Hydrodynamic Limits in Particle Systems

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25th of April, 2022

Disclaimer

- not rigorous at all!
- mostly heuristics
- references at the end

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- slides for experts...

1. The Symmetric Simple Exclusion Process
2. What Are Hydrodynamic Limits
3. Extending the Model

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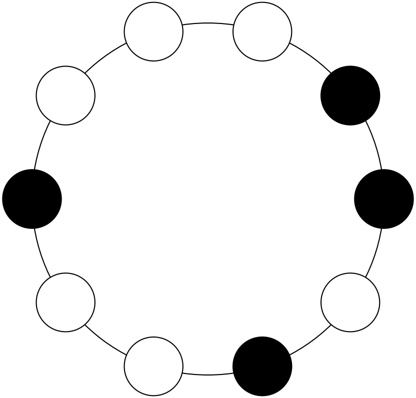
The SSEP



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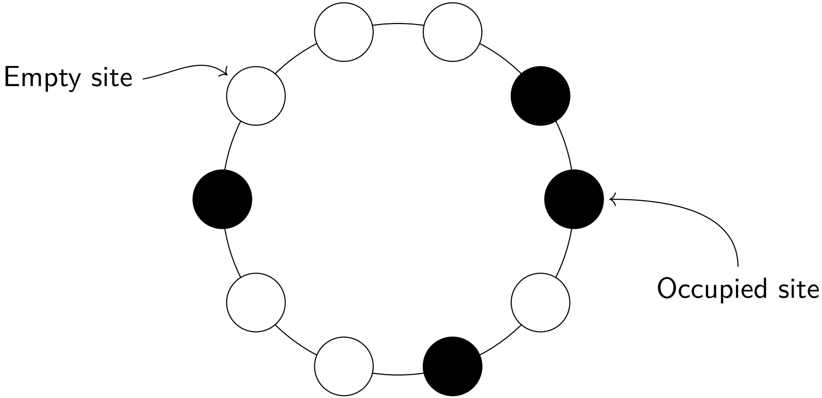

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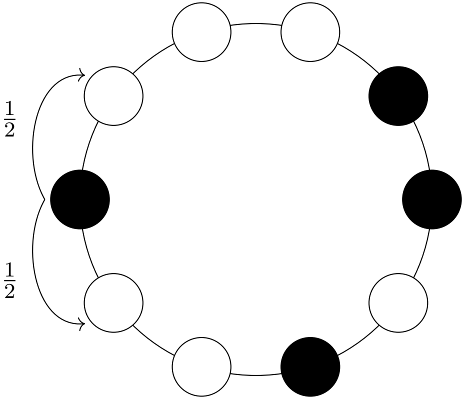

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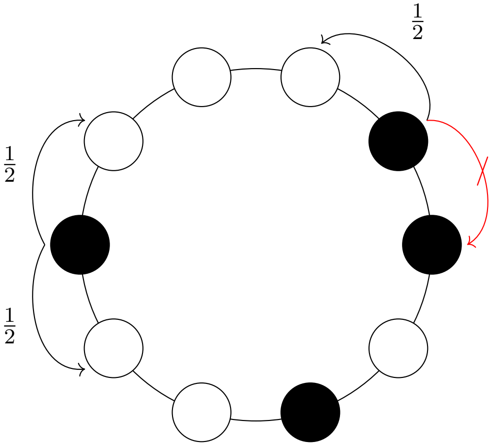

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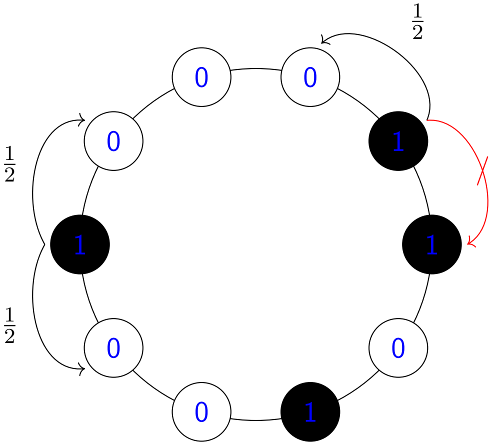

SSEP: Notation

- on torus $\mathbb{T}^N := \mathbb{Z}/N\mathbb{Z}$
- state space: **configurations** $\Omega^N := \{0,1\}^{\mathbb{T}^N}$
- configuration $\eta \in \Omega^N$:

$$\begin{aligned} \eta_x = 0 \quad &\Leftrightarrow \quad \text{site } x \text{ is empty} \\ \eta_x = 1 \quad &\Leftrightarrow \quad \text{site } x \text{ is occupied} \end{aligned}$$

- write $\eta^{x,y}$ for the configuration where sites x and y are swapped:

$$\eta_z^{x,y} = \begin{cases} \eta_y & \text{if } z = x \\ \eta_x & \text{if } z = y \\ \eta_z & \text{otherwise} \end{cases}$$

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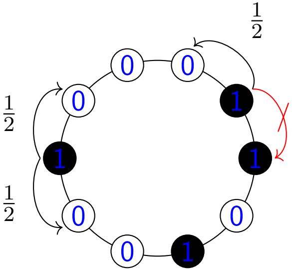

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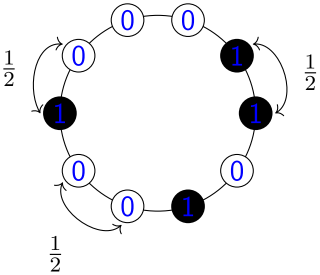

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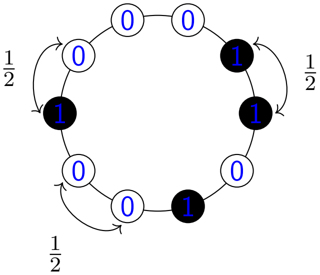

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- Set $\rho_t^N\left(\frac{x}{N}\right) := \mathbb{E}[\eta_t^N(x)]$.
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$$\partial_t \rho_t^N\left(\frac{x}{N}\right) = \mathbb{E}\left[\mathcal{L}^N \eta_t^N(x)\right]$$

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Two descriptions of the world

A microscopic description

- atoms
- molecules
- cars
- microscopic particles

A macroscopic description

- heat diffusion (heat equation)
- fluid dynamics (Navier-Stokes)
- traffic flow (Burger's equation)
- some PDE

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From microscopic to macroscopic

- zoom out: $x \rightsquigarrow \frac{x}{N}$
- keep the global density of particles fixed! $\rightsquigarrow \quad \approx N$ particles
- accelerate time by $\Theta(N)$
- take $N \to +\infty$

$\rightsquigarrow$ **Hydrodynamic Limit**: behaviour as $N \to +\infty$ of

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Instead of the mean density, use the **empirical distribution**:

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Models density:

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LLN: Inside a **small macroscopic** ball around $\frac{x}{N}$ are a lot of particles!

$\rightsquigarrow$ averaging effect

$\rightsquigarrow$ deterministic limit

Mixing: After mixing time, process is at **equilibrium**!

$\rightsquigarrow$ at time scale $\Theta(N)$: **locally at equilibrium**

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The role of time scale $\Theta(N)$

Three regimes:

trivial:	too small $\rightsquigarrow$ no evolution
hydrodynamic:	exactly right $\rightsquigarrow$ local equilibrium
hydrostatic:	too big $\rightsquigarrow$ global equilibrium

LLN $\leftrightarrow$ Mixing:

- Mixing happens in microscopic box of size $\frac{\varepsilon\Theta(N)}{N}$

$\rightsquigarrow$ That means that

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Possbile directions

- extend to $\mathbb{Z}$ (infinite volume)
- add reservoirs (boundary problem)
- add asymmetry (subdiffusive scale)
- add long range interaction (fractional PDE)
- mix the above!

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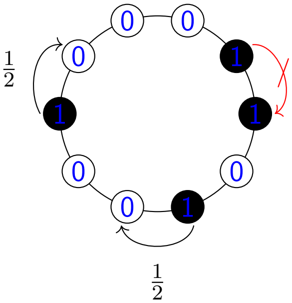

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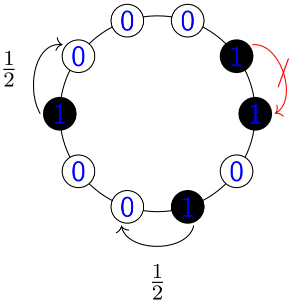

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$\rightsquigarrow$ **Replacement Lemma**:

$$\eta_x \eta_{x+1} \approx \overleftarrow{\eta}^{\ell}_x \cdot \overrightarrow{\eta}^{\ell}_{x+1}$$

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