Introduction to stochastic processes, Winter term 2018/2019

Exercise 1

Modelling a probability space

Construct a sample space $\Omega \neq \emptyset$ reflecting the following experiments. Decide which information is relevant and which you may drop.

- "Throwing a numbered Octahedron (dice with eight faces) n times."
- "Flipping a coin 3 times, where we merely register if all tosses yield the same result or not."
- "All paths from A to E through stations B, C, D, passing each station exactly once. We assume that each station is directly connected with every other station."
- "Playing a slot machine (one-armed bandits) with 22 symbols and 3 rows."
- "We randomly choose an entry from a database, where each entry consists of the three fields gender, weekday of birth and zodiac sign."

Exercise 2

A machine is made from 4 different parts. We want to investigate the state of the machine using the events $A_i =$ "part i is broken" $(i \in \{1,2,3,4\})$. Express the following five events in terms of A_1, A_2, A_3, A_4 using set operations.

1) $A \widehat{=} \{$At least one part is broken.$\}$
2) $B \widehat{=} \{$All parts are broken$\}$
3) $C \widehat{=} \{$All parts are not broken.$\}$
4) $D \widehat{=} \{$At least one part is not broken.$\}$
5) $E \widehat{=} \{$At most part 1 is broken.$\}$

Exercise 3

Let $\Omega = \mathbb{R}^2$ and consider two subsets of Ω denoted by A and B being defined as follows:

$$A = \{(x,y) \in \Omega : \; x + y \leq 1\}, \qquad B = \{(x,y) \in \Omega : \; y \leq 2x + 2\}.$$

Sketch the following sets: $A \cup B,\ A \cap B,\ A \setminus B,\ A^c,\ B^c,\ A^c \cap B^c$.

Exercise 4

Define for $A \subset \Omega$ the map $\mathbb{1}_A : \Omega \to \{0,1\}$ via

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A; \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Prove the following equations.

1) $\mathbb{1}_{A\cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$.
2) $\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \cdot \mathbb{1}_B$.
3) $\mathbb{1}_{A\Delta B} = |\mathbb{1}_A - \mathbb{1}_B|$. (Hint: $A\Delta B := (A \setminus B) \cup (B \setminus A)$.)

Exercise 5

Let a random variable X be the number of points after one roll of a fair dice. State the range of values $\mathcal{X}$ for X. State the range of values for a random variable

$$Y = \left[\frac{X^2}{6}\right],$$

where $[y],\ y \in \mathbb{R}$ is the integer part (or floor) of y. Describe the events $A = \{Y = 0\}$, $B = \{Y < 5\}$, $C = \{Y \geq 5\}$ through the events in $\mathcal{X}$.
Hint: The integer part of y is the maximal integer number n, such that $n \leq y$.

Exercise 6

1. Let $\Omega = \{0, 1, \ldots, n\}, n \geq 1$ and
$$\mathbb{P}(\{k\}) = \begin{cases} cp(1-p), & k = 0 \\ cp^k, & k = 1, \ldots, n \end{cases}.$$
where $c \in \mathbb{R}^+, p \in (0,1)$. Express c in terms of p, n such that $\mathbb{P}$ is a probability measure.
2. In some society every individual owns either $0, 1, 2, 3, 4$ or 5 flats. We denote by $\mathbb{Q}(\{k\})$ the probability that any individual owns k flats and assign:
$$\mathbb{Q}(\{k\}) = \begin{cases} \frac{c'}{4}, & k = 0 \\ \frac{c'}{2^k}, & k = 1, \ldots, 5 \end{cases}$$
Derive the probability that someone in this society owns at least two flats and find the appropriate c'.
3. Now there are infinitely many flats available such that $\tilde{\Omega} = \mathbb{N}$ is a sensible choice for the sample space. Consequently,
$$\tilde{\mathbb{Q}}(\{k\}) = \begin{cases} \frac{\tilde{c}}{4}, & k = 0 \\ \frac{\tilde{c}}{2^k}, & k \geq 1 \end{cases}$$
Derive the appropriate $\tilde{c}$ making $\tilde{\mathbb{Q}}$ a probability measure. How big is the probability that someone owns at least two flats?

Exercise 7

Let X be the number of points after one roll of a fair Icosahedron. State the range of values for the random variable
$$Y = \mathbb{1}_{\{\text{X is divisible by 3}\}},$$
and compute the probability that $Y = 1$.

Exercise 8

Suppose that each morning you roll a fair dice to decide if you go to the university or not. If the number of points on the dice face is greater then 5, you get up, if not then you stay in bed. Let X be a random variable that counts the number of days that you skipped the university out of a five days a week. State the distribution of X and compute $\mathbb{P}(X \leq 3)$, $\mathbb{P}(X \geq 4)$, $\mathbb{P}(X = 5)$, as well as $\mathbb{P}(2 \leq X < 4)$.

Exercise 9

Let (X, Y) be a random vector with values in $\{1, \ldots 6\}^2$ and probability mass function
$$\mathbb{P}_{(X,Y)}\{(i,j)\} = \begin{cases} c & \text{if } 2 < i + j < 6 \\ 0 & \text{otherwise.} \end{cases}$$

1. Draw a chart of the joint distribution.
2. Compute the value of c.
3. Compute the marginal distributions, i.e. the distributions of X and Y.

Good luck!