{18}$$

where the set $\Lambda^+ = \{i \in \mathbb{Z}^d : (\Delta + i) \cap \Lambda \neq \varnothing\}$ is a Δ-enlarged version of the set Λ (recall that Δ is the interaction range of $\mathbf{b}$). $Z_\Lambda(\xi) = \int e^{-H_{\Lambda^+}(\omega)} \Pi_\Lambda^0(\xi, d\omega)$ is the normalising constant, usually called *partition function* in Statistical Mechanics.
Notice that this kernel contains a stochastic integral which is not a priori meaningful. Moreover, it is not trivial why $Z_\Lambda(\xi)$ belongs to $]0, +\infty[$. However, it is the case in our framework, as we show in the next lemma.

Lemma 3.2 *The map $\xi \mapsto \Pi_\Lambda^{H,+}(\xi, \cdot)$ is well-defined for $\mathbf{W}$-almost all ξ. In particular, it is also P-almost surely defined for any probability measure P which is locally absolutely continuous with respect to $\mathbf{W}$.*

Proof. The stochastic integrals with respect to $(\xi_i)_{i \in \Lambda^+ \setminus \Lambda}$ appearing in $\Pi_\Lambda^{H,+}(\xi, .)$ are clearly meaningful $\mathbf{W}$-almost surely. Moreover, by Girsanov theorem, $E_{\mathbf{W}}(Z_\Lambda) = 1$ which ensures that Z_Λ is $\mathbf{W}$-a.s. finite. Since H_Λ is $\mathbf{W}$-almost surely finite, Z_Λ is $\mathbf{W}$-a.s. positive and the lemma is proved. ■

The measurability property of the kernels Π_Λ^H and $\Pi_\Lambda^{H,+}$ is the subject of the following remark.

Remark 3.1 *Define, for $\Lambda \subset \mathbb{Z}^d$, the σ-field $\mathcal{G}_\Lambda = \sigma(X_{\Lambda^c}, X(0))$. It builds a decreasing family when Λ increases and $\Pi_\Lambda^0 = \mathbf{W}(\quad|\mathcal{G}_\Lambda)$ a.s.. Moreover, $\xi \mapsto \Pi_\Lambda^H(\xi, \cdot)$ is $\mathcal{G}_\Lambda \cap \mathcal{F}_{\Lambda^+} = \sigma\{X_{\Lambda^+ \setminus \Lambda}, X_\Lambda(0)\}$-measurable since H_Λ is $\mathcal{F}_{\Lambda^+}$-measurable, and $\xi \mapsto \Pi_\Lambda^{H,+}(\xi, \cdot)$ is $\partial\mathcal{F}_\Lambda$-measurable, where the boundary σ-fields $\partial\mathcal{F}_\Lambda$ are defined by $\partial\mathcal{F}_\Lambda := \mathcal{G}_\Lambda \cap \mathcal{F}_{\Lambda^{++}}$.*

We now present an equilibrium equation - or fixed point property - satisfied by $\bar{P}$ which in fact determines its local specifications, and therefore induces some Gibbsian structure, as we will emphasize in Section 4.

Lemma 3.3 *For any finite subset Λ of $\mathbb{Z}^d$,*

$$\bar{P}(d\omega) \;=\; \int_\Omega \Pi_\Lambda^{H,+}(\xi, d\omega)\, \bar{P}(d\xi). \tag{20}$$

Proof. First, let us note that the right term in (20) is meaningful. Indeed, since the specific entropy of $\bar{P}$ is finite, $\bar{P}$ is locally absolutely continuous with respect to $\mathbf{W}$. Therefore, by Lemma 3.2, $\Pi_\Lambda^{H,+}(\xi,.)$ is well defined for $\bar{P}$-almost all ξ.
We have to prove that

$$\int g(\omega)\bar{P}(d\omega) = \int g(\omega)\Pi_\Lambda^{H,+}(\xi,d\omega)\,\bar{P}(d\xi)$$

holds for any bounded local measurable function g. Denote by Γ a bounded set of $\mathbb{Z}^d$ which includes both the support of g and Λ^{++}. Using standard conditional calculus, it is simple to show that for n large enough assuring that $\Lambda_n \supset \Gamma$, the probability measure P_n satisfies

$$\int g(\omega)P_n(d\omega) = \int g(\omega)\Pi_\Lambda^{H,+}(\xi,d\omega)P_n(d\xi).$$

Noting that $\xi \mapsto \int g(\omega)\Pi_\Lambda^{H,+}(\xi,d\omega)$ is local we have

$$\begin{aligned}
\int g(\omega)\bar{P}(d\omega) &= \lim_n \frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n}\int g(\omega)P_n^{\text{per}}\circ\theta_i^{-1}(d\omega) \\
&= \lim_n \frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n,\theta_i\Gamma\subset\Lambda_n}\int g(\theta_i\omega)P_n(d\omega) \\
&= \lim_n \frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n,\theta_i\Gamma\subset\Lambda_n}\int g(\theta_i\omega)\Pi_\Lambda^{H,+}(\xi,d\omega)P_n(d\xi) \\
&= \lim_n \frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n}\int g(\omega)\Pi_\Lambda^{H,+}(\xi,d\omega)P_n^{\text{per}}\circ\theta_i^{-1}(d\xi) \\
&= \int g(\omega)\Pi_\Lambda^{H,+}(\xi,d\omega)\,\bar{P}(d\xi),
\end{aligned}$$

which is the expected identity. ■

We interpret the identity (20) as follows: Randomizing under $\bar{P}$ the boundary condition ξ of the kernel $\Pi_\Lambda^{H,+}(\xi,\cdot)$ leads back to $\bar{P}$. It implies in particular that

$$\bar{P}(\cdot\,|\mathcal{G}_\Lambda) = \Pi_\Lambda^{H,+} \quad a.s..$$

3.4 $\bar{P}$ minimizes the free energy functional.

For any probability measure $Q \in \mathcal{P}_s(\Omega)$ with finite specific entropy, we define the Q-mixtures of the kernels Π^H_Λ and $\Pi^{H,+}_\Lambda$ by:

$$\Pi^H_{\Lambda,Q}(d\omega) = \int_\Omega \Pi^H_\Lambda(\xi, d\omega)\, Q(d\xi), \quad \Pi^{H,+}_{\Lambda,Q}(d\omega) := \int_\Omega \Pi^{H,+}_\Lambda(\xi, d\omega)\, Q(d\xi).$$

With these notations, the equilibrium equation (20) reads as follows: $\bar{P}$ is a fixed point of the map $Q \mapsto \Pi^{H,+}_{\Lambda,Q}$.

Moreover we define $\mathfrak{I}^{\mathbf{b}}(Q)$, the so-called free energy of Q, as the difference between its specific entropy and its specific energy, namely

$$\mathfrak{I}^{\mathbf{b}}(Q) := \mathfrak{I}(Q) - \mathfrak{I}(Q \circ X(0)^{-1}) - E_Q\Big(\int_0^1 \mathbf{b}_t(X) dX_0(t) - \frac{1}{2}\int_0^1 \mathbf{b}^2_t(X) dt \Big).$$

Note that $\mathfrak{I}^{\mathbf{b}}(Q)$ is well defined although a stochastic integral term occurs. Since Q has a finite specific entropy, by Lemma 3.1, we have that $E_Q(\int_0^1 \mathbf{b}_t(X) dX_0(t))$ is nothing but $E_Q(\int_0^1 \mathbf{b}_t(X)\beta_t(X))dt)$ which is finite because β is in $L^2(dt \otimes dQ)$.

In the proposition below we show that $\mathfrak{I}^{\mathbf{b}}$ is a thermodynamical functional, in the sense that it can be also obtained as limit of rescaled finite-volume relative entropies.

Proposition 3.4 *Consider $Q \in \mathcal{P}_s(\Omega)$ with finite specific entropy. Then*

$$\mathfrak{I}^{\mathbf{b}}(Q) = \lim_n \frac{1}{|\Lambda_n|} \mathcal{I}_{\Lambda_n^+}(Q; \Pi^H_{\Lambda_n,Q}). \tag{21}$$

Proof. By definition of the relative entropy we have

$$\begin{aligned}
\mathcal{I}_{\Lambda_n^+}(Q; \Pi^H_{\Lambda_n,Q}) &= E_Q\Big(\ln \big(\frac{dQ}{d\Pi^H_{\Lambda_n,Q}} \big|_{\Lambda_n^+} \big) \Big) \\
&= E_Q\Bigg(\ln \frac{dQ_{\Lambda_n^+}}{dW^{\otimes \Lambda_n^+}} + \ln \frac{dW^{\otimes \Lambda_n^+}}{d(\int \otimes_{i\in\Lambda_n} W^{\xi_i(0)} Q(d\xi) \otimes W^{\otimes \Lambda_n^+ \setminus \Lambda_n})} \\
&\quad + \ln \frac{d\Big(\int \otimes_{i\in\Lambda_n} W^{\xi_i(0)} Q(d\xi) \otimes W^{\otimes \Lambda_n^+\setminus\Lambda_n}\Big)}{d\Pi^0_{\Lambda_n,Q}\big|_{\Lambda_n^+}} + \ln \frac{d\Pi^0_{\Lambda_n,Q}}{d\Pi^H_{\Lambda_n,Q}}\Big|_{\Lambda_n^+} \Bigg) \\
&= \mathcal{I}_{\Lambda_n^+}(Q; \mathbf{W}) - \mathcal{I}_{\Lambda_n}(Q \circ X(0)^{-1}; m^{\otimes \mathbb{Z}^d}) - \mathcal{I}_{\Lambda_n^+\setminus\Lambda_n}(Q; \mathbf{W}) \\
&\quad + E_Q(H_{\Lambda_n})
\end{aligned} \tag{22}$$

The normalised third term of (22) vanishes: By subadditivity of the relative entropy (see Proposition 15.10 in [G11]),

$$0 \leq \mathcal{I}_{\Lambda_n^+ \setminus \Lambda_n}(Q; \mathbf{W}) \leq \mathcal{I}_{\Lambda_n^+}(Q; \mathbf{W}) - \mathcal{I}_{\Lambda_n}(Q; \mathbf{W})$$

and since $\lim_n |\Lambda_n|/|\Lambda_n^+| = 1$ it follows that

$$\lim_n \frac{1}{|\Lambda_n|} \mathcal{I}_{\Lambda_n^+ \setminus \Lambda_n}(Q; \mathbf{W}) = 0. \tag{23}$$

Let us compute the fourth term of (22). By stationarity of Q and by the definition of H_{Λ_n}, we get

$$E_Q(H_{\Lambda_n}) = -|\Lambda_n| \, E_Q\Big(\int_0^1 \mathbf{b}_t(X) dX_0(t) - \frac{1}{2} \int_0^1 \mathbf{b}_t^2(X) dt \Big). \tag{24}$$

From (23), (24) inserted in (22) we obtain

$$\lim_n \frac{1}{|\Lambda_n|} \mathcal{I}_{\Lambda_n^+}(Q; \Pi^H_{\Lambda_n, Q}) =$$
$$\mathfrak{I}(Q) - \mathfrak{I}(Q \circ X(0)^{-1}) - E_Q\Big(\int_0^1 \mathbf{b}_t(X) dX_0(t) - \frac{1}{2} \int_0^1 \mathbf{b}_t^2(X) dt \Big).$$

■

Now we are ready for proving that the free energy vanishes under $\bar{P}$.

Proposition 3.5 *The probability measure $\bar{P}$ is a zero of the free energy:*

$$\mathfrak{I}^{\mathbf{b}}(\bar{P}) = 0.$$

Proof. The representation (21) implies that the free energy $\mathfrak{I}^{\mathbf{b}}$ is non negative. So the proof of Proposition 3.5 is complete as soon as we can show that $\mathfrak{I}^{\mathbf{b}}(\bar{P}) \leq 0$.
Since $\bar{P}$ is absolutely continuous with respect to $\Pi^H_{\Lambda_n, \bar{P}}$ with a $\mathcal{F}_{\Lambda_n^+}$-measurable density (see Remark 3.1), for any finite set Γ containing Λ_n^+, $\mathcal{I}_\Gamma(\bar{P}; \Pi^H_{\Lambda_n, \bar{P}})$ and $\mathcal{I}_{\Lambda_n^+}(\bar{P}; \Pi^H_{\Lambda_n, \bar{P}})$ are identical. Taking in particular $\Gamma = \Lambda_n^{++}$, one obtains

$$\mathfrak{I}^{\mathbf{b}}(\bar{P}) = \lim_n \frac{1}{|\Lambda_n|} \mathcal{I}_{\Lambda_n^{++}}(\bar{P}; \Pi^H_{\Lambda_n, \bar{P}}). \tag{25}$$

Thanks to Lemma 3.3

$$\begin{aligned}
\mathcal{I}_{\Lambda_n^{++}}(\bar{P};\Pi^H_{\Lambda_n,\bar{P}}) &= E_{\bar{P}}\Big(\ln \frac{d\bar{P}}{d\Pi^0_{\Lambda_n,\bar{P}}}\Big|_{\Lambda_n^{++}} + \ln \frac{d\Pi^0_{\Lambda_n,\bar{P}}}{d\Pi^H_{\Lambda_n,\bar{P}}}\Big|_{\Lambda_n^{++}}\Big) \\
&= E_{\bar{P}}\Big(\ln \frac{d\Pi^{H,+}_{\Lambda_n,\bar{P}}}{d\Pi^0_{\Lambda_n,\bar{P}}}\Big|_{\Lambda_n^{++}} + \ln \frac{d\Pi^0_{\Lambda_n,\bar{P}}}{d\Pi^H_{\Lambda_n,\bar{P}}}\Big|_{\Lambda_n^{++}}\Big) \\
&= -E_{\bar{P}}(H_{\Lambda_n^+}) - E_{\bar{P}}(\ln(Z_{\Lambda_n})) + E_{\bar{P}}(H_{\Lambda_n}) \\
&= |\Lambda_n^+ \backslash \Lambda_n|\, E_{\bar{P}}\Big(\int_0^1 \mathbf{b}_t(X)dX_0(t) - \frac{1}{2}\int_0^1 \mathbf{b}_t^2(X)dt\Big) \\
&\quad -E_{\bar{P}}(\ln(Z_{\Lambda_n})). \qquad (26)
\end{aligned}$$

By (25) and (26) the proof of Proposition 3.5 is completed provided that we show that

$$\lim_n \frac{E_{\bar{P}}(\ln(Z_{\Lambda_n}))}{|\Lambda_n|} \geq 0. \qquad (27)$$

Indeed we have

$$\begin{aligned}
E_{\bar{P}}(\ln(Z_{\Lambda_n})) &= \int \ln\Big(\int e^{-H_{\Lambda_n^+}(\omega_{\Lambda_n}\xi_{\Lambda_n^c})} \otimes_{i\in\Lambda_n} W^{\xi_i(0)}(d\omega)\Big)\,\bar{P}(d\xi) \\
&= \int \ln\Big(\int e^{(H_{\Lambda_n}-H_{\Lambda_n^+})(\omega_{\Lambda_n}\xi_{\Lambda_n^c})} e^{-H_{\Lambda_n}(\omega_{\Lambda_n}\xi_{\Lambda_n^c})} \otimes_{i\in\Lambda_n} W^{\xi_i(0)}(d\omega)\Big)\,\bar{P}(d\xi) \\
&\geq \int (H_{\Lambda_n} - H_{\Lambda_n^+})(\omega)\Pi^H_{\Lambda_n}(\xi,d\omega)\bar{P}(d\xi) \\
&= \int \sum_{i\in{\Lambda_n}^+\backslash\Lambda_n} \int_0^1 \mathbf{b}_t(\theta_i(\omega))\big(d\xi_i(t) - \frac{1}{2}\mathbf{b}_t(\theta_i(\omega))dt\big)\Pi^H_{\Lambda_n}(\xi,d\omega)\bar{P}(d\xi).
\end{aligned}$$

Since

$$\int_0^1 \mathbf{b}_t^2(\theta_i(\omega))dt\,\Pi^H_{\Lambda_n}(\xi,d\omega)\bar{P}(d\xi) \leq \|\mathbf{b}\|^2_\infty \qquad (28)$$

it remains to prove that

$$\inf_n \inf_{i\in\Lambda_n^+\backslash\Lambda_n} \int \Big(\int_0^1 \mathbf{b}_t(\theta_i(\omega))d\xi_i(t)\Big)\,\Pi^H_{\Lambda_n}(\xi,d\omega)\bar{P}(d\xi) > -\infty. \qquad (29)$$

In the following, we show that (29) is a consequence of (28). We use the decomposition of $\xi(t)$ under $\bar{P}$, proved in Section 3.3.1, as a Brownian semimartingale with drift $\beta \in L^2(dt \otimes d\bar{P})$. Therefore for any Λ and any $i \notin \Lambda$

$$
\begin{aligned}
& \int \int_0^1 \mathbf{b}_t(\theta_i(\omega)) d\xi_i(t) \Pi_\Lambda^H(\xi, d\omega) \bar{P}(d\xi) \\
= & \int \int_0^1 \mathbf{b}_t(\theta_i(\omega))(d\xi_i(t) - \beta_t(\theta_i\xi)dt)\, \Pi_\Lambda^H(\xi, d\omega)\bar{P}(d\xi) \\
& \qquad + \int \int_0^1 \mathbf{b}_t(\theta_i(\omega))\beta_t(\theta_i\xi)\, dt\, \Pi_\Lambda^H(\xi, d\omega)\bar{P}(d\xi) \\
= & \int \int_0^1 \mathbf{b}_t(\theta_i(\omega))\beta_t(\theta_i\xi) dt\, \Pi_\Lambda^H(\xi, d\omega)\bar{P}(d\xi) \\
\geq & -\Big(\int \int_0^1 \mathbf{b}_t^2(\theta_i(\omega)) dt\, \Pi_\Lambda^H(\xi, d\omega)\bar{P}(d\xi)\Big)^{1/2} E_{\bar{P}}\Big(\int_0^1 \beta_t^2\, dt\Big)^{1/2} \\
\geq & -\|\mathbf{b}\|_\infty\, E_{\bar{P}}\Big(\int_0^1 \beta_t^2\, dt\Big)^{1/2} > -\infty
\end{aligned}
$$

uniformly in Λ. ■

Therefore the minimum of the free energy is attained on $\bar{P}$:

$$
\mathfrak{I}^{\mathbf{b}}(\bar{P}) = 0 = \min \Big\{ \mathfrak{I}^{\mathbf{b}}(Q),\ Q \in \mathcal{P}_s^*(\Omega) \text{ such that } \mathfrak{I}(Q) < +\infty \Big\},
$$

or, with other words, $\bar{P}$ solves a variational principle.

3.5 $\bar{P}$ is a weak solution of the SDE (1)

We have to identify the initial marginal law of $\bar{P}$ and its drift.

3.5.1 Identification of $\bar{P}$'s marginal law at time 0

Let g be a bounded Γ-local function on $\mathbb{R}^{\mathbb{Z}^d}$, satisfying $g(\omega) = g(\omega_\Gamma)$ for all $\omega \in \mathbb{R}^{\mathbb{Z}^d}$. By shift invariance of μ, for all $n \geq 1$ and $i \in \mathbb{Z}^d$ such that $\theta_i^{-1}\Gamma \subset \Lambda_n$,

$$
\mu_{\Lambda_n} \circ \theta_i^{-1}(g) = \mu \circ \theta_i^{-1}(g) = \mu(g).
$$

$$\begin{aligned}
\text{So} \qquad \bar{P} \circ X(0)^{-1}(g) &= \lim_{n\to\infty} \bar{P}_n \circ X(0)^{-1}(g) \\
&= \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{i\in\Lambda_n} P_n^{\text{per}} \circ \theta_i^{-1} \circ X(0)^{-1}(g) \\
&= \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{i\in\Lambda_n, \theta_i^{-1}\Gamma\subset\Lambda_n} \mu_{\Lambda_n} \circ \theta_i^{-1}(g) = \mu(g),
\end{aligned}$$

which proves that $\bar{P} \circ X(0)^{-1} = \mu$.

3.5.2 Identification of the dynamics under $\bar{P}$

It only remains to identify the unknown drift β of $\bar{P}$. Let us rewrite the free energy functional of $\bar{P}$ inserting β:

$$\begin{aligned}
\mathfrak{I}^{\mathbf{b}}(\bar{P}) &= \mathfrak{I}(\bar{P}) - \mathfrak{I}(\bar{P} \circ X(0)^{-1}) - E_{\bar{P}}\Big(\int_0^1 \mathbf{b}_t(X)(dX_0(t) - \beta_t(X)dt) \\
&\qquad + \int_0^1 \Big(\beta_t(X)\mathbf{b}_t(X) - \frac{1}{2}\mathbf{b}_t^2(X)\Big)dt\Big) \\
&= \mathfrak{I}(\bar{P}) - \mathfrak{I}(\mu) - E_{\bar{P}}\Big(\int_0^1 \Big(\beta_t(X)\mathbf{b}_t(X) - \frac{1}{2}\mathbf{b}_t^2(X)\Big)dt\Big).
\end{aligned}$$

By Proposition 3.5, this quantity vanishes. On the other side, due to Lemma 3.1, $\mathfrak{I}(\bar{P}) \geq \mathfrak{I}(\mu) + \frac{1}{2}E_{\bar{P}}\big(\int_0^1 \beta_t^2(X)dt\big)$. Therefore,

$$\begin{aligned}
0 &\geq E_{\bar{P}}\Big(\frac{1}{2}\int_0^1 {\beta_t}^2(X)dt - \int_0^1 \Big(\beta_t(X)\mathbf{b}_t(X) + \frac{1}{2}\mathbf{b}_t^2(X)\Big)dt\Big) \\
&= \frac{1}{2}E_{\bar{P}}\Big(\int_0^1 \big(\beta_t(X) - \mathbf{b}_t(X)\big)^2 dt\Big),
\end{aligned}$$

which implies that $\beta_t(\omega) = \mathbf{b}_t(\omega)$ for $dt \otimes \bar{P}$-almost all t and ω.

It completes the proof that $\bar{P}$ is an infinite-dimensional Brownian motion with drift $\mathbf{b}$ and initial law μ.

4 On the Gibbs property

In this section, we deal with the Gibbsian structure of solutions of the SDE (1). First recall that the probability measure $\Pi_\Lambda^{H,+}(\xi, .)$ is not always well defined, as remarked in Section 3.3.2. To circumvent this difficulty take $\Pi_\Lambda^{H,+}(\xi,.) \equiv 0$ when the partition function $Z_\Lambda(\xi)$ is not finite or when the stochastic integral with respect to ξ in H_Λ^+ is not defined. In this way the family of kernels $(\Pi_\Lambda^{H,+})_{\Lambda\subset\mathbb{Z}^d}$ builds a local specification as introduced by Preston in [Pr76] (2.10)-(2.14), which allows to define associated Gibbs measures.

Definition 4.1 *A probability measure Q on Ω is a Gibbs measure with respect to the specification $(\Pi_\Lambda^{H,+})_{\Lambda\subset\mathbb{Z}^d}$ if, for all finite subset Λ of $\mathbb{Z}^d$,*

$$Q(d\omega) = \int \Pi_\Lambda^{H,+}(\xi, d\omega)\, Q(d\xi). \tag{30}$$

Note the similarity with equation (20) where Q appears here in place of $\bar{P}$. It follows that $\bar{P}$ is a Gibbs measure with respect to the specification $(\Pi_\Lambda^{H,+})_{\Lambda\subset\mathbb{Z}^d}$. Actually we obtain a more general result.

Theorem 4.1 *Let Q be a probability measure in $\mathcal{P}_s(\Omega)$ with finite specific entropy. Then Q is a Gibbs measure with respect to the specification $(\Pi_\Lambda^{H,+})_{\Lambda\subset\mathbb{Z}^d}$ if and only if Q is a weak solution of the SDE* (1).

Proof.

"$\Leftarrow$": it is similar to the proof of Theorem 2.1. Indeed, in Section 3, for proving that $\bar{P}$ is a weak solution of the SDE (1), we only used the fact that $\bar{P}$ satisfies equation (20) and that its specific entropy is finite.

"$\Rightarrow$": it is straightforward. A similar detailed proof can be found in [DPRZ02], Proposition 1. ■

To complete this section we present the

Proof of Theorem 2.2.

Let us recall that a shift invariant probability measure is ergodic if it is trivial on the σ-field of shift invariant sets.

By previous Theorem 4.1, the set of weak solutions **Sol** is exactly the set of shift invariant Gibbs measures with finite specific entropy. It is known that the set of stationary Gibbs measure admits a representation by mixing of its extremal points which are ergodic (Theorem 2.2 and 4.1 in [Pr76]). Since the specific entropy functional is affine ([G11], Proposition 15.14), this representation remains valid inside of the set of Gibbs measures with finite specific entropy and the first part of the theorem is proved.

Now let μ be an ergodic probability measure in $\mathcal{P}_s(\mathbb{R}^{\mathbb{Z}^d})$ with finite specific entropy. By Theorem 2.1, there exists a weak solution P of the SDE (1) with initial condition μ. Thanks to the above representation, P is a mixing of ergodic weak solutions of the s.d.e. (1). Their initial condition is necessarily μ, by ergodicity. The second part of the theorem is then proved. ■

5 An unbounded drift setting

In this section, we improve the result presented in Theorem 2.1 by adding to the bounded non-regular drift considered above an unbounded regular term. More precisely, the drift decomposes now as follows:

$$\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$$

where

- none of $\mathbf{b}_1$ or $\mathbf{b}_2$ is supposed to be Markov.
- $\mathbf{b}_1$ is Δ-local and uniformly bounded (i.e. satisfies (2) and (3) as in Section 2.3); It is possibly non-regular.
- $\mathbf{b}_2$ is possibly unbounded but local and regular in the sense that it satisfies the following uniform pathwise Lipschitz assumption:

  $\exists L > 0$ such that $\forall \omega, \omega' \in \Omega$,

$$\begin{aligned} \forall t \in [0,1], \qquad |\mathbf{b}_{2,t}(\omega) - \mathbf{b}_{2,t}(\omega')| &\leq L \sup_{s \leq t, i \in \Delta} |\omega_i(s) - \omega'_i(s)| \quad (31) \\ \text{and} \qquad |\mathbf{b}_{2,t}(0)| &\leq L. \quad (32) \end{aligned}$$

(Without loss of generality, we consider the same Δ-locality for both drifts $\mathbf{b}_1$ and $\mathbf{b}_2$). Under these assumptions the drift $\mathbf{b}$ has a sublinear growth, or equivalently

$$\mathbf{b}_t(\omega)^2 \leq C\Big(1 + \sum_{j \in \Delta} \omega_j^*(t)^2\Big) \tag{33}$$

where $C := 2(\|\mathbf{b}_1\|_\infty^2 + L^2)$ and $\omega^*(t) =: \sup_{0 \leq s \leq t} |\omega(s)|$.

A typical example of such a drift, dealt in [RRR10] Equation (20), is

$$\mathbf{b}_{2,t}(\omega) = \int_0^t \alpha(s, \omega_\Delta(s))\, ds,$$

where $\alpha(s,\cdot)$ is a Lipschitz function from $\mathbb{R}^\Delta$ to $\mathbb{R}$.

We can now state our next existence result.

Theorem 5.1 *Fix an initial probability measure* $\mu \in \mathcal{P}_s(\mathbb{R}^{\mathbb{Z}^d})$ *with finite specific entropy, satisfying the integrability condition* $\int x_0^2\,\mu(dx) < +\infty$. *Assume that the drift* $\mathbf{b}$ *admits the decomposition* $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ *with* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *as above. Then the infinite-dimensional SDE* (1) *admits, at least, one shift-invariant weak solution* P *with initial marginal law* μ. *Moreover its specific entropy* $\mathfrak{I}(P)$ *is finite and* $E_P\Big(\sup_{t\in[0,1]} X_0(t)^2\Big) < +\infty$.

Proof.

The proof will have the same structure than the proof of Theorem 2.1. Nevertheless some technical issues will appear in the computation of the upper bounds involved in the tightness and in the minimization of the free energy. This leads us to construct in (34) an infinite-dimensional approximation of (1) for general boundary conditions and deterministic initial condition. In Lemma 5.1 we prove an upper bound for the supremum norm of this approximating process. In particular, it implies (41) which is a crucial uniform integrability property of $\mathbf{b}^2$ under the kernels Π^H_Λ.

In fact we solve the SDE (1) in the Hilbert subspace of $\mathbb{R}^{\mathbb{Z}^d}$ defined as weighted ℓ^2-space, which we now define, following the framework of [SS80]. Take the summable sequence $\gamma_i := \frac{1}{(1+|i|)^{d+1}}, i \in \mathbb{Z}^d$. As usual,

$$\ell^2(\gamma) := \{x \in \mathbb{R}^{\mathbb{Z}^d}, \|x\|_\gamma^2 := \sum_{i\in\mathbb{Z}^d} \gamma_i x_i^2 < +\infty\}.$$

For any finite subset $\Lambda \subset \mathbb{Z}^d$ and $\xi \in \Omega$ a fixed path, we define the Λ-approximation of the SDE (1) with outside frozen configuration ξ_{Λ^c} and initial fixed condition $\xi(0)$ as the solution of

$$\begin{cases} dX_i(t) &= \mathbf{b}_t(\theta_i(X_\Lambda \xi_{\Lambda^c}))\,dt + dB_i(t)\ , \quad i \in \Lambda,\ t \in [0,1], \\ X_\Lambda(0) &= \xi_\Lambda(0), \\ X_{\Lambda^c} &\equiv \xi_{\Lambda^c}. \end{cases} \tag{34}$$

Note that this SDE depends on ξ_Λ only via its initial value $\xi_\Lambda(0)$.

Lemma 5.1 *For any $\xi \in \Omega$, the SDE (34) admits a weak solution $P^{\xi,\Lambda}$. Moreover there exists a constant $K > 0$ which does not depend on Λ such that*

$$E_{P^{\xi,\Lambda}}\Big(\|X^*\|_\gamma^2\Big) \le K\Big(1 + \|\xi_\Lambda(0)\|_\gamma^2 + \|\xi^*_{\Lambda^c}\|_\gamma^2\Big). \tag{35}$$

Proof. First, if the drift $\mathbf{b}$ reduces to its regular part $\mathbf{b}_2$, that is if $\mathbf{b}_1$ vanishes, the Lipschitz continuity (31) and (32) ensures existence and uniqueness of an (even strong) solution to (34), see [RW87] Theorem 11.2. Now, if the non-regular term $\mathbf{b}_1$ does not vanish, since it is bounded, applying Girsanov theory, one obtains a weak solution to (34).
To obtain the upper bound (35), we take our inspiration from (4.18) in [SS80] or Lemma 4.2.9 in [R99] who treated the particular Markovian case. First fix $i \in \Lambda$. By Itô formula applied to $X_i(t)^2$ and (33), one gets

$$X_i^*(t)^2 \le X_i(0)^2 + M_t^* + \int_0^t \Big(X_i^*(s)^2 + C\big(1 + \sum_{k\in\Lambda} X_k^*(s)^2 \mathbb{1}_{k\in i+\Delta} + \sum_{k\in\Lambda^c} \xi_k^*(s)^2 \mathbb{1}_{k\in i+\Delta}\big)\Big) ds + t$$

where M_t is a martingale with quadratic variation $4\int_0^t X_i(s)^2 ds$. Using Doob inequality,

$$E(M_t^*) \le \sqrt{E((M_t^*)^2)} \le 2\sup_{s\le t}\sqrt{E(M_s^2)} \le 1 + \sup_{s\le t} E(M_s^2) \le 1 + 4\int_0^t X_i^*(s)^2 ds.$$

Therefore, denoting by $u_i(t)$ the function $t \mapsto E_{P^{\xi,\Lambda}}(X_i^*(t)^2)$, we obtain

$$\begin{aligned} u_i(t) &\le \xi_i^2(0) + 1 + 4\int_0^t u_i(s) ds \\ &\quad + \int_0^t \Big(u_i(s) + C\big(1 + \sum_{k\in\Lambda} u_k(s)\mathbb{1}_{k\in i+\Delta} + \sum_{k\in\Lambda^c} \xi_k^*(s)^2 \mathbb{1}_{k\in i+\Delta}\big)\Big) ds + t \\ &\le \Big(\xi_i^2(0) + C + 2 + C\sum_{k\in\Lambda^c} (\xi_k^*)^2 \mathbb{1}_{k\in i+\Delta}\Big) + \sum_k Q_{ik}\int_0^t u_k(s) ds \end{aligned} \tag{36}$$

where the matrix Q is given by $Q_{ik} = (5 + C)\mathbb{1}_{k\in\Lambda\cap i+\Delta}$ for $k \in \mathbb{Z}^d$, and $\xi^* := \xi^*(1) = \sup_{0\le s\le 1} |\xi(s)|$.
For $i \in \Lambda^c$, we consider the rough inequality

$$u_i(t) \le (\xi_i^*)^2 + \sum_k Q_{ik}\int_0^t u_k(s) ds. \tag{37}$$

Remark now that there exists a real number $C' > 0$ depending only on Δ but not on Λ, such that

$$\forall k \in \mathbb{Z}^d, \quad \sum_i \gamma_i Q_{ik} \leq C' \gamma_k.$$

Thus, summing over i the inequalities (36) and (37) weighted by γ, we get

$$\begin{aligned}\sum_i \gamma_i u_i(t) \quad &\leq \quad \|\xi_\Lambda(0)\|_\gamma^2 + (C+2) \sum_{i \in \Lambda} \gamma_i + \frac{C'}{5+C} \|\xi^*_{\Lambda^+ \setminus \Lambda}\|_\gamma^2 + \|\xi^*_{\Lambda^c}\|_\gamma^2 \\ &\quad + C' \int_0^t \sum_k \gamma_k u_k(s) ds,\end{aligned}$$

where the term $\|\xi^*_{\Lambda^c}\|_\gamma^2$ could be equal to $+\infty$ if $\xi^* \notin \ell^2(\gamma)$. This leads by Gronwall's lemma to

$$\begin{aligned}E_{P^{\xi,\Lambda}}\Big(\|X^*(t)\|_\gamma^2\Big) \quad &\leq \quad \Big(\|\xi_\Lambda(0)\|_\gamma^2 + (C+2) \sum_{i \in \Lambda} \gamma_i + \frac{C'}{5+C} \|\xi^*_{\Lambda^+ \setminus \Lambda}\|_\gamma^2 + \|\xi^*_{\Lambda^c}\|_\gamma^2\Big) e^{C' t} \\ &\leq \quad K\Big(1 + \|\xi_\Lambda(0)\|_\gamma^2 + \|\xi^*_{\Lambda^c}\|_\gamma^2\Big)\end{aligned}$$

for a constant K which does not depend on Λ and is uniformly bounded for $t \in [0, 1]$. ■

In particular, from the upper bound (35) we deduce, under the assumptions $\xi^* \in \ell^2(\gamma)$ that, for any $j \in \mathbb{Z}^d$,

$$E_{P^{\xi,\Lambda}}\big((X_j^*)^2\big) \leq \gamma_j^{-1} K\Big(1 + \|\xi^*\|_\gamma^2\Big) < +\infty. \tag{38}$$

Note that this upper bound is uniform in Λ but not in j. This issue is solved below thanks to the stationarity.

As in Section 3.1, we define P_n as the marginal law

$$P_n = \int P_{\Lambda_n}^{\xi_{\Lambda_n} 0_{\Lambda_n^c}, \Lambda_n} \mu_{\Lambda_n}(d\xi_{\Lambda_n}(0)).$$

With other words, P_n is a weak solution on Λ_n of SDE (34) with vanishing outside configuration and random initial condition following the law μ_{Λ_n}. The definition of the space-averaged $\bar{P}_n$ is done by (10) too. The bound (33) for the growth of the drift together with (38) implies that

$$
\begin{aligned}
&\sup_n \sup_{i\in\Lambda_n} E_{P_n}\Big(\int_0^1 \mathbf{b}_t^2(\theta_i(X_{\Lambda_n}0_{\Lambda_n^c}))\,dt\Big)\\
= \;& \sup_n \sup_{i\in\Lambda_n} \int\int_0^1 \mathbf{b}_t^2(\theta_i\omega)\,dt P^{\xi_{\Lambda_n}0_{\Lambda_n^c},\Lambda_n}(d\omega)\,\mu_{\Lambda_n}(d\xi_{\Lambda_n}(0))\\
= \;& \sup_n \sup_{i\in\Lambda_n} \int\int_0^1 \mathbf{b}_t^2(\omega)\,dt P^{\theta_i(\xi_{\Lambda_n}0_{\Lambda_n{}^c}),\theta_i\Lambda_n}(d\omega)\,\mu_{\Lambda_n}(d\xi_{\Lambda_n}(0))\\
\le \;& C + CK\left(1+\int_{\mathbb{R}^{\Lambda_n}} \|\theta_i x_{\Lambda_n})\|_\gamma^2\,\mu_{\Lambda_n}(dx_{\Lambda_n})\right)\sum_{j\in\Delta}\gamma_j^{-1}\\
\le \;& C + CK\Bigg(1+(\sum_{j\in\mathbb{Z}^d}\gamma_j)\int_{\mathbb{R}^{\mathbb{Z}^d}} x_0^2\,\mu(dx)\Bigg)\sum_{j\in\Delta}\gamma_j^{-1}, \qquad (39)
\end{aligned}
$$

where the last inequality comes from the stationarity of μ. From (39) we deduce an uniform bound, as in (13) Section 3.2, which implies the tightness of the sequence $(\bar{P}_n)_n$ and the existence of an accumulation point, denoted by $\bar{P}$.
The structure of the limit point $\bar{P}$ is similar as for bounded drift (see Section 3.3). Note that, by the convergence of $(\bar{P}_n)_n$ to $\bar{P}$ for the local topology, we also obtain that

$$
E_{\bar{P}}(\|X^*\|_\gamma^2) < +\infty \quad \text{and} \quad E_{\bar{P}}\left(\int_0^1 \mathbf{b}_t^2(X)dt\right) < +\infty, \qquad (40)
$$

which means that the finiteness of the second moment and of the specific entropy propagates through the dynamics.

Some more technical problems appear to generalize the results obtained in Section 3.4. First, the free energy $\mathfrak{I}^{\mathbf{b}}(Q)$ is not a priori defined for any Q with finite specific entropy, but only for Q satisfying $E_Q(\int_0^1 \mathbf{b}_t^2(X)dt) < +\infty$. Thanks to (40), it is the case for $Q = \bar{P}$, which is exactly what is needed in the following. Thus the proof of the variational principle (Proposition 3.5 for bounded drift) works as soon as we gain the following boundedness:

$$
\sup_n \sup_{i\notin\Lambda_n} \int_\Omega \int_0^1 \mathbf{b}_t^2(\theta_i\omega)\,dt\,\Pi_{\Lambda_n}^H(\xi, d\omega)\bar{P}(d\xi) < +\infty, \qquad (41)
$$

which is the generalization of (28).
Recall that, for any ξ, $\Pi_{\Lambda_n}^H(\xi, .)$ defined in (17) corresponds to a weak solution of (34) with fixed initial condition $\xi_\Lambda(0)$ and frozen path outside ξ_{Λ^c}.

Therefore we deduce from the stationarity of $\bar{P}$ and inequalities (33), (38) and (40)

$$\begin{aligned}
\sup_n \sup_{i\notin\Lambda_n} \quad & \int_\Omega \int_0^1 \mathbf{b}_t^2(\theta_i(\omega))\, dt\, \Pi^H_{\Lambda_n}(\xi, d\omega)\bar{P}(d\xi) \\
& = \sup_n \sup_{i\notin\Lambda_n} \int_\Omega \int_0^1 \mathbf{b}_t^2(\omega)\, dt\, P^{\xi,\theta_i\Lambda_n}(d\omega)\bar{P}(d\xi) \\
& \le \sup_n \sup_{i\notin\Lambda_n} \int_\Omega \int_0^1 C\Big(1+\sum_{j\in\Delta}\omega_j^*(t)^2\Big)\, dt\, P^{\xi,\theta_i\Lambda_n}(d\omega)\bar{P}(d\xi) \\
& \le C + CK\Big(1+E_{\bar{P}}(\|X^*\|_\gamma^2)\Big)\sum_{j\in\Delta}\gamma_j^{-1} < +\infty.
\end{aligned}$$

Now, to complete the proof of Theorem 5.1 we only need to identify the drift and the initial distribution of $\bar{P}$. This can be done in a very similar way as in Section 3.5. ■

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