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Reciprocal class of random walks on a Abelian group

GIOVANNI CONFORTI[1,2,*] and SYLVIE ROELLY[2,**]

[1] *Università degli Studi di Padova, Dipartimento di Matematica Pura e Applicata. Via Trieste 63, 35121 Padova, Italy. E-mail:* [*] giovanniconfort@gmail.com

[2] *Institut für Mathematik der Universität Potsdam , Am Neuen Palais 10, 14469 Potsdam, Germany. E-mail:* [**] roelly@math.uni-potsdam.de *(Corresponding author)*

Abstract

Processes having the same bridges as a given reference Markov process constitute its *reciprocal class*. In this paper we study the reciprocal class of a continuous time random walk with values in a countable Abelian group, compute explicitly its reciprocal characteristics and present an integral characterization of it. Our main tool is a new iterated version of the celebrated Mecke's formula from the point process theory, which allows us to study, as transformation of the path space, the addition of random loops. Thanks to the lattice structure of the set of loops, we even obtain a sharp characterization.

At the end, we discuss several examples to illustrate the richness of reciprocal classes. We observe how their structure depends on the algebraic properties of the underlying group.

Keywords: Reciprocal class, stochastic bridge, random walk on Abelian group.

Contents

Introduction

Given a reference Markov probability on the time interval $[0,1]$, its *reciprocal class* is the set of all probability measures which can be written as a mixture of its bridges. All the elements of the class enjoy the reciprocal property, which is a weaker version of the Markov property since it is a (time) Markov field property. For a detailed comparison between the Markov property and the reciprocal one, we refer to the recent survey [10]. Reciprocal probabilities were first introduced by Schrödinger in [18] to study the dynamics of a Brownian particle with prescribed laws at times at the initial and the final times, see e.g. [9]. Bernstein then highlighted in [1] the importance of non Markovian reciprocal processes. Jamison initiated later in a series of papers [6],[7],[8] a rigorous mathematical study of reciprocal processes, by partitioning them into classes and underlining the importance of invariants called *reciprocal characteristics*. The problem of computing, interpreting reciprocal characteristics and then using them to characterize reciprocal classes has attracted the attention of many authors in the context of diffusions (see e.g. [2],[19],[20], [16]).
The study of reciprocal classes of concrete jump processes has been started recently by Murr with the case of counting processes, see [12] and [5]. Then results concerning the characterization of the reciprocal classes of a compound Poisson process have been obtained in [3], in the particular case where (i) the state space is $\mathbb{R}^d$ and (ii) the support of the jump measure is a finite set of A different types of jumps. There, the approach is to study separately the jump-times of the reciprocal paths and their type distribution. In particular, the reciprocal characteristics come out analysing the behaviour of the Poisson random vector $\mathbf{N} \in \mathbb{N}^A$ under shifts, where $\mathbf{N}$ describes the number of jumps of a path during the time interval $[0,1]$, classified by types.

In this paper we propose to characterize reciprocal classes in the following more general framework: The state space of the random walk is a *countable Abelian group* G. See e.g. [17] for a review on random walks on groups.

Our tool is, by working directly at the level of the path space, to exhibit for each reciprocal class a family of integral equations which characterize it. The equations we obtain -see (14)- can be viewed as a twofold generalisation of Mecke's formula, which characterizes Poisson random measures via transformations which consist in adding one point to the canonical point process. Indeed, first we iterate this procedure adding several jumps to the canonical process, and secondly we work under the constraint that the added paths are loops, that is they should have as initial and final value the identity element. However, our method is efficient only if one can assure that the set of loop paths is rich enough to allow to transform any given path of the random walk into any other one having the same initial and final value, only by adding a finite number of well chosen elementary loops. This assumption on the support $G_\nu \subseteq G$ of the jump measure ν is formalised through (H1) and (H2), see Section 2.2.

As an interesting byproduct of our integral characterisation of the reciprocal classes of a ν-random walk, we get the identification thanks equation (14), of the associated family of reciprocal characteristics (15). These quantities remain unchanged on the full set of random walks having the same bridges, see Corollary 16.

The paper is organized as follows. In Section 1 we set up the necessary definitions and notations regarding random walks on groups, and provide an iterated formula on the path space satisfied by them. In Section 2, after having defined the concept of reciprocal classes, we state and prove our main result: the integral formula derived before on the path space is in fact a efficient way to characterize the complete reciprocal class of a random walk. In the last section we present several examples to illustrate the richness of the class of processes we are dealing with.

1. Random walk on Abelian groups

1.1. The random walk as Poisson random measure

Let $(G,+)$ be a countable Abelian group with identity element e. We denote by $\mathbb{D}([0,1],G)$ the space of càdlàg paths for the topology induced by the discrete metric in G. Note that, because of the existence of left and right limits, paths in $\mathbb{D}([0,1],G)$ have finitely many jumps. $\mathbb{D}([0,1],G)$ is equipped with its canonical sigma-algebra $\mathcal{F}$ and its canonical filtration $(\mathcal{F}_t)_{t\in[0,1]}$.

For any ν non negative finite measure on G, we call *ν-random walk* on G a Markov probability on $\mathbb{D}([0,1],G)$ denoted by $\mathbb{P}_\nu$ whose infinitesimal generator is given by:

$$(\mathscr{G}\phi)(g) := \sum_{g'\in G} \nu(g')\Big(\phi(g+g') - \phi(g)\Big),$$

for any ϕ bounded function. In the rest of the paper $G_\nu \subseteq G$ denotes the support of ν, that is the set of allowed jumps of the ν-random walk. The path space $\Omega \subset \mathbb{D}([0,1],G)$ is the set of paths with jumps in G_ν.

Mecke proved in [11] an integral characterization of Poisson point processes on general spaces which we recall in Proposition 2 in a form adapted to our framework. In the spirit of Murr [13] and Privault ([14], Section 6.4.4) who studied real-valued Processes with Independent Increments, we turn Mecke's formula into a characterization of random walks on G in Proposition 6. Let us first introduce some notations.

For a measurable space $\mathcal{X}$ we denote by $\mathcal{P}(\mathcal{X})$ the set of probability measures on $\mathcal{X}$ and by $\mathcal{M}(\mathcal{X})$ the set of finite point measures, that is

$$\mathcal{M}(\mathcal{X}) := \Big\{ \sum_{i=1}^{N} \delta_{x_i} : x_i \in \mathcal{X},\, N \in \mathbb{N}\Big\}.$$

$\mathcal{B}^+(\mathcal{X})$ denotes the set of non-negative bounded measurable functions over $\mathcal{X}$. We will often choose $\mathcal{X}$ to be the following product space of time-space elements:

$$\Gamma := [0,1] \times G \quad \ni \gamma = (t,g).$$

We identify trajectories in $\mathbb{D}([0,1],G)$ and point measures in $\mathcal{M}(\Gamma)$ via the following canonical bijective map M :

$$X \mapsto M_X := \sum_{0\leq t\leq 1} \sum_{g\in G} \delta_{(t,g)} \mathbb{1}_{\{\Delta X_t = g\}} \tag{1}$$

A useful observation is that the image measure of the random walk $\mathbb{P}_\nu$ under M is a Poisson random measure on Γ with intensity the finite measure $dt \otimes \nu$.

1.2. An integral characterization

Mecke's original idea was to characterize Poisson random measures by mean of an integral formula (see Satz 3.1 in [11]), via the change of measures which consists to add one (random) atom to the initial point measure, as in the right-hand side of Equation (3). Adapted to our context it reads as follows.

Proposition 2. *For $\tilde{\mathbb{P}} \in \mathcal{P}(\mathcal{M}(\Gamma))$ the following assertions are equivalent:*

i) $\tilde{\mathbb{P}}$ is the Poisson random measure with intensity measure $\rho = dt \otimes \nu$ on Γ.
ii) For all $\Phi \in \mathcal{B}^+(\mathcal{M}(\Gamma)\times\Gamma)$,

$$\int\int_\Gamma \Phi(\mu,\gamma)\ \mu(d\gamma)\tilde{\mathbb{P}}(d\mu) = \int\int_\Gamma \Phi(\mu+\delta_\gamma,\gamma)\,\rho(d\gamma)\tilde{\mathbb{P}}(d\mu). \tag{3}$$

Remark that the left-hand side of (3) also reads $\int\sum_{\gamma\in\mu}\Phi(\mu,\gamma)\tilde{\mathbb{P}}(d\mu)$ where the notation $\gamma \in \mu$ means that the points $\gamma \in \Gamma$ build the support of the point measure μ: one integrates the function Φ under the Campbell measure associated with $\tilde{\mathbb{P}}$. Thus (3) determines the Campbell measure of a Poisson random measure as the shifted product measure of itself with its intensity.
Let us adapt this tool to $\mathbb{D}([0,1],G)$. First, for $\gamma = (t,g) \in \Gamma$, let us denote by χ^γ the corresponding simple step function $\chi^\gamma := g\,\mathbb{1}_{[t,1]} \in \mathbb{D}([0,1],G)$. Then define the transformation Ψ_γ on the path space which consists in adding the jump g at time t.

Definition 4. *For $\gamma = (t,g) \in \Gamma$,* $\quad \Psi_\gamma X := X + \chi^\gamma, \quad X \in \mathbb{D}([0,1],G)$.

Notice that, under any probability $\mathbb{P} \in \mathcal{P}(\mathbb{D}([0,1],G))$ satisfying $\mathbb{P}(X_t = X_{t^-}) = 1$ for all $t \in [0,1]$, one has:

$$M_{\Psi_\gamma X} = M_X + \delta_\gamma \quad \mathbb{P}-a.s.. \tag{5}$$

Let us rewrite Proposition 2 in the language of random walks.

Proposition 6. *For $\mathbb{P} \in \mathcal{P}(\mathbb{D}([0,1],G))$ the following assertions are equivalent:*

i) *$\mathbb{P}$ is a ν-random walk on G.*
ii) *For all $F \in \mathcal{B}^+(\mathbb{D}([0,1],G) \times \Gamma)$,*

$$\mathbb{P}\Big(\int_\Gamma F(X,\gamma)\ M_X(d\gamma)\Big) = \mathbb{P}\Big(\int_\Gamma F(\Psi_\gamma X,\gamma)\ \rho(d\gamma)\Big), \tag{7}$$

where M_X is defined through (1).

Proof. *i)* $\Rightarrow$ *ii)*.
Since $\mathbb{P}_\nu$ is ν-random walk, M_X is a Poisson random measure with intensity $dt \otimes \nu$. Then Mecke's formula holds for $\tilde{\mathbb{P}} := \mathbb{P} \circ M^{-1}$. Since M is invertible and its inverse is measurable we can plug into (3) test functions Φ of the form $F(X,\gamma)$ and the conclusion follows.

ii) $\Rightarrow$ *i)*.
Let $\mathbb{P} \in \mathcal{P}(\mathbb{D}([0,1],G))$ satisfying (7). We define $\tilde{\mathbb{P}} := \mathbb{P} \circ M^{-1} \in \mathcal{P}(\mathcal{M}(\Gamma))$. Then, by considering test functions of the form $\Phi = F(M_X,\gamma)$ and using the fact that $M_{\Psi_\gamma X} = M_X + \delta_\gamma\ \mathbb{P} \otimes \rho - a.s.$, we deduce that $\tilde{\mathbb{P}}$ is a Poisson random measure with intensity $\rho = dt \otimes \nu$ by Proposition 2. Observing that

$$X_t = \sum_{g \in G} g\, M_X([0,t] \times \{g\})$$

the conclusion follows using (5). □

1.3. An iterated formula

To prepare the characterization of bridges which we will present in the next section, we now consider a generalisation of the formula (7) obtained by iteration. For this purpose, we define n-dimensional analogous of the objects appearing in Equation (7), $n \geq 1$ fixed.

$$\text{For } \gamma = (\gamma_1, ..., \gamma_n) \in \Gamma^n, \Psi_\gamma := \Psi_{\gamma_n} \circ ... \circ \Psi_{\gamma_1}, \quad \chi^\gamma = \sum_{i=1}^n \chi^{\gamma_i}.$$

Then Ψ_γ is the transformation that adds to X the n-step function χ^γ, which corresponds in the framework of point measures to add n new atoms to the original measure. It remains to specify the set over which we will integrate the point measure $M_X^{\otimes n}$. Indeed, we remove from Γ^n the set Δ_n, union of all diagonals:

$$\Delta_n := \{\gamma \in \Gamma^n :\ \exists i \neq j, \gamma_i = \gamma_j\}.$$

With these definitions, we can state the iterated formula satisfied under the ν-random walk.

Proposition 8. *Let $\mathbb{P}_\nu$ be a ν-random walk on G. Then, for any test function $F \in \mathcal{B}^+(\mathbb{D}([0,1],G)\times\Gamma^n)$,*

$$\mathbb{P}_\nu\Big(\int_{\Gamma^n\setminus\Delta_n} F(X,\gamma)\, M_X^{\otimes n}(d\gamma)\Big) = \mathbb{P}_\nu\Big(\int_{\Gamma^n} F(\Psi_\gamma X,\gamma)\ \rho^{\otimes n}(d\gamma)\Big). \tag{9}$$

Remark 10. *In general it is not true that $\int_{\Delta_n} F(X,\gamma) M_X^{\otimes n}(d\gamma) = 0$. Indeed, if γ is an atom of M_X then $\underbrace{(\gamma,...,\gamma)}_{n\,times}$ is an atom of $M_X^{\otimes n}$ which belongs to Δ_n.*

Proof. We first prove a preliminary result.

Lemma 11. *Define for $\gamma\in\Gamma$, $A_\gamma := \{\bar\gamma \in \Gamma^n : \bar\gamma_i = \gamma \text{ for some } 1\le i\le n\}$. Then*

$$M_{\Psi_\gamma X}^{\otimes n}(\cdot) = M_X^{\otimes n}(\cdot) + M_{\Psi_\gamma X}^{\otimes n}(A_\gamma\cap\cdot), \quad \mathbb{P}_\nu\otimes\rho \ a.e..$$

Proof. We denote by $\mathcal{E}(X)\subset\Gamma$ the set of atoms of M_X and by $\mathcal{E}^n(X)$ the set of atoms of $M_X^{\otimes n}$. Clearly $\mathcal{E}^n(X) = (\mathcal{E}(X))^n \subset \Gamma^n$. From (5),

$$\mathcal{E}(\Psi_\gamma X) = \mathcal{E}(X)\cup\{\gamma\} \quad \mathbb{P}_\nu\otimes\rho \ a.e.$$

so that the atoms of $M_{\Psi_\gamma X}^{\otimes n}$ are $\mathcal{E}^n(\Psi_\gamma X) = \Big(\mathcal{E}(X)\cup\{\gamma\}\Big)^n$ $\mathbb{P}_\nu\otimes\rho$ *a.e.*. In this case $\mathcal{E}^n(\Psi_\gamma X)\setminus\mathcal{E}^n(X) = \mathcal{E}^n(\Psi_\gamma X)\cap A_\gamma$, which leads to the conclusion. □

To prove the proposition by induction, we adopt the following notation: we decompose any element $\gamma = (\gamma_1,...,\gamma_{n+1})$ of Γ^{n+1} into $(\tilde\gamma,\gamma)$ where $\tilde\gamma$ is its projection on Γ^n, and $\gamma = \gamma_{n+1}$ is its last coordinate.
Proof of the statement for $n = 1$: It is exactly Proposition 6.
Let us now assume that the statement is true for n, that is (9) holds true for all test functions $f \in \mathcal{B}^+(\mathbb{D}([0,1],G)\times\Gamma^n)$. Let us now consider $v\in\mathcal{B}^+(\Gamma)$ and prove that (9) holds for any function of the form $F(X,\gamma) = f(X,\tilde\gamma)v(\gamma)$. The extension to a general $F\in\mathcal{B}^+(\mathbb{D}([0,1],G)\times\Gamma^{n+1})$ will be then standard. We have:

$$\mathbb{P}_\nu\Big(\int_{\Gamma^{n+1}} F(\Psi_\gamma X,\gamma)\rho^{\otimes n+1}(d\gamma)\Big) = \mathbb{P}_\nu\Big(\int_\Gamma\int_{\Gamma^n} f(\Psi_\gamma(\Psi_{\tilde\gamma}X),\tilde\gamma)\rho^{\otimes n}(d\tilde\gamma)v(\gamma)\ \rho(d\gamma)\Big)$$

Exchanging the order of integration, and applying the inductive hypothesis to $f(\Psi_\gamma X,\tilde\gamma)$ allows to rewrite the right hand side of the last identity as:

$$\int_\Gamma \mathbb{P}_\nu\Big(\int_{\Gamma^n\setminus\Delta_n} f(\Psi_\gamma X,\tilde\gamma) M_X^{\otimes n}(d\tilde\gamma)\ \Big)v(\gamma)\rho(d\gamma)$$

We can apply Lemma 11 to rewrite the former integral as:

$$\int_\Gamma \mathbb{P}_\nu\Big(\int_{(\Gamma^n\setminus\Delta_n)\setminus A_\gamma} f(\Psi_\gamma X,\tilde\gamma)\ M_{\Psi_\gamma X}^{\otimes n}(d\tilde\gamma)\ \Big)v(\gamma)\rho(d\gamma).$$

We apply Proposition 6 to $(X,\gamma)\mapsto\int_{(\Gamma^n\setminus\Delta_n)\setminus A_\gamma} f(X,\tilde\gamma)M_X^n(d\tilde\gamma)\,v(\gamma)$ and we obtain

$$\begin{aligned}&\mathbb{P}_\nu\Big(\int_\Gamma\Big(\int_{(\Gamma^n\setminus\Delta_n)\setminus A_\gamma} f(X,\tilde\gamma)M_X^{\otimes n}(d\tilde\gamma)\Big)v(\gamma)M_X(d\gamma)\Big)\\ &=\mathbb{P}_\nu\Big(\int_{\{\gamma:\gamma\in\Gamma,\tilde\gamma\in(\Gamma^n\setminus\Delta_n)\setminus A_\gamma\}} F(X,\gamma)M_X^{\otimes n+1}(d\gamma)\Big)\end{aligned}$$

It is easy to see that $\{\gamma:\gamma\in\Gamma,\tilde\gamma\in(\Gamma^n\setminus\Delta_n)\setminus A_\gamma\}=\Gamma^n\setminus\Delta_{n+1}$ and the conclusion follows. $\square$

2. The reciprocal class and its characterization

2.1. Bridges of the random walk and their mixtures

First, consider the set of pairs $(x,y)\in G^2$ for which the bridge of the ν-random walk is meaningful:

$$S(\nu):=\{(x,y)\in G^2:\mathbb{P}_\nu(X_1=y|X_0=x)>0\}.$$

Then, for $(x,y)\in S(\nu)$ the *bridge* $\mathbb{P}_\nu^{xy}$ between x and y is defined by

$$\mathbb{P}_\nu^{xy}(\cdot):=\frac{\mathbb{P}_\nu(\cdot\cap\{X_0=x,X_1=y\})}{\mathbb{P}_\nu(X_0=x,X_1=y)}.$$

We now define the reciprocal class associated with ν as the set of all possible mixtures of bridges of $\mathbb{P}_\nu$:

$$\mathcal{R}ec\ (\nu)=\Big\{\mathbb{Q}\in\mathcal{P}(\Omega):\mathbb{Q}=\int_{S(\nu)}\mathbb{P}_\nu^{xy}\,\mathbb{Q}_{01}(dxdy)\Big\},$$

where $\mathbb{Q}_{01}$ denotes the joint marginal law of $\mathbb{Q}$ at times 0 and 1. Let us note that for (x,y) fixed in $S(\nu)$, the bridge $\mathbb{P}_\nu^{xy}$ belongs to the reciprocal class $\mathcal{R}ec\ (\nu)$. For a recent review on reciprocal processes, reciprocal classes and the analysis of typical examples we refer the reader e.g. to [10] or [15].

2.2. Loops and their skeletons

We call *loop* a path in $\mathbb{D}([0,1],G)$ that starts and ends at the identity element e. We define, for any path $X\in\Omega$ its *skeleton* as the application $\varphi_X:G_\nu\to\mathbb{N}$ defined by:

$$\varphi_X(g):=M_X([0,1]\times\{g\}).$$

Thus $\varphi_X(g)$ counts how many times the jump g occurred along the path X.
If X is a loop, we observe that

$$\sum_{g\in G_\nu}\varphi_X(g)\,g=e.$$

Therefore, as X varies in the set of all possible loops, φ_X varies in the set

$$\mathscr{L}^+ := \{\varphi \in \mathbb{N}^{G_\nu} : \sum_{g\in G_\nu} \varphi(g)\, g = e,\, \ell(\varphi) < +\infty\} \tag{12}$$

where $\ell(\varphi) := \sum_{g\in G_\nu} |\varphi(g)|$ is the *length* of φ. Enlarging this set to the maps φ with negatives values by considering

$$\mathscr{L} := \{\varphi \in \mathbb{Z}^{G_\nu} : \sum_{g\in G_\nu} \varphi(g)\ g = e,\, \ell(\varphi) < +\infty\},$$

one recovers for $\mathscr{L}$ a lattice structure, which will be very useful. In particular $\mathscr{L}$ admits a *basis* $\mathscr{B}$. Suppose now that one can choose $\mathscr{B} \subset \mathscr{L}^+$, which is the case if the following assumption (H1) is satisfied:

$$Span(\mathscr{L}^+) = \mathscr{L} \tag{H1}$$

where $Span(\mathscr{L}^+)$ is, as usual, the set of all integer combinations of elements of $\mathscr{L}^+$. From now on, we fix such a basis $\mathscr{B}$.
To any $\varphi^* \in \mathscr{B}$ we can associate the - non empty - set of loops whose skeleton is φ^*:

$$\Omega_{e,\varphi^*} := \{X \in \Omega : X_0 = X_1 = e \text{ and } \varphi_X = \varphi^*\}.$$

These paths have exactly $\varphi^*(g)$ jumps of type g, for all $g \in G_\nu$. Furthermore, we have to assume that each jump in G_ν belongs to (at least) the skeleton of one loop, that is, the following assumption holds:

$$\forall g \in G_\nu \text{ there exists } \varphi \in \mathscr{L} \text{ such that } \varphi(g) > 0. \tag{H2}$$

Note that w.l.o.g. we can assume that this skeleton φ belongs indeed to the basis $\mathscr{B}$.
As we shall see in Section 2.4, assumptions (H1) and (H2) allow a fruitful decomposition of the path space Ω. Heuristically, one can transform one path into any other one having the same initial and final values, by subsequently adding and removing loops whose skeleton belongs to $\mathscr{B}$. However, let us first state our main result.

2.3. Main result: an integral characterization of the reciprocal class

In the next theorem we state that the identity (9) proved in Proposition 8 is not only valid over the whole reciprocal class $\mathcal{R}ec\ (\nu)$ but indeed characterizes it, if one restricts the set of test functions F to some well chosen subset.
For each skeleton φ^* in the basis $\mathscr{B}$, consider the following set of test functions:

$$\mathscr{H}_{\varphi^*} := \left\{ F \in \mathcal{B}^+(\mathbb{D}([0,1], G) \times \Gamma^{\ell(\varphi^*)}) : F(X,\gamma) \equiv \mathbb{1}_{\{\chi^\gamma \in \Omega_{e,\varphi^*}\}} F(X,\gamma) \right\}.$$

Therefore, we will restrict our attention to perturbations of the sample paths consisting in adding a loop χ^γ whose skeleton is equal to φ^*. Now we are ready for stating and proving the main result.

Theorem 13. *The probability measure $\mathbb{Q} \in \mathcal{P}(\Omega)$ belongs to the reciprocal class $\mathcal{R}ec\,(\nu)$ if and only if for any skeleton φ^* in the basis $\mathscr{B}$ and for all test functions $F \in \mathscr{H}_{\varphi^*}$, we have:*

$$\mathbb{Q}\Big(\int_{\Gamma^n\setminus\Delta_n} F(X,\gamma)M_X^{\otimes n}(d\gamma)\Big) = \Phi^\nu_{\varphi^*}\,\mathbb{Q}\Big(\int_{\Gamma^n} F(\Psi_\gamma X,\gamma)\,(dtd\Lambda)^{\otimes n}\Big), \tag{14}$$

where $n = \ell(\varphi^*)$, $\Lambda := \sum_{g\in G}\delta_g$ is the counting measure on G and

$$\Phi^\nu_{\varphi^*} := \prod_{g\in G_\nu} \nu(g)^{\varphi^*(g)}. \tag{15}$$

In particular, if (14) holds true under $\mathbb{Q}$ satisfying $\mathbb{Q}(X_0 = x, X_1 = y) = 1$ for some $(x,y) \in S(\nu)$, then $\mathbb{Q}$ is nothing else but the bridge $\mathbb{P}^{xy}_\nu$.

The positive number $\Phi^\nu_{\varphi^*}$ is called the **reciprocal characteristics** associated to the jump measure ν and the skeleton φ^*.

Corollary 16. *The reciprocal characteristics are invariants of the reciprocal class in the following sense. Let ν and μ two non negative finite measures on G with the same support. The reciprocal classes $\mathcal{R}ec\,(\nu)$ and $\mathcal{R}ec\,(\mu)$ coincide if and only if their family of reciprocal characteristics coincide:*

$$\Phi^\mu_{\varphi^*} = \Phi^\nu_{\varphi^*}, \quad \forall\varphi^* \in \mathscr{B}. \tag{17}$$

In that case the bridges of both ν- and μ-random walk on G coincide too.

Remark 18. *There is a remarkable probabilistic interpretation of the reciprocal characteristics $\Phi^\nu_{\varphi^*}$ as the leading factor, in the short-time expansion, of the probability that the ν-random walk follows a loop with skeleton φ^*. This is proven for Markov processes on graphs in the forthcoming paper [4].*

2.4. Proof of the main theorem

Proof. $(\Rightarrow)$
We use, as main argument, the specific form of the density with respect to $\mathbb{P}_\nu$ of any probability measure in its reciprocal class, as it was proved in Proposition 1.5 in [3]: If $\mathbb{Q} \in \mathcal{R}ec\,(\nu)$ then

$$\mathbb{Q} << \mathbb{P}_\nu, \quad \textit{and } \frac{d\mathbb{Q}}{d\mathbb{P}_\nu} = h(X_0,X_1) \text{ for some } h : G\times G \to \mathbb{R}^+.$$

Take now any $F \in \mathscr{H}_{\varphi^*}$. Then, using Identity (9), the definition of $\mathscr{H}_{\varphi^*}$ and the fact that $(\Psi_\gamma X)_0 = X_0$, $(\Psi_\gamma X)_1 = X_1 + e = X_1$, one gets

$$\begin{aligned}
\mathbb{Q}\Big(\int_{\Gamma^n} F(\Psi_\gamma X,\gamma)(dtd\nu)^{\otimes n}\Big) &= \mathbb{Q}\Big(\int_{\Gamma^n} F(\Psi_\gamma X,\gamma)\prod_{g\in G_\nu}\nu(g)^{\varphi_{X^\gamma}(g)}(dtd\Lambda)^{\otimes n}\Big)\\
&\overset{F\in\mathscr{H}_{\varphi^*}}{=} \Phi^\nu_{\varphi^*}\mathbb{Q}\Big(\int_{\Gamma^n} F(\Psi_\gamma X,\gamma)(dtd\Lambda)^{\otimes n}\Big)\\
&= \Phi^\nu_{\varphi^*}\mathbb{P}_\nu\Big(h(X_0,X_1)\int_{\Gamma^n} F(\Psi_\gamma X,\gamma)(dtd\Lambda)^{\otimes n}\Big)\\
&= \Phi^\nu_{\varphi^*}\mathbb{P}_\nu\Big(\int_{\Gamma^n} h((\Psi_\gamma X)_0,(\Psi_\gamma X)_1)F(\Psi_\gamma X,\gamma)(dtd\Lambda)^{\otimes n}\Big)\\
&= \mathbb{P}_\nu\Big(\int_{\Gamma^n\setminus\Delta_n} h(X_0,X_1)F(X,\gamma)M_X^{\otimes n}(d\gamma)\Big)\\
&= \mathbb{Q}\Big(\int_{\Gamma^n\setminus\Delta_n} F(X,\gamma)M_X^{\otimes n}(d\gamma)\Big)
\end{aligned}$$

which completes the proof of the first implication.

($\Leftarrow$)
The converse implication is more sophisticated and needs several steps.
Let us introduce the set of paths which correspond to the support of ey-bridges, $y \in G$:

$$\Omega_y := \{X \in \Omega : X_0 = e, X_1 = y\}.$$

Now we decompose Ω_y according to the skeleton of its elements:

$$\Omega_y = \bigcup_{\varphi\in\mathscr{L}_y^+}\Omega_{y,\varphi}, \quad \Omega_{y,\varphi} := \Omega_y \cap \{X \in \Omega : \varphi_X = \varphi\}$$

$$\text{where } \mathscr{L}_y^+ = \{\varphi \in \mathbb{N}^{G_\nu} : \sum_{g\in G_\nu} g\varphi(g) = y, \ell(\varphi) < +\infty\},$$

and thus obtain a partition.
In order to discretize the time, we introduce a mesh $h \in \mathbb{N}^*$ and partition once more $\Omega_{y,\varphi}$ by specifying the number of jumps of each type occurred in each h-dyadic interval. That is, we consider functions $\theta : \{0,..,2^h-1\}\times G_\nu \longrightarrow \mathbb{N}$ and we look for paths which have $\theta(k,g)$ jumps of type g during the time interval $I_k^h := (2^{-h}k, 2^{-h}(k+1)]$, for each k and each $g \in G_\nu$. We define therefore, for each skeleton φ, the set

$$\Theta^h_\varphi := \Big\{\theta : \{0,..,2^h-1\}\times G_\nu \longrightarrow \mathbb{N}, \sum_{0\le k\le 2^h-1}\theta(k,g) = \varphi(g), \quad \forall g \in G_\nu\Big\}$$

of all possible h-dyadic time repartition of the jumps, compatible with the skeleton φ. We thus obtain $\Omega_{y,\varphi} = \bigcup_{\theta\in\Theta^h_\varphi}\Omega^{h,\theta}_{y,\varphi}$ where

$$\Omega^{h,\theta}_{y,\varphi} := \{X \in \Omega_y : M_X(I_k^h \times \{g\}) = \theta(k,g),\ 0 \le k < 2^h,\ g \in G_\nu\}.$$

Consider the set

$$V := \Big\{ v = (\varphi, \theta) \ \text{ such that } \ \varphi \in \mathscr{L}_y^+, \theta \in \Theta_\varphi^h \Big\} \tag{19}$$

of pairs of skeletons connecting e to y and h-dyadic time repartition of their jumps. Elements of this set are discrete versions of paths of Ω: the spatial structure of the path is given by the skeleton φ, and the time structure is approximated by θ. One equips V with the following l^1-metric:

$$d(v, \tilde{v}) := \sum_{(k,g) \in \{0,\cdots,2^h-1\} \times G_\nu} |\theta - \tilde{\theta}|(k,g) \in \mathbb{N}, \quad v = (\varphi, \theta), \tilde{v} = (\tilde{\varphi}, \tilde{\theta}) \in V.$$

Take now two paths $X, X' \in \Omega_y$ and their trace v, v' on V. Our aim is to find a way to transform X into X' (resp. v into v') by adding or canceling a finite number of loops whose skeletons belong to the basis $\mathscr{B}$. Let us introduce the following relation:

$$v_1 = (\varphi_1, \theta_1) \hookrightarrow v_2 = (\varphi_2, \theta_2) \text{ if } \varphi_2 \in \varphi_1 + \mathscr{B} \ \text{ and } \ \theta_2 - \theta_1 \in \Theta^h_{\varphi_2 - \varphi_1}.$$

We shall now use assumptions (H1) and (H2).

Lemma 20. *For each v and $\tilde{v} \neq v \in V$ on can construct a connecting finite sequence $v_1, \cdots, v_N = \tilde{v}_{\tilde{N}}, \tilde{v}_{\tilde{N}-1}, \cdots, \tilde{v}_1$ in V such that*

$$v \hookrightarrow v_1 \hookrightarrow \cdots \hookrightarrow v_N = \tilde{v}_{\tilde{N}} \hookleftarrow \tilde{v}_{\tilde{N}-1} \cdots \hookleftarrow \tilde{v}_1 \hookleftarrow \tilde{v}.$$

Proof. We distinguish two cases:

Case i) The skeletons φ and $\tilde{\varphi}$ coincide.
In this case, it is sufficient to show that we can construct v_1 and $\tilde{v}_1$ in V such that $v \hookrightarrow v_1$, $\tilde{v} \hookrightarrow \tilde{v}_1$, $\varphi_1 = \tilde{\varphi}_1$ and $d(v_1, \tilde{v}_1) \leq d(v, \tilde{v}) - 1$. The conclusion would then follow by iterating this procedure until $d(v_K, \tilde{v}_K) = 0$, i.e. $v_K = \tilde{v}_K$. At this point, we have constructed a chain from v to v_K, and another one from $\tilde{v}$ to $\tilde{v}_K$. Joining them, we obtain a chain from v to $\tilde{v}$ and the conclusion follows.
Therefore, let us indicate how to construct v_1 and $\tilde{v}_1$. Since $\theta \neq \tilde{\theta}$ but $\varphi = \tilde{\varphi}$ there exists a jump $g \in G_\nu$ and two time intervals I^h_k and I^h_l such that $\theta(k,g) \geq \tilde{\theta}(k,g) + 1$ and $\theta(l,g) \leq \tilde{\theta}(l,g) - 1$. Moreover, thanks to (H2) there exists at least one skeleton φ^* in the basis $\mathscr{B}$ containing the jump g: $\varphi^*(g) > 0$. Consider now any time repartition $\theta_1 \in \Theta^h_{\varphi^*}$ such that $\theta_1(l,g) \geq 1$. We then construct $\tilde{\theta}_1$ as follows:

$$\tilde{\theta}_1 = \theta_1 + \mathbb{1}_{\{(k,g)\}} - \mathbb{1}_{\{(l,g)\}}.$$

It is simple to check that $v_1 := (\varphi + \varphi^*, \theta + \theta_1)$, $\tilde{v}_1 := (\tilde{\varphi} + \varphi^*, \tilde{\theta} + \tilde{\theta}_1)$ fulfil the desired requirements. By construction, $v \hookrightarrow v_1$, $\tilde{v} \hookrightarrow \tilde{v}_1$ and $v_1, \tilde{v}_1$ have the same skeleton. Moreover

$$|\theta + \theta_1 - (\tilde{\theta} + \tilde{\theta}_1)| = |\theta - \tilde{\theta}| - \mathbb{1}_{\{(k,g),(l,g)\}}$$

so that $d(v_1, \tilde{v}_1) = d(v, \tilde{v}) - 2$.

Case ii) The skeletons φ and $\tilde{\varphi}$ differ.
We first observe that, if $\varphi, \tilde{\varphi} \in \mathscr{L}^+_y$ thus $\varphi - \tilde{\varphi} \in \mathscr{L}$. Since $\mathscr{B}$ is a basis of the lattice $\mathscr{L}$ (see (H1)), there exist $(\varphi^*_j)_{j=1}^K, (\tilde{\varphi}^*_i)_{i=1}^{\tilde{K}} \subseteq \mathscr{B}$ such that

$$\varphi + \sum_{j=1}^{K} \varphi^*_j = \tilde{\varphi} + \sum_{i=1}^{\tilde{K}} \tilde{\varphi}^*_i.$$

Let us now choose for all j and i a time repartition $\theta_j \in \Theta^h_{\tilde{\varphi}^*_j}$ and $\tilde{\theta}_i \in \Theta^h_{\tilde{\varphi}_i}$. It is straightforward to control that, if we define

$$v_0 = v,\ v_j := (\varphi + \sum_{j'=1}^{j} \varphi^*_{j'}, \theta + \sum_{j'=1}^{j} \theta_{j'}),\ \tilde{v}_0 = \tilde{v}, \tilde{v}_i := (\tilde{\varphi} + \sum_{i'=1}^{i} \tilde{\varphi}^*_{i'}, \tilde{\theta}^* + \sum_{i'=1}^{i} \tilde{\theta}_{i'})$$

then $(v_j)_{j=0}^K$,$(\tilde{v}_i)_{i=0}^{\tilde{K}}$ are two sequences connecting v to v_K and $\tilde{v}$ to $\tilde{v}_{\tilde{K}}$. By construction $v_K, \tilde{v}_{\tilde{K}}$ have the same skeleton and one can use case i) again.

□

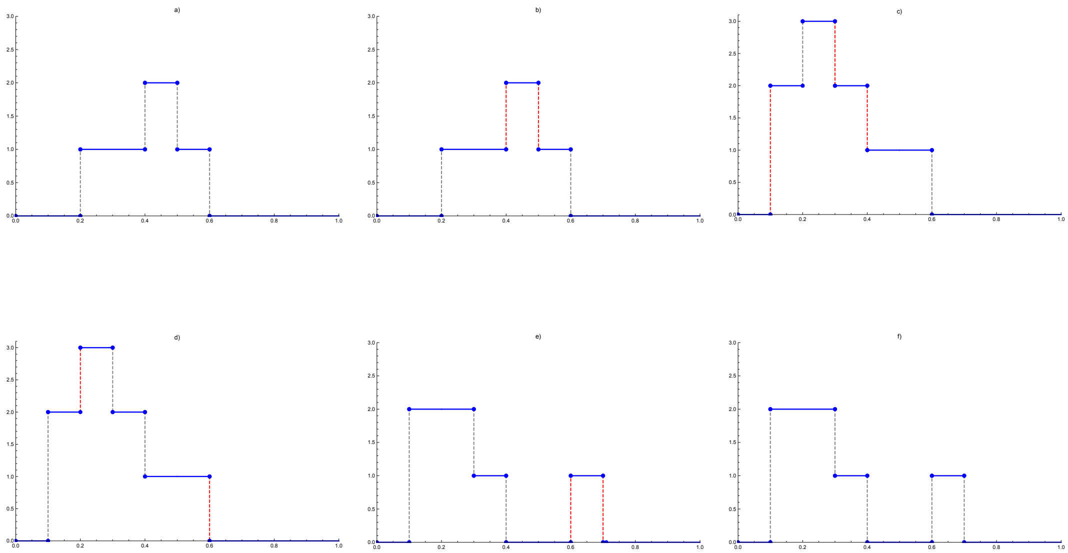

Figure 1. In this picture we illustrate by an example the proof of Lemma 20. Take $G = (\mathbb{Z}, +)$, and $G_\nu = \{-1, 1, 2\}$, situation which is treated in Section 3.1.1. It is easy to see that a basis fulfilling H1) and H2) is $\mathscr{B} = \{\varphi_1, \varphi_2\}$, with $\varphi_1 = \mathbb{1}_1 + \mathbb{1}_{-1}$, $\varphi_2 = \mathbb{1}_2 + 2\,\mathbb{1}_{-1}$. We show in the picture how to transform the path in a) in the path in f) by mean of addition and cancellation of loops whose skeleton belongs to $\mathscr{B}$. All loops that are either added or removed are denoted by red dashed lines, which correspond to their jumps. At first, following Case ii), we have to modify the loop a) to make its skeleton match with that of the one in f). Therefore in b) we remove a loop with skeleton φ_1, then in c) add back a loop with skeleton φ_2. The skeleton is now the desired one. Now we follow Case i): We need to shift one jump of height -1 and one of height 1 further right. Since those two jumps form a loop with skeleton φ_1 we simply delete them in d) and add a new loop with the same skeleton but now with the desired jump times in e).

In the next lemma we compare the probability of the paths in $\Omega^{h,\theta}_{y,\varphi}$ and those obtained by adding a loop with skeleton $\varphi^* \in \mathscr{B}$, under $\mathbb{Q}$ and under $\mathbb{P}_\nu$.

Lemma 21. *Let $y \in G, h \in \mathbb{N}^*, \varphi \in \mathscr{L}^+_y, \theta \in \Theta^h_\varphi$ be fixed. Suppose (14) holds under $\mathbb{Q}$. Then, for any $\varphi^* \in \mathscr{B}$ and $\theta^* \in \Theta^h_{\varphi^*}$,*

$$\frac{\mathbb{Q}(\Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*})}{\mathbb{P}_\nu(\Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*})} = \frac{\mathbb{Q}(\Omega^{h,\theta}_{y,\varphi})}{\mathbb{P}_\nu(\Omega^{h,\theta}_{y,\varphi})}. \tag{22}$$

Proof. Take an arbitrary ordering of the support of θ^*: $(k_1, g_1), ..., (k_N, g_N)$. To simplify the notation, we write θ_j (resp. θ^*_j) for $\theta(k_j, g_j)$ (resp $\theta^*(k_j, g_j)$). Consider the test function $F \in \mathscr{H}_{\varphi^*}$ of the form $F(X,\gamma) = f(X)\; v(\gamma)$, where

$$f = \mathbb{1}_{\Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*}}, \text{ and } v(\gamma) = \mathbb{1}_{\Omega^{h,\theta^*}_{e,\varphi^*}}(\chi^\gamma).$$

It is straightforward that

$$f \circ \Psi_\gamma(X)\; v(\gamma) = \mathbb{1}_{\Omega^{h,\theta}_{y,\varphi}}(X)\; v(\gamma) \quad \mathbb{Q} \otimes \rho^n \; a.e..$$

Therefore, since $F \in \mathscr{H}_{\varphi^*}$, (14) holds and its right hand side rewrites as

$$\Phi^\nu_{\varphi^*} \; \Big(\int_{\Gamma^n} v(\gamma) d\gamma \Big) \; \mathbb{Q}(\Omega^{h,\theta}_{y,\varphi}).$$

Concerning the left hand side, let us first rewrite it as

$$\mathbb{Q}\Big(f(X) \int_{\Gamma^n \setminus \Delta_n} v(\gamma) dM^{\otimes n}_X(d\gamma) \Big).$$

Our aim is to show that the (discrete) stochastic integral $\int_{\Gamma^n \setminus \Delta_n} v(\gamma) dM^{\otimes n}_X(d\gamma)$ is actually constant for that choice of v if $X \in \Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*}$ by computing it explicitly.
First, we observe that an atom $\gamma \in \Gamma^n \setminus \Delta_n$ of $M^{\otimes n}_X$ contributes (with the value 1) to the integral if and only if $\chi^\gamma \in \Omega^{h,\theta^*}_{e,\varphi^*}$, that is

$$\sharp\{i : \gamma_i \in I^h_{k_j} \times \{g_j\}\} = \theta^*_j, \quad 1 \le j \le N. \tag{23}$$

We then need to count the atoms of $M^{\otimes n}_X$ satisfying (23). This is equivalent to count all ordered list of $n = \ell(\varphi^*)$ atoms of M_X verifying that
1) the list contains no repetitions
2) for all $1 \le j \le N$, the number of elements in the list which belong to $I^h_{k_j} \times \{g_j\}$ is θ^*_j.
Therefore, we first choose, for each j, a subset of cardinality θ^*_j among $\theta_j + \theta^*_j$ elements (recall that $X \in \Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*}$). To do that we have $\binom{\theta_j+\theta^*_j}{\theta^*_j}$ choices. Then we should decide how to sort the list, and for this, there are $n!$ possibilities.
Therefore

$$\mathbb{1}_{\Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*}}(X) \int_{\Gamma^n \setminus \Delta_n} v(\gamma) d\gamma = \mathbb{1}_{\Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*}}(X)\, n! \; \prod_{j=1}^N \binom{\theta_j + \theta^*_j}{\theta^*_j}$$

and (14) rewrites as

$$\Phi^\nu_{\varphi^*} \int_{\Gamma^n} v(\gamma) d\gamma \; \mathbb{Q}(\Omega_{y,\varphi}) = n! \prod_{j=1}^N \binom{\theta_j + \theta^*_j}{\theta^*_j} \mathbb{Q}(\Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*}). \tag{24}$$

Since equation (14) holds under $\mathbb{P}_\nu$, equation (24) holds under $\mathbb{P}_\nu$ as well. Since $\mathbb{P}_\nu$ gives positive probability to both events $\Omega_{y,\varphi}$ and $\Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*}$, the identity (22) follows. $\square$

Remark that, with the notation of the above lemma, if we define $v := (\varphi, \theta)$ and $w := (\varphi + \varphi^*, \theta + \theta^*)$, then $v \hookrightarrow w$.

Lemma 20 allows us to extend the conclusion of Lemma 21 to the full set of skeletons, as we will prove now.

Lemma 25. *Let $y \in G, h \in \mathbb{N}^*, \varphi, \tilde{\varphi} \in \mathscr{L}_y^+, \theta \in \Theta^h_\varphi, \tilde{\theta} \in \Theta^h_{\tilde{\varphi}}$ be fixed. Suppose (14) holds under $\mathbb{Q}$. Then,*

$$\frac{\mathbb{Q}(\Omega^{h,\tilde{\theta}}_{y,\tilde{\varphi}})}{\mathbb{P}_\nu(\Omega^{h,\tilde{\theta}}_{y,\tilde{\varphi}})} = \frac{\mathbb{Q}(\Omega^{h,\theta}_{y,\varphi})}{\mathbb{P}_\nu(\Omega^{h,\theta}_{y,\varphi})}. \tag{26}$$

Proof. We observe that $v = (\varphi, \theta)$ and $\tilde{v} = (\tilde{\varphi}, \tilde{\theta})$ are elements of V. As proved above, there exist a connecting sequence $(v_i)_{i=0}^K := (\varphi_i, \theta_i)_{i=0}^K$, with $v_0 = v, v_K = \tilde{v}$, linking v to $\tilde{v}$, and such that either $v_i \hookleftarrow v_{i+1}$ or $v_i \hookrightarrow v_{i+1}$. This entitles us to apply recursively Lemma 21 to any pair v_i, v_{i+1} and obtain

$$\frac{\mathbb{Q}(\Omega^{h,\theta_{i+1}}_{y,\varphi_{i+1}})}{\mathbb{P}_\nu(\Omega^{h,\theta_{i+1}}_{y,\varphi_{i+1}})} = \frac{\mathbb{Q}(\Omega^{h,\theta_i}_{y,\varphi_i})}{\mathbb{P}_\nu(\Omega^{h,\theta_i}_{y,\varphi_i})} = ... = \frac{\mathbb{Q}(\Omega^{h,\theta}_{y,\varphi})}{\mathbb{P}_\nu(\Omega^{h,\theta}_{y,\varphi})}.$$

The conclusion follows with $i = N - 1$. □

We can now complete the proof of the converse implication of the main theorem.

Fix, $x, y \in G$ with $\mathbb{Q}(X_0 = x, X_1 = y) > 0$. W.l.o.g. we assume that $x = e$, the general case following with minor modifications. Thanks to Lemma 25 we know that for any mesh h, there exists a positive constant c_h such that

$$\mathbb{Q}(\Omega^{h,\theta}_{y,\varphi}) = c_h \mathbb{P}_\nu(\Omega^{h,\theta}_{y,\varphi}), \quad \forall \varphi \in \mathscr{L}_y^+, \theta \in \Theta^h_\varphi.$$

Now we shall show that the proportionality constant does not depend on the scale of the time discretisation: $c_h = c_{h+1}$. To this aim, let us observe that:

$$\mathbb{Q}(\Omega_y) = \sum_{(\varphi,\theta)\in V} \mathbb{Q}(\Omega^{h,\theta}_{y,\varphi}) = \sum_{(\varphi,\theta)\in V} c_h \mathbb{P}_\nu(\Omega^{h,\theta}_{y,\varphi}) = c_h \, \mathbb{P}_\nu(\Omega_y).$$

In the same way one gets $\mathbb{Q}(\Omega_y) = c_{h+1}\mathbb{P}_\nu(\Omega_y)$ which implies that $c_h = c_{h+1}$. Therefore, there exists a constant $c > 0$ such that

$$\mathbb{Q}(\Omega^{h,\theta}_{y,\varphi}) = c \, \mathbb{P}_\nu(\Omega^{h,\theta}_{y,\varphi}), \quad \forall h \in \mathbb{N}^*, \varphi \in \mathscr{L}_y^+, \theta \in \Theta^h_\varphi.$$

By standard approximation arguments this implies the equality between $\mathbb{Q}$ and $c \, \mathbb{P}_\nu$ on $\Omega_y \cap \mathcal{F}$ which then implies $\mathbb{Q}^{ey} = \mathbb{P}^{ey}_\nu$. The conclusion follows. □

Remark 27. *Consider the identities (14) for $G = \mathbb{R}^d$ and compute them for particular test functions F which only depend on the skeleton of the paths. These equations, indexed* *by the skeletons in $\mathscr{B}$, then characterize the (marginal) distribution of the random vector defined as the number of jumps of any type occurred during the time interval $[0,1]$, as it was done in [3]. Note that for the unconstrained random walk the distribution of this random vector is a multivariate Poisson law, see e.g. [3] Section 2.2.1.*

3. Examples

In this section, we present several examples of finite and infinite Abelian groups G, on which various random walks are defined.
For each example, we control if assumptions (H1) and (H2) are satisfied by computing a basis $\mathscr{B}$ of skeleton of loops. We also give explicitely the associated reciprocal characteristics (15). In some cases, we also write down the integral formula (14), highlighting how it is influenced by the geometrical properties of the underlying group G.
Finally, for a fixed random walk $\mathbb{P}_\nu$ on G, we address the question of finding all random walks $\mathbb{P}_\mu$ which have the same bridges than $\mathbb{P}_\nu$, that is, using Corollary 16, we solve equation (17) and identify the set of probability measures:

$$\mathcal{R}ec\ (\nu) \cap \{\mathbb{P}_\mu : \mu \text{ finite measure on } G_\nu\}.$$

We will see that, in some cases, this set reduces to the singleton $\mathbb{P}_\nu$ and in other cases, this set is non trivial.

3.1. The group $G = \mathbb{Z}$ is infinite

3.1.1. The finite support G_ν of the jump measure contains $\{-1,1\}$.

For any $z \in G_\nu \setminus \{1\}$ we define on G_ν the non negative map φ_z as follows:

$$\varphi_z = \mathbb{1}_z + |z|\, \mathbb{1}_{-sgn(z)}.$$

It corresponds to the skeleton of paths with one jump of type z and $|z|$ jumps of type $-sgn(z)$. As candidate for the lattice basis of $\mathscr{L}$, we propose

$$\mathscr{B} := \{\varphi_z\}_{z \in G_\nu \setminus \{1\}}.$$

Assumption (H2) is trivially satisfied and it is clear that the elements of $\mathscr{B}$ are linearly independent. Therefore we only need to check if $\mathscr{B}$ spans $\mathscr{L}$, that is, if for each $\phi \in \mathscr{L}$, there exist integer coefficients $\alpha_z \in \mathbb{Z}, z \in G_\nu \setminus \{1\}$, such that

$$\forall \bar{z} \in G_\nu, \quad \phi(\bar{z}) = \sum_{\substack{z \in G_\nu \\ z \neq 1}} \alpha_z \varphi_z(\bar{z}). \tag{28}$$

We now verify that the following choice is the right one:

$$\alpha_z = \phi(z), \text{ if } z \in G_\nu \setminus \{-1,+1\}, \text{ and } \alpha_{-1} = \phi(-1) - \sum_{\substack{z\in G_\nu \\ z>1}} z\phi(z).$$

- $\bar{z} \notin \{-1,+1\}$. Since $\varphi_{\bar{z}}$ is the only element of $\mathscr{B}$ whose support contains $\bar{z}$, we have

$$\phi(\bar{z}) = \alpha_{\bar{z}}\, \varphi_{\bar{z}}(\bar{z}) = \sum_{\substack{z\in G_\nu \\ z\neq 1}} \alpha_z\, \varphi_z(\bar{z})$$

- $\bar{z} = -1$. Notice that -1 belongs to the support of any φ_z, as soon as $z > 1$. Therefore

$$\phi(-1) = \sum_{\substack{z\in G_\nu \\ z>1}} \phi(z)z + \alpha_{-1} = \sum_{\substack{z\in G_\nu \\ z>1}} \alpha_z\varphi_z(-1) + \alpha_{-1}\ \varphi_{-1}(-1) = \Big(\sum_{\substack{z\in G_\nu \\ z\neq 1}} \alpha_z\varphi_z\Big)(-1).$$

- $\bar{z} = 1$. Notice that $+1$ belongs to the support of any φ_z, as soon as $z \leq -1$. Recall that $\phi \in \mathscr{L}$. Therefore

$$\begin{aligned}\phi(1) = -\sum_{\substack{z\in G_\nu \\ z\neq 1}} \phi(z)z &= \sum_{\substack{z\in G_\nu \\ z<1}} -\phi(z)z + \phi(-1) \\ &= \sum_{\substack{z\in G_\nu \\ z\leq -1}} \alpha_z\, \varphi_z(1) = \Big(\sum_{\substack{z\in G_\nu \\ z\neq 1}} \alpha_z\varphi_z\Big)(1).\end{aligned}$$

Let us now compute the reciprocal characteristics associated to each skeleton in $\mathscr{B}$:

$$\Phi^\nu_{\varphi_z} = \nu(-sgn(z))^{|z|}\ \nu(z), \quad z \in G_\nu \setminus \{1\}.$$

Finally, thanks to Corollary 16, we obtain

$$\begin{aligned}\mathbb{P}_\mu \in \mathcal{R}ec\ (\nu) \quad &\Leftrightarrow \quad \forall z \in G_\nu \setminus \{1\}, \quad \mu(-sgn(z))^{|z|}\ \mu(z) = \nu(-sgn(z))^{|z|}\ \nu(z) \\ &\Leftrightarrow \quad \exists\ \alpha > 0 \text{ such that } \frac{d\mu}{d\nu}(z) = \alpha^z.\end{aligned}$$

Example 29. Simple random walks: $G_\nu = \{-1, 1\}$.
Due to the above computations, the basis $\mathscr{B}$ of the lattice $\mathscr{L}$ reduces to the singleton $\{\varphi_{-1}\}$ and the unique reciprocal characteristics is given by

$$\Phi^\nu_{\varphi_{-1}} = \nu(-1)\nu(1).$$

Therefore the only loops we need in the integral characterization (14) have length $n = \ell(\varphi_{-1}) = 2$.
Test functions of the form

$$F(X, (\gamma_1, \gamma_2)) = f(X)\mathbb{1}_{\{g_1=1, g_2=-1\}} h(t_1, t_2)$$

belong to $\mathscr{H}_{\varphi_{-1}}$. Such functions are supported on the pairs (γ_1, γ_2) such that Ψ_{γ_1,γ_2} is a transformation adding to the paths a jump +1 at time t_1 and a jump -1 at time t_2. The identity (14) now reads as:

$$\mathbb{Q}\Big(f(X) \sum_{\substack{(t_1,t_2):\Delta X_{t_1}=1 \\ \Delta X_{t_2}=-1}} h(t_1,t_2)\Big) = \nu(-1)\nu(1) \int_{[0,1]^2} \mathbb{Q}\big(f(\Psi_{\gamma_1,\gamma_2}X)\big) h(t_1,t_2) dt_1 dt_2.$$

In particular, if we consider, as in Remark 27, test functions f which only depend on the skeletons of the paths, $f(X) = v(\varphi_X)$, we obtain that the distribution $\chi(dn_{-1}, dn_1) \in \mathcal{P}(\mathbb{N}^2)$ of the number n_{-1} (resp. n_1) of negative (resp. positive) jumps is characterized by the system of equations: For all $v \in \mathcal{B}^+(\mathbb{N}^2)$,

$$\begin{aligned} \int v(n_{-1}, n_1)\ n_{-1}\ n_1\, \chi(dn_{-1}, dn_1) &= \nu(-1)\nu(1) \int v(n_{-1}+1, n_1+1)\chi(dn_{-1}, dn_1), \\ \chi(n_1 = n_{-1}) &= 1. \end{aligned}$$

This result coincides with [3], Example 2.18.

3.1.2. $G_\nu = \{1, 2\}$.

In that case, since -1 does not belong to the support of the jump measure, it leads to a counterexample where Assumption (H2) is not satisfied. It is straightforward to prove that the lattice $\mathscr{L}$ is one-dimensional and equal to $\{z\varphi^*, z \in \mathbb{Z}\}$ where

$$\varphi^*(1) = 2, \quad \varphi^*(2) = -1.$$

Clearly $\mathscr{L}$ does not admit a non negative basis.

3.2. G is the cyclic group $\mathbb{Z}/N\mathbb{Z}$

We now consider the finite cyclic group $G := \mathbb{Z}/N\mathbb{Z} =: \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \cdots, \mathbf{N-1}\}$.

3.2.1. The support G_ν of the jump measure reduces to $\{-\mathbf{1}, \mathbf{1}\}$.

This case corresponds to nearest neighbour random walks. The following choice for a non negative basis $\mathscr{B}$ is suitable:

$$\mathscr{B} = \{\varphi_{N-1}, \varphi^*\}$$

where

$$\varphi_{N-1} = \mathbb{1}_{\mathbf{1}} + \mathbb{1}_{\mathbf{N-1}} = \mathbb{1}_{\mathbf{1}} + \mathbb{1}_{-\mathbf{1}}, \quad \varphi^* = N\mathbb{1}_{\mathbf{1}}.$$

The associated reciprocal characteristics are

$$\Phi^\nu_{\varphi_{N-1}} = \nu(\mathbf{1})\nu(-\mathbf{1}), \quad \Phi^\nu_{\varphi^*} = \nu(\mathbf{1})^N.$$

The existence of the second invariant $\Phi^\nu_{\varphi^*}$ corresponding to the loop around the cycle $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \cdots, \mathbf{N-1}\}$ implies that $\mathbb{P}_\nu$ is the unique nearest neighbour random walk of its reciprocal class $\mathcal{R}ec\ (\nu)$. This differs from the nearest neighbour random walk on $\mathbb{Z}$, treated in Example 29: We proved there that any random walk $\mathbb{P}_\mu$, with μ satisfying $\mu(-1)\mu(1) = \nu(-1)\nu(1)$, induces the same reciprocal class.

The distribution χ of the random vector $(n_{-\mathbf{1}}, n_{\mathbf{1}})$ under the $\mathbf{00}$-bridge is given by the following system of integral equations, satisfied for any test function v on $\mathbb{N}^2$:

$$\begin{aligned}
\int v(n_{-\mathbf{1}}, n_{\mathbf{1}}) n_{-\mathbf{1}}\ n_{\mathbf{1}}\, \chi(dn_{-\mathbf{1}}, dn_{\mathbf{1}}) &= \nu(-\mathbf{1})\nu(\mathbf{1}) \int v(n_{-\mathbf{1}}+1, n_{\mathbf{1}}+1)\chi(dn_{-\mathbf{1}}, dn_{\mathbf{1}}),\\
\int v(n_{-\mathbf{1}}, n_{\mathbf{1}}) n_{\mathbf{1}} \cdots (\ n_{\mathbf{1}} - (N-1))\, \chi(dn_{-\mathbf{1}}, dn_{\mathbf{1}}) &= \nu(\mathbf{1})^N \int v(n_{-\mathbf{1}}, n_{\mathbf{1}}+N)\chi(dn_{-\mathbf{1}}, dn_{\mathbf{1}}),\\
\chi(n_{\mathbf{1}} - n_{-\mathbf{1}} \in N\mathbb{Z}) &= 1.
\end{aligned}$$

3.2.2. The support G_ν of the jump measure covers $\mathbb{Z}/N\mathbb{Z}$.

We now consider a random walk on $\mathbb{Z}/N\mathbb{Z}$ which can jump anywhere: $G_\nu = \mathbb{Z}/N\mathbb{Z} \setminus \{\mathbf{0}\}$. Here, we focus for simplicity on the case $N = 4$, which is the first non trivial example, and desintegrate the jump measure ν as follows:

$$\nu = \nu(\mathbf{1})\ \delta_{\mathbf{1}} + \nu(\mathbf{2})\ \delta_{\mathbf{2}} + \nu(\mathbf{3})\ \delta_{\mathbf{3}},\ \nu(i) > 0.$$

It can be proven along the same lines as in the previous examples, that a suitable non negative basis for the lattice $\mathscr{L}$ is given by $\mathscr{B} = \{\varphi^*, \eta^*, \xi^*\}$ where

$$\varphi^* = \mathbb{1}_{\mathbf{1}} + \mathbb{1}_{\mathbf{3}}, \quad \eta^* = 4\mathbb{1}_{\mathbf{1}}, \quad \xi^* = 2\mathbb{1}_{\mathbf{1}} + \mathbb{1}_{\mathbf{2}}.$$

Hence the associated reciprocal characteristics are:

$$\Phi^\nu_{\varphi^*} = \nu(\mathbf{1})\nu(\mathbf{3}), \Phi^{\eta^*}_\nu = \nu(\mathbf{1})^4, \Phi^\nu_{\xi^*} = \nu(\mathbf{1})^2\nu(\mathbf{2}).$$

We now turn our attention to the integral formula (14). Simple functions $F \in \mathscr{H}_{\xi^*}$ are of the form:

$$F(X, \gamma_1, \gamma_2, \gamma_3) = f(X)\mathbb{1}_{\{g_1 = g_2 = \mathbf{1}, g_3 = \mathbf{2}\}} h(t_1, t_2, t_3).$$

$(\gamma_1, \gamma_2, \gamma_3)$ is in the support of F if two jumps of value $\mathbf{1}$ happen at times t_1, t_2 and one jump of value $\mathbf{2}$ at time t_3, leading to a global null displacement since $\mathbf{4} = \mathbf{0}$. The formula (14) reads:

$$\mathbb{Q}\Big(f(X) \sum_{\substack{(t_1,t_2,t_3): t_1 \neq t_2,\\ \Delta X_{t_1} = \Delta X_{t_2} = \mathbf{1}\\ \Delta X_{t_3} = \mathbf{2}}} h(t_1, t_2, t_3)\Big) = \nu(\mathbf{1})^2\nu(\mathbf{2})\mathbb{Q}\Big(\int_{[0,1]^3} f\big(\Psi_{\gamma_1,\gamma_2,\gamma_3} X\big) h(t_1, t_2, t_3)\Big) dt_1 dt_2 dt_3.$$

The distribution of the random vector (n_1, n_2, n_3) under the **00**-bridge is given by the following identities, valid for any $v : \mathbb{N}^3 \to \mathbb{R}$:

$$\int v(n_1,n_2,n_3)n_1n_3\,\chi(dn_1,dn_2,dn_3) = \nu(\mathbf{1})\nu(\mathbf{3})\int \big(v(n_1+1,n_2,n_3+1)\chi(dn_1,dn_2,dn_3)$$
$$\int v\; n_1(n_1-1)(n_1-2)(n_1-3)\chi(dn_1,dn_2,dn_3) = \nu(\mathbf{1})^4 \int v(n_1+4,n_2,n_3)\chi(dn_1,dn_2,dn_3)$$
$$\int v(n_1,n_2,n_3)n_1(n_1-1)n_2\chi(dn_1,dn_2,dn_3) = \nu(\mathbf{1})^2\nu(\mathbf{2}) \int v(n_1+2,n_2+1,n_3)\chi(dn_1,dn_2,dn_3)$$
$$\chi(n_1\mathbf{1}+n_2\mathbf{2}+n_3\mathbf{3}=\mathbf{0}) = 1.$$

In this situation, again $\mathbb{P}_\nu$ is the unique random walk of its reciprocal class.

3.3. The state space is a product group

Consider the product of two groups, say G and G' and two non negative finite measures on them, say ν and ν', such that in both cases (H1) and (H2) are satisfied. Then, the product group $G \times G'$ equipped with the product measure $\nu \otimes \nu'$ also fulfils Assumptions (H1) and (H2). The key idea is as follows: if $\mathscr{B}$ and $\mathscr{B}'$ are suitable basis of G and G' then we can define for all $\eta \in \mathscr{B}$,

$$\varphi_\eta : G_\nu \times G_{\nu'} \to \mathbb{N}, \quad \varphi_\eta(g,g') = \eta(g)$$

and for all $\eta' \in \mathscr{B}'$,

$$\varphi_{\eta'} : G_\nu \times G_{\nu'} \to \mathbb{N}, \quad \varphi_{\eta'}(g,g') = \eta'(g').$$

The set $\mathscr{B}_\otimes = \{\varphi_\eta\}_{\eta\in\mathscr{B}} \cup \{\varphi_{\eta'}\}_{\eta'\in\mathscr{B}'}$ is an appropriate basis for the lattice of skeletons defined on the product group.

Example 30. Random walk on the d-dimensional discrete hypercube $(\mathbb{Z}/2\mathbb{Z})^d$.
The d-dimensional discrete hypercube is the d-times product of the cyclic group with two elements. We denote by $e_i \in (\mathbb{Z}/2\mathbb{Z})^d$ the basis element with i-th coordinate equal to 1, $1 \le i \le d$.
A random walk on the hypercube is defined uniquely through its jump measure $\nu = \sum_{i=1}^d \nu(i)\delta_{e_i}$. Since it can be realized as the product of d random walks on $\mathbb{Z}/2\mathbb{Z}$, we can choose $\mathscr{B}$ as follows: $\mathscr{B} = \{\varphi_i^*\}_{1\le i\le d}\,, \quad \varphi_i^* = 2\mathbb{1}_{e_i}$.
For the integral characterisation it is enough to consider loops of length $\ell = 2$. However, we have here d different skeletons to consider. Test functions of the form

$$F(X,\gamma) = f(X)\mathbb{1}_{\{g_1=g_2=e_i\}}h(t_1,t_2), \quad 1 \le i \le d$$

belongs to $\mathscr{H}_{\varphi_i^*}$. For i fixed, (14) reads as:

$$\mathbb{Q}\Big(f(X)\sum_{\substack{(t_1,t_2):t_1\neq t_2,\\ \Delta X_{t_1}=\Delta X_{t_2}=e_i}} h(t_1,t_2)\Big) = \nu(i)^2\int_{[0,1]^2}\mathbb{Q}\Big(f\big(\Psi_{\gamma_1,\gamma_2}X\big)\Big)h(t_1,t_2)dt_1dt_2$$

Concerning the distribution of the random vector $(n_{e_1}, \cdots, n_{e_d})$*, it has independent marginals* χ_i *who are characterized through the system of equations:*
for all $v \in \mathcal{B}^+(\mathbb{N})$ *and* $1 \leq i \leq d$*,*

$$\begin{aligned} \int v(n)n(n-1)\chi_i(dn) &= \nu(i)^2 \int v(n+1)\chi_i(dn), \\ \chi_i(n \in 2\mathbb{N}) &= 1. \end{aligned}$$

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