Consensus Dynamics and Exclusion processes

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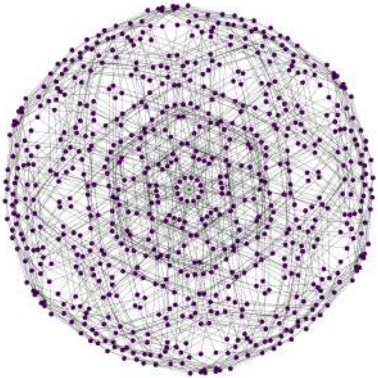

What does symmetry teach us about exclusion processes?

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Change over time $t \in \mathbb{N}$ is induced by interaction of individuals defined by **interaction graphs** $G_t = (V, E_t)$ and some **rule**.

Consensus in Networks

Inspired by Henry, Prałat and Zhang (2011)

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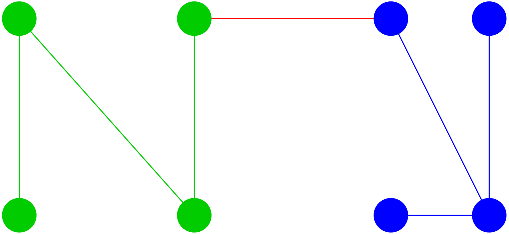

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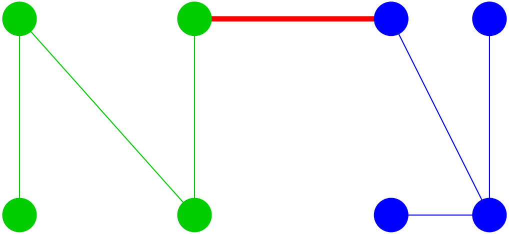

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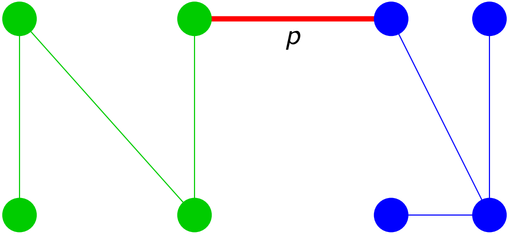

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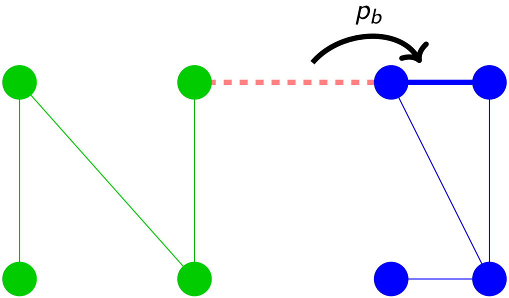

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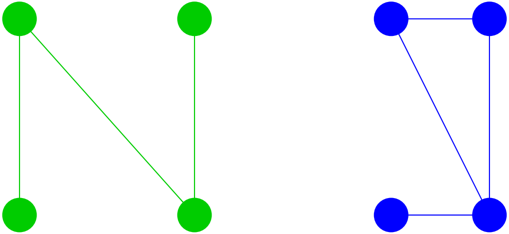

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Goal: Consensus in Networks

Find formalism for arbitrary opinions and graphs!

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(Assume edges move always for now.)

Line graph as state space

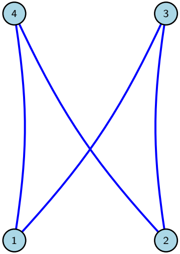

Figure: Translation of existing edges (blue) in G into occupied sides (blue) in the line graph L_{G_c}.

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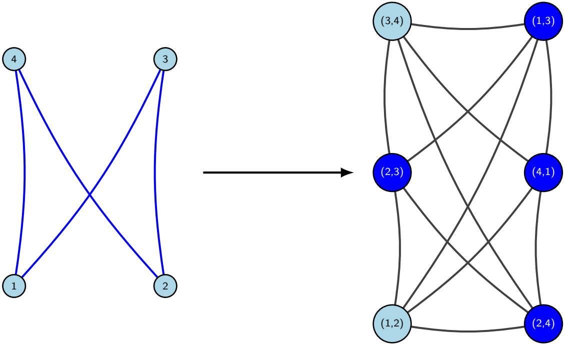

Figure: Translation of existing edges (blue) in G into occupied sides (blue) in the line graph L_{G_c}. Existing edges are interpreted as particles occupying sides on L_{G_c}.

Strongly regular graphs

Definition

Let $\bar{n}, \bar{d}, \alpha, \beta \in \mathbb{N}$. A graph $L = (V, E)$ is called $(\bar{n}, \bar{d}, \alpha, \beta)$ strongly regular iff

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- for $v, w \in V$ with $\{v, w\} \in E$ there are $u_1, \ldots, u_\alpha \in V$ s.t. $\{v, u_i\} \in E, \{w, u_i\} \in E$ for all $i = 1, \ldots, \alpha$,

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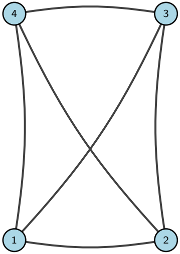
- for $v, w \in V$ with $\{v, w\} \notin E$ there are $p_1, \ldots, p_\beta \in V$ s.t. $\{v, p_i\} \in E, \{w, p_i\} \in E$ for all $i = 1, \ldots, \beta$.

Example: Strongly regular graphs

Consider a complete graph $G_c = (V_c, E_c)$ with $|V_c| = n$. Then its line graph $L(G)$ is a $\left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right)$ stronlgy regular graph.



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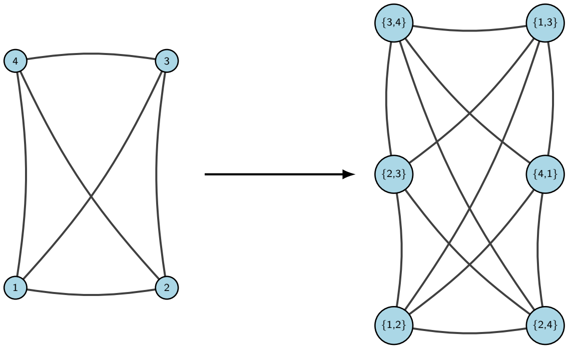

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Exclusion process on strongly-regular graphs

Consider a graph $L = (V, E)$. Define for $v \in V$ its neighborhood N_v as

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Definition

Let $L = (V, E)$ be a strongly regular connected graph with parameters $(\bar{n}, \bar{d}, \alpha, \beta)$. Then we define the exclusion process $\eta_k := (\eta_{k;t})_{t \in \mathbb{N}}$ containing $k \in \{1, \dots, \bar{n} - 1\}$ particles on L as follows.

- $\forall t \in \mathbb{N} : \eta_{k;t} \subset V$,
- at any time $t \in \mathbb{N}$ do:
  1. draw uniformly $U_t \in \eta_{k;t}$ and $W_t \in (N_{U_t} \backslash \eta_{k;t}) \cup \{U_t\}$
  2. set $\eta_{k;t+1} = (\eta_{k;t} \backslash \{U_t\}) \cup \{W_t\}$.

What is the actual state space of η_k?

Goal

Construct a graph $\tilde{\mathcal{H}}_k^r = (\tilde{\mathcal{V}}_k^r, \tilde{\mathcal{E}}_k^r)$ which satisfies the following conditions:

- $\mathfrak{v} \in \tilde{\mathcal{V}}_k^r$ satisfies $|\mathfrak{v}| = k$
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- $\mathfrak{v} \in \tilde{\mathcal{V}}_k^r$ satisfies $|\mathfrak{v}| = k$
  (k particles),
- if $\{\mathfrak{v}, \mathfrak{w}\} \in \tilde{\mathcal{E}}_k^r$, then $\mathfrak{v} \triangle \mathfrak{w} = \{v, w\} \in E$
  (one particle moves at a time.).

Goal: State space $\tilde{\mathcal{H}}_k^r$ for $n = 5$, $k = 3$

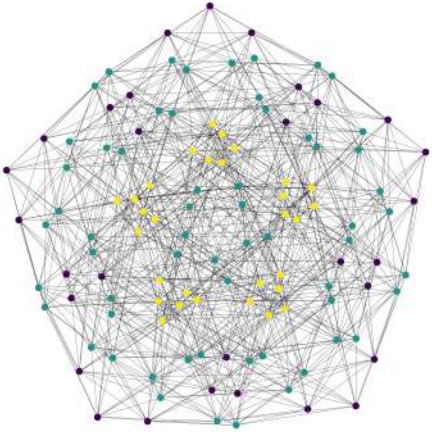

Figure: State space of a Markov chain associated to η_k.

Cartesian Product of graphs

Definition (Cartesian Product of Graphs)

Let $G_1, \dots, G_k$ be connected graphs with $G_i = (V(G_i), E(G_i))$. Then their Cartesian product $H_k = (V_k, E_k)$ is the graph

$$H_k := G_1 \times \dots \times G_k$$

with vertex set $V_k = \{v = (v_1, \dots, v_k) | v_i \in V(G_i)\}$. Two vertices v, v' are adjacent, whenever there is exactly one index $1 \leq i \leq k$ such that $\{v_i, v_i'\} \in E(G_i)$ and $v_j = v_j'$ for $j \neq i$.

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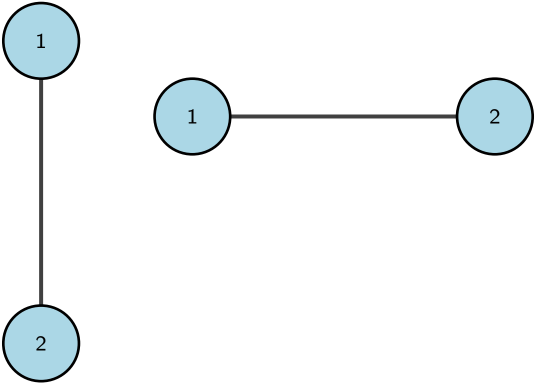

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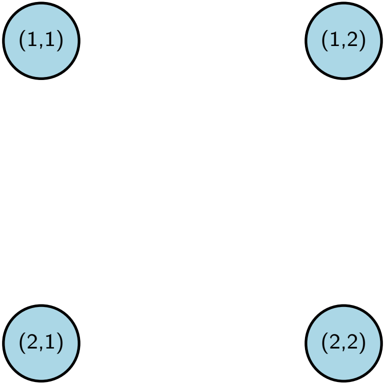

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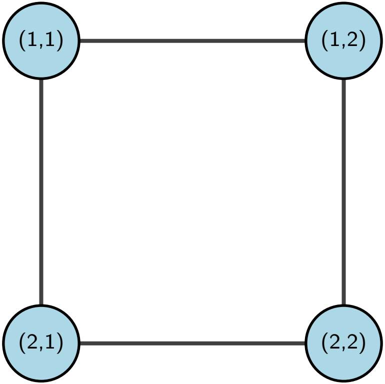

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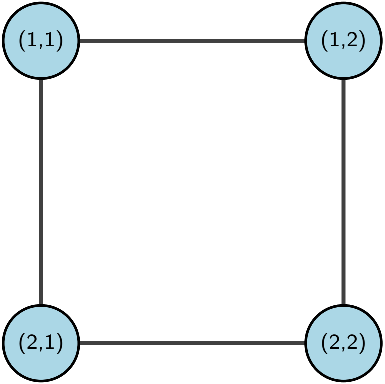

Remark: $(1,2) \neq (2,1)$!

Quotient graphs

Let $G = (V, E)$ be a graph and $\leftrightarrow$ an equivalence relation on V. The quotient graph ${}^{G}/_{\leftrightarrow} = G^{\leftrightarrow} = (V^{\leftrightarrow}, E^{\leftrightarrow})$ is defined as

- $V^{\leftrightarrow} = \{[v]^{\leftrightarrow} | v \in V\}$;
- $\{[v]^{\leftrightarrow}, [w]^{\leftrightarrow}\} \in E^{\leftrightarrow} \Leftrightarrow \exists \hat{v} \in [v]^{\leftrightarrow}, \hat{w} \in [w]^{\leftrightarrow} : \{\hat{v}, \hat{w}\} \in E$.

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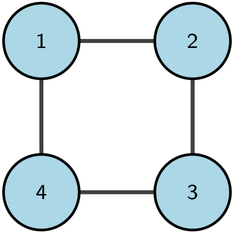

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State space $\tilde{\mathcal{H}}_k^r$ for η_k

Consider $L = (V, E)$. For fixed $k \in \{1, \ldots, \bar{n} - 1\}$ denote by $\mathcal{S}_k$ the symmetric group of order k. Let $\mathfrak{R}_k = \{\tilde{v} \in V^k | \exists i \neq j : v_i = v_j\}$.

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$$\begin{aligned} E_k^r &= \{\{u, w\} \in E_k | u, w \in V_k^r\}, \\ \mathcal{H}_k^r &:= (V_k^r, E_k^r). \end{aligned}$$

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Finally set $\tilde{\mathcal{H}}_k^r = {}^{\mathcal{H}_k^r}/{}_{\mathcal{S}_k}$.

Example: State space $\tilde{\mathcal{H}}_k^r$ for η_k; $n = 5$, $k = 3$

Start with complete graph K_5.

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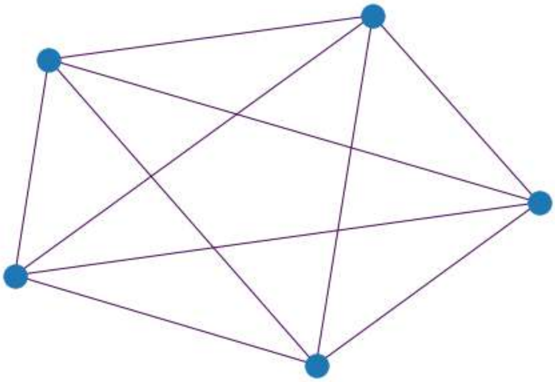

Figure: Complete graph on 5 vertices.

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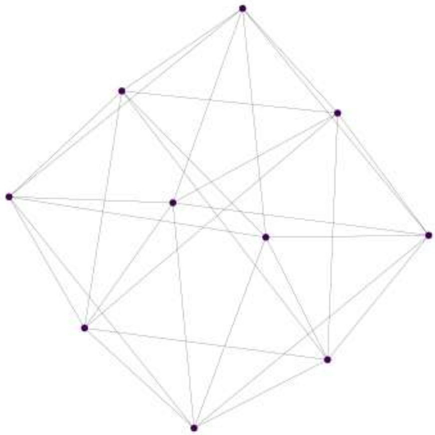

Figure: Line graph of complete graph on 5 vertices.

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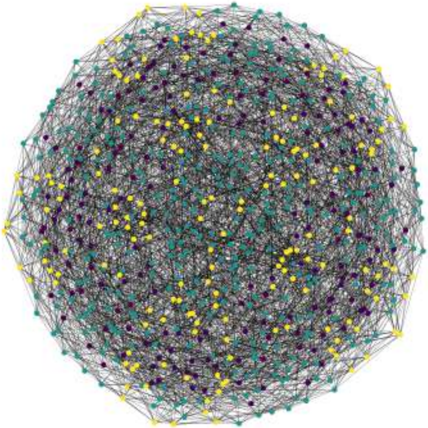

Figure: Associated graph $\mathcal{H}_k^r$.

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Set $\tilde{\mathcal{H}}_3^r = {}^{\mathcal{H}_3^r}/_{\mathcal{S}_3}$.

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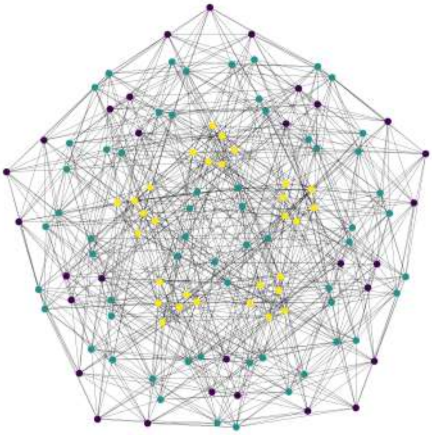

Figure: Quotient graph $\tilde{\mathcal{H}}_k^r$ of $\mathcal{H}_k^r$.

η_k as a Markov chain on $\tilde{\mathcal{H}}_k^r$

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Theorem

Let $k \in \{1, \dots, \bar{n} - 1\}$, L a strongly regular and η_k the k particle exclusion process on L. Then there is a Markov chain $\mathfrak{S}_k$ with transition Matrix $P_k^{\triangle}$ on $\tilde{\mathcal{H}}_k^r$ such that for any $\mathfrak{v} \subset V$ and $t \in \mathbb{N}$ the equation

$$\mathbb{P}[\eta_{k;t} = \mathfrak{v} | \eta_{k;0}] = \mathbb{P}[\mathfrak{S}_{k;t} = \mathfrak{v} | \mathfrak{S}_{k;0} = \eta_{k;0}]$$

is satisfied.

η_k as a Markov chain on $\tilde{\mathcal{H}}_k^r$

Theorem

Let $k \in \{1, \dots, \bar{n}-1\}$, $\mathfrak{v} \subset V$, $|\mathfrak{v}| = k$ denote by $L_{\mathfrak{v}}$ the vertex induced sub-graph of $\mathfrak{v}$ in L and for $v \in \mathfrak{v}$ write $\deg^{L_{\mathfrak{v}}}(v)$ the degree of v in $L_{\mathfrak{v}}$. The transition matrix $P_k^{\triangle}$ of $\mathfrak{S}_k$ satisfies

$$p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} = \begin{cases} \frac{1}{k}\frac{1}{\bar{d} - \deg^{L_{\mathfrak{v}}}(v) + 1}, & \mathfrak{v} \triangle \mathfrak{w} = \{v, w\} \text{ with } v \sim^L w, \\ \sum_{v \in \mathfrak{v}} \frac{1}{k}\frac{1}{\bar{d} - \deg^{L_{\mathfrak{v}}}(v) + 1}, & \mathfrak{v} = \mathfrak{w}, \\ 0, & \text{otherwise}. \end{cases}$$

Properties of $\mathfrak{S}_k$

The Markov chain $\mathfrak{S}_k$ is irreducible, aperiodic and, therefore, ergodic.

Reversibility of $\mathfrak{S}_k$

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Lemma

Define for $\mathfrak{v}, \mathfrak{w} \in \tilde{\mathcal{V}}_k^r$ *the value*

$$\Psi_{\mathfrak{v}}(\mathfrak{w}) = \begin{cases} \prod\limits_{\substack{v \in \mathcal{N}^{\mathfrak{v}}(\bar{v}) \setminus \mathcal{N}^{\mathfrak{w}}(\bar{w}) \\ v \neq \bar{v}}} \dfrac{\bar{d} - \deg^{L_{\mathfrak{v}}}(v) + 2}{\bar{d} - \deg^{L_{\mathfrak{v}}}(v) + 1}, & \mathfrak{v} \sim^r \mathfrak{w}, \bar{v} \sim^L \bar{w} \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

and $\pi(\mathfrak{v}) = \prod_{v \in \mathfrak{v}} (\bar{d} - \deg^{L_{\mathfrak{v}}}(v) + 1)$. *Then for any* $\mathfrak{v}, \mathfrak{w} \in \tilde{\mathcal{V}}_k^r$ *the following property holds true.*

$$\pi(\mathfrak{v}) p^{\triangle}_{k;\mathfrak{v},\mathfrak{w}} \Psi_{\mathfrak{v}}(\mathfrak{w}) = \pi(\mathfrak{w}) p^{\triangle}_{k;\mathfrak{w},\mathfrak{v}} \Psi_{\mathfrak{w}}(\mathfrak{v}). \tag{2}$$

Reversibility of $\mathfrak{S}_k$

The Markov chain $\mathfrak{S}_k^{\leftrightarrow}$ is in general not reversible. ¡2-¿Apply Kolmogorov's criterion for reversibility.

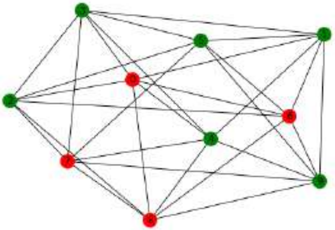

Figure: Graph L with assigned labels between 0 and 9. A vertex colored in red represents a vertex occupied by a particle.

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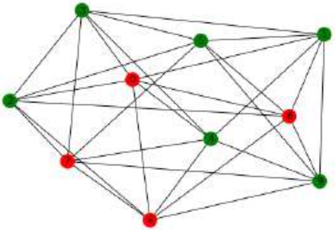

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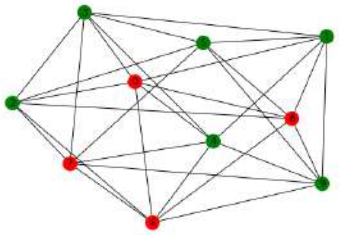

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Consider the following closed path

$$\phi = (\{0,6,7,8\},\{0,6,7,9\},\{6,7,8,9\},\{5,6,7,8\},\{0,5,7,8\},\{0,6,7,8\}).$$

Reversibility of $\mathfrak{S}_k$

The Markov chain $\mathfrak{S}_k$ is reversible for $k \in \{1, 2, \bar{n} - 2, \bar{n} - 1\}$.

Conjecture: $\mathfrak{S}_k$ is reversible if and only if $\mathfrak{S}_{\bar{n}-k}$ is reversible.

Outlook

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- Examine exclusion process in random environments.

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Why stop there?

Outlook

Why stop there?

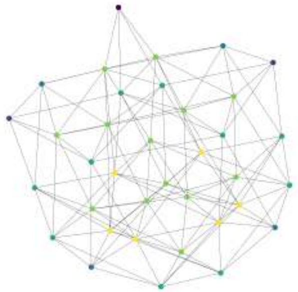

Figure: Quotient graph $\tilde{\mathcal{H}}_2^r$ when underlying graph is line graph of a $\mathcal{G}(6, 0.5)$.