Bridges of Markov counting processes. Reciprocal classes and duality formulas [*]

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Abstract

Processes having the same bridges are said to belong to the same *reciprocal class*. In this article we analyze reciprocal classes of Markov counting processes by identifying their *reciprocal invariants* and we characterize them as the set of counting processes satisfying some duality formula.

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Introduction

The theory of reciprocal processes is rooted in an early work of Schrödinger [15] on constrained Brownian particles. It was developed in the framework of diffusion processes by Jamison, Krener, Thieullen and Zambrini among other contributors, see [12, 9] for recent reviews on this topic. After Jamison [8], it was clear that a central notion of the theory was that of *reciprocal class* of a given stochastic process, which is the set of all processes that share their bridges with this process.

The aim of the present article is to investigate the reciprocal class of the easiest possible processes with jumps, namely the counting processes. Although based on simple processes, it appears that this reciprocal structure is interesting.

These simple processes with jumps, which we call *nice Markov counting* (NMC, for short) processes and include the standard Poisson process, are introduced in the first section together with their reciprocal classes. It is proved at Theorem 1.11 that two NMC processes belong to the same reciprocal class, i.e. have the same bridges, if and only if some specific time-space functions derived from their intensities of jumps coincide. This common function is called the *reciprocal invariant* of the reciprocal class. We

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derive at Lemma 2.8 a duality relation between some stochastic integral and a derivative operator built on variations of the instants of jump, which holds for any NMC process. It will lead us to our main result which states at Theorem 2.11 a characterization of the reciprocal class of a given NMC process by means of a duality formula involving both the reciprocal invariant of the class and small variations of the times of jumps.

This article is partly based on the third author's PhD thesis [11]. Our results are analogous to those of Thieullen and the last author [13, 14] that were obtained in the framework of Brownian diffusion processes. However, the proofs of the present paper differ significantly from those of these earlier works. Recently, similar results have been obtained in [4] for compound Poisson processes, using variations based on space perturbations, rather than time perturbations as in the present article.

1   Counting processes and their reciprocal classes

The basic object of this paper, called *nice Markov counting* process (NMC process, for short) is described, the reciprocal class of an NMC process is defined and its relation with some h-transforms is made precise. Theorem 1.11 states a characterization of the reciprocal class of an NMC process in terms of its reciprocal invariant.

Framework, definitions and notation

The sample path space Ω of the counting processes consists of all càdlàg step functions with finitely many jumps with amplitude $+1$ and an initial value in $\mathbb{Z}$. Any path $\omega \in \Omega$ is described by the collection $(x; t_1, \ldots, t_n)$ of its initial position $x \in \mathbb{Z}$ and its $n = \omega_1 - \omega_0$ instants of jumps $0 < t_1 < \cdots < t_n < 1$. It is practical to set $t_i = 1$, for all $i > n$ and identify ω with $[x; (t_i)_{i \geq 1}]$. We denote $X_0(\omega) := x$ and $T_i(\omega) := t_i$ the i-th instant of jump of ω.

The canonical counting process is denoted by $X = (X_t)_{0 \leq t \leq 1}$ and $\mathcal{P}(\mathbf{E})$ denotes the space of all probability measures on a measurable space $\mathbf{E}$. For any $Q \in \mathcal{P}(\Omega)$ and $t \in [0, 1]$, the marginal law of Q at time t is denoted by $Q_t := Q \circ X_t^{-1} \in \mathcal{P}(\mathbb{Z})$ and $Q_{01} := Q \circ (X_0, X_1)^{-1} \in \mathcal{P}(\mathbb{Z}^2)$ is the endpoint marginal law of Q.

Any $Q \in \mathcal{P}(\Omega)$ admits an increasing process denoted by $A : [0,1] \times \Omega \to \mathbb{R}_+$ such that $Q(A(0) = 0) = 1$ and $t \mapsto X_t - X_0 - A(t)$ is a local Q-martingale, (see Jacod [7, Thm. 2.1], for instance) which characterizes the dynamics of Q. When the compensator is absolutely continuous, we call its derivative the *intensity* of Q.

Let us introduce the set of the reference processes of this paper.

Definition 1.1. *Let $\ell : [0,1] \times \mathbb{Z} \to (0, \infty)$ be a positive and upper bounded function: $0 < \ell(t, z) \leq \bar{\lambda}$, for some $\bar{\lambda} > 0$ and for all $(t, z) \in [0, 1] \times \mathbb{Z}$, such that for each $z \in \mathbb{Z}$, $t \mapsto \ell(t, z)$ is in $\mathcal{C}^1([0, 1])$. Suppose moreover that its time-derivative is uniformly bounded: $|\partial_t \ell(t, z)| \leq \bar{\lambda}, \forall (t, z) \in [0, 1] \times \mathbb{Z}$.*
Such a function ℓ is called a nice Markov counting *intensity, an NMC intensity for short. A counting process with an NMC intensity is called an NMC process.*

Each NMC process belongs to $\mathcal{P}(\Omega)$ and the strict positivity of the intensity implies that two NMC are absolutely continuous with respect to each other.

Let ℓ be an NMC intensity. For any $x \in \mathbb{Z}$, we define $R_\ell^x \in \mathcal{P}(\Omega)$ as the law of the NMC process starting from x with intensity ℓ and we denote for any probability measure $\mu \in \mathcal{P}(\mathbb{Z})$, $P_\ell^\mu(\cdot) := \int_{\mathbb{Z}} R_\ell^x(\cdot) \, \mu(dx) \in \mathcal{P}(\Omega)$ the law of the counting process with intensity ℓ and initial law μ. When the initial law μ is not relevant, we drop the corresponding superscript and simply write P_ℓ. Let us introduce the kernel

$$\left(R_\ell^{xy} := R_\ell^x(\cdot \mid X_1 = y); x \leq y \in \mathbb{Z} \right) \tag{1.1}$$

which is a family in $\mathcal{P}(\Omega)$ indexed by the endpoints (x, y). Clearly, R_ℓ^{xy} is a regular version of the xy-bridge of the unbounded σ-finite measure $R_\ell := \sum_{x \in \mathbb{Z}} R_\ell^x$ which is defined everywhere, i.e. for all $x \leq y \in \mathbb{Z}$. Of course, $P_\ell^{xy} = R_\ell^{xy}$ and $P_\ell^\mu(\cdot \mid X_0 = x) = R_\ell^x$. The kernel (1.1) is the basic dynamical object that will be used later on. It is uniquely determined by the intensity ℓ.

Example 1.2. *A fundamental subclass of $\mathcal{P}(\Omega)$ is the set of Poisson processes characterized by two parameters: a constant intensity, say $\alpha > 0$, and the distribution $\mu \in \mathcal{P}(\mathbb{Z})$ of the random initial value. We denote it by $\mathbf{P}_\alpha^\mu$. If $\alpha = 1$ we drop the index and simply write $\mathbf{P}^\mu$ or $\mathbf{P}$. The bridges $R_{\alpha=1}^{xy}$ are denoted by $\mathbf{R}^{xy}$.*

Proposition 1.3 (Girsanov formula). *Let $P_\ell, P_k \in \mathcal{P}(\Omega)$ be two NMC processes sharing the same initial distribution. Then they are absolutely continuous with respect to each other and*

$$\frac{dP_\ell}{dP_k} = \exp\Big(- \int_{[0,1]} (\ell(s, X_{s-}) - k(s, X_{s-})) ds \Big) \prod_{i:T_i<1} \frac{\ell(T_i, X_{T_i^-})}{k(T_i, X_{T_i^-})}, \tag{1.2}$$

In particular, the density of P_ℓ with respect to a standard Poisson process with the same initial distribution is

$$G_\ell := \frac{dP_\ell^\mu}{d\mathbf{P}^\mu} = \exp\Big(- \int_{[0,1]} \Big(\ell(s, X_{s-}) - 1 \Big) ds \Big) \prod_{i:T_i<1} \ell(T_i, X_{T_i^-}). \tag{1.3}$$

Since $X_{1-} = X_1, P_\ell$-a.s., "$i : T_i < 1$" indicates that only the effective jumps are taken into account.
Remark that $G_\ell > 0$, $\mathbf{P}^\mu$-a.s.

Proof. It follows from the boundedness of the intensity and [7, Thm. 5.1]. ∎

Reciprocal classes of Markov counting processes

Let us define the reciprocal class of an NMC intensity.

Definition 1.4. *Let ℓ be an NMC intensity. The* reciprocal class $\mathfrak{R}(\ell)$ *associated to the intensity ℓ is the set of all probability measures on Ω obtained as mixtures of the bridges R_ℓ^{xy}, see (1.1), that is*

$$\mathfrak{R}(\ell) := \Big\{ Q \in \mathcal{P}(\Omega) : Q(\cdot) = \int_{\mathbb{Z}^2} R_\ell^{xy}(\cdot) \, Q_{01}(dxdy) \Big\}. \tag{1.4}$$

The integral in (1.4) makes sense since Q_{01} is concentrated on $\{x \leq y \in \mathbb{Z}\}$.
Any element Q of $\mathfrak{R}(\ell)$ possesses the following interesting time symmetry property, called *reciprocal property*: For any $0 \leq s \leq u \leq 1$ and any $X_{[s,u]}$-measurable A,

$$Q(A \mid X_{[0,s]}, X_{[u,1]}) = Q(A \mid X_s, X_u).$$

Bernstein put forward this property in [1]. The reciprocal property is weaker than the Markov one. We will encounter in the rest of the paper path measures which are not Markov but are reciprocal.
A better understanding of the dynamics of the processes in $\mathfrak{R}(\ell)$ will follow from next result.

Proposition 1.5. *The path measure $Q \in \mathcal{P}(\Omega)$ belongs to $\mathfrak{R}(\ell)$ if and only if there exists a nonnegative measurable function $h : \mathbb{Z}^2 \to \mathbb{R}_+$ such that*

$$Q = h(X_0, X_1) \, R_\ell. \tag{1.5}$$

Recall that $R_\ell := \sum_{x \in \mathbb{Z}} R_\ell^x$.

Proof. See [9]. □

Remark 1.6. *If the density $h = dQ/dR_\ell$ is a function of the final state X_1 only, one identifies Q as an h-transform of R_ℓ in the sense of Doob [5].*

Example 1.7. *Let $\mathbf{P}$ be the standard Poisson process with unit intensity. Recall the notation of Example 1.2.*

(a) *Any Poisson process $\mathbf{P}_\alpha$ belongs to the reciprocal class $\mathfrak{R}(1)$ associated with the unit intensity $\ell = 1$, whatever is its intensity α. Indeed, for each $x \in \mathbb{Z}$, the identity $\mathbf{P}_\alpha^x = e^{1-\alpha} \alpha^{X_1} \mathbf{P}^x$ implies that for any integer $x \le y$, $\mathbf{P}_\alpha^{xy} = \mathbf{R}^{xy}$.*

(b) *Let μ be any probability measure on $\mathbb{N}$. The unique probability measure Q in $\mathfrak{R}(1)$ with endpoint distribution $Q_{01} = \delta_{\{0\}} \otimes \mu$ is given by*

$$Q = e\, X_1!\, \mu(X_1)\, \mathbf{P}^0.$$

In next section, we will obtain an alternate proof of this former assertion using reciprocal invariants.

Reciprocal invariant

We show that the reciprocal class of an NMC intensity is characterized by a function which is called its *reciprocal invariant*. This new result extends previous studies by Clark in the framework of Brownian diffusions. It was proved in [3, Thm. 1] that reciprocal classes of Brownian diffusions are characterized by some reciprocal invariants that are functions of the drift. Rather than adapting Clark's proof, we follow the strategy of Zambrini and the second author [10] in the setting of diffusion processes.

Let us describe how the intensity of a counting process behaves through a change of measure involving the final state only.

Proposition 1.8 (*h-transform*). *Let P_ℓ be the law of an NMC process and $h : \mathbb{Z} \to (0, \infty)$ be any positive function such that $E_\ell\, h(X_1) = 1$. Then the process on Ω whose law is given by*

$$Q := h(X_1)\, P_\ell$$

is Markov and its intensity $k : [0,1] \times \mathbb{Z} \to [0, \infty)$ satisfies:

$$k(t, X_{t^-}) = \frac{h(t, X_{t^-} + 1)}{h(t, X_{t^-})}\, \ell(t, X_{t^-}) \quad dt \otimes Q\text{-}a.s., \tag{1.6}$$

where $h(t, z) := E_\ell(h(X_1) | X_t = z)$.

Proof. This is a special easy case of a Doob h-transform, [5]. First note that $h(t, z)$ is time-differentiable (a standard semigroup argument) and space-time harmonic, that is

$$\partial_t h(t, z) + \ell(t, z)\, [h(t, z+1) - h(t, z)] = 0, \quad P_\ell \circ X_t^{-1}\text{-a.e., for all } t \in [0, 1]. \tag{1.7}$$

Denote $\psi(t, z) := \log h(t, z)$ which is $P_\ell \circ X_t^{-1}$-a.e. well defined. It satisfies

$$\partial_t \psi(t, z) = \ell(t, z)\big(1 - e^{\psi(t, z+1) - \psi(t, z)}\big).$$

By Itô formula, for any $0 \le t \le 1$,

$$
\begin{aligned}
\psi(t, X_t) &= \int_{]0,t]} \partial_t \psi(s, X_{s^-})\, dr + \sum_{i : T_i \le t} \left(\psi(T_i, X_{T_i}) - \psi(T_i, X_{T_i^-}) \right) \\
&= \int_{]0,t]} \left(1 - \frac{h(s, X_{s^-} + 1)}{h(s, X_{s^-})} \right) \ell(s, X_{s^-})\, ds + \sum_{i : T_i \le t} \log \frac{h(T_i, X_{T_i^-} + 1)}{h(T_i, X_{T_i^-})}.
\end{aligned}
$$

Since $h(X_1) = e^{\psi(1, X_1)}$, this implies that

$$h(X_1) = \exp\left(-\int_{(0,1]}\left(\frac{h(s, X_{s^-} + 1)}{h(s, X_{s^-})} - 1\right)\ell(s, X_{s^-})ds\right)\prod_{i: T_i < 1}\frac{h(T_i, X_{T_i^-} + 1)}{h(T_i, X_{T_i^-})}.$$

and

$$h(X_1)G_\ell = \exp\left(-\int_{(0,1]}\left(\ell(s, X_{s^-})\frac{h(s, X_{s^-} + 1)}{h(s, X_{s^-})} - 1\right)ds\right)\prod_{i: T_i < 1}\ell(T_i, X_{T_i^-})\frac{h(T_i, X_{T_i} + 1)}{h(T_i, X_{T_i^-})}.$$

With (1.3) we see that Q admits the intensity k as defined in (1.6). $\qquad\square$

Example 1.9. *Take $x \le y \in \mathbb{Z}$. The bridge $\mathbf{P}_\alpha^{xy}$ of the Poisson process with intensity α starting from x is given by*

$$\mathbf{P}_\alpha^{xy} = \frac{\mathbf{1}_{\{X_1 = y\}}}{\mathbf{P}_\alpha^x(X_1 = y)}\mathbf{P}_\alpha^x = e^\alpha\frac{(y - x)!}{\alpha^{y-x}}\mathbf{1}_{\{X_1 = y\}}\mathbf{P}_\alpha^x.$$

Since

$$\mathbf{P}_\alpha^x(X_1 = y \mid X_t = z) = e^{-\alpha(1-t)}\frac{(\alpha(1-t))^{y-z}}{(y - z)!},$$

this implies that the intensity k of the bridge $\mathbf{P}_\alpha^{xy}$ is given for $t \in [0, 1)$ by

$$k(t, z) = \frac{y - z}{1 - t}\mathbf{1}_{\{y > z\}},$$

whose interpretation is as follows. If the process is located at z at time t, it will undergo a kind of mean velocity $(y - z)/(1 - t)$ to reach the state y during the remaining duration $1 - t$. Note the similarity with the drift of the Brownian bridge. The intensity explodes at time 1 and it does not depend on α, as already mentioned in Example 1.7-(a).

We give at Theorem 1.11 below a necessary and sufficient condition for the equality of the reciprocal classes associated with two distinct NMC intensities.

Definition 1.10 (Reciprocal invariant). *For any NMC intensity ℓ, we define the map*

$$\Xi_\ell(t, z) := \partial_t \log \ell(t, z) + \ell(t, z + 1) - \ell(t, z) \tag{1.8}$$

and call it the reciprocal invariant *of the class $\mathfrak{R}(\ell)$.*

This terminology is justified by the following result.

Theorem 1.11. *Let ℓ and k be two NMC intensities. Then, $\mathfrak{R}(\ell) = \mathfrak{R}(k)$ if and only if $\Xi_\ell = \Xi_k$.*

Proof. Note that for any $x \in \mathbb{Z}$, the measures P_k^x and P_ℓ^x are absolutely continuous with respect to each other. Assume that $P_k \in \mathfrak{R}(\ell)$ and fix $x \in \mathbb{Z}$. By Proposition 1.5, there exists a positive function h such that $P_k^x = h(X_1)P_\ell^x$. Therefore k and ℓ are related by (1.6):

$$k(t, z) = e^{\psi(t, z+1) - \psi(t, z)}\ell(t, z),$$

where $\psi(t, z) := \log h(t, z)$. This leads to

$$
\begin{aligned}
\partial_t(\log k(t, z) - \log \ell(t, z)) &= \partial_t(\psi(t, z + 1) - \psi(t, z)) \\
&= \ell(t, z + 1)\left(1 - e^{\psi(t, z+2) - \psi(t, z+1)}\right) - \ell(t, z)\left(1 - e^{\psi(t, z+1) - \psi(t, z)}\right) \\
&= \left(\ell(t, z + 1) - \ell(t, z)\right) - \left(k(t, z + 1) - k(t, z)\right)
\end{aligned}
$$

which implies the equality of Ξ_k and Ξ_ℓ.

Let us prove the converse statement. Suppose that $\Xi_k = \Xi_\ell$. This implies that there exists a regular function c which is only space dependent such that

$$\log \frac{k}{\ell}(t,z) = \phi(t, z+1) - \phi(t, z) + c(z) \text{ where } \phi(t,z) := \int_0^t [\ell(s,z) - k(s,z)] \, ds.$$

By Equation (1.2), for any $x \in \mathbb{Z}$, we have

$$\frac{dP_k^x}{dP_\ell^x} = \exp\Big(\int_{(0,1]} [\ell(t, X_{t^-}) - k(t, X_{t^-})] \, dt \Big) \prod_{i:T_i<1} \frac{k}{\ell}(T_i, X_{T_i^-}),$$

where only the random instants $T_i < 1$ of jumps are taken into account in the last product. Therefore,

$$\begin{aligned}
\frac{dP_k^x}{dP_\ell^x} &= \exp\Big(\int_{(0,1]} \partial_t \phi(t, X_{t^-}) \, dt + \sum_{i:T_i<1} \log \frac{k}{\ell}(T_i, X_{T_i^-}) \Big) \\
&= \exp\Big(\int_{(0,1]} \partial_t \phi(t, X_{t^-}) \, dt + \sum_{i:T_i<1} [\phi(T_i, X_{T_i}) - \phi(T_i, X_{T_i^-})] + \sum_{i:T_i<1} c(X_{T_i^-}) \Big) \\
&= \exp\big(\phi(1, X_1) - \phi(0, x) + C(x, X_1) \big)
\end{aligned}$$

where $\sum_{i:T_i<1} c(X_{T_i^-}) = c(x) + \cdots + c(X_1 - 1) =: C(x, X_1)$ only depends on x and X_1. Thus, thanks to Proposition 1.5 we deduce that for any $x \in \mathbb{Z}$, $P_k^x \in \mathfrak{R}(\ell)$ and it follows that $P_k \in \mathfrak{R}(\ell)$. □

Remark 1.12. *Defining the reciprocal invariant Ξ_ℓ requires positivity of the intensity ℓ. In particular, any bridge of an NMC process is ruled out since its intensity vanishes at the terminal state. This is in some sense a weakness of Definition 1.4 in the present framework of jump processes. It is in contrast with the diffusion process setting where reciprocal invariants of bridges are always well defined, see [13, 14]. However, we will present in Theorem 2.11 a significant improvement of Theorem 1.11.*

Example 1.13. (a) *It is immediate to see that for any constant intensities $\alpha, \beta > 0$, we have $\mathfrak{R}(\alpha) = \mathfrak{R}(\beta)$, meaning that any Poisson process with a constant intensity has the same bridges as the standard Poisson process. This was the content of Example 1.7-(a).*

(b) *Consider two time-homogeneous intensities $\ell(x)$ and $k(x)$. Their associated reciprocal invariants are also time-homogeneous: $\Xi_\ell(t, x) = \ell(x+1) - \ell(x)$. Then*

$$\mathfrak{R}(\ell) = \mathfrak{R}(k) \iff \forall x \in \mathbb{Z}, \, \ell(x+1) - \ell(x) = k(x+1) - k(x),$$

which means that $\ell - k$ is a 1-periodic function on $\mathbb{Z}$. This implies that the function k is equal to ℓ up to some additive constant λ: $k(x) \equiv \ell(x) + \lambda$.
Therefore, if $\lambda > 0$, this means that P_k is the law of the sum (or superposition) of P_ℓ with an independent Poisson process with intensity λ.

(c) *Consider two space-homogeneous deterministic intensities $\ell(t)$ and $k(t)$. Their associated reciprocal invariants are space-homogeneous too: $\Xi_\ell(t, x) = (\log \ell)'(t)$. By Theorem 1.11, $\mathfrak{R}(\ell) = \mathfrak{R}(k)$ if and only if $k(t) \equiv c \, \ell(t)$ for some constant $c > 0$.*
Therefore, if $0 < c \leq 1$, P_k is obtained by c-thinning P_ℓ as follows. Let $X_0 + \sum_i \delta_{T_i}$ be the point process with law P_ℓ. Then P_k is the law of $X_0 + \sum_i \xi_i \delta_{T_i}$ where $(\xi_i)_i$ is an iid sequence of Bernoulli(c) random variables that is independent of $(T_i)_i$.
For a general c, decompose $c = n + r$ with $n \in \mathbb{N}$ its integer part and $0 \leq r < 1$. Superposing n independent copies of P_ℓ and a c-thinning of it gives the process P_k.

2   Characterization of reciprocal classes by duality formulas

The main result of the article is Theorem 2.11 below. We show that each reciprocal class coincides with the set of random processes for which a duality relation holds between some stochastic integral and some derivative operator on the path space. This is in the spirit of the results obtained by Thieullen and the fourth author for Brownian diffusion processes. Indeed, it is shown in [13] for one-dimensional case and in [14] for the multidimensional case that any Brownian diffusion process satisfies an integration by parts formula expressed in terms of its reciprocal invariants that fully characterizes its reciprocal class.

We introduce a derivative operator by perturbing the time parametrization of the path and prove a duality formula that holds for any counting process. Then, we extend these results to bridges of NMC processes and their mixtures, to obtain finally a characterization of the reciprocal class of an NMC intensity at Theorem 2.11.

Directional derivative on Ω

In the framework of Malliavin calculus, Carlen and Pardoux [2] introduced a directional derivative on the Poisson space by considering infinitesimal changes of the time parametrization (see also Elliott & Tsoi [6]). We follow a similar approach in the context of general counting processes.

The perturbation operator

The perturbation operator is defined in terms of a change of time.

Definition 2.1 (The set $\mathcal{U}$ of perturbation functions). *The set $\mathcal{U}$ of perturbation functions consists of all $\mathcal{C}^1$-functions $u : [0, 1] \to \mathbb{R}$ such that $u(0) = u(1) = 0$.*

For any function $u \in \mathcal{U}$ and $\varepsilon > 0$ small enough, we define the change of time $\theta_u^\varepsilon : [0, 1] \to [0, 1]$ by

$$\theta_u^\varepsilon(t) = t + \varepsilon\, u(t).$$

The boundedness of the derivative $\dot{u}$ of u and the property $u(0) = u(1) = 0$ ensure that for any ε small enough, θ_u^ε is indeed a change of time with $\theta_u^\varepsilon(0) = 0$ and $\theta_u^\varepsilon(1) = 1$.

The perturbation operator is defined for any path $\omega \in \Omega$ by

$$\Theta_u^\varepsilon(\omega) = \omega \circ \theta_u^\varepsilon. \tag{2.1}$$

Note that the operator Θ_u^ε keeps the initial and final values of the path unchanged.

The derivative operator

We are now ready to give the definition of a derivative in the direction of the elements of $\mathcal{U}$, in the spirit of the stochastic calculus of variations.

Definition 2.2 (The derivative $\mathcal{D}_u\Phi$). *Let Φ be a measurable real function on Ω and $u \in \mathcal{U}$ a perturbation function. We define*

$$\mathcal{D}_u\Phi := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\Phi \circ \Theta_u^\varepsilon - \Phi \right), \tag{2.2}$$

provided that this limit exists.

Let us remark that we slightly changed the notations introduced by Carlen and Pardoux in the context of Malliavin calculus: we write $\mathcal{D}_u$ instead of $\mathcal{D}_{\dot{u}}$ in [2].

Definition 2.3 (The set $\mathcal{S}$ of simple functions). *We say that $\Phi : \Omega \to \mathbb{R}$ belongs to the set $\mathcal{S}$ of simple functions if there exists $m \geq 1$ such that $\Phi = \varphi(X_0; T_1, \ldots, T_m)$ for some $\varphi : \mathbb{Z} \times [0,1]^m \to \mathbb{R}$ such that for all $x \in \mathbb{Z}$, the partial functions $\varphi(x; \cdot)$ are $\mathcal{C}^\infty$-differentiable.*

These functions are differentiable on the Poisson space in a natural way, as proved in [2, Thm. 1.3].

Lemma 2.4. *Let $\Phi \in \mathcal{S}$ be a simple function. It is differentiable in the direction of any $u \in \mathcal{U}$ and one has*

$$
\begin{aligned}
\mathcal{D}_u \Phi &= \mathcal{D}_u \varphi(X_0; T_1, \ldots, T_m) = -\sum_{j=1}^m \partial_{t_j} \varphi(X_0; T_1, \ldots, T_m) u(T_j) & (2.3) \\
&= -\int_{[0,1]} \Big(\sum_{j=1}^m \partial_{t_j} \varphi(X_0; T_1, \ldots, T_m) \mathbf{1}_{[0,T_j]}(t) \Big) \dot{u}(t) dt.
\end{aligned}
$$

Duality formula. Markov counting process

Proposition 2.5 below states two duality relations between the derivative operator $\mathcal{D}$ and some stochastic integrals.

Proposition 2.5. *The NMC process P_ℓ satisfies the following duality formulas. For all $\Phi \in \mathcal{S}$ and $u \in \mathcal{U}$,*

$$
E_\ell(\mathcal{D}_u \Phi) = E_\ell\Big(\Phi \int_{[0,1]} \frac{\partial_t(\ell u)}{\ell} \big[dX_t - \ell(t, X_{t^-}) \, dt \big] \Big) \tag{2.4}
$$

$$
\text{and} \quad E_\ell(\mathcal{D}_u \Phi) = E_\ell\Big(\Phi \int_{[0,1]} \big[\dot{u}(t) + \Xi_\ell(t, X_{t^-}) u(t) \big] \, dX_t \Big). \tag{2.5}
$$

We do not make the initial distribution μ precise since it does not play any role.

Proof. Consider the left hand side of (2.4)

$$
E_\ell(\mathcal{D}_u \Phi) = E_\ell\Big(\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \big(\Phi \circ \Theta_u^\varepsilon - \Phi \big) \Big).
$$

Because of the smoothness of Φ and the boundedness of $\mathcal{D}_u \Phi$, see (2.3), the expectation $E_\ell(\mathcal{D}_u \Phi)$ is well defined and we can exchange limit and expectation. Therefore

$$
E_\ell(\mathcal{D}_u \Phi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(E_\ell\big(\Phi \circ \Theta_u^\varepsilon \big) - E_\ell(\Phi) \Big) = \lim_{\varepsilon \to 0} E_\ell\Big(\Phi \frac{G^{\varepsilon,u} - 1}{\varepsilon} \Big) \tag{2.6}
$$

where $G^{\varepsilon,u} := dP_\ell \circ (\Theta_u^\varepsilon)^{-1} / dP_\ell$.

Now $P_\ell \circ (\Theta_u^\varepsilon)^{-1}$ is a counting process with uniformly bounded intensity $\ell(\theta_u^\varepsilon, .) \dot{\theta}_u^\varepsilon = \ell(\theta_u^\varepsilon, .)(1 + \varepsilon \dot{u})$. Therefore, taking into account the regularity of ℓ assumed in Definition 1.1, the Girsanov density of $P_\ell \circ (\Theta_u^\varepsilon)^{-1}$ with respect to P_ℓ verifies:

$$
\begin{aligned}
G^{\varepsilon,u} &= \exp{-\Big(\int_{[0,1]} \big[\ell(\theta_u^\varepsilon(t), X_{t^-})(1 + \varepsilon \dot{u}(t)) - \ell(t, X_{t^-}) \big] \, dt \Big) \prod_{i:T_i < 1} \frac{\ell(\theta_u^\varepsilon(T_i), X_{T_i^-})(1 + \varepsilon \dot{u}(T_i))}{\ell(T_i, X_{T_i^-})}} \\
&= \exp\Big(-\varepsilon \int_{[0,1]} \partial_t(\ell u)(t, X_{t^-}) dt + o(\varepsilon) \Big) \prod_{i:T_i < 1} \Big(1 + \varepsilon \big[\frac{\partial_t \ell}{\ell}(T_i, X_{T_i^-}) u(T_i) + \dot{u}(T_i) \big] + o(\varepsilon) \Big) \\
&= 1 + \varepsilon \int_{[0,1]} \frac{\partial_t(\ell u)}{\ell} \big[-\ell(t, X_{t^-}) \, dt + dX_t \big] + o(\varepsilon), \quad P_\ell\text{-a.s.}
\end{aligned}
$$

which implies that

$$(G^{\varepsilon,u} - 1)/\varepsilon \quad = \quad \int_{[0,1]} \frac{\partial_t(\ell u)}{\ell} \big[dX_t - \ell(t, X_{t^-}) \, dt \big] + o(1), \ P_\ell\text{-a.s.}$$

which leads us to (2.4). In particular, under the standard Poisson process $\mathbf{P}$,

$$\mathbf{E}(\mathcal{D}_u \Phi) = \mathbf{E}\Big(\Phi \int_{[0,1]} \dot{u}(t) \big(dX_t - dt \big) \Big) \tag{2.7}$$

as it appears in [2].

To prove the duality formula (2.5) satisfied by P_ℓ, we take advantage of its mutual absolute continuity with the Poisson measure $\mathbf{P}$ with the same initial distribution as P_ℓ. The density $G_\ell := dP_\ell/d\mathbf{P}$, given at (1.3), is differentiable in the direction of any $u \in \mathcal{U}$ and

$$\mathcal{D}_u G_\ell = - G_\ell \int_{[0,1]} \Xi_\ell(t, X_{t^-}) u(t) \, dX_t. \tag{2.8}$$

Indeed, one can write G_ℓ as a function of the jump times: $G_\ell = \varphi(X_0; T_1, \dots)$ where

$$\varphi(x; t_1, \dots) = \exp \Big(- \sum_{i \geq 0 : t_i < 1} \int_{[t_i, t_{i+1} \wedge 1)} (\ell(t, x+i) - 1) \, dt + \sum_{i \geq 1 : t_i < 1} \ln \ell(t_i, x+i-1) \Big)$$

with the convention that $t_0 = 0$. Since the Poisson process performs almost surely finitely many jumps, we are allowed to invoke the first identity in (2.3) to obtain (2.8). Then, for any $\Phi \in \mathcal{S}$,

$$
\begin{aligned}
E_\ell\Big(\Phi \int_{[0,1]} \dot{u}(t) dX_t \Big) \quad &= \quad \mathbf{E}\Big(G_\ell \, \Phi \int_{[0,1]} \dot{u}(t) dX_t \Big) \\
&= \quad \mathbf{E}\Big(G_\ell \, \Phi \int_{[0,1]} \dot{u}(t)(dX_t - dt) \Big) \\
&\overset{(2.7)}{=} \quad \mathbf{E}\big(\mathcal{D}_u(G_\ell \, \Phi) \big) \\
&\overset{\checkmark}{=} \quad \mathbf{E}\big(G_\ell \, \mathcal{D}_u \Phi \big) + \mathbf{E}\big(\Phi \, \mathcal{D}_u G_\ell \big) \\
&\overset{(2.8)}{=} \quad E_\ell\big(\mathcal{D}_u \Phi \big) - E_\ell\Big(\Phi \int_{[0,1]} \Xi_\ell(t, X_{t^-}) u(t) \, dX_t \Big),
\end{aligned}
$$

where we used at the marked equality the rule for the derivative of a product of functions, as proved in [2, Thm. 1.4] . Therefore

$$E_\ell\big(\mathcal{D}_u \Phi \big) = E_\ell\Big(\Phi \int_{[0,1]} \big(\dot{u}(t) + \Xi_\ell(t, X_{t^-}) u(t) \big) dX_t \Big),$$

which is the desired result. $\qquad\square$

Example 2.6. *Let P_ℓ be a non-homogeneous Poisson process with exponential intensity $\ell(t) = e^{\lambda t}, \lambda > 0$. Then its reciprocal invariant is the constant λ and the duality formula (2.5) reduces to*

$$E_\ell(\mathcal{D}_u \Phi) = E_\ell^{xy}\Big(\Phi \int_{[0,1]} (\dot{u}(t) + \lambda u(t)) dX_t \Big) = E_\ell\Big(\Phi \int_{[0,1]} \dot{u}(t)(dX_t - \lambda X_t dt) \Big) \tag{2.9}$$

for any $\Phi \in \mathcal{S}$ and $u \in \mathcal{U}$.

Remark 2.7. *(a) The Markov property of the intensity ℓ does not play any role in the previous argumentation. Identity (2.4) extends to a large class of predictable regular intensities.*

(b) Take as intensity ℓ a constant number $\alpha > 0$. Equation (2.4) reduces to

$$\forall \Phi \in \mathcal{S}, \forall u \in \mathcal{U}, \quad \mathbf{E}_\alpha(\mathcal{D}_u \Phi) = \mathbf{E}_\alpha\Big(\Phi \int_{[0,1]} \dot{u}(t)(dX_t - \alpha\, dt)\Big) \tag{2.10}$$

Relaxing the definition of $\mathcal{D}_u$ for functions u which do not vanish at time $t = 1$, one can obtain a characterization of the set $\{\mathbf{P}_\alpha^\mu; \mu \in \mathcal{P}(\mathbb{Z})\}$ of all Poisson processes with intensity α.

Duality formula. Reciprocal class

Our main result Theorem 2.11 states a characterization of a reciprocal class in terms of the duality formula (2.5). On the way to its proof, we start noting in Lemma 2.8 that the identity (2.5) remains true for any bridge R_ℓ^{xy} associated with the NMC intensity ℓ.

Lemma 2.8. *Let ℓ be an NMC intensity and $x \leq y \in \mathbb{Z}$. Then the duality formula*

$$E_\ell^{xy}(\mathcal{D}_u \Phi) = E_\ell^{xy}\Big(\Phi \int_{[0,1]} \big[\dot{u}(t) + \Xi_\ell(t, X_{t-})u(t)\big]\, dX_t\Big) \tag{2.11}$$

holds for all $\Phi \in \mathcal{S}$ and all $u \in \mathcal{U}$.

Proof. Take the identity (2.5) and apply it with a test function of the form $f_0(X_0)f_1(X_1)\Phi$ where $f_0, f_1 : \mathbb{Z} \to \mathbb{R}$ and $\Phi \in \mathcal{S}$. Noting that $\mathcal{D}_u f_0(X_0) = \mathcal{D}_u f_1(X_1) = 0$, we deduce that the identity (2.11) is valid for any bridge R_ℓ^{xy}. □

Next result is concerned with the Poissonian case. It shows that any Poisson bridge is characterized by a simple duality formula.

Proposition 2.9. *Let $Q \in \mathcal{P}(\Omega)$ admit the endpoint marginal $Q_{01} = \delta_{(x,y)}$ with $x \leq y \in \mathbb{Z}$. The process Q is the Poisson bridge between x and y if and only if*

$$E_Q(\mathcal{D}_u \Phi) = E_Q\Big(\Phi \int_{[0,1]} \dot{u}(t)\, dX_t\Big) \tag{2.12}$$

holds for any $\Phi \in \mathcal{S}$ and any $u \in \mathcal{U}$.

Proof. The direct part is a particular case of Lemma 2.8.

Let us prove the converse statement by computing the jump intensity of Q by means of a Nelson stochastic derivative.

Fix $t \in [0,1)$. We are going to apply (2.12) with Φ an $X_{[0,t)}$-measurable function and $u \in \mathcal{U}$ such that $\dot{u} = \frac{1}{\varepsilon}\mathbf{1}_{[t,t+\varepsilon]} - \frac{1}{1-(t+\varepsilon')}\mathbf{1}_{[t+\varepsilon',1]}$ where $0 < \varepsilon < \varepsilon'$. We obtain the following equality

$$E_Q\Big(\frac{1}{\varepsilon}\int_{[t,t+\varepsilon]} dX_r \mid X_{[0,t)}\Big) = E_Q\Big(\frac{1}{1-(t+\varepsilon')}\int_{[t+\varepsilon',1]} dX_r \mid X_{[0,t)}\Big)$$

for every small enough ε'. Remark that both sides of the equality are constant as functions of ε and ε'. In particular, for almost every t the stochastic derivative

$$a(t) := \lim_{\varepsilon \to 0} E_Q\Big(\frac{1}{\varepsilon}\int_{[t,t+\varepsilon]} dX_r \mid X_{[0,t)}\Big)$$

exists (and is equal to the right hand side). This shows that Q admits the $dt\, Q(d\omega)$-almost everywhere defined process $(t,\omega) \mapsto a(t,\omega)$ as its intensity. Letting ε' tend to zero gives

$$a(t) = E_Q\Big(\frac{1}{1-t}(X_1 - X_t) \mid X_{[0,t)}\Big) = \frac{y - X_t}{1-t}. \tag{2.13}$$

We recognize the intensity of a Poisson bridge at time t with final condition y. □

By randomizing the endpoint marginal of Q in (2.12), we obtain the following

Corollary 2.10. *If for any $\Phi \in \mathcal{S}$ and any $u \in \mathcal{U}$ the duality formula*

$$E_Q(\mathcal{D}_u \Phi) = E_Q\Big(\Phi \int_{[0,1]} \dot{u}(t)\, dX_t\Big)$$

holds under $Q \in \mathcal{P}(\Omega)$ where $E_Q\big(X_1 - X_0\big) < +\infty$, then Q belongs to the reciprocal class $\mathfrak{R}(1)$ of the Poisson process.

Next result emphasizes that the duality formula (2.11) characterizes the reciprocal class of any NMC intensity ℓ. A natural idea would be to follow the guideline of the proof of Proposition 2.9. Unfortunately, this leads to an implicit equation for the intensity, in contrast with the special Poissonian case where the reciprocal invariant $\Xi_{\ell=1}$ vanishes and gives (2.13). However, a fruitful method consists in relying on the last corollary and the fact that any $Q \in \mathcal{P}(\Omega)$ is dominated by some Poisson process.

Theorem 2.11. *Let $Q \in \mathcal{P}(\Omega)$ be such that $E_Q\big(X_1 - X_0\big) < +\infty$. Then Q is in $\mathfrak{R}(\ell)$ if and only if the duality formula*

$$E_Q(\mathcal{D}_u \Phi) = E_Q\Big(\Phi \int_{[0,1]} \big[\dot{u}(t) + \Xi_\ell(t, X_{t-})u(t)\big]\, dX_t\Big) \tag{2.14}$$

holds for any $\Phi \in \mathcal{S}$ and any $u \in \mathcal{U}$.

Note that each term in (2.14) is meaningful since Ξ_ℓ is bounded and $X_1 - X_0 \in L^1(Q)$.

Proof. The direct statement follows from Definition 1.4 and Lemma 2.8.
Let us prove its converse. We define the event $A_n = \{X_1 - X_0 = n\}$ and consider n such that $Q(A_n) > 0$. We note that (2.14) is satisfied by the measure $Q^n := \frac{\mathbf{1}_{A_n}}{Q(A_n)}Q$ as well. By (2.8), $G_\ell := dP_\ell/d\mathbf{P}$ given at (1.3) is differentiable in any direction $u \in \mathcal{U}$. We define the probability measure $\widetilde{Q}^n$ as follows

$$d\widetilde{Q}^n := c\, G_\ell^{-1} dQ^n \tag{2.15}$$

where c is the normalising constant. Since G_ℓ^{-1} is uniformly bounded from above and below on A_n, $\widetilde{Q}^n$ is well defined. With (2.8), our assumption (2.14) leads us to

$$E_{Q^n}\mathcal{D}_u(G_\ell^{-1}\Phi) = E_{Q^n}\Big(G_\ell^{-1}\Phi \int_0^1 [\dot{u}_t + \Xi_\ell(t)u_t]\, dX_t\Big)$$

$$= E_{Q^n}\Big(G_\ell^{-1}\Phi \int_0^1 \dot{u}_t\, dX_t\Big) + E_{Q^n}\Big(\mathcal{D}_u(G_\ell^{-1})\Phi\Big). \tag{2.16}$$

Hence,

$$E_{\widetilde{Q}^n}(\mathcal{D}_u\Phi) = c\, E_{Q^n}\big(G_\ell^{-1}\mathcal{D}_u\Phi\big) = -c\, E_{Q^n}(\mathcal{D}_u(G_\ell^{-1})\Phi) + c\, E_{Q^n}\mathcal{D}_u(G_\ell^{-1}\Phi)$$

$$\overset{(2.16)}{=} c\, E_{Q^n}\Big(G_\ell^{-1}\Phi \int_{[0,1]} \dot{u}(t)dX_t\Big) = E_{\widetilde{Q}^n}\Big(\Phi \int_{[0,1]} \dot{u}(t)dX_t\Big).$$

It follows from Corollary 2.10 that $\widetilde{Q}^n \in \mathfrak{R}(1)$. By Proposition 1.5 there exists h such that $d\widetilde{Q}^n = h(X_0, X_1)\,\mathbf{P}$. But this implies that

$$dQ^n = c^{-1}\, G_\ell\, d\widetilde{Q}^n = c^{-1}\, G_\ell\, h(X_0, X_1)\, d\mathbf{P} = c^{-1}\, h(X_0, X_1)\, dP_\ell$$

and therefore $Q^n \in \mathfrak{R}(\ell)$. By integrating with respect to n, we obtain that $Q \in \mathfrak{R}(\ell)$ which is the desired result. □

Theorem 2.11 improves Theorem 1.11 significantly because (i) it is not required a priori that the process which stands in the reciprocal class is an NMC process and (ii) no explicit expression of its intensity of jump is required.

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