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ASYMPTOTIC PROPERTIES OF VARIOUS STOCHASTIC CUCKER-SMALE DYNAMICS

Laure Pédèches

Institut de Mathématiques de Toulouse
118, route de Narbonne
F-31062 Toulouse Cedex 9, France

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Abstract. Starting from the stochastic Cucker-Smale model introduced in [14], we look into its asymptotic behaviours for different kinds of interaction. First in term of ergodicity, when t goes to infinity, seeking invariant probability measures and using Lyapunov functionals. Second, when the number N of particles becomes large, leading to results about propagation of chaos.

Introduction. Phenomena in which a large number of agents reaches a consensus without a hierarchical structure have been widely studied in recent years, as they occur in numerous scientific fields. Indeed, can be considered as such events as diverse as the emergence of a new language in a primitive society, the belief in a price system in an active market or the collective motions of a population. This last instance encompasses itself very different situations: amongst others, the behaviours of school of fish, flights of birds, bacterial populations or even human groups.

The so-called Cucker-Smale model is one of many attempts at representing such phenomena. It was introduced in 2007 by Cucker and Smale in [11] and [12]. It is a mean-field kinetic deterministic model describing a N-particle system evolving in $\mathbb{R}^d$ through the position x_i and the velocity v_i of particle number i for $i \in \{1, ..., N\}$:

$$\begin{cases} x_i'(t) &= v_i(t) \\ v_i'(t) &= -\frac{1}{N}\sum_{j=1}^{N} \psi(x_j(t), x_i(t))\,(v_i(t) - v_j(t)), \end{cases} \tag{1}$$

where ψ is a positive, symmetric function called *communication rate*. Typically, in Cucker and Smale works, it is of the form

$$\psi(x, y) = \bar{\psi}(x - y) \text{ with } \bar{\psi}(u) = \frac{\lambda}{(1 + |u|^2)^\gamma},$$

where λ is a positive constant, representing the intensity of this interaction.

A fundamental property of this model, due to the symmetry of the communication rate, is that the center of mass of the velocities, $v_c = \frac{1}{N}\sum_{j=1}^{N} v_j$, is constant at all times: that is, for every t, $v_c(t) = v_c(0)$. Thus, if the initial velocities $v_i(0)$ are all equal, then the velocities are constant and so equal at all times: for every i and t, $v_i(t) = v_c(0)$. This is an equilibrium situation, towards which tends the system.

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Indeed, the main result of Cucker and Smale is related to flocking, a phenomenon in which self-propelled individuals or particles organize themselves to reach a motion with global coherence, characterized in a mathematical sense by both velocity alignment and formation of a group structure. More precisely, here is the definition for (deterministic) flocking:

Definition 0.1. Flocking happens for a set of N particles if, for all $i \in \{1, ...N\}$,

- $\lim_{t\to\infty} |v_i(t) - v_c(t)|^2 = 0$ with $v_c(t) = \frac{1}{N}\sum_{j=1}^N v_j(t)$;
- $\sup_{0\le t<\infty} |x_i(t) - x_c(t)|^2 < \infty$ where $x_c(t) = \frac{1}{N}\sum_{j=1}^N x_j(t)$.

Cucker and Smale main result is that, whatever the initial conditions, there is flocking when $\gamma < 1/2$ and that it still occurs under some condition depending only on the initial configuration $(x_i(0), v_i(0))_i$ otherwise; it was later shown (see [16]) that there is always flocking if $\gamma = 1/2$ too.

This result is a major reason why, since then, various authors have studied properties of such models, giving also alternative proofs of the results (for instance in [15]) and proposed refined, more reality-compliant versions of the model: hierarchical leadership is presented by Shen ([22]), a collision-avoiding model is introduced by Cucker and Dong ([9]), the idea of a *vision cone* for the agents (birds in this case) is studied by Agueh, Illner and Richardson ([1]), amongst many others.

There has been a fair number of attempts (including by Cucker and Mordecki in [10], where is added smooth noise, Ahn and Ha in [2] or Ton, Linh and Yagi in [24]) to introduce a random component in this model. Indeed in the above system the effects of the environment are neglected: what about the effects of some (very) localized ocean currents or wind gusts, for fishes or birds respectively? What of the free will of each individual? And why should the trajectory of a particle be totally predetermined by its initial configuration?

In this paper, we first focus on the model presented in [14] by Ha, Lee and Levy in 2009, the main difference with the system (1) consisting in the addition of a stochastic noise, which takes the form of a Brownian motion:

$$\left\{\begin{array}{rcl} dx_i(t) &=& v_i(t)\,dt \\ dv_i(t) &=& -\frac{1}{N}\sum_{j=1}^N \psi(x_j(t), x_i(t))\,(v_i(t) - v_j(t))\,dt + \sqrt{D}\,dW_i(t) \end{array}\right. \tag{2}$$

for every $i \in \{1, ..., N\}$, where D is a non-negative number, representing the intensity of the noise, $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+^*$ is positive symmetric function, and $W_1, ..., W_N$ are d-dimensional independent standard Brownian motions.

Here the choice of stochastic noise is such that one can interpret the W_i, random and independent from each other, as a way of representing the degree of freedom of each individual. In order to model the wind, for example, one should consider the same Brownian motion for all particles: the behaviour of such models is studied in [8]. Notice that, here, because of this choice of noise, and contrary to the deterministic model, the equality of all initial velocities $v_i(0)$ does not imply the equality of all velocities at any given time. That is, having $v_i(0) = v_c(0)$ for every i does not mean that $v_i(t) = v_c(t)$ at time $t > 0$. Thus, the model is no more a perturbation of some equilibrium.

The first question that comes to mind is whether it is possible to obtain results similar to those proven by Cucker and Smale. However, one must first determine what should be the stochastic equivalent of flocking, especially concerning the velocities.

This is a complex problem: in [14], stochastic flocking is defined as (deterministic) flocking for the expectations of the velocities and positions of the particles. This is a relatively weak condition. Alternatively, one can define an almost sure type of flocking, as in [2], if definition 0.1 holds almost surely, or a $\mathcal{L}^p$-flocking (see [24]) if the difference between the velocity of a particle and the center of mass goes to 0 in $\mathcal{L}^p$ or even some kind of weak flocking. For more detailed explanations about the different types of flocking and their appearances in conjunction with different types of stochastic noise, see e.g. [8].

Here, the independence of the W_i rules out almost sure- and $\mathcal{L}^p$-flockings. We thus focus on the asymptotic behaviour of the system (2).

In order to study these stochastic dynamics, we will decompose it in two different parts, as is done in [14], corresponding to two different scales:

- On the one hand, we consider a macroscopic (or coarse-scale) system represented by the center of mass x_c of the positions x_i, and its velocity v_c (which, incidentally, is also the center of mass of the velocities v_i): $x_c = \frac{1}{N}\sum_{i=1}^N x_i$ and $v_c = \frac{1}{N}\sum_{i=1}^N v_i$.

  From (2), we deduce the stochastic differential equations satisfied by x_c and v_c:
$$\begin{cases} dx_c(t) &= v_c(t)\,dt \\ dv_c(t) &= \sqrt{D}\,dW_c(t) \end{cases}$$
  where $W_c(t) = \frac{1}{N}\sum_{i=1}^N W_i(t)$ is a $\mathbb{R}^d$-valued Gaussian process, with expectation 0 and covariance matrix $\frac{1}{N}\,t\,I_d$, for every $t \geq 0$.

  This system can therefore be explicitly solved:
$$\begin{cases} x_c(t) &= x_c(0) + t\,v_c(0) + \sqrt{D}\int_0^t W_c(s)\,ds \\ v_c(t) &= v_c(0) + \sqrt{D}\,W_c(t). \end{cases} \tag{3}$$

- On the other hand, we consider a microscopic (or fine-scale) system described by the relative fluctuations of the positions and velocities, around the center of mass and its velocity, $\hat{x}_i = x_i - x_c$ and $\hat{v}_i = v_i - v_c$. Notice that for every positive t,
$$\sum_{i=1}^N \hat{x}_i(t) = \sum_{i=1}^N \hat{v}_i(t) = 0. \tag{4}$$
  Assume that ψ is of the form $\psi(x,y) = \bar{\psi}(x-y)$, with $\bar{\psi} : \mathbb{R}^d \to \mathbb{R}_+^*$ a positive even function. Therefore, the relative fluctuations satisfy for all $i \in \{1, ...N\}$,
$$\begin{cases} d\hat{x}_i(t) &= \hat{v}_i(t)dt \\ d\hat{v}_i(t) &= -\frac{1}{N}\sum_{i=1}^N \psi(\hat{x}_i(t), \hat{x}_j(t))\,(\hat{v}_i(t) - \hat{v}_j(t))\,dt + \sqrt{D}\,d\widehat{W}_i(t), \end{cases} \tag{5}$$
  setting $\widehat{W}_i = W_i - W_c$.

  Studying these equations will prove to be much more challenging, especially with some nondescript communication rate ψ. We will mainly focus on this relative (or microscopic) system in this work.

We first turn our attention to the particular – nonsensical from a biological point of view but computation-friendly – case of a constant communication rate. We study the time-asymptotic behaviour of system (5) in this setting and in a modified setting obtained by the addition of an attractive, linear, input of the positions in the velocity equations, in the same vein as [9]. In this case, as proven in section 3, there exists an invariant probability measure for the pair position-velocity and the system is exponentially ergodic.

Then, in two particular settings, we obtain in section 4 the existence of an invariant probability measure and ergodicity for non-constant communication rates. On the one hand we prove a polynomial ergodicity result. On the other hand, using the cluster expansion method presented in [20], we prove exponential ergodicity for general drifts with finite delay.

In section 5, we give further results on stationarity, based on Itô-Nisio result (see [17]): in particular, we obtain stationary solutions for a larger class of communication rates. This approach requires moment controls as presented in the final part. In section 6 system (2) is investigated when the number N of agents goes to infinity to obtain propagation of chaos results, after proving the uniqueness of the associated non-linear stochastic differential system. The results we present in this section can be deduced from those obtained in [6]; however our proof, purely based on probability theory, is very different from [6], where transport equations methods are used.

1. The basic stochastic Cucker-Smale model with a constant communication rate.

From here to the end of Section 4, we place ourselves on $\Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R}^{2d})$, the canonical continuous $\mathbb{R}^{2d}$-valued path space, with $\mathcal{F}$ the canonical Borel σ-field on Ω.

As it shall not have any impact on any of the results presented in this paper, we set, for the sake of simplicity, $D = 1$ (except in subsection 3.1).

Suppose first that the communication rate is constant, that is $\psi = \lambda$ for a certain positive constant λ. This means that whatever the distance between two particles, the interaction between them will have the same intensity. This assumption is quite unrealistic, but mathematically tractable.

In the first paragraph, we recall and present in a clearer way results from [14] that we are going to use later; in subsection 1.2, we prove the existence of a invariant measure – with infinite mass – for the microscopic system. In the following paragraph, we obtain a central limit theorem for the behaviour of the microscopic positions, derived from a result of [7]. Finally, in paragraph 1.4, we exhibit a reversible measure for the global velocities.

1.1. Explicit expression and distribution for the relative velocities.

From observation (4), the microscopic system (5) becomes for every positive t and for every $i \in \{1, ...N\}$,

$$\begin{cases} d\hat{x}_i(t) &= \hat{v}_i(t)\, dt \\ d\hat{v}_i(t) &= -\lambda \hat{v}_i(t)\, dt + d\widehat{W}_i(t). \end{cases} \tag{6}$$

Remark that the second equation is autonomous in $\hat{v}_i$; moreover it is an Ornstein-Uhlenbeck type equation.

Proposition 1. *For every $t \geq 0$ and for every $i \in \{1, ..., N\}$,*

$$\hat{v}_i(t) = e^{-\lambda t}\, \hat{v}_i(0) + \int_0^t e^{-\lambda(t-s)}\, d\widehat{W}_i(s).$$

Proof. Apply Itô's formula to $t \mapsto f(t, \hat{v}_i(t)) = e^{\lambda t}\, \hat{v}_i(t)$. Then,

$$e^{\lambda t}\hat{v}_i(t) = \hat{v}_i(0) + \int_0^t e^{\lambda s}\, d\widehat{W}_i(s) + \int_0^t \left(\lambda\, e^{\lambda s}\, \hat{v}_i(s) - e^{\lambda s}\, \lambda\, \hat{v}_i(s)\right)\, ds. \qquad \square$$

Furthermore, if the initial value $(\hat{v}_1(0),...,\hat{v}_N(0))$ is Gaussian then

$$\hat{v}(t) = (\hat{v}_1(t),...,\hat{v}_N(t)) \in \mathbb{R}^{Nd}$$

is a Gaussian process entirely determined by its expectation and covariance matrix. Setting $\Pi_{N,d}$ the following square block matrix of size Nd,

$$\Pi_{N,d} = \begin{pmatrix} (1-\frac{1}{N})I_d & -\frac{1}{N}I_d & \cdots & -\frac{1}{N}I_d \\ -\frac{1}{N}I_d & (1-\frac{1}{N})I_d & \cdots & -\frac{1}{N}I_d \\ \vdots & \ddots & \ddots & \vdots \\ -\frac{1}{N}I_d & -\frac{1}{N}I_d & \cdots & (1-\frac{1}{N})I_d \end{pmatrix}, \tag{7}$$

the proposition below holds.

Proposition 2. *For every $t \geq 0$,*

$$\hat{v}(t) \sim \mathcal{N}\left(e^{-\lambda t}\,\mathbb{E}[\hat{v}(0)], \Lambda_{N,d}(t)\right),$$

where $\Lambda_{N,d}(t) = \frac{1}{2\lambda}\,(1-e^{-2\lambda t})\,\Pi_{N,d}$.

Remark 1. Notice that the eigenvalues of $\Pi_{N,d}$ are 0 with multiplicity d and 1 with multiplicity $(N-1)d$. Thus, the matrix $\Pi_{N,d}$ is not invertible and the law of $\hat{v}$ is degenerate at all time. In particular, it is not absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{Nd}$, which will prove problematic in later stages.

1.2. **Invariant probability measure for $\hat{v}$.** Let μ be the Gaussian distribution $\mathcal{N}\left(0, \frac{1}{2\lambda}\,\Pi_{N,d}\right)$ on $\mathbb{R}^{Nd}$.

Firstly, we prove that μ is a reversible (and thus, invariant) probability measure for the vector $\hat{v}$ of the relative velocities with respect to the center of mass v_c.

Proposition 3. *The process $\hat{v}(.)$ admits $\mu = \mathcal{N}\left(0, \frac{1}{2\lambda}\,\Pi_{N,d}\right)$ as its unique reversible probability measure.*

For every $i \in \{1,...,N\}$, we write $\hat{x}_i^\alpha(t)$ (resp. $\hat{v}_i^\alpha(t)$) for $\alpha \in \{1,...,d\}$ the α^{th}-component of the $\mathbb{R}^d$ vector $\hat{x}_i(t)$ (resp. $\hat{v}_i(t)$).

Proof. Since μ does not admit a probability density with respect to the Lebesgue measure we cannot use the usual characterization of reversible measures involving the infinitesimal generator associated with the process in $\mathcal{L}^2(\mu)$.

μ is invariant if and only if for every n, $0 < t_1 < ... < t_n$, $\tau > 0$, $(\hat{v}(t_1),...,\hat{v}(t_n))$ and $(\hat{v}(t_1+\tau),...,\hat{v}(t_n+\tau))$ have the same distribution under $\mathbb{P}_\mu$, the law with initial distribution μ. As both are Gaussian processes, it is sufficient to show that they have same expectation and covariance matrix.

For every $i \in \{1,...,N\}$, $t \geq 0$,

$$\mathbb{E}_\mu[\hat{v}_i(t)] = 0,$$

and for every $\alpha, \beta \in \{1,...,d\}$, $i, j \in \{1,...,N\}$, $t \geq 0$,

$$cov_\mu(\hat{v}_i^\alpha(t), \hat{v}_j^\beta(t)) = \delta_{\alpha,\beta}\,\frac{1}{2\lambda}\,(\delta_{i,j} - \frac{1}{N}).$$

Thus, for every non negative t and τ, $v(t)$ and $v(t+\tau)$ have the same distribution under $\mathbb{P}_\mu$.

Furthermore, for $t_1 < t_2$,

$$\mathbb{E}_\mu[\hat{v}_i^\alpha(t_1+\tau)\hat{v}_j^\beta(t_2+\tau)] = e^{-\lambda(t_1+t_2+2\tau)}\ \mathbb{E}_\mu[\hat{v}_i^\alpha(0)\ \hat{v}_j^\beta(0)]$$
$$+ \delta_{\alpha,\beta}\ (\delta_{i,j} - \frac{1}{N})\ \mathbb{E}_\mu\left[\int_0^{t_1+\tau} e^{-\lambda(t_1+\tau-s)}\ d\widehat{W}_i^{\alpha}(s) \int_0^{t_2+\tau} e^{-\lambda(t_2+\tau-s)}\ d\widehat{W}_j^{\beta}(s)\right]$$
$$= e^{-\lambda(t_1+t_2+2\tau)}\ \delta_{\alpha,\beta}\ \frac{1}{2\lambda}(\delta_{i,j} - \frac{1}{N}) + \delta_{\alpha,\beta}\ (\delta_{i,j} - \frac{1}{N})\ \int_0^{t_1+\tau} e^{-\lambda(t_1+t_2+2\tau-s)}\ ds$$
$$= \delta_{\alpha,\beta}\ \frac{1}{2\lambda}\ (\delta_{i,j} - \frac{1}{N})\ \left(e^{-\lambda(t_1+t_2+2\tau)} + e^{-\lambda(t_1+t_2+2\tau)}\ (e^{2\lambda(t_1+\tau)} - 1)\right)$$
$$= \delta_{\alpha,\beta}\ \frac{1}{2\lambda}\ (\delta_{i,j} - \frac{1}{N})\ e^{-\lambda(t_2-t_1)} = \mathbb{E}_\mu[\hat{v}_i^\alpha(t_1)\ \hat{v}_j^\beta(t_2)].$$

Hence, $cov_\mu(\hat{v}_i^\alpha(t_1+\tau), \hat{v}_j^\beta(t_2+\tau)) = cov_\mu(\hat{v}_i^\alpha(t_1), \hat{v}_j^\beta(t_2))$.

$(\hat{v}(t_1), ..., \hat{v}(t_n))$ and $(\hat{v}(t_1+\tau), ..., \hat{v}(t_n+\tau))$ have then the same distribution under $\mathbb{P}_\mu$.

In the same manner, by computing the first and second order moments of $\hat{v}(.)$ and $\hat{v}(\tau - .)$, one deduce that they have the same distribution under $\mathbb{P}_\mu$. □

Secondly, setting $\hat{x} = (\hat{x}_1, ..., \hat{x}_N)$, we show that the pair $(\hat{x}, \hat{v})$ does not admit an invariant probability measure but only an invariant σ-finite measure.

Proposition 4. *The measure $dx \otimes \mu$ is invariant for $(\hat{x}, \hat{v})$, with dx denoting the Lebesgue measure on $\mathbb{R}^{Nd}$.*

Proof. Using Itô's formula, we obtain the infinitesimal generator $\hat{L}$ associated with the system (6): for a function f regular enough and $x, v \in \mathbb{R}^{Nd}$,

$$\hat{L}f(x,v) = \hat{L}_x f(x,v) + \hat{L}_v f(x,v)$$

where

$$\hat{L}_x f(x,v) = \sum_{i=1}^N v_i \nabla_{x_i} f$$

and

$$\hat{L}_v f(v) = -\lambda \sum_{i=1}^N v_i.\nabla_{v_i} f + \frac{1}{2}\sum_{i=1}^N \left(\Delta_{v_i} f - \frac{1}{N}\sum_{j=1}^N \sum_{\alpha=1}^d \partial^2_{v_i^\alpha v_j^\alpha} f\right).$$

As μ is invariant for $\hat{v}$,

$$\int \left(\int \hat{L}_v f(v)\ d\mu(v)\right) dx = \int 0\ dx = 0.$$

Besides, denoting by $\bar{x_i^\alpha}$ the vector x missing its (i,α)-component,

$$\begin{aligned} \int \hat{L}_x f\ dx \otimes \mu &= \sum_{i,\alpha} \int \left(\int v_i^\alpha\ \partial_{x_i^\alpha} f(x,v)\ dx_i^\alpha\right) d\bar{x_i^\alpha}\ d\mu(v) \\ &= -\sum_{i,\alpha} \int \left(\int f(x,v)\ \partial_{x_i^\alpha} v_i^\alpha\right) dx\ d\mu(v) = 0 \end{aligned}$$

because $\partial_{x_i^\alpha} v_i^\alpha = 0$.

It follows that $\int \hat{L}f\ dx \otimes \mu = 0$. □

As $dx \otimes \mu$ is a measure with infinite mass, there is no invariant probability measure for the random system $(\hat{x}(.), \hat{v}(.))$.

1.3. **Behaviour of $\hat{x}$ and central limit theorem.** In this subsection, we prove a new central limit theorem for the asymptotic behaviour of the relative positions $\hat{x}_1(t)$, ..., $\hat{x}_N(t)$, when t goes to infinity, applying a result by Cattiaux, Chafaï and Guillin from [7].

Recall that

$$\hat{x}(t) = \hat{x}(0) + \int_0^t \hat{v}(s)\, ds.$$

Using the ergodic theorem,

$$\frac{1}{t}\, \hat{x}(t) = \frac{1}{t}\left(\int_0^t \hat{v}(s)\, ds + \hat{x}(0)\right) \underset{t\to\infty}{\longrightarrow} \int \hat{v}\, \mu(d\hat{v}) = 0 \quad a.s.$$

Looking for a more precise result, we prove the following convergence result:

Proposition 5. *The central limit theorem below holds:*

$$\frac{1}{\sqrt{t}}\, \hat{x}(t) \xrightarrow[t\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{\lambda^2}\, \Pi_{N,d}\right).$$

Proof. We start with a lemma about the variance of the components of $\hat{x} = (\hat{x}_i^\alpha)_{\alpha\in\{1,\dots,d\}, i\in\{1,\dots,N\}}$.

Lemma 1.1. *For every $\alpha \in \{1, ..., d\}$ and $i \in \{1, ..., N\}$,*

$$\frac{1}{t}\, var_\mu\left(\int_0^t \hat{v}_i^\alpha(s)\, ds\right) \underset{t\to+\infty}{\longrightarrow} \frac{1}{\lambda^2}\left(1 - \frac{1}{N}\right).$$

Proof. We prove this lemma using the method, based on the invariance of the probability measure μ, used in the proof of Lemma 2.3 of [7].

$$\begin{aligned} var_\mu\left(\int_0^t \hat{v}_i^\alpha(s)\, ds\right) &= \mathbb{E}_\mu\left[\left(\int_0^t \hat{v}_i^\alpha(s)\, ds\right)^2\right] = 2\, \mathbb{E}_\mu\left[\int_0^t \int_0^s \hat{v}_i^\alpha(s)\hat{v}_i^\alpha(u)\, du\, ds\right] \\ &= 2\int_0^t \int_0^s \mathbb{E}_\mu[\hat{v}_i^\alpha(s-u)\hat{v}_i^\alpha(0)]\, du\, ds \text{ by invariance of } \mu. \end{aligned}$$

Moreover, as the initial conditions and the Brownian motions are independent, it follows that

$$\mathbb{E}_\mu[\hat{v}_i^\alpha(s-u)\, \hat{v}_i^\alpha(0)] = \mathbb{E}_\mu[e^{-\lambda(s-u)}\, \hat{v}_i^\alpha(0)^2] = e^{-\lambda(s-u)} \frac{1}{2\lambda}\left(1 - \frac{1}{N}\right)$$

and

$$\begin{aligned} \tfrac{1}{t}\, var_\mu\left(\int_0^t \hat{v}_i^\alpha(s)\, ds\right) &= \tfrac{1}{\lambda t}\left(1 - \tfrac{1}{N}\right) \int_0^t e^{-\lambda s}\left(\int_0^s e^{\lambda u}\, du\right) ds \\ &= \tfrac{1}{\lambda^2}\left(1 - \tfrac{1}{N}\right)\left(1 - \tfrac{1}{t}\left(1 - e^{-\lambda t}\right)\right) \\ &\underset{t\to+\infty}{\longrightarrow} \tfrac{1}{\lambda^2}\left(1 - \tfrac{1}{N}\right). \end{aligned}$$

□

Therefore, according to Theorem 3.3 of [7], under $\mathbb{P}_\mu$, for all $i \in \{1, ..., N\}$ and $\alpha \in \{1, ..., d\}$

$$\frac{1}{\sqrt{t}} \int_0^t \hat{v}_i^\alpha(s) ds \xrightarrow[t\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{\lambda^2}(1 - \frac{1}{N})\right). \tag{8}$$

The only convergence left to prove is that for $i \neq j$, α, β,

$$\frac{1}{t}\, cov\left(\int_0^t \hat{v}_i^\alpha(s)\, ds, \int_0^t \hat{v}_j^\beta(s)\, ds\right) \underset{t\to\infty}{\longrightarrow} -\, \delta_{\alpha,\beta}\, \frac{1}{\lambda^2 N}.$$

A direct computation leads to:

$$\frac{1}{t}\, cov \left(\int_0^t \hat{v}_i^\alpha(s)\, ds, \int_0^t \hat{v}_j^\beta(s)\, ds \right) = \frac{1}{t}\, (1-e^{-\lambda t})^2\, \mathbb{E}[\hat{v}_i^\alpha(0)\, \hat{v}_j^\beta(0)]$$

$$+ \frac{1}{t}\, \mathbb{E}\left[\int_0^t \int_0^s e^{-\lambda(s-u)}\, d\widehat{W}_i^\alpha(u)\, ds \int_0^t \int_0^s e^{-\lambda(s-u)}\, d\widehat{W}_j^\beta(u)\, ds \right]$$

$$= -\,\delta_{\alpha,\beta}\, \frac{1}{2\lambda N t}\, (1-e^{-\lambda t})^2 - \delta_{\alpha,\beta}\, \frac{1}{Nt}\, \mathbb{E}\left[\left(\int_0^t \int_0^s e^{-\lambda(s-u)} dW_i^\alpha(u)\, ds \right) \right]$$

$$= -\,\delta_{\alpha,\beta} \left(\frac{1}{2\lambda N t}\, (1-e^{-\lambda t})^2 + \frac{1}{Nt} \int_0^t \left(\int_u^t e^{-\lambda(s-u)}\, ds \right)^2 ds \right)$$

$$\xrightarrow[t\to\infty]{} -\,\delta_{\alpha,\beta}\, \frac{1}{N} \left(\int_u^\infty e^{-\lambda(s-u)}\, ds \right)^2 = -\delta_{\alpha,\beta}\, \frac{1}{\lambda^2 N}.$$

□

This asymptotic behaviour of $\hat{x}$ confirms the result of the previous subsection: there is no invariant probability measure for the relative system $(\hat{x}, \hat{v})$ associated with the model (2). Indeed the particles do not particularly tend to come closer from each other, and thus there is not formation of a group structure.

We will come back to this later, but first we briefly turn our attention to the global process (x, v) and its behaviour when t is large.

1.4. **Back to the original system.** In the setting of a constant communication rate, the stochastic differential equations (2) become:

$$\begin{cases} dx_i(t) &= v_i(t)\, dt \\ dv_i(t) &= -\lambda\, (v_i(t) - v_c(t))\, dt + dW_i(t). \end{cases} \tag{9}$$

As was shown previously, $x = \hat{x} + x_c$ and $v = \hat{v} + v_c$ have a known explicit expression. v_c is a Brownian motion (and thus admits the Lebesgue measure as an invariant measure); therefore v cannot admits an invariant measure with finite mass. Nevertheless, we can find an invariant (and even symmetric) measure for the vector of the velocity v.

Proposition 6. *The measure ν with infinite mass given by*

$$d\nu(v) = exp\left(-\lambda \sum_{\alpha=1}^{d} \sum_{i=1}^{N} (v_i^\alpha - v_c^\alpha)^2 \right) dv$$

is reversible for v defined in (9).

The proof of this proposition follows from a classical result on gradient diffusions.

Lemma 1.2. *If X is solution in $\mathbb{R}^n$ of*

$$dX_t = \eta\, dW_t - \nabla F(X_t)\, dt,$$

where W_t is a n-dimensional standard Brownian motion, F a smooth function and η a real number, then X admits $\rho(dx) = e^{-\frac{2}{\eta^2}F(x)}\, dx$ as reversible measure.

Proof. The associated infinitesimal generator is defined by

$$Lf = -\sum_{i=1}^{n} \partial_i f\, \partial_i F + \frac{\eta^2}{2} \sum_{i=1}^{n} \partial_i^2 f.$$

To prove the reversibility of ρ, we have to show that for every smooth f and g, $\int gLf\, d\rho = \int fLg\, d\rho$.

$$\begin{aligned}
&\int f(x)Lg(x)\, d\rho(x) - \int g(x)Lf(x)\, d\rho(x)\\
&= -\sum_{i=1}^{n} \int (f\partial_i g - g\partial_i f)\, \partial_i F e^{-\frac{2}{\eta^2}F(x)} dx \;+\; \frac{\eta^2}{2}\sum_{i=1}^{n}\int \left(f\partial_i^2 g - g\partial_i^2 f\right) e^{-\frac{2}{\eta^2}F(x)} dx\\
&= -\sum_{i=1}^{n}\int \left(\left(\partial_i g\, \partial_i f + g\, \partial_i^2 f - \partial_i f\, \partial_i g - f\, \partial_i^2 g\right)\frac{\eta^2}{2} + \frac{\eta^2}{2}\left(f\, \partial_i^2 g - g\, \partial_i^2 f\right)\right) d\rho(x)\\
&=0.
\end{aligned}$$

□

Proposition 6 follows by applying this lemma with $n = Nd$ for

$$F(v) = -\frac{\lambda}{2} \sum_{\alpha=1}^{d}\sum_{i=1}^{N} (v_i^\alpha - v_c^\alpha)^2\,.$$

2. **Introducing x in the v-equation.** As mentioned previously, we would like to modify the above linear model into a more realistic one. A simple idea is to add a linear attractive term in $x_i - x_j$, as is done for instance with kinetic models: indeed the velocity dynamics are not solely determined by their differences, but also by the relative positions of the particles.

After the addition of this pull-back force, the projection of the system on a certain hyperplane will now admit an invariant probability measure. Proposition 8 is the main result of subsection 2.1. In subsection 2.2, we use Lyapunov function theory to prove in Theorem 2.2 the exponential ergodicity of these dynamics towards its invariant probability measure.

For $i \in \{1, ..., N\}$, we now consider

$$\left\{\begin{array}{lcl} dx_i(t) &=& v_i(t)\, dt\\ dv_i(t) &=& -\lambda\, (v_i(t) - v_c(t))\, dt - \beta\, (x_i(t) - x_c(t))\, dt + dW_i(t), \end{array}\right. \tag{10}$$

where β is a positive parameter coding the intensity of this new interaction.

As in the previous section, we divide the system in two parts, a "macroscopic" one and a "microscopic" one. The center of mass (x_c, v_c) is subject to exactly the same dynamics and the expressions of x_c and v_c are still given by (3).

Changes appear for the relative fluctuations however, and instead of (6) one now obtains, for every i in $\{1, ..., N\}$,

$$\left\{\begin{array}{lcl} d\hat{x}_i(t) &=& \hat{v}_i(t)\, dt\\ d\hat{v}_i(t) &=& -\lambda\hat{v}_i(t)\, dt - \beta\hat{x}_i(t)\, dt + d\widehat{W}_i(t). \end{array}\right. \tag{11}$$

We now focus on finding an invariant probability measure for the random dynamics given by (11).

2.1. **Invariant probability measure on a "d-hyperplane" for the relative fluctuations.** The introduction of this new interaction will lead to the existence of an invariant probability measure for the microscopic system. Such an occurrence is impossible for the global system (x, v); however, one can easily check the validity of the following proposition.

Proposition 7. *The measure* μ_β *defined on* $(\mathbb{R}^d \times \mathbb{R}^d)^N$ *by*

$$d\mu_\beta(x,v) = exp\left(-\lambda \left[\sum_{i=1}^N |v_i - v_c|^2 + \beta \sum_{i=1}^N |x_i - x_c|^2\right]\right) dx\, dv$$

is invariant for the original dynamics (10). Nevertheless, it has an infinite mass.

Proof. To verify the invariance of μ_β, it suffices to check that – with L_β the infinitesimal generator associated with (10) – for every function f regular enough, $\int L_\beta f\, d\mu_\beta = 0$. □

Now, introduce the subspace

$$\mathcal{H} = \{(x,v) \in \mathbb{R}^{Nd} | x_1 + ... + x_N = 0 \text{ and } v_1 + ... + v_N = 0\}$$

of codimension $2d$. The projection of the system on $\mathcal{H}$ admits an invariant probability measure.

Proposition 8. *Define* $\phi(z_1, ..., z_{N-1}) = \sum_{i=1}^{N-1} |z_i|^2 + \left|\sum_{i=1}^{N-1} z_i\right|^2$.

The probability measure $\hat{\mu}_\beta$ *on* $(\mathbb{R}^d \times \mathbb{R}^d)^{N-1}$ *whose density is*

$$\hat{f}_\beta(\hat{x}_1, ..., \hat{x}_{N-1}, \hat{v}_1, ..., \hat{v}_{N-1}) = \frac{1}{Z} exp\left(-\lambda \left(\phi(\hat{x}_1, ..., \hat{x}_{N-1}) + \beta\, \phi(\hat{v}_1, ..., \hat{v}_{N-1})\right)\right),$$

where Z *is a renormalisation constant, is invariant for the projection on* $\mathcal{H}$ *of the stochastic dynamics defined in (11).*

Proof. To find $\hat{\mu}_\beta$, we start from the measure μ_β introduced in Proposition 7, we make the substitution of variables $(x_1, ..., x_N) \longrightarrow (\hat{v}_1, ..., \hat{v}_{N-1}, N v_c)$ and then project on $\mathcal{H}$.

To prove the invariance of $\hat{\mu}_\beta$, we proceed as previously. □

Having found this elusive invariant probability measure, we determine the rate of convergence of its associated semi-group towards $\hat{\mu}_\beta$ using Lyapunov function theory.

2.2. **Lyapunov functions and ergodicity.** First we specify what we mean by Lyapunov function.

Definition 2.1. A positive, continuous, smooth enough function V is called Lyapunov function for the Markov process associated with the infinitesimal generator L if there exists $K \geq 0$ such that, outside of a certain compact set U,

$$LV \leq -K\, V.$$

By exhibiting a Lyapunov function, we obtain the exponential ergodicity of the system, in what is the main result of this paragraph.

Theorem 2.2. *Let* P_t^β *be the semi-group associated with the system (11). Assume that* $\lambda^2 > 2\beta$.

For all $(\tilde{x}, \tilde{v}) \in (\mathbb{R}^d \times \mathbb{R}^d)^{N-1}$, $P_t^\beta((\tilde{x}, \tilde{v}), .)$ *converges exponentially towards* μ_β *for the total variation distance: there exists* $\rho > 0$ *and* $C > 0$ *such that for all* t:

$$\|P_t^\beta((\tilde{x}, \tilde{v}), .) - \mu_\beta\|_{TV} \leq C\, V(\tilde{x}, \tilde{v})\, e^{-\rho\, t}$$

where V *is the Lyapunov function defined in (12), associated with the stochastic system (11).*

In this case, we shall say that μ_β *is exponentially ergodic.*

Proof. The proof rests on a theorem, due to Down, Meyn and Tweedie ,see [13] and also [4]. It explicits the link between Lyapunov functions and ergodicity: an irreducible process admitting an invariant probability measure is exponentially ergodic as soon as there exists a Lyapunov function for the associated infinitesimal generator.

The difficulty now is to find an explicit Lyapunov function. We solve it and propose the function V defined by

$$V(x,v) = exp\left(\sum_i (\frac{1}{2}\beta\lambda\ |x_i|^2 + \beta\ x_i v_i + \frac{1}{2}\lambda\ |v_i|^2)\right) \tag{12}$$

with $\lambda^2 > 2\beta$.

Let us prove that it is a Lyapunov function for the system (11). Indeed, the infinitesimal generator associated with these dynamics satisfies for f regular enough,

$$L_\beta f(x,v) = \frac{1}{2}\sum_{i,\alpha}\left(\partial^2_{v_i^\alpha} f(x,v) - \frac{1}{N}\sum_j \partial^2_{v_i^\alpha v_j^\alpha} f(x,v)\right) + \sum_{i,\alpha} v_i^\alpha \partial_{x_i^\alpha} f(x,v) - \sum_{i,\alpha}(\lambda v_i^\alpha + \beta x_i^\alpha)\partial_{v_i^\alpha} f(x,v),$$

where $i \in \{1,...,N\}$ and $\alpha \in \{1,...,d\}$.

We compute $L_\beta V$ for every $(x,v) \in \mathbb{R}^{2Nd}$:

$$\begin{aligned} L_\beta V(x,v) &= \tfrac{1}{2}\ \textstyle\sum_{i,\alpha}\Big[\lambda + (\beta\ x_i^\alpha + \lambda\ v_i^\alpha)^2 \\ &\qquad - \tfrac{1}{N}\textstyle\sum_j (\beta\ x_i^\alpha + \lambda\ v_i^\alpha)\left(\beta\ x_j^\alpha + \lambda\ v_j^\alpha\right)\Big] V(x,v) \\ &\quad + \textstyle\sum_{i,\alpha}(\beta(v_i^\alpha)^2 + \lambda v_i^\alpha x_i^\alpha)\ V(x,v) - \sum_{i,\alpha}(\lambda v_i^\alpha + \beta x_i^\alpha)^2\ V(x,v) \\ &= \Big[\tfrac{1}{2}\lambda N d - \tfrac{1}{2}\textstyle\sum_{i,\alpha}(\lambda v_i^\alpha + \beta x_i^\alpha)^2 - \tfrac{1}{N}\sum_\alpha\left(\sum_i (\beta x_i^\alpha + \lambda v_i^\alpha)\right)^2 \\ &\qquad + \textstyle\sum_{i,\alpha}(\beta(v_i^\alpha)^2 + \lambda v_i^\alpha x_i^\alpha)\Big] V(x,v) \\ &\le \Big[\tfrac{1}{2}\lambda N d - \tfrac{1}{2}\textstyle\sum_{i,\alpha}\left(-\lambda^2 (v_i^\alpha)^2 - \beta^2 (x_i^\alpha)^2 + 2\beta (v_i^\alpha)^2\right)\Big] V(x,v) \\ &= -\tfrac{1}{2}\left((\lambda^2 - 2\beta)|v|^2 + \beta^2 |x|^2 - \lambda N d\right) V(x,v). \end{aligned}$$

Thus, if $\lambda^2 > 2\beta$, setting $K = \min((\lambda^2 - 2\beta), \beta^2)$, when $|x|^2 + |v|^2$ is large enough,

$$L_\beta V(x,v) \le -K\ V(x,v).$$

□

3. **Non-constant communication rate: two particular cases.** We go back to the stochastic Cucker-Smale model (2): we now turn our attention to more realistic non-constant communication rates.

First, in subsection 3.1, we again apply Lyapunov function theory to obtain ergodicity for a two particle system: we prove the semi-group converges at polynomial speed towards its invariant probability measure.

Second, in subsection 3.2, we consider as communication rate a small perturbation of a constant one. Applying results from [20], based on the cluster expansion method, from statistical physics, we obtain some exponential ergodicity for general drifts with finite delay.

3.1. **One (or two) particle(s) along the real line.** Consider (x_1, v_1) and (x_2, v_2) satisfying equation (2) in which is added the term in β introduced in the previous section.

If we set $x = x_1 - x_2$ and $v = v_1 - v_2$, then (x, v) solves the stochastic differential system:

$$\begin{cases} dx_t &= v_t\, dt \\ dv_t &= -\frac{\lambda\, v_t}{(1+x_t^2)^\gamma}\, dt - \beta x_t\, dt + \sqrt{D} dW_t. \end{cases} \tag{13}$$

One can also consider these equations as the modelization of a single particle moving along the real line, according to some version of the modified stochastic Cucker-Smale model (10), studied in the previous section.

We look for the asymptotic behaviour of the system: even though there does not exist an explicit solution, we are able to exhibit a Lyapunov functional which is the key of the following convergence, the main result of this paragraph.

Theorem 3.1. *We define the function ϕ_γ by*

$$\phi_\gamma(t) = \begin{cases} t^{-\frac{1-\gamma}{\gamma}} & \text{for } \gamma \le \frac{1}{2} \\ t^{-\frac{1}{4\gamma-1}} & \text{for } \gamma \ge \frac{1}{2}. \end{cases}$$

The Markov process (x_t, v_t) solution of (13) admits an invariant probability measure, called μ_γ.

Moreover, its semi-group converges towards μ_γ for the total variation distance and the convergence rate is at least ϕ_γ.

To prove this result, we first give a criterion for the existence of an invariant probability measure, before applying it to system (13). Then, we prove the polynomial ergodicity.

3.1.1. *A criterion for the existence of an invariant probability measure.* We first recall in the proposition below a sufficient condition for the existence of an invariant probability measure. This result is a direct adaptation to continuous time of Theorem 12.3.4 in Meyn and Tweedie's book [19].

Proposition 9. *Let $(U_t)_{t\ge 0}$ be a Feller Markov process on Ω, whose infinitesimal generator L is such that there exists a non-negative function V, a positive b and a compact set C satisfying, for every u,*

$$LV(u) \le -1 + b\, \mathbf{1}_C(u). \tag{14}$$

Then, the process $(U_t)_{t\ge 0}$ admits an invariant probability measure on Ω.

3.1.2. *Application to system (13).* The infinitesimal generator associated with (13) is the differential operator defined by

$$L_{x,v} = D\, \partial_v^2 + v\, \partial_x - \frac{\lambda v}{(1+x^2)^\gamma}\, \partial_v - \beta x\, \partial_v.$$

Set

$$V_\gamma : (x, v) \mapsto \beta x^2 + \lambda\, f_\gamma(x)\, v + v^2,$$

where f_γ is the primitive of $\psi_\gamma : x \mapsto \frac{1}{(1+x^2)^\gamma}$ that vanishes at 0 (which exists by continuity of ψ_γ).

Applying the generator L to V_γ, we obtain

$$LV_\gamma(x, v) = D - \frac{\lambda v^2}{(1+x^2)^\gamma} - \frac{\lambda^2 v}{(1+x^2)^\gamma}\, f_\gamma(x) - \lambda\beta\, x f_\gamma(x). \tag{15}$$

The proposition below guarantees that Proposition 9 can be applied.

Proposition 10. *For a positive large enough* R *, if* $\max(|x|,|v|)>R$*, then*

$$LV_\gamma(x,v)\leq -1.$$

Proof. We begin by giving a few properties of f_γ: as ψ_γ is positive, f_γ is increasing with $f_\gamma(x)$ positive (resp. negative) if x is positive (resp. negative). In particular, for every x, $xf_\gamma(x)$ is non-negative. Furthermore ψ_γ is an even function, making f_γ an odd one.

ψ_γ is integrable on $\mathbb{R}$ if and only if $\gamma>\frac{1}{2}$; in this case f_γ tends towards the finite number $\int_0^\infty \frac{1}{(1+u^2)^\gamma}\,du$ when x goes to infinity. When $\gamma<\frac{1}{2}$ there exists a positive C_γ such that, for x large, $f_\gamma(x)\sim C_\gamma x^{1-2\gamma}$.

Suppose that $\max(|x|,|v|)>R$.

- If $|v|<R$, from (15),
$$\begin{aligned} LV_\gamma(x,v) &\leq 2D-\lambda\, f_\gamma(x)\left(\beta\, x+\tfrac{\lambda\, v}{(1+x^2)^\gamma}\right)\\ &\leq 2D-\lambda\,|f_\gamma(x)|\left(\beta\,|x|-\tfrac{\lambda\,|v|}{(1+x^2)^\gamma}\right)\\ &\leq 2D-\lambda\,|f_\gamma(x)|\left(\beta\, R-\tfrac{\lambda\, R}{(1+R^2)^\gamma}\right).\end{aligned}$$
  We briefly notice that $\frac{\lambda\, R}{(1+R^2)^\gamma}\sim_{R\to\infty}\lambda\, R^{1-2\gamma}$ and $1-2\gamma<1$ if and only if $\gamma<0$.

  Whatsoever,
$$\beta\, R\geq \frac{\lambda\, R}{(1+R^2)^\gamma}\Leftrightarrow (1+R^2)^\gamma\geq\frac{\lambda}{\beta}\Leftrightarrow R\geq\sqrt{\left(\frac{\lambda}{\beta}\right)^{1/\gamma}-1}.$$
  For $R\geq\sqrt{\left(\frac{\lambda}{\beta}\right)^{1/\gamma}-1}$, $LV_\gamma(x,v)\leq 2D-\lambda\,|f_\gamma(R)|\left(\beta\, R-\frac{\lambda\, R}{(1+R^2)^\gamma}\right)$.

  Hence, if R is such that $R\geq\sqrt{1+(\lambda/\beta)^\gamma}$ and $2D-\lambda\;|f_\gamma(R)|\;(\beta\;R-\frac{\lambda\, R}{(1+R^2)^\gamma})\leq -1$, we have
$$LV_\gamma(x,v)\leq -1.$$
- If $|x|<R$, for every positive γ, $|f_\gamma(x)|$ is negligible with respect to x; therefore, for R sufficiently large,
$$|v|>\lambda\,|f_\gamma(x)| \text{ and } \lambda\,|f_\gamma(x)| \text{ is negligible w.r.t. } |v|. \tag{16}$$
  Accordingly, on the one hand, if $\gamma<1$,
$$\begin{aligned} LV_\gamma(x,v) &\leq 2D-\tfrac{\lambda\, v}{(1+x^2)^\gamma}\,(v+\lambda\, f_\gamma(x))\\ &\leq 2D-\tfrac{\lambda\,|v|}{(1+x^2)^\gamma}\,(|v|-\lambda\,|f_\gamma(x)|)\\ &\leq 2D-\tfrac{\lambda\, R}{(1+R^2)^\gamma}\,(R-\lambda\, f_\gamma(R))\\ &\sim -\lambda\, R^{2(1-\gamma)} \text{ for } 2(1-\gamma)>0, \text{ that is } \gamma<1\end{aligned}$$
  We can thus find R such that $LV_\gamma(x,v)\leq -1$.
  On the other hand, when $\gamma\geq 1$, keeping (16) in mind,
$$LV_\gamma(x,v)\leq 2D-\frac{\lambda\,|v|}{(1+x^2)^\gamma}\,(|v|-\lambda\,|f_\gamma(x)|)-\lambda\beta\, xf_\gamma(x).$$
  (i) If $v\leq |x|^{2\gamma}$, $LV_\gamma(x,v)\leq 2D-\lambda\beta\, f_\gamma(R^{\frac{1}{2\gamma}})R^{\frac{1}{2\gamma}}\leq -1$ for R large enough.

(ii) Otherwise, if $v > |x|^{2\gamma}$, $LV_\gamma(x,v) \le 2D - \frac{\lambda\ |v|\ (|v| - \lambda\ |f_\gamma(x)|)}{(1+v^{\gamma^{-1}})^\gamma}$.
For $|v|$ sufficiently large, the fraction is equivalent to λv, which allows us to conclude.

- If $|x| > R$ and $|v| > R$, we will use Young's inequality: for every positive real numbers p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$,

$$\frac{\lambda^2\ |v|}{(1+x^2)^\gamma}\ |f_\gamma(x)| \le \frac{1}{p}\frac{\lambda^p\ |v|^p}{(1+x^2)^{\gamma p}} + \frac{1}{q}\lambda^q\ |f_\gamma(x)|^q.$$

Subsequently,

$$\begin{array}{rcl} LV_\gamma(x,v) & \le & 2D - \left(\frac{\lambda\ v^2}{(1+x^2)^\gamma} - \frac{1}{p}\frac{\lambda^p\ |v|^p}{(1+x^2)^{\gamma p}}\right) - \left(\lambda\beta\ x f_\gamma(x) - \frac{1}{q}\lambda^q\ |f_\gamma(x)|^q\right) \\ & = & 2D - a_1(x,v) - a_2(x,v). \end{array}$$

Roughly,

$$a_1(x,v) \approx \frac{\lambda\ v^2}{|x|^{2\gamma}} - \frac{\lambda^p}{p}\frac{|v|^p}{|x|^{2\gamma p}}$$

and

$$a_2(x,v) \approx C_\gamma \lambda\beta\ |x|^{1+\max(0,1-2\gamma)} - C_\gamma^q\frac{\lambda^q}{q}\ |x|^{q\max(0,1-2\gamma)}$$

where C_γ is a positive real number depending only on γ (α_γ if $\gamma < \frac{1}{2}$, else $f_\gamma(\infty)$).

We would like to exhibit p and q such that a_1 and a_2 tends to infinity when x and v do, regardless of the ratio x/v.

We will find them if each of the following assumptions is satisfied:

(a) $\frac{1}{p} + \frac{1}{q} = 1$
(b) $p < 2$
(c) $2\gamma < 2\gamma p$
(d) $q\max(0, 1-2\gamma) < 1 + \max(0, 1-2\gamma)$.

Conditions (b) and (c) come from the expression of a_1, (d) from the expression of a_2.

As $\gamma \ne 0$, (b) and (c) can be summed up by (e): $1 < p < 2$.

If $1 - 2\gamma > 0$, ie $\gamma < \frac{1}{2}$,

$$q < \frac{2(1-\gamma)}{1-2\gamma} \Leftrightarrow 1 - \frac{1}{p} > \frac{1-2\gamma}{2(1-\gamma)} \Leftrightarrow \frac{1}{p} < \frac{1}{2(1-\gamma)} \Leftrightarrow p > 2(1-\gamma).$$

Thus, for every $\gamma > 0$, $p \in (\max(1, 2(1-\gamma)), 2)$.

When $\gamma < \frac{1}{2}$ we set $p = 2 - \gamma$, hence $q = \frac{2-\gamma}{1-\gamma}$; otherwise we choose $p = \frac{3}{2}$ and $q = 3$.

Next, we check that we indeed observe the behaviour we were looking for:

If $\gamma > \frac{1}{2}$, $a_1(x,v) \le \frac{\lambda v^2}{(1+x^2)^\gamma} - \frac{3\lambda^{\frac{3}{2}}}{2}\frac{|v|^{\frac{3}{2}}}{(1+x^2)^{\frac{3\gamma}{2}}}$ is non-negative for R sufficiently large and $a_2(x,v) = \lambda\beta\ x f_\gamma(x) - \frac{\lambda^3}{3} f_\gamma(\infty)^3 \ge 2D + 1$ when x is large enough, and we are therefore able to conclude in this situation.

A similar verification can be done when $\gamma < \frac{1}{2}$.

Finally, for every positive real number γ, there exists some positive real number R such that

$$LV_\gamma \le -1 + \delta \mathbf{1}_{B_\gamma}$$

where $B_\gamma = \{(x,v) \in \mathbb{R}^2 |\max(|x|, |v|) \le R\}$ and δ a real number. □

B_γ being a compact set of $\mathbb{R}^2$ and V_γ being bounded on $\bar{B}(0,R)$, the stochastic dynamical system (13) admits an invariant probability measure, thanks to Proposition 9.

There exists an invariant measure μ_γ; we now determine the convergence rate of the semi-group associated with (x_t, v_t) towards this probability measure.

3.1.3. *Polynomial ergodicity.* The proof of the second part of Theorem 3.1 follows from Theorem 1.2 of [4] and the following proposition.

Proposition 11. *Let H be the function defined on $\mathbb{R}^+$ by $H(u) = |u|^{1-\gamma}$ if $\gamma \leq \frac{1}{2}$ and $H(u) = |u|^{\frac{1}{4\gamma}}$ if $\gamma \geq \frac{1}{2}$.*

For a positive, large enough, R , if $\max(|x|,|v|) > R$, then

$$\forall (x,v) \in \mathbb{R}^2, \quad LV_\gamma(x,v) \leq -K_\gamma \, H(V_\gamma(x,v))) \tag{17}$$

where K_γ is a positive constant depending only on γ.

Proof. It will follow a pattern similar to the one of Proposition 10.

Notice that H is a non-negative, increasing and concave map.

Suppose that $\max(|x|,|v|) > R$.

- If $\gamma \leq \frac{1}{2}$, we would like to prove that there exists K_γ such that for R large enough,

$$K_\gamma(\beta x^2 + \lambda f_\gamma(x)v + v^2)^{1-\gamma} \leq -D + \frac{\lambda v^2}{(1+x^2)^\gamma} + \frac{\lambda^2 v}{(1+x^2)^\gamma} f_\gamma(x) + \lambda\beta x f_\gamma(x). \tag{18}$$

  - Suppose that $|x| < |v|$ and $|v| > R$.
    On the one hand, for R such that $\frac{\lambda f_\gamma(R)}{R} < 1$,

$$(\beta x^2 + \lambda f_\gamma(x)v + v^2)^{1-\gamma} = |v|^{2(1-\gamma)}\left(1 + \frac{\beta x^2}{v^2} + \frac{\lambda f_\gamma(x)}{v}\right)^{1-\gamma} \leq (2+\beta)^{1-\gamma}|v|^{2(1-\gamma)}.$$

    On the other hand, if R is sufficiently large,

$$-D + \frac{\lambda v^2}{(1+x^2)^\gamma} + \frac{\lambda^2 v}{(1+x^2)^\gamma} f_\gamma(x) + \lambda\beta x f_\gamma(x) \geq -D + \frac{\lambda v^2}{2(1+x^2)^\gamma} \geq \frac{\lambda v^2}{4(1+x^2)^\gamma}.$$

    Furthermore, if $R \geq 1$,

$$\frac{\lambda v^2}{4(1+x^2)^\gamma} \geq \frac{\lambda v^2}{4 \times 2^\gamma \max(1,x^2)^\gamma} \geq \frac{1}{8}\lambda v^2 \min(1, x^{-2\gamma})$$
$$\geq \frac{1}{8}\lambda v^2 \min(1, |v|^{-2\gamma}) \geq \frac{1}{8}\lambda |v|^{2(1-\gamma)}.$$

    Thus, with $K_\gamma = \frac{1}{16(2+\beta)^{1-\gamma}}$ (18) is satisfied.
  - Suppose that $|v| \leq |x|$ and $|x| > R$.
    We proceed in exactly the same way, swapping x and v, to obtain the inequality

$$(\beta x^2 + \lambda f_\gamma(x)v + v^2)^{1-\gamma} \leq (2+\beta)^{1-\gamma}|x|^{2(1-\gamma)}.$$

    Besides, with similar arguments as those previously used,

$$-D + \frac{\lambda v^2}{(1+x^2)^\gamma} + \frac{\lambda^2 v}{(1+x^2)^\gamma} f_\gamma(x) + \lambda\beta x f_\gamma(x) \geq -D + \frac{1}{2}\lambda\beta|x| \times C_\gamma|x|^{1-2\gamma}$$

    where C_γ is a positive constant such that $f_\gamma(|x|) \sim C_\gamma|x|^{1-2\gamma}$ when $|x|$ is quite large.

Thus,

$$-D+\frac{\lambda v^2}{(1+x^2)^\gamma}+\frac{\lambda^2 v}{(1+x^2)^\gamma}f_\gamma(x)+\lambda\beta x f_\gamma(x)\geq \frac{1}{4}\lambda\beta C_\gamma|x|^{2(1-\gamma)} \geq K_\gamma(2+\beta)^{1-\gamma}|x|^{2(1-\gamma)}$$

for R big enough and K_γ below $\frac{\lambda\beta C_\gamma}{4(2+\beta)^{1-\gamma}}$, which implies inequality (18).

- If $\gamma \geq \frac{1}{2}$, we aim to show that we can find a positive constant K_γ such that for R large enough,

$$K_\gamma(\beta x^2+\lambda f_\gamma(x)v+v^2)^{\frac{1}{4\gamma}} \leq -D+\frac{\lambda v^2}{(1+x^2)^\gamma}+\frac{\lambda^2 v}{(1+x^2)^\gamma}f_\gamma(x)+\lambda\beta x f_\gamma(x). \quad (19)$$

  - Suppose that $|v|\leq |x|$ and $|x|>R$.
    Then, for R large enough,

$$(\beta x^2+\lambda f_\gamma(x)v+v^2)^{\frac{1}{4\gamma}} \leq (2+\beta)^{\frac{1}{4\gamma}}|x|^{\frac{1}{2\gamma}}.$$

    Moreover,

$$-D+\frac{\lambda v^2}{(1+x^2)^\gamma}+\frac{\lambda^2 v}{(1+x^2)^\gamma}f_\gamma(x)+\lambda\beta x f_\gamma(x)\geq \frac{1}{4}\lambda\beta C_\gamma|x|$$

    when R is large enough, with $C_\gamma=\lim_{|x|\to\infty} f_\gamma(|x|)$.
    Hence (19) holds with $K_\gamma \leq \frac{\lambda\beta C_\gamma}{4(2+\beta)^{\frac{1}{\gamma}}}$.
  - Suppose that $|x|<|v|$ and $|v|>R$. We have, by analogy with previous assumptions,

$$(\beta x^2+\lambda f_\gamma(x)v+v^2)^{\frac{1}{4\gamma}} \leq (2+\beta)^{\frac{1}{4\gamma}}|v|^{\frac{1}{2\gamma}}.$$

    Furthermore,

$$-D+\frac{\lambda v^2}{(1+x^2)^\gamma}+\frac{\lambda^2 v}{(1+x^2)^\gamma}f_\gamma(x)+\lambda\beta x f_\gamma(x)\geq \frac{\lambda v^2}{2(1+x^2)^\gamma}+\frac{1}{2}\lambda\beta C_\gamma|x|.$$

    (i) If $|x|^{2\gamma}\leq |v|$ then

$$\frac{\lambda v^2}{2(1+x^2)^\gamma}+\frac{1}{2}\lambda\beta C_\gamma|x| \geq \frac{\lambda v^2}{2(1+v^{\frac{1}{\gamma}})^\gamma} \geq \frac{\lambda v^2}{2(2v^{\frac{1}{\gamma}})^\gamma} \geq \frac{1}{8}|v| \geq K_\gamma(2+\beta)^{\frac{1}{4\gamma}}|v|^{\frac{1}{2\gamma}}$$

    with $K_\gamma=\frac{1}{8(2+\beta)^{\frac{1}{4\gamma}}}$, as $2\gamma\leq 1$ and (19) is satisfied.

    (ii) If $|x|^{2\gamma}>|v|$ we have

$$\frac{\lambda v^2}{2(1+x^2)^\gamma}+\frac{1}{2}\lambda\beta C_\gamma|x| \geq \frac{1}{2}\lambda\beta C_\gamma|x| \geq \frac{1}{2}\lambda\beta C_\gamma|v|^{\frac{1}{2\gamma}} \geq K_\gamma(2+\beta)^{\frac{1}{4\gamma}}|v|^{\frac{1}{2\gamma}}$$

    for $K_\gamma=\frac{\lambda\beta C_\gamma}{2((2+\beta)^{\frac{1}{4\gamma}})}$ and R large enough, ensuring us of the validity of inequality (19).

□

For two different values of γ, we illustrate in figures 1 and 2 the veracity of the proposition we have just proven. These graphics were realised with Matlab; in blue/dark is LV_γ, in green/light is $-K_\gamma H(V_\gamma)$.

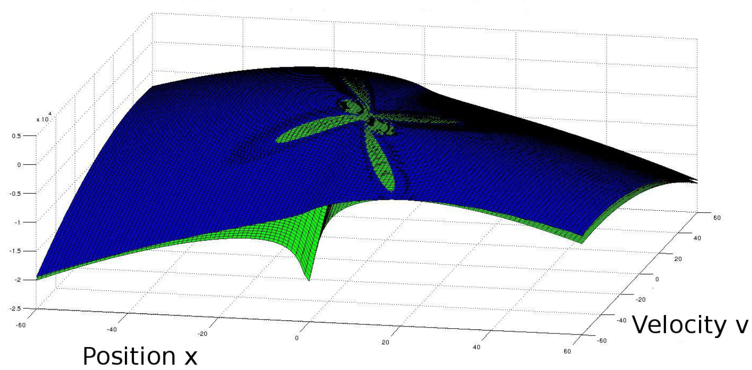

FIGURE 1. Case $\gamma = 0.2$, with $D = 0.1$, $\lambda = 5$, $\beta = 2$ and $K_\gamma = 8.3$.

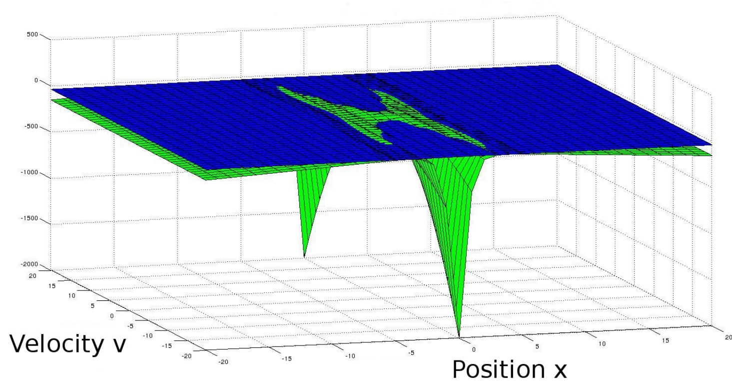

FIGURE 2. Case $\gamma = 2$, with $D = 0.1$, $\lambda = 5$, $\beta = 2$ and $K_\gamma = 17$.

We indeed observe that, in both situations, when we are far enough from the origin, the green/lighter surface is under the blue/darker one, which illustrates the drift condition shown in this section.

Thanks to Theorem 1.2 of [4], we know that the semi-group associated with the Markov process converges towards the invariant measure; furthermore we obtain a precise statement about the convergence rate, hence ϕ_γ.

Remark 2. Results of this subsection are only valid for $d = 1$. Unfortunately, we are not able to find explicit Lyapunov functions in higher dimensions.

3.2. **Ergodicity for small perturbations: The cluster expansion method.** In this section, we apply the cluster expansion method to the system considered: we start from a well-known symmetric diffusion, the case of the constant communication rate. The aim is to disrupt it through a small perturbation with finite delay t_0, to obtain a perturbation of a stochastic Cucker-Smale model whose drift has a finite delay t_0.

For the sake of simplicity, computations are done here in the case $d = 1$. The symmetry of the system ensures it is possible without loss of generality on the final result, even though some constants depend on d.

We saw in subsection 1.2 that the system corresponding to the communication rate $\psi = \lambda$ admits a reversible probability measure, $\mu = \mathcal{N}\left(0, \frac{1}{2\lambda}\,\Pi_N\right)$, setting $\Pi_N = \Pi_{N,1}$ defined in (7).

In this subsection, we apply the cluster expansion method established in [20] to obtain ergodicity for small perturbations of the drift in this model.

Consider the dynamics:

$$d\hat{v}(t) = -\lambda\hat{v}(t)dt + \Pi_N\; dW(t), \quad t \in \mathbb{R}_+ \tag{20}$$

where $\hat{v} \in \mathbb{R}^N$ and W is a N-dimensional standard Brownian motion.

The law of the Ornstein-Uhlenbeck process studied in section 1 is degenerate: the $N \times N$ matrix Π_N is not invertible (see Remark 1), nor is $\Pi_N\Pi_N^*$ (and a projection on the first $N-1$ coordinates of Π_N satisfies neither of these requirements). Thus, in order to apply results from [20] we have to project the system on an ad hoc subspace, where the process will be elliptic, once we have established an adequate orthonormal basis on it. We then introduce the perturbation on this subspace.

As previously mentioned, the vector of the microscopic velocities $(\hat{v}_1, ..., \hat{v}_N)$ is living on the hyperplane $H = \{\ v \in \mathbb{R}^N \mid v_1 + ... + v_N = 0\ \}$ whose orthonormal basis $(e_i)_{i\in\{1,...,N-1\}}$ is given by

$$e_i^j = \sqrt{\frac{i}{i+1}}\left(\frac{1}{i}\delta_{j\le i} - \delta_{j=i+1}\right)$$

for every $j \in \{1, ..., N\}$.

In what follows, we set $\alpha_i = \sqrt{\frac{i}{i+1}}$. Let e_N be the vector of $\mathbb{R}^N$ such that $e_N^j = \frac{1}{\sqrt{N}}$ for every j. Then $(e_i)_{i\in\{1,...,N\}}$ is an orthonormal basis of $\mathbb{R}^N$.

The microscopic system $\hat{v}(t)$ has in the basis $(e_i)_{i\in\{1,...,N\}}$ the coordinates $u_i(t) = e_i^*\hat{v}(t)$, $i = 1, ..., N$.

Thus,

$$u_N(t) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\hat{v}_k(t)$$

and for $i \in \{1, ...N-1\}$,

$$u_i(t) = \alpha_i\left(\frac{1}{i}\sum_{k=1}^{i}\hat{v}_k - \hat{v}_{i+1}\right).$$

This means that, on the one hand

$$u_N(t) = \sqrt{\frac{1}{N}}\sum_{k=1}^{N}\widehat{W}_k(t) = 0$$

and on the other hand

$$du_i(t) = \alpha_i\left(\frac{1}{i}\sum_{k=1}^{i}\left(-\lambda\hat{v}_k(t)\; dt + (dW_k(t) - \frac{1}{N}\sum_{j=1}^{N}\; dW_j(t)\right)\right)$$

$$+\lambda \hat{v}_{i+1}(t)\, dt - (dW_{i+1}(t) - \frac{1}{N}\sum_{j=1}^{N} dW_j(t))\Bigg)$$

so that

$$du_i(t) = -\lambda u_i(t)\, dt + \alpha_i \left(\frac{1}{i}\sum_{k=1}^{i} dW_k(t) - dW_{i+1}(t)\right).$$

Setting $U = (u_1, ..., u_{N-1})$, it satisfies in $\mathbb{R}^{N-1}$

$$dU(t) = -\lambda U(t)\, dt + \sigma\, dW(t) \tag{21}$$

with σ the $(N-1)\times N$ matrix whose j-th row is $\alpha_j\ e_j^*$. The system is now non degenerate: $\sigma\sigma^*$ is invertible.

U is another Ornstein-Uhlenbeck type process, different from (20).

Proposition 12. *For every $t \geq 0$,*

$$U(t) = e^{-\lambda t}\, U(0) + \int_0^t e^{-\lambda(t-s)}\, \sigma\, dW(s),$$

$$U(t) \sim \mathcal{N}\left(e^{-\lambda t}\, \mathbb{E}[U(0)]\ ,\ \frac{1}{2\lambda}\,(1-e^{-2\lambda t})\, I_{N-1}\right).$$

Finally, $\rho = \mathcal{N}\left(0, \frac{1}{2\lambda}\, I_{N-1}\right)$ is a reversible probability measure for U ; since all the hypotheses required for Theorem 2 in [20] are satisfied, Theorem 3.2 holds:

Theorem 3.2. *Assume that $b : \mathcal{C}(\mathbb{R}_+, \mathbb{R}^{N-1}) \to \mathbb{R}^{N-1}$ is a measurable function bounded by 1, which is moreover local in the sense that there exists $t_0 > 0$ such that, for any $u \in \Omega$, $b(u) = b((u)_{t-t_0}^t)$. Then, when β is small enough, the system with delay*

$$dZ(t) = (-\lambda Z(t) + \beta\, b((Z)_{t-t_0}^t)\, dt + \sigma\, dW(t),$$

where $(Z)_{t-t_0}^t$ is the trajectory of Z between times $t-t_0$ and t, admits a unique weak stationary solution Q on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^{N-1})$.

Moreover, there is exponential ergodicity: there exist $\theta > 0$ and $C : \mathbb{R}^{N-1} \to \mathbb{R}_+$ such that for t and t' large enough, for every $z \in \mathbb{R}^{N-1}$, for every bounded measurable function f,

$$|\mathbb{E}_Q[f(Z(t))|Z(0)=z] - \mathbb{E}_Q[f(Z(t'))|Z(0)=z]| \leq C(z)\, e^{-\theta\, |t-t'|}.$$

Finally, we go back to the canonical basis.

Let, for $b = (b_1, ..., b_{N-1}) : \mathcal{C}(\mathbb{R}_+, \mathbb{R}^{N-1}) \to \mathbb{R}^{N-1}$, the function $B = (B_1, ..., B_{N-1}) : \mathcal{C}(\mathbb{R}_+, \mathbb{R}^{N-1}) \to \mathbb{R}^{N-1}$ be given by

$$B_i(.) = \sum_{j=i}^{N-1} \frac{1}{\sqrt{j(j-1)}}\, b_j(P\,.) - \sqrt{\frac{i-1}{i}}\, b_{i-1}(P\,.)$$

with $P = (P_{ij})_{i,j\in\{1,...,N\}}$ the square matrix of size N such that, for all $j \in \{1, ..., N\}$,

$$P_{ij} = \sqrt{\tfrac{i}{i+1}}\ \left(\tfrac{1}{i}\,\delta_{j\leq i} - \delta_{i=j+1}\right) \text{ if } i < N \quad \text{and} \quad P_{Nj} = \tfrac{1}{\sqrt{N}}.$$

Corollary 1. *Assume that b is as in Theorem 3.2 and B as defined just above. Then, if β is small enough, the dynamics*

$$d\hat{v}(t) = \big(-\lambda \hat{v}(t) + \beta\, B\big((\hat{v})_{t-t_0}^t\big)\big)\, dt + \Pi_N\, dW(t)$$

admits a weak stationary solution and there is exponential ergodicity.

Remark 3. Following Scheutzow in [21] (see Theorem 3), one already knows that there exists a unique invariant probability measure for such dynamics. The novelty here is the explicit rate of convergence.

Remark 4. That we consider a finite delay instead of an unbounded one, as in the original Cucker-Smale model, is not that much of a stretch: indeed,it is realistic to suppose that the behaviour of a particle at time t depends on the difference of the positions at times t and $t-t_0$, for a certain t_0.

4. **Stationarity solutions and moment controls.** Here we obtain a more general result about the existence of stationary solutions and thus of a certain form of invariant probability measures by applying results from Itô and Nisio ([17]).

First, however, we introduce a few hypotheses:

- (H1): There exists a even, positive, function $\bar{\psi}:\mathbb{R}^d\to\mathbb{R}$ such that, for all x and y, $\psi(x,y)=\bar{\psi}(x-y)$.
- (H2): There exists two constants ψ_1 and ψ_2 such that, for all $s\in\mathbb{R}^d$, $0<\psi_1\le\bar{\psi}(s)\le\psi_2$.
- (H3): $\bar{\psi}$ is bounded and Lipschitz continuous.

4.1. **Stationarity results.** We place ourselves in the general case of the microscopic velocities of the stochastic Cucker-Smale system (5) seen as a delayed equation autonomous in $\hat{v}$, with unbounded delay:

$$d\hat{v}_i(t)=-\frac{1}{N}\sum_{j=1}^{N}\widetilde{\psi}\left((\hat{v}_i)_0^t,(\hat{v}_j)_0^t\right)(\hat{v}_i(t)-\hat{v}_j(t))\,dt+d\widehat{W}_i(t),\quad i\in\{1,...,N\}\tag{22}$$

where $(\hat{v})_0^t=(\hat{v}_s)_{s\in(0,t]}$ and $\widetilde{\psi}$ is defined by

$$\widetilde{\psi}\left((\hat{v}_i)_0^t,(\hat{v}_j)_0^t\right):=\psi(\hat{x}_i(t),\hat{x}_j(t))=\psi\left(\hat{x}_i(0)+\int_0^t\hat{v}_i(s)\,ds,\hat{x}_j(0)+\int_0^t\hat{v}_j(s)\,ds\right).$$

Theorem 4.1. *Assume (H1) and (H2). Then the delayed equation (22) admits at least one stationary solution.*

Proof. The key ingredient of the proof is Theorem 3 of [17]: it states that for the stochastic differential equation

$$d\hat{v}_t=a((\hat{v})_0^t)\,dt+b((\hat{v})_0^t)\,dB_t,\quad t\in\mathbb{R},\tag{23}$$

which satisfy the following three assumptions:

- (H4): $a(f)$ and $b(f)$ are continuous on the space of the continuous functions on $\mathbb{R}_-$;
- (H5): there exist $M>0$ and a bounded measure K with compact support on $\mathbb{R}$ such that for every continuous f,

$$|a(f)|^2+|b(f)|^2\le M+\int_0^0|f(t)|^2\,dK_t;$$

- (H6): there is a uniform control of the second-order moments:

$$\sup_{t\in\mathbb{R}_+}\mathbb{E}[\hat{v}_t^2]<+\infty,$$

then equation (23) admits a stationary solution, that is a solution that is invariant under the time shift. An argument of weak compactness is central in its proof.

In this case, one can easily verify that (H4) is satisfied.

Furthermore, so is (H5), when ψ is bounded by ψ_2, with $M = N$ and $dK_t = 4N^2\psi_2^2\ \delta_0(dt)$: indeed, here $b_i(f) = 1$ and $a_i(f) = -\frac{1}{N}\ \sum_{j=1}^N \widetilde{\psi}\,(f_i, f_j)\,(f_i(0) - f_j(0))$, so that $|a_i(f)|^2 \leq 4N\,\psi_2^2\,|f(0)|^2$. Thus

$$|a(f)|^2 + |b(f)|^2 \leq N + 4N^2\,\psi_2^2\,|f(0)|^2.$$

The crucial point to apply this result is the hypothesis (H6): we will show in Proposition 13 that

$$\mathbb{E}\left[|\hat{v}(t)|^2\right] \leq \mathbb{E}\left[|\hat{v}(0)|^2\right] + \frac{dN}{2\psi_1}, \tag{24}$$

if there exists a positive constant ψ_1 such that for all non-negative s, $0 < \psi_1 \leq \bar{\psi}(s)$, which is the case, as assumption (H2) holds.

Thus, we have the existence of a stationary solution for this particular class of communication rates. □

Remark 5. To the best of our knowledge, we cannot conclude anything about the uniqueness of such stationary solutions.

We now prove the necessary results to obtain the upper bound (24), as well as other moment controls that will be useful in the last section of this paper.

4.2. **Various controls of first and second order moments.** *Proposition 13 is actually very close to Theorem 3.5 of* [14].

Assume (H1) and (H2). First we truly conclude the proof of Theorem 4.1 with the crucial result that brings about the inequality (24).

Proposition 13. *Suppose that (H1) and (H2) are satisfied, and that the initial law has a finite second order moment. Then, for all $t \geq 0$,*

$$\sum_{i=1}^N \mathbb{E}[|\hat{v}_i(t)|^2] \leq \sum_{i=1}^N \mathbb{E}[|\hat{v}_i(0)|^2]\ e^{-2\psi_1 t} + \frac{d(N-1)}{2\psi_1}\ (1 - e^{-2\psi_1 t}).$$

Proof. The lemma below is a generalization to stopping times of Lemma 3.4 of [14]; we do not give its proof here.

Lemma 4.2. *Suppose that (H1) and (H2) are satisfied, and that the initial law has a finite second order moment. Let t be any (stopping) time. Then, almost surely,*

a) $|\hat{x}(t)|^2 \leq |\hat{x}(0)|^2 + 2\ \int_0^t \sqrt{|\hat{x}(s)|^2\ |\hat{v}(s)|^2}\, ds$.

b) $|\hat{v}(t)|^2 \leq |\hat{v}(0)|^2 - 2\psi_1\ \int_0^t\ |\hat{v}(s)|^2\, ds + d(N-1)\, t + 2\sum_{i=1}^N \int_0^t \hat{v}_i(s)\, d\widehat{W}_i(s)$.

We introduce $T_k = \inf\ \{\ u \geq 0 \mid |\hat{v}(u)|^2 \geq k\ \} \wedge t$. Then, with part b) of lemma 4.2,

$$\begin{aligned} \mathbb{E}[|\hat{v}(T_k)|^2] &\leq \mathbb{E}[|\hat{v}(0)|^2] - 2\,\psi_1\,\mathbb{E}\left[\int_0^{T_k}\ |\hat{v}(s)|^2\, ds\right] + d(N-1)\,\mathbb{E}[T_k] \\ &\leq |\hat{v}(0)|^2 + d(N-1)\, t. \end{aligned}$$

Hence, when k goes to infinity, we obtain the finiteness of $\mathbb{E}[|\hat{v}(t)|^2]$.

Furthermore,

$$\mathbb{E}[|\hat{v}(t)|^2] = \mathbb{E}[|\hat{v}(0)|^2] - \frac{1}{N}\ \int_0^t\ \mathbb{E}\left[\sum_{i,j=1}^N\ \bar{\psi}(\hat{x}_i - \hat{x}_j)\ |\hat{v}_i - \hat{v}_j|^2\right]\ ds + d(N-1)\, t,$$

so that by differentiation,

$$\frac{d}{dt}\left(\mathbb{E}[|\hat{v}(t)|^2] - \frac{d(N-1)}{2\psi_1}\right) = -\frac{1}{N}\,\mathbb{E}\left[\sum_{i,j=1}^{N}\ \bar{\psi}(\hat{x}_i - \hat{x}_j)\ |\hat{v}_i - \hat{v}_j|^2\right] + d(N-1)$$
$$\leq d(N-1) - 2\psi_1\,\mathbb{E}[|\hat{v}(t)|^2]$$

Thus, by Gronwall's lemma,

$$\mathbb{E}[|\hat{v}(t)|^2] \leq \mathbb{E}[|\hat{v}(0)|^2]\ e^{-2\psi_1 t} + \tfrac{d(N-1)}{2\psi_1}\ (1 - e^{-2\psi_1\ t}).$$

□

We now focus on results that will be needed in the next section, dealing with propagation of chaos, adding exchangeability to the assumptions.

We recall (see for instance [3]) that particles are said to be exchangeable if every permutation of these particles has the same law: that is $(X_1, ..., X_n)$ are exchangeable if for any permutation σ of $\{1, ..., n\}$, $(X_1, ..., X_n)$ and $(X_{\sigma(1)}, ..., X_{\sigma(n)})$ have same law.

Proposition 14. *Suppose that (H1) and (H2) are satisfied and that the particles are exchangeable at time $t = 0$; in particular particles have the same initial law. Assume also that this initial law has a finite second order moment.*

Then, for all $i \in \{1, ...N\}$,

$$\sup_{t\geq 0}\mathbb{E}\left[|\hat{v}_i(t)|^2\right] \leq \mathbb{E}\left[|\hat{v}_i(0)|^2\right] + \frac{d}{2\psi_1}.$$

Proof. We have previously seen that:

$$\mathbb{E}\left[\textstyle\sum_{i=1}^N\ |\hat{v}_i(t)|^2\right] \leq \mathbb{E}\left[\textstyle\sum_{i=1}^N\ |\hat{v}_i(0)|^2\right]\ e^{-2\psi_1\ t} + \tfrac{d(N-1)}{2\psi_1}\ (1 - e^{-2\psi_1\ t}).$$

Exchangeability leads to:

$$\mathbb{E}\left[|\hat{v}_i(t)|^2\right] \leq \mathbb{E}\left[|\hat{v}_i(0)|^2\right]\ e^{-2\psi_1\ t} + \tfrac{d(N-1)}{2\psi_1 N}\ (1 - e^{-2\psi_1\ t})$$

which brings the conclusion of the proof. □

Corollary 2. *Suppose that (H1) and (H2) are satisfied and that the particles are exchangeable at time $t = 0$. Assume also that the common initial law has a finite second order moment.*

Then, for all non-negative t, there exists a positive constant M_t, such that

$$\sup_{i\in\{1,\dots,N\}}\mathbb{E}\left[\|(\hat{x}_i(t), \hat{v}_i(t))\|\right] \leq M_t.$$

Proof. Let $C_1 = \mathbb{E}\left[|\hat{v}_i(0)|^2\right] + \frac{d}{2\psi_1}$. Then

$$|\hat{x}_i(t)|^2 = |\hat{x}_i(0)|^2 + \left|\textstyle\int_0^t \hat{v}_i(s)\ ds\right|^2 + 2\ \hat{x}_i(0)\ \textstyle\int_0^t \hat{v}_i(s)\ ds.$$

Thus by multiple uses of Cauchy-Schwarz inequality,

$$\mathbb{E}\left[|\hat{x}_i(t)|^2\right] \leq \mathbb{E}\left[|\hat{x}_i(0)|^2\right] + C_1\ t^2 + 2\ \sqrt{C_1\ \mathbb{E}\left[|\hat{x}_i(0)|^2\right]}\ t$$

Then, choosing $M_t = \sqrt{C_1 + \mathbb{E}\,[|\hat{x}_i(0)|^2] + C_1\, t^2 + 2\,\sqrt{C_1\,\mathbb{E}\,[|\hat{x}_i(0)||^2]}\; t}$, by Proposition 14,

$$\mathbb{E}\,[|(\hat{x}_i(t), \hat{v}_i(t))|] \leq \sqrt{\mathbb{E}\,[|(\hat{x}_i(t), \hat{v}_i(t))|^2]} \leq M_t.$$

□

We give another moment control, involving a single particle and a stopping time. This will be useful to apply Aldous criterion to obtain tightness in Section 5.

We restrict the trajectories to a finite time interval: we place ourselves on $\Omega_T = \mathcal{C}([0,T], \mathbb{R}^{2d})$, the canonical continuous $\mathbb{R}^{2d}$-valued path space, where T is a fixed positive time.

Proposition 15. *Suppose that (H1) and (H2) are satisfied and that the particles are exchangeable at time $t = 0$. Assume also that the common initial law has a finite second order moment.*

Then, there exist two constants C and K independent of N, such that for two stopping times τ_1 and τ_2 on Ω_T satisfying $\tau_1 \leq \tau_2 \leq (\tau_1 + \theta) \wedge T$,

$$\sup_{i \in \{1,\dots,N\}} \mathbb{E}\left[\left| (\hat{x}_i(\tau_2) - \hat{x}_i(\tau_1), \hat{v}_i(\tau_2) - \hat{v}_i(\tau_1)) \right|^2 \right] \leq K\,\theta + C\,\theta^2.$$

Proof. We apply Itô's formula:

$$\begin{aligned} |\hat{v}_i(\tau_2) - \hat{v}_i(\tau_1)|^2 = M_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\left(1 - \frac{1}{N}\right)\, du \\ - \frac{2}{N} \int_{\tau_1}^{\tau_2} (\hat{v}_i(u) - \hat{v}_i(\tau_1)) \sum_{j=1}^{N} \psi(\hat{x}_i(u), \hat{x}_j(u))\, (\hat{v}_i(u) - \hat{v}_j(u))\, du, \end{aligned}$$

where $M_\tau^{\tau+u}$ is a martingale and satisfies $\mathbb{E}[M_\tau^{\tau+u}] = 0$ for every u.

This leads to

$$\begin{aligned} \mathbb{E}[|\hat{v}_i(\tau_2) - \hat{v}_i(\tau_1)|^2] &= d\left(1 - \frac{1}{N}\right)\, \mathbb{E}[\tau_2 - \tau_1] \\ &\quad - \frac{2}{N}\, \mathbb{E}\left[\int_{\tau_1}^{\tau_2} (\hat{v}_i(u) - \hat{v}_i(\tau_1)) \sum_{j=1}^{N} \psi(\hat{x}_i(u), \hat{x}_j(u))\, (\hat{v}_i(u) - \hat{v}_j(u))\, du \right] \\ &\leq d\left(1 - \frac{1}{N}\right)\, \theta + \frac{2\psi_2}{N} \sum_{j=1}^{N} \left(\mathbb{E}\left[\int_{\tau_1}^{\tau_2} |\hat{v}_i(u) - \hat{v}_i(\tau_1)|\, |\hat{v}_i(u) - \hat{v}_j(u)|\, du \right] \right) \\ &\leq d\left(1 - \frac{1}{N}\right) \theta + \frac{2\psi_2}{N} \sum_{j=1}^{N} \int_0^{\theta} \sqrt{\mathbb{E}[|\hat{v}_i(\tau_1 + u) - \hat{v}_i(\tau_1)|^2]} \\ &\qquad \times \sqrt{\mathbb{E}[|\hat{v}_i(\tau_1 + u) - \hat{v}_j(\tau_1 + u)|^2]}\, du, \end{aligned}$$

thanks to Cauchy-Schwarz inequality.

According to Lemma 4.2, for all τ (stopping) time smaller than $T + \theta$,

$$\begin{aligned} \mathbb{E}[|\hat{v}(\tau)|^2] \leq \mathbb{E}[|\hat{v}(0)|^2] - 2\psi_1\, \mathbb{E}\left[\int_0^{\tau} |\hat{v}(s)|^2\, ds \right] + d(N-1)\, \mathbb{E}[\tau] \\ \leq \mathbb{E}[|\hat{v}(0)|^2] + 2d(N-1)T. \end{aligned}$$

Using the exchangeability, for all i,

$$\mathbb{E}[|\hat{v}_i(\tau)|^2] \leq \mathbb{E}[|\hat{v}_i(0)|^2] + 2d\left(1 - \frac{1}{N}\right) T \leq \mathbb{E}[|\hat{v}_i(0)|^2] + 2dT =: C.$$

It means that

$$\mathbb{E}[|\hat{v}_i(\tau_2) - \hat{v}_i(\tau_1)|^2] \leq d\left(1 - \frac{1}{N}\right)\theta + 2\psi_2 \int_0^\theta 4\, C\, du \leq K\,\theta,$$

with

$$K = d + 8\psi_2\, C.$$

Besides,

$$|\hat{x}_i(\tau_2) - \hat{x}_i(\tau_1)|^2 = \left|\int_{\tau_1}^{\tau_2} \hat{v}_i(u)\, du\right|^2 \leq \left(\int_{\tau_1}^{\tau_2} |\hat{v}_i(u)|\, du\right)^2 \leq (\tau_2 - \tau_1) \int_{\tau_1}^{\tau_2} |\hat{v}_i(u)|^2\, du,$$

using once again Cauchy-Schwarz inequality.

Thus, the proof is concluded, as

$$\mathbb{E}[|\hat{x}_i(\tau_2) - \hat{x}_i(\tau_1)|^2] \leq \theta \int_0^\theta \mathbb{E}\left[|\hat{v}_i(\tau_1 + u)|^2\right] du \leq C\,\theta^2.$$

□

5. **Propagation of chaos.** Into the behaviour of the system for a very large number N of particles: with these mean-field dynamics, there is propagation of chaos, as introduced by Sznitman [23] in the late 1980s. This is known, in a more general case but for a simpler diffusion coefficient, see Bolley, Cañizo and Carrillo in [6].

One should note however that the two approaches are very different; because of the non-independence of the diffusion coefficients of the particles in our model, the coupling method used in [6] cannot be employed. Furthermore, the proof we propose is entirely probabilistic, in constrast with the more analytical one of [6] where transport equation method are used.

From now on we assume that hypotheses (H1), (H2) and (H3)introduced at the beginning of Section 4 are satisfied: we recall in particular that there exist two constants ψ_1 and ψ_2 such that, for all $s \in \mathbb{R}^d$, $0 < \psi_1 \leq \bar{\psi}(s) \leq \psi_2$ and that $\bar{\psi}$ is k-Lipschitz continuous.

Recall that $\Omega_T = \mathcal{C}([0,T], \mathbb{R}^{2d})$ is the canonical continuous $\mathbb{R}^{2d}$-valued path space, with $\mathcal{F}$ the canonical Borel σ-field on Ω_T.

First, we recall the definition of chaoticity.

Definition 5.1. We consider E a Polish space, Q a probability measure on E and for $N \in \mathbb{N}$, Q_N a probability measure on E^N. The sequence $(Q_N)_{N\geq 1}$ is Q-chaotic if for any fixed integer $k \geq 1$ and any continuous bounded functions $f_1, ..., f_k$ on E,

$$\lim_{N\to\infty} \int f_1(x_1) \dots f_k(x_k)\, dQ_N(x_1, ..., x_N) = \prod_{i=1}^k \int f_i(x_i)\, dQ(x_i).$$

In other words, it means that when N goes towards infinity, any fixed finite number of coordinates become independent with the same distribution Q.

The objective here is to show the convergence, in law and in probability, of the empirical measure, in N, associated with the N-particle system (5) towards a limit η, and to prove that a chaotic behaviour appears.

Remember that the system (5) is, for every $i \in \{1, ...N\}$,

$$\begin{cases} d\hat{x}_i^N(t) &= \hat{v}_i^N(t)\, dt \\ d\hat{v}_i^N(t) &= -\frac{1}{N}\sum_{i=1}^N \psi(\hat{x}_i^N(t), \hat{x}_j^N(t))\, (\hat{v}_i^N(t) - \hat{v}_j^N(t))\, dt + d\widehat{W}_i^N(t). \end{cases}$$

When there is no risk of confusion, we will forego the exponent N.
If there is chaoticity, the "natural" limit would be the non linear system:

$$\begin{cases} \mathbf{x}_t &= \mathbf{x}_0 + \int_0^t \mathbf{v}_s\, ds \\ \mathbf{v}_t &= \mathbf{v}_0 + W_t - \int_0^t \int \psi(\mathbf{x}_s, x)(\mathbf{v}_s - v)\, \mathbf{Q}_s(dx, dv)\, ds \\ \mathbf{Q}_t &= \mathcal{L}(\mathbf{x}_t, \mathbf{v}_t). \end{cases} \tag{25}$$

At this point, we need to introduce a few notations.

Let, for every integer N larger than 1, $\eta_N(\omega) = \frac{1}{N}\sum_{i=1}^N \delta_{(\hat{x}_i^N, \hat{v}_i^N)(\omega)}$ be the empirical measure on Ω_T associated with the N-particle system defined by (5), and π_N its law on $\mathcal{P}(\Omega_T)$.

We then introduce the martingale problems, associated respectively with systems (5) and (25):

1. A probability measure Q^N on $\mathcal{C}([0,T], \mathbb{R}^{2dN})$ is a solution of the martingale problem $(\mathcal{P}_N)$ if for all Φ in $\mathcal{C}_b^2(\mathbb{R}^{2dN})$, $M_t^N(\Phi)$ defined by

$$M_t^N(\Phi) = \Phi(\hat{x}(t), \hat{v}(t)) - \Phi(\hat{x}(0), \hat{v}(0)) - \int_0^t \widehat{L}_N \Phi(\hat{x}(s), \hat{v}(s))\, ds \tag{26}$$

is a Q^N-martingale such that,

$$< M_t^N(\Phi) >= \sum_{i=1}^N \int_0^t |\nabla_{v_i}\Phi(\hat{x}(s), \hat{v}(s)) - \frac{1}{N}\sum_{j=1}^N \nabla_{v_j}\Phi(\hat{x}_s, \hat{v}_s)|^2\, ds,$$

where $\widehat{L}_N$ is the infinitesimal generator associated with (5), that is

$$\widehat{L}_N\Phi(\hat{x}, \hat{v}) = \sum_{i=1}^N \hat{v}_i.\nabla_{\hat{x}_i}\Phi - \frac{1}{N}\sum_{i,j=1}^N \psi(\hat{x}_i, \hat{x}_j)\, (\hat{v}_i - \hat{v}_j).\nabla_{\hat{v}_i}\Phi + \frac{1}{2}\sum_{i=1}^N \left(\Delta_{\hat{v}_i}\Phi - \frac{1}{N}\sum_{j=1}^N \sum_{\alpha=1}^d \partial^2_{\hat{v}_i^\alpha\, \hat{v}_j^\alpha}\Phi\right).$$

When $\Phi(\hat{x}, \hat{v}) = \phi(\hat{x}_i, \hat{v}_i)$ with ϕ in $\mathcal{C}_b^2(\mathbb{R}^{2d})$, we set $M_t^{N,i}(\phi) := M_t^N(\Phi)$.

2. A probability measure Q on $\Omega_T = \mathcal{C}([0,T], \mathbb{R}^{2d})$ is a solution of the martingale problem $(\mathcal{P}_\infty)$ if for all ϕ in $\mathcal{C}_b^2(\mathbb{R}^{2d})$,

$$M_t^\phi(Q) = \phi(\mathbf{x}_t, \mathbf{v}_t) - \phi(\mathbf{x}_0, \mathbf{v}_0) - \int_0^t \nabla_x\phi(\mathbf{x}_s, \mathbf{v}_s)\, \mathbf{v}_s\, ds + \int_0^t \int \psi(\mathbf{x}_s, x)\, \nabla_v\phi(\mathbf{x}_s, \mathbf{v}_s).(\mathbf{v}_s - v)\, Q_s(dx, dv)\, ds - \frac{1}{2}\int_0^t \Delta_v\phi(\mathbf{x}_s, \mathbf{v}_s)\, ds, \tag{27}$$

where Q_s is defined by $Q_s = Q \circ (\mathbf{x}_s, \mathbf{v}_s)^{-1}$, is a Q-martingale such that

$$< M_t^\phi >= \int_0^t |\nabla_v\phi(\mathbf{x}_s, \mathbf{v}_s)|^2\, ds.$$

The main result is the following theorem:

Theorem 5.2. *Assume (H1), (H2) and (H3). Suppose that the particles are exchangeable at time* $t = 0$. *Assume also that the initial law* η_0 *on* $\mathbb{R}^d \times \mathbb{R}^d$ *has a finite second order moment and that for* $a = kT\,(1+T)\,e^{\psi_2 T}$, $\mathbb{E}[e^{a|v_0|}] < \infty$.

The sequence of the empirical measures $(\eta_N)_{N\geq 1}$ *converges in law and in probability to* η, *the unique solution of (27), if* $\eta_N(0)$ *converges in probability towards* η_0 *when* N *goes to infinity*

Remark 6. Notice that, while the uniqueness of the solution of (27) will be established as we prove the theorem, its existence derives from the convergence and will be a consequence of the proof.

To obtain the chaoticity of the system, we apply the following proposition whose proof can be found in [23] or in [18].

Proposition 16. *If* $(Q_N)_{N\geq 1}$ *is a sequence of exchangeable probability measures on* E^N, *it is Q-chaotic if and only if the associated empirical measure converges in law - and in probability - as* $\mathcal{P}(E)$*-valued variables under* Q_N, *towards the probability measure* Q.

Corollary 3. *Under the hypotheses of Theorem 5.2, the sequence* $(\eta_N)_{N\geq 1}$ *is* η*-chaotic.*

Remark 7. If we consider a fixed number of particles among a large amount, they behave independently from each other, which seems quite far from the concept of flocking.

To prove Theorem 5.2, we will follow a classical procedure and proceed in three steps presented in the next three subsections of this work:

1. tightness of $(\pi_N)_{N\geq 1}$ in $\mathcal{P}(\mathcal{P}(\Omega_T))$;
2. the link between the accumulation points of $(\pi_N)_{N\geq 1}$ and a martingale problem;
3. uniqueness of the solution of (27), coming from the uniqueness of the solution of the limit process (25).

We actually start with the third step.

5.1. Uniqueness of the non-linear equation and of the associated martingale problem.

Consider the non-linear stochastic differential system, on $[0, T]$,

$$(\mathcal{S}_W)\begin{cases} \mathbf{x}_t &= \mathbf{x}_0 + \int_0^t \mathbf{v}_s\, ds \\ \mathbf{v}_t &= \mathbf{v}_0 + W_t - \int_0^t \int \psi(\mathbf{x}_s, x)(\mathbf{v}_s - v)\, \mathbf{Q}_s(dx, dv)\, ds \\ \mathbf{Q}_t &= \mathcal{L}(\mathbf{x}_t, \mathbf{v}_t), \end{cases}$$

Recall in particular that $\psi(x, y) = \bar{\psi}(x - y)$ with $\bar{\psi}$ an even, k-Lipschitz continuous function such that $0 < \bar{\psi}(x) \leq \psi_2$ for all $x \in \mathbb{R}^d$.

Theorem 5.3. *Assume (H1) and (H3). For a fixed initial condition* (x_0, v_0) *with*

- *a finite second order moment*
- $\mathbb{E}[e^{a|v_0|}] < \infty$ *for* $a = kT\,(1+T)\,e^{\psi_2 T}$,

the non-linear stochastic system $(\mathcal{S}_W)$ *admits at most one strong solution.*

Corollary 4. *Assume (H1) and (H3). For any fixed initial condition* (x_0, v_0) *with a finite second order moment and such that,* $\mathbb{E}[e^{a|v_0|}] < \infty$, *the martingale problem (27) associated with* $(\mathcal{S}_W)$ *admits at most one solution.*

Let $(W_t)_{t\in[0,T]}$ and $(\widetilde{W}_t)_{t\in[0,T]}$ be two independent standard $\mathbb{R}^d$-valued Brownian motions. We will sometimes construct W (resp. $(\widetilde{W})$) on the first (resp. second) component of the product space $\Omega_T \times \Omega_T$, and denote, in this subsection alone, by $\mathbb{E}$ (resp. $\widetilde{\mathbb{E}}$) the expectation with respect to the first coordinate (resp. the second coordinate) of this product space.

5.1.1. *Reformulation of the problem.* System $(\mathcal{S}_W)$ can also be seen as:

$$\left\{\begin{array}{lcl} \mathbf{x}_t &=& \mathbf{x}_0 + \int_0^t \mathbf{v}_s\, ds \\ \mathbf{v}_t &=& \mathbf{v}_0 + W_t - \int_0^t \widetilde{\mathbb{E}}[\psi(\mathbf{x}_s, \widetilde{\mathbf{x}}_s)(\mathbf{v}_s - \widetilde{\mathbf{v}}_s)]\, ds, \end{array}\right. \tag{28}$$

where $(\widetilde{\mathbf{x}}_t, \widetilde{\mathbf{v}}_t)_{t\in[0,T]}$ is an independent copy of $(\mathbf{x}_t, \mathbf{v}_t)_{t\in[0,T]}$ and satisfies system $(\mathcal{S}_{\widetilde{W}})$, that is

$$\left\{\begin{array}{lcl} \widetilde{\mathbf{x}}_t &=& \widetilde{\mathbf{x}}_0 + \int_0^t \widetilde{\mathbf{v}}_s\, ds \\ \widetilde{\mathbf{v}}_t &=& \widetilde{\mathbf{v}}_0 + \widetilde{W}_t - \int_0^t \int \psi(\widetilde{\mathbf{x}}_s, x)(\widetilde{\mathbf{v}}_s - v)\, \mathbf{Q}_s(dx, dv)\, ds \\ \mathbf{Q}_t &=& \mathcal{L}(\widetilde{\mathbf{v}}_t, \widetilde{\mathbf{v}}_t). \end{array}\right.$$

Suppose now that there exist two strong solutions of $\mathcal{S}_W$ on Ω_T, $(\mathbf{x}, \mathbf{v})$ and $(\mathbf{x}', \mathbf{v}')$, with the same initial condition $(\mathbf{x}_0, \mathbf{v}_0)$ and respective laws $\mathbf{Q}$ and $\mathbf{Q}'$. Considering the processes, in $\Omega_T \times \Omega_T$, $((\mathbf{x}, \mathbf{v}), (\widetilde{\mathbf{x}}, \widetilde{\mathbf{v}}))$ and $((\mathbf{x}', \mathbf{v}'), (\widetilde{\mathbf{x}}', \widetilde{\mathbf{v}}'))$ – defined as in equation (28), we will show that they are almost surely equal, hence the strong uniqueness.

We can write:

$$\mathbf{v}_t = \mathbf{v}_0 + W_t - \int_0^t \widetilde{\mathbb{E}}[\psi(\mathbf{x}_s, \widetilde{\mathbf{x}}_s)(\mathbf{v}_s - \widetilde{\mathbf{v}}_s)]\, ds, \tag{29}$$

$$\mathbf{v}'_t = \mathbf{v}_0 + W_t - \int_0^t \widetilde{\mathbb{E}}[\psi(\mathbf{x}'_s, \widetilde{\mathbf{x}}'_s)(\mathbf{v}'_s - \widetilde{\mathbf{v}}'_s)]\, ds.$$

5.1.2. *Control of the trajectories.* First we track an upper bound for $\sup_{t\in[0,T]} |\mathbf{v}_t|$ and $\sup_{t\in[0,T]} |\mathbf{v}'_t|$.

$$\begin{aligned} \mathbb{E}[|\mathbf{v}_t|] = |\mathbf{v}_0| + \mathbb{E}[|W_t|] + \int_0^t \mathbb{E}\left[\widetilde{\mathbb{E}}[\psi(\mathbf{x}_s, \widetilde{\mathbf{x}}_s)(\mathbf{v}_s - \widetilde{\mathbf{v}}_s)]\right]\, ds \\ \le |\mathbf{v}_0| + \sqrt{\frac{2T}{\pi}} + 2\psi_2 \int_0^t \mathbb{E}[|\mathbf{v}_s|]\, ds. \end{aligned}$$

We can then apply Gronwall's lemma: for every $t < T$,

$$\mathbb{E}[|\mathbf{v}_t|] \le \left(|\mathbf{v}_0| + \sqrt{\frac{2T}{\pi}}\right) e^{2\psi_2 T}.$$

From there, keeping in mind that $\mathbb{E}[|\mathbf{v}_t|] = \widetilde{\mathbb{E}}[|\widetilde{\mathbf{v}}_t|]$

$$\begin{array}{rcl} |\mathbf{v}_t| &\le& |\mathbf{v}_0| + \psi_2 \int_0^t |\mathbf{v}_s|\, ds + \psi_2 \int_0^t \widetilde{\mathbb{E}}[|\widetilde{\mathbf{v}}_s|]\, ds + \sup_{t\in[0,T]} |W_t| \\ &\le& |\mathbf{v}_0| + \psi_2 T \left(|\mathbf{v}_0| + \sqrt{\frac{2T}{\pi}}\right) e^{2\psi_2 T} + \sup_{t\in[0,T]} |W(t)| + \psi_2 \int_0^t |\mathbf{v}_s| ds. \end{array}$$

Thus, thanks again to Gronwall's lemma,

$$\sup_{t\in[0,T]} |\mathbf{v}_t| \le C_W, \tag{30}$$

where the random variable C_W satisfies

$$C_W = \left(|\mathbf{v}_0| + \psi_2 T \left(|\mathbf{v}_0| + \sqrt{\frac{2T}{\pi}} \right) e^{2\psi_2 T} + \sup_{t\in[0,T]} |W(t)| \right) e^{\psi_2 T}.$$

Besides, as $\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{v}_s \, ds$,

$$\sup_{t\in[0,T]} |\mathbf{x}_t| \leq |\mathbf{x}_0| + T\, C_W.$$

In a similar way, with

$$C_{\widetilde{W}} = \left(|\mathbf{v}_0| + \psi_2 T \left(|\mathbf{v}_0| + \sqrt{\frac{2T}{\pi}} \right) e^{2\psi_2 T} + \sup_{t\in[0,T]} |\widetilde{W}(t)| \right) e^{\psi_2 T},$$

one has

$$\sup_{t\in[0,T]} |\widetilde{\mathbf{v}}_t| \leq C_{\widetilde{W}}. \tag{31}$$

Note that we also have $\sup_{t\in[0,T]} |\mathbf{v}'_t| \leq C_W$ and $\sup_{t\in[0,T]} |\widetilde{\mathbf{v}}'_t| \leq C_{\widetilde{W}}$.

5.1.3. *Lipschitz continuity.* Let x, x', v and v' be in $\mathbb{R}^d$.

If we suppose that $|v| \leq M$, then,

$$\begin{aligned} |\bar\psi(x)\, v - \bar\psi(x')\, v'| &\leq |\bar\psi(x) - \bar\psi(x')|\, |v| + \bar\psi(x')\, |v - v'| \leq kM\, |x - x'| + \psi_2 |v - v'| \\ &\leq (kM + \psi_2)\, (|x - x'| + |v - v'|). \end{aligned}$$

In particular, using (30) and (31), as $\bar\psi$ is bounded by ψ_2 and k-Lipschitz continuous,

$$\begin{aligned} &|\bar\psi(\mathbf{x}_s - \widetilde{\mathbf{x}}_s)\, (\mathbf{v}_s - \widetilde{\mathbf{v}}_s) - \bar\psi(\mathbf{x}'_s - \widetilde{\mathbf{x}}'_s)\, (\mathbf{v}'_s - \widetilde{\mathbf{v}}'_s)| \\ &\quad \leq (k\, (C_W + C_{\widetilde{W}}) + \psi_2)\, (|\mathbf{x}_s - \widetilde{\mathbf{x}}_s - \mathbf{x}'_s + \widetilde{\mathbf{x}}'_s| + |\mathbf{v}_s - \widetilde{\mathbf{v}}_s - \mathbf{v}'_s + \widetilde{\mathbf{v}}'_s|) \\ &\leq (K_W + K_{\widetilde{W}})\, (|\mathbf{x}_s - \mathbf{x}'_s| + |\mathbf{v}_s - \mathbf{v}'_s|) + (K_W + K_{\widetilde{W}})\, (|\widetilde{\mathbf{x}}_s - \widetilde{\mathbf{x}}'_s| + |\widetilde{\mathbf{v}}_s - \widetilde{\mathbf{v}}'_s|), \end{aligned} \tag{32}$$

setting the random variables $K_W = k\, C_W + \frac{\psi_2}{2}$ and $K_{\widetilde{W}} = k\, C_{\widetilde{W}} + \frac{\psi_2}{2}$.

5.1.4. *Computations towards the uniqueness.* Using (29) and (32),

$$\begin{aligned} |\mathbf{v}_t - \mathbf{v}'_t| &\leq - \int_0^t \widetilde{\mathbb{E}}[|\psi(\mathbf{x}_s, \widetilde{\mathbf{x}}_s)(\mathbf{v}_s - \widetilde{\mathbf{v}}_s) - \psi(\mathbf{x}'_s, \widetilde{\mathbf{x}}'_s)(\mathbf{v}'_s - \widetilde{\mathbf{v}}'_s)]\, ds \\ &\leq \int_0^t \widetilde{\mathbb{E}}[(K_W + K_{\widetilde{W}})\, (|\mathbf{x}_s - \mathbf{x}'_s| + |\mathbf{v}_s - \mathbf{v}'_s|)]\, ds \\ &\quad + \int_0^t \widetilde{\mathbb{E}}[(K_W + K_{\widetilde{W}})\, (|\widetilde{\mathbf{x}}_s - \widetilde{\mathbf{x}}'_s| + |\widetilde{\mathbf{v}}_s - \widetilde{\mathbf{v}}'_s|)]\, ds \\ &= \widetilde{\mathbb{E}}[K_W + K_{\widetilde{W}}]\, \int_0^t (|\mathbf{x}_s - \mathbf{x}'_s| + |\mathbf{v}_s - \mathbf{v}'_s|)\, ds \\ &\qquad + \int_0^t \widetilde{\mathbb{E}}[(K_W + K_{\widetilde{W}})\, (|\widetilde{\mathbf{x}}_s - \widetilde{\mathbf{x}}'_s| + |\widetilde{\mathbf{v}}_s - \widetilde{\mathbf{v}}'_s|)]\, ds. \end{aligned}$$

Thus, setting

$$\begin{aligned} S_W(s) &= \sup_{u\in[0,s]} |\mathbf{x}_u - \mathbf{x}'_u| + \sup_{u\in[0,s]} |\mathbf{v}_u - \mathbf{v}'_u|, \\ S_{\widetilde{W}}(s) &= \sup_{u\in[0,s]} |\widetilde{\mathbf{x}}_u - \widetilde{\mathbf{x}}'_u| + \sup_{u\in[0,s]} |\widetilde{\mathbf{v}}_u - \widetilde{\mathbf{v}}'_u|, \end{aligned}$$

we can affirm that

$$\sup_{u\in[0,t]} |\mathbf{v}_u - \mathbf{v}'_u| \le \widetilde{\mathbb{E}}[K_W + K_{\widetilde{W}}] \int_0^t S_W(s)\, ds \; + \int_0^t \widetilde{\mathbb{E}}[(K_W + K_{\widetilde{W}})\, S_{\widetilde{W}}(s)]\, ds.$$

As $\sup_{u\in[0,t]} |\mathbf{x}_u - \mathbf{x}'_u| \le T \; \sup_{u\in[0,t]} |\mathbf{v}_u - \mathbf{v}'_u|$, we have

$$S_W(t) \le (1+T)\, \widetilde{\mathbb{E}}[K_W + K_{\widetilde{W}}] \int_0^t S_W(s)\, ds + (1+T) \int_0^t \widetilde{\mathbb{E}}[(K_W + K_{\widetilde{W}})\, S_{\widetilde{W}}(s)]\, ds.$$

Applying a generalized version of Gronwall's inequality,

$$S_W(t) \le c_W \int_0^t \widetilde{\mathbb{E}}[(K_W + K_{\widetilde{W}})\, S_{\widetilde{W}}(s)]\, ds,$$

with $c_W = (1+T)\, e^{\,T(1+T)\, \widetilde{\mathbb{E}}[K_W + K_{\widetilde{W}}]}$.

In order to bound $t \mapsto \mathbb{E}[(K_W + K_{\widetilde{W}})\, S_W(t)]$, we again apply Gronwall's lemma. One can notice that both the following equalities are true:

$$\mathbb{E}[(K_W + K_{\widetilde{W}})\, S_W(t)] = K_{\widetilde{W}}\, \mathbb{E}[S_W(t)] + \mathbb{E}[K_W\, S_W(t)],$$

$$\widetilde{\mathbb{E}}[(K_W + K_{\widetilde{W}})\, S_{\widetilde{W}}(t)] = K_W\, \mathbb{E}[S_W(t)] + \mathbb{E}[K_W\, S_W(t)].$$

From there,

$$\begin{aligned}
\mathbb{E}[(K_W + K_{\widetilde{W}})\, S_W(t)] &\le \mathbb{E}\left[(K_W + K_{\widetilde{W}})\, c_W \int_0^t \widetilde{\mathbb{E}}[(K_W + K_{\widetilde{W}})\, S_{\widetilde{W}}(s)]\, ds\right] \\
\le \mathbb{E}[c_W (K_W + K_{\widetilde{W}})\, K_W\,] & \int_0^t \mathbb{E}[S_W(s)]\, ds + \mathbb{E}[c_W (K_W + K_{\widetilde{W}})] \int_0^t \mathbb{E}[K_W\, S_W(s)]\, ds \\
&\le \mathbb{E}[c_W (K_W + K_{\widetilde{W}})^2\,] \int_0^t \mathbb{E}[S_W(s)]\, ds + \mathbb{E}\left[c_W (K_W + K_{\widetilde{W}}) \left(1 + \frac{K_W}{K_{\widetilde{W}}}\right)\right] \\
&\qquad \times \int_0^t \mathbb{E}[K_W\, S_W(s)]\, ds,
\end{aligned}$$

which finally leads to

$$\begin{aligned}
&\mathbb{E}[(K_W + K_{\widetilde{W}})\, S_W(t)] \\
&\qquad \le \mathbb{E}\left[c_W\, (K_W + K_{\widetilde{W}}) \left(1 + \frac{K_W}{K_{\widetilde{W}}}\right)\right] \int_0^t \mathbb{E}[(K_W + K_{\widetilde{W}})\, S_W(s)]\, ds.
\end{aligned}$$

Thus, by Gronwall's inequality,

$$\mathbb{E}[(K_W + K_{\widetilde{W}})\, S_W(t)] = 0 \quad a.s.$$

for all $t \in [0,T]$, which implies, as all terms are non-negative, that

$$(K_W + K_{\widetilde{W}})\, S_W(T) = 0 \quad a.s.$$

By definition of S_W, it means that, for all $t \in [0,T]$, $(\mathbf{x}_t, \mathbf{v}_t) = (\mathbf{x}'_t, \mathbf{v}'_t)$ a.s., which completes the proof.

5.2. **Tightness of the** $(\pi_N)_N$**.** The following lemma can be found in [18], and we will admit it:

Lemma 5.4. *The tightness of* $(\pi_N)_{N\geq 1}$ *in* $\mathcal{P}(\mathcal{P}(\Omega_T))$ *is equivalent to the tightness of the law of* $(\hat{x}_1^N, \hat{v}_1^N)_{N\geq 1}$ *in* $\mathcal{P}(\Omega_T)$*.*

In order to prove the tightness of the law of $(\hat{x}_1^N, \hat{v}_1^N)$, we use Aldous criterion.

We start by proving the tightness of the law of $(\hat{x}_1^N(t), \hat{v}_1^N(t))$ for a.e. t. Take $\epsilon > 0$.

$$\mathbb{P}(|(\hat{x}_1^N(t), \hat{v}_1^N(t))| > \alpha) \leq \tfrac{1}{\alpha}\, \mathbb{E}[|(\hat{x}_1^N(t), \hat{v}_1^N(t))|] \leq \tfrac{M_t}{\alpha},$$

according respectively to Markov's inequality and Corollary 2. Thus, for $\alpha = \frac{M_t}{\epsilon}$, $\mathbb{P}((\hat{x}_1^N(t), \hat{v}_1^N(t)) \in \bar{B}(0,\alpha)) > 1-\epsilon$.

We fix $\varepsilon, \eta > 0$. According to Aldous criterion (see [5])we need to show that there exist $\delta > 0$ and an integer N_0 such that

$$\sup_{N\geq N_0} \sup_{\substack{\tau_1,\tau_2\in\mathbf{T} \\ \tau_1\leq\tau_2\leq(\tau_1+\delta)\wedge T}} \mathbb{P}(|(\hat{x}_1^N(\tau_2) - \hat{x}_1^N(\tau_1), \hat{v}_1^N(\tau_2) - \hat{v}_1^N(\tau_1))| > \varepsilon) \leq \eta,$$

where $\mathbf{T}$ is the set of stopping times on Ω_T.

Again thanks to Markov's inequality and Proposition 15 we have:

$$\begin{aligned} &\mathbb{P}(|(\hat{x}_1^N(\tau_2) - \hat{x}_1^N(\tau_1), \hat{v}_1^N(\tau_2) - \hat{v}_1^N(\tau_1))| > \varepsilon) \\ &\qquad\qquad \leq \frac{1}{\varepsilon^2}\, \mathbb{E}[|(\hat{x}_1^N(\tau_2) - \hat{x}_1^N(\tau_1), \hat{v}_1^N(\tau_2) - \hat{v}_1^N(\tau_1))|^2] \leq \frac{K\delta + C\delta^2}{\varepsilon^2}. \end{aligned}$$

Thus, δ such that $K\delta + C\delta^2 = \eta\, \varepsilon^2$, which is $\delta = \frac{-K+\sqrt{K^2+4C\eta\,\varepsilon^2}}{2C}$, provides the solution, and allows us to conclude to the tightness of $(\hat{x}_1^N, \hat{v}_1^N)$, as K and C are independent from N (but depend on T).

5.3. **The accumulation points of** $(\pi_N)_N$**.** We now know that the sequence $(\pi_N)_{N\geq 1}$ is tight; hence its relative compactness, thanks to Prokhorov's theorem.

Let π_∞ be one of its accumulation points; we still denote by $(\pi_N)_{N\geq 1}$ the subsequence that converges towards it. We show that under π_∞, for almost every Q in $\mathcal{P}(\Omega_T)$,

$$\mathbb{E}_Q[M_t^\phi(Q) - M_s^\phi(Q)|\mathcal{F}_s] = 0,$$

with M_t^ϕ defined in (27), this shall mean that Q is a solution of the martingale problem $(\mathcal{P}_\infty)$.

For $q \in \mathbb{N}^*$, $0 \leq s_1 < ... < s_q \leq s \leq t \leq T$ and $g_1, ..., g_q \in \mathcal{C}_b(\mathbb{R}^{2d})$, we define

$$F_{s,t}(Q) = \int_{\Omega_T} (M_t^\phi(Q) - M_s^\phi(Q))\, g_1(x_{s_1}, v_{s_1}) \ldots g_q(x_{s_q}, v_{s_q})\, dQ(x,v).$$

Lemma 5.5. *For every* $q \in \mathbb{N}$*,* $0 \leq s_1 < ... < s_q \leq s \leq t \leq T$ *and* $g_1, ..., g_q \in \mathcal{C}_b(\mathbb{R}^{2d})$*,*

$$\int_{\mathcal{P}(\Omega_T)} |F_{s,t}(Q)| \pi_\infty(dQ) = 0.$$

Proof. For the sake of simplicity, we forego here exponent N for the $\hat{x}_i$ and the $\hat{v}_i$.

Recall that π_N is the law on $\mathcal{P}(\Omega_T)$ of $\eta_N = \frac{1}{N}\sum_{i=1}^N \delta_{(\hat{x}_i,\hat{v}_i)}$, the empirical measure on Ω_T associated with the N-particle system defined by (5). It immediately follows that

$$\int F_{s,t}^2(Q)\, \pi_N(dQ) = \mathbb{E}[F_{s,t}(\eta_N)^2].$$

As $F_{s,t}(\eta_N) = \frac{1}{N}\sum_{i=1}^{N}(M_t^{N,i}(\phi) - M_s^{N,i}(\phi))g_1(\hat{x}_i(s_1), \hat{v}_i(s_1))...g_q(\hat{x}_i(s_q), \hat{v}_i(s_q))$,

$$\begin{aligned}
&\int F_{s,t}^2(Q)\pi_N(dQ) \\
&\quad = \frac{1}{N}\,\mathbb{E}\left[(M_t^{N,1}(\phi) - M_s^{N,1}(\phi))^2(g_1(\hat{x}_1(s_1), \hat{v}_1(s_1))...g_q(\hat{x}_1(s_q), \hat{v}_1(s_q)))^2\right] \\
&\qquad + \frac{N(N-1)}{N^2}\,\mathbb{E}\Big[(M_t^{N,1}(\phi) - M_s^{N,1}(\phi))(M_t^{N,2}(\phi) - M_s^{N,2}(\phi)) \\
&\quad \times g_1(\hat{x}_1(s_1), \hat{v}_1(s_1))...g_q(\hat{x}_1(s_q), \hat{v}_1(s_q))g_1(\hat{x}_2(s_1), \hat{v}_2(s_1))...g_q(\hat{x}_2(s_q), \hat{v}_2(s_q))\Big].
\end{aligned}$$

The first part goes to zero when N tends towards infinity because $g_1, ..., g_q$ are bounded, and for $t \in [0, T]$, the expectation of $M_t^{N,1}(\phi)^2$ is uniformly bounded in N, according to the estimates on the second order moment proven in Proposition 14.

As for the second term,

$$\begin{aligned}
&< M^{N,1}(\phi), M^{N,2}(\phi) > \\
&\qquad = \frac{1}{2}(< M^{N,1}(\phi) + M^{N,2}(\phi) > - < M^{N,1}(\phi) > - < M^{N,2}(\phi) >) = 0.
\end{aligned}$$

Thus, we have $\lim_{N\to\infty}\int F_{s,t}^2(Q)\pi_N(dQ) = 0$ which implies

$$\lim_{N\to\infty}\int |F_{s,t}(Q)|\pi_N(dQ) = 0.$$

$(\pi_N)_{N\geq 1}$ is a sequence of probability measures converging towards π_∞, thus the uniform integrability of $(F_{s,t}(\eta_N))$ (by virtue of being bounded in L^2) allows us to affirm that

$$\int |F_{s,t}(Q)|\pi_\infty(dQ) = 0$$

by inverting limit and integral. □

Then, for every $q \in \mathbb{N}$, $0 \leq s_1 < ... < s_q \leq s \leq t$ and $g_1, ..., g_q \in \mathcal{C}_b(\mathbb{R}^{2d})$, for π_∞-a.e. Q in $\mathcal{P}(\Omega_T)$, $F_{s,t}(Q) = 0$. Using the pathwise continuity, we conclude that for π_∞-a.e. Q, $(M_t^\phi(Q))_{t\geq 0}$ is a Q-martingale.

This means that if π_∞ is some limiting point of $(\pi_N)_{N\geq 1}$, then every Q in $\mathcal{P}(\Omega_T)$ which is in the support of π_∞ is solution of (27).

Thanks to corollary 4, we know that there exists a unique probability measure η on Ω_T such that $\pi_\infty = \delta_\eta$; furthermore, π_∞ is entirely determined, and subsequently, unique.

As δ_η is a Dirac measure, this convergence in law implies the convergence in probability. And so, Theorem 5.2 holds.

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E-mail address: laure.pedeches@math.univ-toulouse.fr