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Exponential a.s. synchronization of one-dimensional diffusions with non-regular coefficients

Abstract

We study the asymptotic behaviour of a real-valued diffusion whose non-regular drift is given as a sum of a dissipative term and a bounded measurable one. We prove that two trajectories of that diffusion converge a.s. to one another at an exponential explicit rate as soon as the dissipative coefficient is large enough. A similar result in L_p is obtained.

Keywords: stochastic differential equation (SDE); singular drift; synchronization.

1 Introduction

The main object of our study is a one-dimensional stochastic differential equation (SDE) of the type

$$\begin{cases} dX_t = (-\lambda X_t + a(X_t)) dt + \sigma(X_t) dw_t, & t > 0, \\ X_0 \equiv x, \end{cases}$$

where λ is a positive real number, the drift a is *measurable*, the diffusion coefficient σ is a Lipschitz continuous non-degenerate function, and $(w_t)_{t \geq 0}$ is a Wiener process.

Thanks to the celebrated transform method, Zvonkin proved in [15] that this SDE admits a unique strong solution, which we will denote by $(X_t^x)_{t > 0}$. Moreover, it was proved during the last decade that due to the presence of noise, the family of processes $(X_t^x)_{t \geq 0, x \in \mathbb{R}}$ shows good spatial regularity

properties even if the drift function is discontinuous, see for example [2, 3, 4, 5, 6, 8, 11, 12, 9].

Concerning the time asymptotic stability of the process $(X_t^x)_{t \geq 0}$ there are much less results in the literature. In case λ is large enough, which corresponds to a strong attraction of the dynamics towards 0 and a strong dissipativity, it is natural to expect that, asymptotically in time, X_t^x will forget its initial position x . Indeed, under Lipschitz continuity assumption on the drift function a , it is proved e.g. in [10], that the L_p -distance between X_t^x and $X_t^y, y \neq x$, vanishes as t tends to $+\infty$, but no rate is available. In [7] the stabilisation is shown as a convergence in probability of $X_t^x - X_t^y$ towards 0, under \mathcal{C}^1 -regularity assumption on the drift function via the negativity of the associated top Lyapunov exponent. For diffusions whose drift function is not differentiable but admits a finite variation, an explicit representation of the Sobolev derivative of $x \mapsto X_t^x$ can be found in [2]. This representation makes it possible to find an exponential decreasing rate for $|X_t^x - X_t^y|, y \neq x$ as $t \rightarrow \infty$, when a stationary distribution exists. Recently, such asymptotic stability was obtained in a multidimensional framework, for diffusions whose drift function admits jump discontinuities concentrated along a hyperplane, see [1].

In the present paper, we address and solve the question of *almost sure synchronization* - see the exact definition in (2) - in high dissipative regime (λ large) for a wide class of SDEs with *irregular drift* functions: the function a is only supposed to be the sum of a Lipschitz function and of a bounded *measurable* one. Furthermore, we exhibit an explicit *exponential convergence rate* to 0 for $|X_t^x - X_t^y|$, both *almost surely* (see (29) and (30)) and in L_p . To our knowledge it is the first result of that type under such general assumptions.

Note that in absence of noise ($\sigma \equiv 0$), there is no reason to expect synchronization of $(X_t^x)_{t \geq 0, x \in \mathbb{R}}$ asymptotically in time. Indeed, consider the ODE $u'(t) = -\lambda u(t) + \text{sgn}(u(t))$ with initial condition x , whose unique solution

is given by $u(t) = \frac{\operatorname{sgn}(x)}{\lambda} + \left(x - \frac{\operatorname{sgn}(x)}{\lambda}\right) e^{-\lambda t}$. Thus $\lim_{t \rightarrow \infty} u(t) = \frac{\operatorname{sgn}(x)}{\lambda}$ which exhibits a clear discontinuity at the point $x = 0$, which corresponds in fact to the (unique) discontinuity point of the drift function $a = \operatorname{sgn}$. We are thus in presence of a phenomena known in the literature as *synchronization by noise*, see [7].

In the spirit of Zvonkin, our approach is based on an accurately chosen space-transform in such a way that the transformed SDE - written via the new coordinate - has a simpler structure. A similar method could theoretically be used in more general context - multidimensional diffusions or SDEs with Lévy-noise. However, the construction of corresponding transforms requires the investigation of elliptic equations whose solution is a non-trivial problem.

The paper is organized as follows. The main results are formulated in Section 2 and the proofs are presented in Section 3.

2 Main results

First we study the asymptotic behavior with respect to its initial condition of the strong solution of an SDE with regular dissipative drift term. Though the result seems to be well known, we failed to find an exact reference. Besides, the proof is instructive itself.

Proposition 1. *Consider the SDE*

$$dY_t = b(Y_t)dt + \sigma(Y_t)dw_t, \quad t > 0, \quad (1)$$

where $(w_t)_{t \geq 0}$ is a Wiener process. Suppose that the following assumptions hold:

(\mathcal{H}_1) *The drift b is continuous and satisfies a dissipative condition:*

$$\exists D_b > 0 \quad \forall x, y \in \mathbb{R} \quad (b(y) - b(x))(y - x) \leq -D_b(y - x)^2;$$

(\mathcal{H}_2) The diffusion coefficient σ is a global Lipschitz continuous function :

$$\exists L_\sigma > 0 \quad \forall x, y \in \mathbb{R} \quad |\sigma(y) - \sigma(x)| \leq L_\sigma |y - x|.$$

Then, denoting by $(Y_t^x)_{t \geq 0}$ the unique strong solution of (1) starting in $x \in \mathbb{R}$, the following almost sure synchronization at exponential rate holds: for any $c < D_b$,

$$\forall x, y \in \mathbb{R}, \quad \lim_{t \rightarrow +\infty} |Y_t^y - Y_t^x| e^{ct} = 0 \quad a.s. \quad (2)$$

Moreover, if $c_{p,b,\sigma} := D_b - \frac{p-1}{2} L_\sigma^2$ is positive, the following bound holds in $L_p, p \geq 2$:

$$\forall x, y \in \mathbb{R}, \quad \forall t > 0, \quad \|Y_t^y - Y_t^x\|_p \leq |y - x| e^{-c_{p,b,\sigma} t}. \quad (3)$$

The main result of the paper, which follows, concerns the asymptotic behavior of the solution of an SDE generalising (1), whose drift function b is the sum of a linear dissipative term, a globally Lipschitz term β and a *non regular* bounded term α .

Theorem 1. Consider the SDE

$$\begin{cases} dX_t = (-\lambda X_t + \beta(X_t) + \alpha(X_t)) dt + \sigma(X_t) dw_t, & t > 0, \\ X_0 = x, \end{cases} \quad (4)$$

where $(w_t)_{t \geq 0}$ is a Wiener process. Suppose that the following assumptions hold:

(\mathcal{A}_1) The function β is global Lipschitz continuous:

$$\exists L_\beta \geq 0 \quad \forall x, y \in \mathbb{R} \quad |\beta(y) - \beta(x)| \leq L_\beta |y - x|;$$

(\mathcal{A}_2) The function σ is global Lipschitz continuous:

$$\exists L_\sigma \geq 0 \quad \forall x, y \in \mathbb{R} \quad |\sigma(y) - \sigma(x)| \leq L_\sigma |y - x|$$

and it is uniformly elliptic:

$$\exists c_\sigma > 0 \quad \forall x \in \mathbb{R} \quad \sigma^2(x) \geq c_\sigma.$$

Assume also that one of the following two conditions is satisfied:

(\mathcal{A}_3) the function α is bounded measurable with compact support

or

(\mathcal{A}'_3) the function α is measurable and its absolute value is a.s. bounded by a bounded global Lipschitz function $g \in L_1(\mathbb{R})$; moreover the functions β and σ are supposed to be bounded too.

Then, in high dissipative regime - λ large enough - the strong solutions of (4) X_t^x and X_t^y starting at different positions x and y almost sure synchronize at exponential rate, i.e., there exists λ_0 such that for any $\lambda > \lambda_0$ there exists a positive constant c_λ given explicitly in (28), (29), (30) such that

$$\forall x, y \in \mathbb{R} \quad \lim_{t \rightarrow \infty} |X_t^y - X_t^x| e^{c_\lambda t} = 0 \quad a.s.. \quad (5)$$

Moreover, the following bound holds in $L_p, p \geq 2$:

$$\exists C > 0, c_{\lambda,p} > 0 \quad \forall x, y \in \mathbb{R}, \quad \forall t \geq 0, \quad \|X_t^y - X_t^x\|_p \leq C |y - x| e^{-c_{\lambda,p} t}. \quad (6)$$

3 Proofs

Proof of Proposition 1. Notice first that, applying [14, Proposition 2.1], assumptions (\mathcal{H}_1)-(\mathcal{H}_2) provide the existence of a unique global strong solution to (1), denoted here by $(Y_t^x)_{t \geq 0}$.

Fix any $y > x$ and define the stopping time $\tau := \inf\{t \geq 0 \mid Y_t^x = Y_t^y\}$. Due to the continuity of trajectories, one has for $t \in [0, \tau)$, $Y_t^y > Y_t^x$ a.s.

So, applying Itô's formula to $\ln(Y_t^y - Y_t^x)$ we obtain

$$d(\ln(Y_t^y - Y_t^x)) = \left(\frac{b(Y_t^y) - b(Y_t^x)}{Y_t^y - Y_t^x} - \frac{(\sigma(Y_t^y) - \sigma(Y_t^x))^2}{2(Y_t^y - Y_t^x)^2} \right) dt + \frac{\sigma(Y_t^y) - \sigma(Y_t^x)}{Y_t^y - Y_t^x} dw_t, \quad t \in [0, \tau]. \quad (7)$$

Moreover, according to [14, Proposition 2.1] the processes $(Y_t^x)_{t \geq 0, x \in \mathbb{R}}$ satisfy a coalescence property which means that as soon as two solutions meet in a point then they stay together forever: $Y_t^x = Y_t^y, t \geq \tau$ a.s.

Therefore, by (\mathcal{H}_1) , the following inequality holds at any time:

$$\ln(Y_t^y - Y_t^x) \leq \ln(y - x) - D_b t + \int_0^t \frac{\sigma(Y_s^y) - \sigma(Y_s^x)}{Y_s^y - Y_s^x} 1_{s < \tau} dw_s \quad \text{a.s.} \quad (8)$$

where $\ln 0 := -\infty$. Note that the expression under the integral sign is bounded because the function σ is Lipschitz continuous.

Further, the martingale $\int_0^t \frac{\sigma(Y_s^y) - \sigma(Y_s^x)}{Y_s^y - Y_s^x} 1_{s < \tau} dw_s$ can be represented as a Brownian motion computed at the random time $\int_0^t \left(\frac{\sigma(Y_s^y) - \sigma(Y_s^x)}{Y_s^y - Y_s^x} \right)^2 1_{s < \tau} ds$ which is uniformly bounded by $L_\sigma^2 t$. Thus the law of iterated logarithm yields

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{\sigma(Y_s^y) - \sigma(Y_s^x)}{Y_s^y - Y_s^x} 1_{s < \tau} dw_s = 0 \quad \text{a.s.} \quad (9)$$

Now the decreasing rate of $|Y_t^y - Y_t^x|$ announced in (2) follows from (8) and (9).

Let us now prove the L_p -bound, $p \geq 2$. For any constant k , it follows

from Itô's formula

$$\begin{aligned}
|Y_t^y - Y_t^x|^p e^{kt} &= |y - x|^p \\
&+ \int_0^t \left(k|Y_s^y - Y_s^x|^2 + p(b(Y_s^y) - b(Y_s^x))(Y_s^y - Y_s^x) \right. \\
&+ \left. \frac{p(p-1)}{2} (\sigma(Y_s^y) - \sigma(Y_s^x))^2 \right) |Y_s^y - Y_s^x|^{p-2} e^{ks} ds \\
&+ \int_0^t p(\sigma(Y_s^y) - \sigma(Y_s^x)) \operatorname{sgn}(Y_s^y - Y_s^x) |Y_s^y - Y_s^x|^{p-1} e^{ks} dw_s.
\end{aligned}$$

Using the dissipativity of b and the Lipschitzianity of σ we get

$$\begin{aligned}
|Y_t^y - Y_t^x|^p e^{kt} &\leq |y - x|^p \\
&+ \int_0^t \left(k - pD_b + \frac{p(p-1)}{2} L_\sigma^2 \right) |Y_s^y - Y_s^x|^p e^{ks} ds \\
&+ \int_0^t p(\sigma(Y_s^y) - \sigma(Y_s^x)) \operatorname{sgn}(Y_s^y - Y_s^x) |Y_s^y - Y_s^x|^{p-1} e^{ks} dw_s. \quad (10)
\end{aligned}$$

The stochastic Gronwall lemma proved in [13] allows to deduce that - see also the stochastic Gronwall-Lyapunov inequality in [9] which improves considerably the former results -

$$\forall T \geq 0 \quad \forall x, y \in \mathbb{R} \quad \forall p \geq 2 : \sup_{t \in [0, T]} \mathbb{E} |Y_t^y - Y_t^x|^p < +\infty.$$

So, since σ is Lipschitz continuous, the stochastic integral in the rhs of (10) is not only a local martingale but also an integrable centered martingale. Now, as soon as $k \leq pD_b - \frac{p(p-1)}{2} L_\sigma^2$,

$$\mathbb{E} (|Y_t^y - Y_t^x|^p) e^{kt} \leq |y - x|^p$$

which implies (3). □

Proof of Theorem 1. Notice first that assumptions $(\mathcal{A}_1), (\mathcal{A}_2)$ and the boundedness of α provide the existence of a unique strong solution to equation (4).

This result follows from [15] via a localization method.

Now, since the function α appearing in the drift is not regular we cannot apply directly Proposition 1. Our first step will then consist to follow Zvonkin's idea and transform the dynamics of (4) into an SDE with regular drift. Unfortunately by removing only the irregular term α , we do not obtain a transformed dynamics satisfying the dissipative assumption (\mathcal{H}_1) . We then introduce a bounded, Lipschitz continuous, integrable *intermediate function* γ , whose exact choice will be done later, see (24) and (26). A partial Zvonkin's transform to remove the drift $\alpha - \gamma$ will yield the SDE (18), whose drift $\tilde{b} := -\lambda \tilde{d} + \tilde{\beta} + \tilde{\gamma}$ is indeed dissipative for λ large enough, as we will prove.

So we rewrite equation (4) as follows:

$$\begin{cases} dX_t = (-\lambda X_t + (\beta(X_t) + \gamma(X_t)) + (\alpha(X_t) - \gamma(X_t))) dt + \sigma(X_t) dw_t, t \geq 0, \\ X_0 \equiv x. \end{cases}$$

To eliminate the non-regular term $\alpha - \gamma$, we define the (partial) scale function s on \mathbb{R} by

$$s(x) := \int_0^x \exp\left(-2 \int_0^y \frac{\alpha(z) - \gamma(z)}{\sigma^2(z)} dz\right) dy, \quad x \in \mathbb{R}. \quad (11)$$

It is differentiable and

$$s'(x) = \exp\left(-2 \int_0^x \frac{\alpha(z) - \gamma(z)}{\sigma^2(z)} dz\right) \quad (12)$$

which is uniformly bounded from below and above as follows:

$$0 < \frac{1}{L_s} \leq s'(x) \leq L_s < +\infty \quad \text{where} \quad L_s := \exp\left(2 \int_{-\infty}^{\infty} \frac{|\alpha(z) - \gamma(z)|}{\sigma^2(z)} dz\right). \quad (13)$$

The finiteness (resp. positivity) of L_s is due to the integrability of both α and γ , combined with the uniform lower bound of σ .

Moreover, the second derivative of s exists for almost all x and satisfies

$$s''(x) = 2 \frac{\gamma(x) - \alpha(x)}{\sigma^2(x)} s'(x). \quad (14)$$

Due to (13), s is a *bilateral Lipschitz continuous* function:

$$\forall x, y \in \mathbb{R}, \quad \frac{1}{L_s} |y - x| \leq |s(y) - s(x)| \leq L_s |y - x|. \quad (15)$$

Since (14) yields a uniform bound on s'' , we get that s' is also global Lipschitz continuous:

$$\forall x, y \in \mathbb{R}, \quad |s'(y) - s'(x)| \leq L_{s'} |y - x| \text{ where } L_{s'} := 2 \frac{\|\gamma - \alpha\|_\infty}{c_\sigma} L_s. \quad (16)$$

The derivative of s being positive, the function s is strictly increasing. Moreover, since $s(\mathbb{R}) = \mathbb{R}$, it admits an inverse function s^{-1} defined on \mathbb{R} and being a *bilateral Lipschitz continuous* function too:

$$\forall x, y \in \mathbb{R}, \quad \frac{1}{L_s} |y - x| \leq |s^{-1}(y) - s^{-1}(x)| \leq L_s |y - x|. \quad (17)$$

The process $s(X_t^x)$ satisfies the following Itô's formula:

$$\begin{aligned} ds(X_t^x) &= s'(X_t^x) dX_t^x + \frac{1}{2} s''(X_t^x) \sigma^2(X_t^x) dt \\ &= s'(X_t^x) (-\lambda X_t^x + \beta(X_t^x) + \gamma(X_t^x)) dt + s'(X_t^x) \sigma(X_t^x) dw_t. \end{aligned}$$

Note that s'' may not exist on a negligible set. However, the applicability of Itô's formula is justified, see e.g. [15, Theorem 3].

Denote the process $s(X_t^x)$ by \tilde{X}_t^x . It solves the SDE:

$$\begin{cases} d\tilde{X}_t^x = \left(-\lambda \tilde{X}_t^x + \tilde{\beta}(\tilde{X}_t^x) + \tilde{\gamma}(\tilde{X}_t^x) \right) dt + \tilde{\sigma}(\tilde{X}_t^x) dw_t, t > 0, \\ \tilde{X}_0 \equiv s(x), \end{cases} \quad (18)$$

where the coefficients are given by

$$\tilde{id} := s' \circ s^{-1} \cdot s^{-1}, \tilde{\beta} := s' \circ s^{-1} \cdot \beta \circ s^{-1}, \tilde{\gamma} := s' \circ s^{-1} \cdot \gamma \circ s^{-1}, \tilde{\sigma} := s' \circ s^{-1} \cdot \sigma \circ s^{-1}.$$

We underline that the irregular drift term α disappeared from the dynamics.

Next step in the proof of the theorem is to check that, for λ large enough, the new drift

$$\tilde{b} := -\lambda \tilde{id} + \tilde{\beta} + \tilde{\gamma}$$

appearing in the transformed SDE (18) satisfies assumption (\mathcal{H}_1) in order to apply Proposition 1 to the process \tilde{X}_t .

Regularity of the three terms composing the drift \tilde{b} .

The next lemma is straightforward.

Lemma 1. *If f and $g : \mathbb{R} \rightarrow \mathbb{R}$ are two Lipschitz continuous functions with respective constant L_f and L_g , their composition $f \circ g$ is also a continuous Lipschitz function with constant $L_f L_g$. If additionally f and g are bounded, then the product fg is a Lipschitz continuous function too with constant $\|f\|_\infty L_g + \|g\|_\infty L_f$.*

It follows from (17) and Lemma 1 that the functions $s' \circ s^{-1}, \beta \circ s^{-1}, \gamma \circ s^{-1}, \sigma \circ s^{-1}$ are Lipschitz continuous, with respective Lipschitz constants $L_{s'} L_s, L_\beta L_s, L_\gamma L_s, L_\sigma L_s$. Then the function \tilde{id} appearing as first term in \tilde{b} is locally Lipschitz continuous.

Since the function γ we will construct will be bounded and Lipschitz continuous, by Lemma 1 the function $\tilde{\gamma}$ is Lipschitz continuous with constant

$$L_{\tilde{\gamma}} = (L_s L_\gamma + \|\gamma\|_\infty L_{s'}) L_s. \quad (19)$$

Let us now construct the function γ such that $\tilde{\beta}$ and $\tilde{\sigma}$ are global Lipschitz continuous. We distinguish both cases, depending on the assumption satisfied by the measurable function α .

- Assumption (\mathcal{A}'_3) holds, i.e. β and σ are bounded.

Then, by Lemma 1, $\tilde{\beta}$ and $\tilde{\sigma}$ are Lipschitz continuous functions with respective constants

$$L_{\tilde{\beta}} = (L_s L_\beta + \|\beta\|_\infty L_{s'}) L_s \quad \text{and} \quad L_{\tilde{\sigma}} = (L_s L_\sigma + \|\sigma\|_\infty L_{s'}) L_s. \quad (20)$$

- Assumption (\mathcal{A}_3) holds, i.e. α has compact support, denoted by $[-N_\alpha, N_\alpha]$.

Since β and σ are not a priori bounded, one can not directly apply Lemma 1 to obtain the regularity of $\tilde{\beta}$ and $\tilde{\sigma}$. It will be possible to construct γ with compact support included in $[-N_\alpha - 1, N_\alpha + 1]$. Since the function $x \mapsto s(x)$ is then linear for $|x| \geq N_\alpha + 1$, by checking the increments of $\tilde{\beta}$ (resp. $\tilde{\sigma}$) separately on the intervals $(-\infty, s(-N_\alpha - 1)]$, $[s(-N_\alpha - 1), s(N_\alpha + 1)]$ and $[s(N_\alpha + 1), +\infty)$ one gets that $\tilde{\beta}$ and $\tilde{\sigma}$ are global Lipschitz continuous with respective constant

$$L_{\tilde{\beta}} = (L_s L_\beta + \|\beta\|_{N_\alpha+1} L_{s'}) L_s \quad \text{and} \quad L_{\tilde{\sigma}} = (L_s L_\sigma + \|\sigma\|_{N_\alpha+1} L_{s'}) L_s, \quad (21)$$

where the following notation is used: $\|f\|_{N_\alpha+1} := \sup_{|x| \leq N_\alpha+1} |f(x)|$.

Notice that all the above Lipschitz constants $L_{\tilde{\beta}}, L_{\tilde{\gamma}}, L_{\tilde{\sigma}}$ may depend on the intermediate drift function γ but not on the real coefficient λ .

Dissipative property of the drift \tilde{b} for λ large enough:

We now show that for λ large enough, the function $\tilde{b} = -\lambda \tilde{id} + \tilde{\beta} + \tilde{\gamma}$ is dissipative and compute its dissipative constant denoted by $D_{\tilde{b}}$. To this aim, we will prove that the slope of the function \tilde{id} is bounded from below by 1/2:

$$\forall x, y \in \mathbb{R}, \quad \frac{\tilde{id}(y) - \tilde{id}(x)}{y - x} \geq \frac{1}{2}. \quad (22)$$

With other words \tilde{id} satisfies a one-sided Lipschitz property. As soon as (22) is proved, it is straightforward to deduce that

$$D_{\tilde{b}} \geq \frac{\lambda}{2} - L_{\tilde{\beta}} - L_{\tilde{\gamma}}. \quad (23)$$

So, for any $\lambda > 2(L_{\tilde{\beta}} + L_{\tilde{\gamma}})$, the drift \tilde{b} is dissipative.

Let us now construct a bounded, Lipschitz continuous, integrable intermediate function γ in such a way that (22) holds true. It is enough to prove that the derivative of $\tilde{id}(= s' \circ s^{-1} \cdot s^{-1})$, which exists almost everywhere, is bounded from below by $\frac{1}{2}$. In fact, for a.a. x ,

$$(\tilde{id})'(x) = \frac{s'' \circ s^{-1}(x)}{s' \circ s^{-1}(x)} s^{-1}(x) + s' \circ s^{-1}(x) \frac{1}{s' \circ s^{-1}(x)} = \left(\frac{s''(u)}{s'(u)} u + 1 \right) \Big|_{u=s^{-1}(x)}.$$

Recall that, since s' is an absolutely continuous function, s'' exists almost everywhere on \mathbb{R} . It follows from (15) that mappings s and s^{-1} push sets of Lebesgue measure zero to sets of Lebesgue measure zero. Thus $s''(s^{-1}(x))$ is independent of a modification of s'' on a negligible set.

Taking into account (14), we get

$$\frac{s''(u)}{s'(u)} u + 1 = 2 \frac{\gamma(u) - \alpha(u)}{\sigma^2(u)} u + 1 \quad \text{for a.a. } u.$$

Let us separate both cases \mathcal{A}_3 and \mathcal{A}'_3 .

- If assumption (\mathcal{A}_3) holds, we denote the compact support of the function α as above by $[-N_\alpha, N_\alpha]$. Fix a positive number $\delta < \|\alpha\|_\infty$ and define an *odd* function γ as follows (see Figure 1):

$$\gamma(u) = \begin{cases} \|\alpha\|_\infty \frac{u}{\delta}, & u \in [0, \delta], \\ \|\alpha\|_\infty, & u \in [\delta, N_\alpha], \\ \|\alpha\|_\infty (N_\alpha + 1 - u), & u \in [N_\alpha, N_\alpha + 1], \\ 0, & u \in [N_\alpha + 1, +\infty), \\ -\gamma(-u), & u \in \mathbb{R}_-. \end{cases} \quad (24)$$

Such function is clearly bounded, Lipschitz continuous and integrable.

Moreover, since by construction $(\gamma(u) - \alpha(u))u \geq 0$ for any $|u| \geq \delta$, $u \mapsto \frac{s''(u)}{s'(u)} u + 1$ is a.a. bounded from below by 1 on that domain.

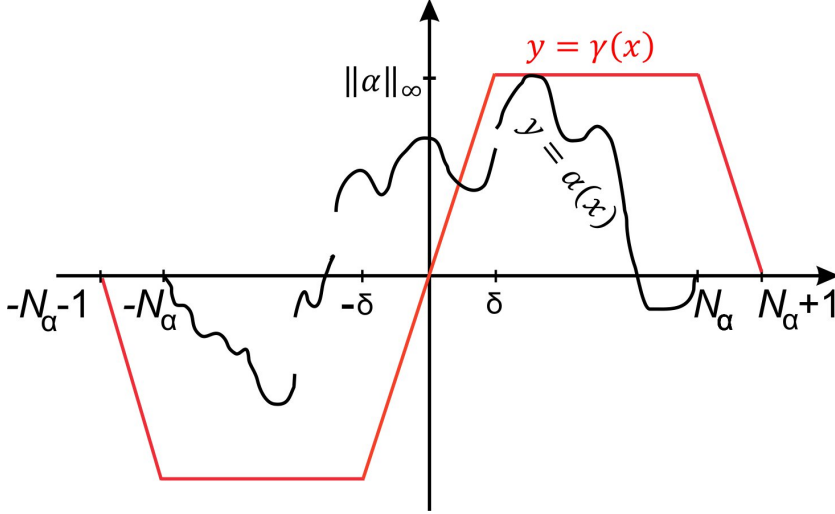


Figure 1:

Inside of the interval $[-\delta, +\delta]$, since $\gamma(u)u \geq 0$, one has:

$$2 \frac{\gamma(u) - \alpha(u)}{\sigma^2(u)} u + 1 \geq -2 \frac{\alpha(u)}{\sigma^2(u)} u + 1 \geq -2 \frac{\|\alpha\|_\infty}{c_\sigma} \delta + 1. \quad (25)$$

Choose $\delta = \frac{c_\sigma}{4\|\alpha\|_\infty}$; one then obtains that $u \mapsto \frac{s''(u)}{s'(u)} u + 1$ is bounded from below by $1/2$ on $[-\delta, +\delta]$. To summarize, we were able to construct a function γ such that uniformly $(\tilde{id})' \geq 1/2$.

- If assumption (\mathcal{A}'_3) is fulfilled, there exists a bounded integrable Lipschitz continuous function g such that $g(u) > |\alpha(u)|, u \in \mathbb{R}$. Without loss of generality we may assume that g is an even function. In this case, set as above $\delta := \frac{c_\sigma}{4\|\alpha\|_\infty}$ and define the odd function γ as follows (see Figure 2):

$$\gamma(u) = \begin{cases} g(\delta) \frac{u}{\delta}, & u \in [0, \delta], \\ g(u), & u \in [\delta, +\infty), \\ -\gamma(-u), & u \in \mathbb{R}_-. \end{cases} \quad (26)$$

By the same argumentation as in the first case, the function $(\tilde{id})'$ is bounded

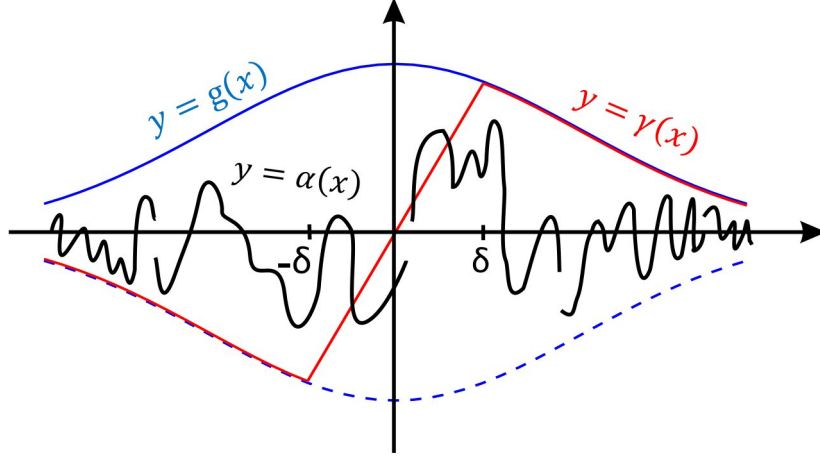


Figure 2:

from below by $1/2$.

Last steps of the proof of Theorem 1.

Applying now Proposition 1 to the process $(\tilde{X}_t)_{t \geq 0}$, thanks to (23), one gets that for $\lambda > 2(L_{\tilde{\beta}} + L_{\tilde{\gamma}})$, the following a.s. synchronization holds

$$\forall x, y \in \mathbb{R}, \quad \lim_{t \rightarrow +\infty} |\tilde{X}_t^y - \tilde{X}_t^x| e^{ct} = 0 \quad \text{a.s.} \quad (27)$$

for any $c < c_\lambda := \frac{\lambda}{2} - L_{\tilde{\beta}} - L_{\tilde{\gamma}} \leq D_{\tilde{b}}$.

To deduce the a.s. synchronization of the process $(X_t)_{t \geq 0}$ from (27) we use the Lipschitz continuity of the function s^{-1} . The exponential rate of convergence for both processes is then identical.

Hence, we may select

$$\lambda_0 := 2(L_{\tilde{\beta}} + L_{\tilde{\gamma}}) \quad \text{and} \quad c_\lambda := \frac{\lambda}{2} - L_{\tilde{\beta}} - L_{\tilde{\gamma}}. \quad (28)$$

We now compute an explicit upper bound for $L_{\tilde{\beta}} + L_{\tilde{\gamma}}$ using only the parameters of the SDE, and not γ .

- If assumption (\mathcal{A}_3) holds, one chooses γ as in (24). Therefore, by (13), one has

$$L_s \leq \exp\left(8 \frac{\|\alpha\|_\infty (N_\alpha + 1)}{c_\sigma^2}\right)$$

and by (16),

$$L_{s'} \leq \frac{4\|\alpha\|_\infty}{c_\sigma} L_s \leq \frac{4\|\alpha\|_\infty}{c_\sigma} \exp\left(8 \frac{\|\alpha\|_\infty (N_\alpha + 1)}{c_\sigma^2}\right).$$

Therefore, using the definition (21),

$$L_{\tilde{\beta}} \leq \left(L_\beta + \|\beta\|_{N_\alpha+1} \frac{4\|\alpha\|_\infty}{c_\sigma}\right) \exp\left(16 \frac{\|\alpha\|_\infty (N_\alpha + 1)}{c_\sigma^2}\right)$$

and

$$\begin{aligned} L_{\tilde{\gamma}} &\leq \left(L_\gamma + \|\gamma\|_\infty \frac{4\|\alpha\|_\infty}{c_\sigma}\right) \exp\left(16 \frac{\|\alpha\|_\infty (N_\alpha + 1)}{c_\sigma^2}\right) \\ &\leq \|\alpha\|_\infty \left(1 + \frac{4\|\alpha\|_\infty}{c_\sigma}\right) \exp\left(16 \frac{\|\alpha\|_\infty (N_\alpha + 1)}{c_\sigma^2}\right). \end{aligned}$$

So

$$\begin{aligned} &L_{\tilde{\beta}} + L_{\tilde{\gamma}} \\ &\leq \left(\|\alpha\|_\infty + L_\beta + (\|\alpha\|_\infty + \|\beta\|_{N_\alpha+1}) \frac{4\|\alpha\|_\infty}{c_\sigma}\right) \exp\left(16 \frac{\|\alpha\|_\infty (N_\alpha + 1)}{c_\sigma^2}\right) \end{aligned} \tag{29}$$

- If assumption (\mathcal{A}'_3) holds, one chooses γ as in (26). By (13), one has $L_s \leq \exp\left(4 \frac{\|g\|_1}{c_\sigma^2}\right)$ and by (16),

$$L_{s'} \leq \frac{4\|g\|_\infty}{c_\sigma} L_s \leq \frac{4\|g\|_\infty}{c_\sigma} \exp\left(4 \frac{\|g\|_1}{c_\sigma^2}\right).$$

Therefore, using the definition (20),

$$L_{\tilde{\beta}} \leq \left(L_{\beta} + \|\beta\|_{\infty} \frac{4\|g\|_{\infty}}{c_{\sigma}} \right) \exp \left(8 \frac{\|g\|_1}{c_{\sigma}^2} \right)$$

and

$$\begin{aligned} L_{\tilde{\gamma}} &\leq \left(L_{\gamma} + \|\gamma\|_{\infty} \frac{4\|g\|_{\infty}}{c_{\sigma}} \right) \exp \left(8 \frac{\|g\|_1}{c_{\sigma}^2} \right) \\ &\leq \left(L_g + \frac{8\|g\|_{\infty}^2}{c_{\sigma}} \right) \exp \left(8 \frac{\|g\|_1}{c_{\sigma}^2} \right). \end{aligned}$$

So in that case,

$$L_{\tilde{\beta}} + L_{\tilde{\gamma}} \leq \left(L_{\beta} + L_g + (\|\beta\|_{\infty} + 2\|g\|_{\infty}^2) \frac{4\|g\|_{\infty}}{c_{\sigma}} \right) \exp \left(8 \frac{\|g\|_1}{c_{\sigma}^2} \right). \quad (30)$$

The L_p -synchronization of $(X_t)_{t \geq 0}$ is a direct consequence from the fact that $(\tilde{X}_t)_{t \geq 0}$ satisfies the L_p -bounds (3): take $C = L_s$ and $c_{\lambda,p} = pc_{\lambda} - \frac{p(p-1)}{2} L_{\tilde{\sigma}}^2$. Indeed, under assumption (\mathcal{A}_3) ,

$$L_{\tilde{\sigma}} \leq \left(L_{\sigma} + \|\sigma\|_{N_{\alpha+1}} \frac{4\|\alpha\|_{\infty}}{c_{\sigma}} \right) \exp \left(16 \frac{\|\alpha\|_{\infty} (N_{\alpha} + 1)}{c_{\sigma}^2} \right)$$

and under assumption (\mathcal{A}'_3)

$$L_{\tilde{\sigma}} \leq \left(L_{\beta} + \|\sigma\|_{\infty} \frac{4\|g\|_{\infty}}{c_{\sigma}} \right) \exp \left(8 \frac{\|g\|_1}{c_{\sigma}^2} \right).$$

The constant $c_{\lambda,p}$ can also be estimated explicitly as function of the parameters of the SDE. This completes the proof. \square

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