Introduction to Stochastic Processes WS2018/19

Angelo Valleriani, Tetiana Kosenkova

1. (13 points) Consider a discrete time Markov chain X on the state space $\sigma = \{0, 1, 2\}$. States $\{0, 1\}$ are transient and state $\{2\}$ is an absorbing state. Assume that $P_{0,i} \neq 0$ for all $i \in \sigma$ and that $P_{1,i} \neq 0$ for all $i \in \sigma$.

   (a) (3 points) Compute explicitly the mean time to absorption for $X(0) = 0$ and for $X(0) = 1$.

   (b) (2 points) For $X(0) = 0$, write the probability, in terms of the P_{ij}, that the process visits state $\{1\}$ at least once before absorption.

   (c) (2 points) For $X(0) = 1$ write the probability that the process returns in state $\{1\}$ before absorption.

   (d) (6 points) For $X(0) = 0$ write the probability distribution of the number of visits in state $\{1\}$ before absorption.

2. Solution:

   (a) The mean time to absorption can be computed using the first step analysis. The set of equations is given by first defining $\nu_i = E[T \mid X(0) = i]$:

$$\begin{aligned} \nu_0 &= 1 + P_{00}\nu_0 + P_{01}\nu_1 \\ \nu_1 &= 1 + P_{10}\nu_0 + P_{11}\nu_1 \\ \nu_2 &= 0 \end{aligned} \tag{1}$$

   with solution

$$\nu_0 = \frac{1 - P_{11} + P_{01}}{(1 - P_{00})(1 - P_{11}) - P_{01}P_{10}} \tag{2}$$

   and

$$\nu_1 = \frac{1 - P_{00} + P_{10}}{(1 - P_{00})(1 - P_{11}) - P_{01}P_{10}} \tag{3}$$

   (b) This can be computed promoting state $\{1\}$ into an absorbing state (i.e. $P_{11} = 1$ and $P_{10} = P_{12} = 0$). The absorption probability u_{01} in state $\{1\}$ starting from state $\{0\}$ is computed then from:

$$u_{01} = P_{00}u_{01} + P_{01} \tag{4}$$

leading to the probability

$$p_0 = P_{01}/(1 - P_{00})$$

to visit state $\{1\}$ before absorption.

(c) This can be computed by first creating an additional state $1'$ such that $P_{1',i} = P_{1,i}$ for $i = 0, 2$ $P_{1',1'} = 0$ and $P_{1',1} = P_{11}$ and then promoting state $\{1\}$ into an absorbing state (i.e. $P_{11} = 1$ and $P_{10} = P_{12} = 0$). The question now translates into asking the probability p_1 of absorption in $\{1\}$ starting from $\{1'\}$. This can be solved by setting

$$p_1 = P_{10}p_0 + P_{11} \tag{5}$$

where p_0 is the one computed in the previous step. Therefore we have

$$p_1 = \frac{P_{01}P_{10}}{1 - P_{00}} + P_{11}\,. \tag{6}$$

(d) Let Z be the random variable that gives the number of visits to state $\{1\}$ for a process starting in state $\{0\}$. We have:

$$\Pr\{Z = 0\} = 1 - p_0\,, \tag{7}$$

then

$$\Pr\{Z = 1\} = p_0(1 - p_1)\,, \tag{8}$$

the

$$\Pr\{Z = 2\} = p_0 p_1(1 - p_1)\,, \tag{9}$$

and generalizing we have

$$\Pr\{Z = k\} = p_0 p_1^{k-1}(1 - p_1)\,. \tag{10}$$

We see also that

$$\sum_{k=0}^{\infty} \Pr\{Z = k\} \;=\; 1 - p_0 + p_0(1 - p_1)/(1 - p_1) \;=\; 1\,. \tag{11}$$

3. (13 points) Consider a random walk on a 1-dimensional line (discrete space and discrete time). The random walk moves stochastically according to the following (unusual) rules. First: The probabilities to make a step to the right or to the left are dependent on whether the previous step was to the right or to the left. Second: After each step (left or right) the walker could fall off the line and disappear with probabilities that are dependent on whether the last step was to the right or to the left.

   (a) (6 points) Describe this process in terms of a Markov chain in discrete time (i.e. define the state space and the one-step transition probability matrix).
   (b) (3 points) Compute explicitly the mean number of steps before falling off when the first step was to the right with probability q and to the left with probability $1 - q$.
   (c) (2 points) Assume that the process starts with a step to the right (i.e. $q = 1$), what is the probability that the walker falls off the line before making any step to the left?
   (d) (2 points) Assume that the process starts with a step to the right (i.e. $q = 1$), what is the probability that the walker falls off the line before making any further step to the right?

4. Solution:

   (a) If we call state $\{0\}$ is the event *step to the right*, state $\{1\}$ is the event *step to the left* and state $\{2\}$ is the event *fall off the line*, then the process is described by $\sigma = \{0, 1, 2\}$ with states $\{0, 1\}$ transient and state $\{2\}$ absorbing. The transition probabilities are: P_{00} for right-after-right, P_{01} for left-after-right, P_{02} for falloff-after-right. Similarly for the probabilities P_{1i}.
   (b) The mean time to absorption can be computed using the first step analysis. The set of equations is given by first defining $\nu_i = E[T \mid X(0) = i]$:

$$\begin{aligned} \nu_0 &= 1 + P_{00}\nu_0 + P_{01}\nu_1 \\ \nu_1 &= 1 + P_{10}\nu_0 + P_{11}\nu_1 \\ \nu_2 &= 0 \end{aligned} \tag{12}$$

   with solution

$$\nu_0 = \frac{1 - P_{11} + P_{01}}{(1 - P_{00})(1 - P_{11}) - P_{01}P_{10}} \tag{13}$$

   and

$$\nu_1 = \frac{1 - P_{00} + P_{10}}{(1 - P_{00})(1 - P_{11}) - P_{01}P_{10}} \tag{14}$$

   The question is answered by considering $\nu = q\nu_0 + (1 - q)\nu_1$.
   (c) This can be computed promoting state $\{1\}$ into an absorbing state (i.e. $P_{11} = 1$ and $P_{10} = P_{12} = 0$). The absorption probability u_{01} in state $\{1\}$ starting from state $\{0\}$ is computed then from:

$$u_{01} = P_{00}u_{01} + P_{01} \tag{15}$$

leading to the probability

$$p_0 = P_{01}/(1 - P_{00})$$

to visit state $\{1\}$ before absorption. The probability to fall off the line before visiting state 1 (i.e. before a step to the left) is therefore $1 - p_0$.

(d) (see previous problem).

5. (8 points) Let us imagine, that in the nearest future there will be a possibility to send an expedition to Mars and they could spend there some years. Imagine that they need to take potatoes to grow there. Let us assume, that one potato gives harvest according to a geometric law $r_j = a(1-a)^j,\ j \in \mathbb{N}$.

   (a) (1 point) Find the pgf of the harvest from one potato.

   (b) (2 points) A usual growth duration for a potato plant is 4 month. Assume, that at the beginning, you've planted one potato and then you planted all potatoes from each harvest. What is the probability, that after a year you will have no harvest?

   (c) (2 points) Assume, that $a = 1/4$. Is the reproduction law sub-(super-)critical?

   (d) (3 points) What is the minimal number N of the potatoes, that you should plant, such that the probability to have no harvest after a year is less then 5%?

   Solution

   (a) $f(s) = \frac{a}{1-(1-a)s}$.

   (b) We need $f_3(0)$. For that

$$f_2(s) = f(f(s)) = \frac{a - a(1-a)s}{1-(1-a)(s+a)},$$

   and

$$f_3(s) = \frac{a(1-(1-a)s) - a^2(1-a)}{(1-(1-a)a)(1-(1-a)s) - (1-a)a}.$$

   Then

$$f_3(0) = \frac{a - a^2(1-a)}{1 - 2(1-a)a}.$$

   (c) For $a = 1/4$ $m = 1 - a/a = 3 > 1$ thus the law is supercritical.

   (d) If we start with N potatoes the probability to have no harvest will be $(f_3(0))^N = \left(\frac{13}{40}\right)^N$. The condition on N is as follows:

$$\left(\frac{13}{40}\right)^N < 0,05 \quad \Leftrightarrow \quad N > 2,6 \quad \Leftrightarrow \quad N > 3.$$

6. (12 points) Assume that we observe a cell, which in one life cycle can divide with probability $p \in (0,1)$ or die. Assume that at each life cycle one cell of this kind can arrive additionally with probability $q \in (0,1)$ independently of the reproduction of the first cell.

   (a) (2 points) Find the pgf of the cell population in one cycle. Find the mean number of cells in one cycle.

   (b) (3 points) Assume that the second cell arrives with probability $q \in (0,1)$ only provided that the first cell does not die. How does it affect the pgf of the population in one cycle? Find the mean number of cells in one cycle for this evolution mechanism.

   (c) (3 points) For this evolution mechanism find the condition on p and q for the reproduction to be sub-(super-)critical. Find the probability of extinction when $p = 2/3$ and $q = 1/4$.

   (d) (4 points) Let us assume that we observe a random number Γ of independent cells and for each cell works the mechanism from part (b) and the reproduction is supercritical. Let $\mathbb{P}(\Gamma = k) := a_k,\ k \in \mathbb{N} \setminus \{0\}$. What is the probability of extinction for this population?

Solution

(a) For this mechanism there can happen four different outcomes: the first cell dies, no external cell comes proba=(1-p)(1-q), the first cell dies, one external cell arrives proba=(1-p)q, the first cell divides, no external cell comes proba=p(1-q), the first cell divides, one external cell comes proba=pq. Thus the pgf is

$$f(s) = (1-p)(1-q) + (1-p)qs + p(1-q)s^2 + pqs^3.$$

The mean is $m = (1-p)q + 2p(1-q) + 3pq = q + 2p$.

(b) In this case the evolution allows only three possible outcomes: the first cell dies with proba=(1-p), the first cell divides, no external cell comes proba=p(1-q), the first cell dies, one external cell comes proba=pq. The pgf is

$$f(s) = 1 - p + p(1-q)s^2 + pqs^3.$$

The mean is $m = 2p(1-q) + 3pq = 2p + pq$.

(c) To find the probability of extinction we need to know what kind of reproduction do we have: for $m \leq 1$ ($p \leq \frac{1}{q+2}$) we will have that $d = 1$. For the supercritical case, where $p > \frac{1}{q+2}$ we need to solve the equation $f(s) = s$.

$$1 - p - s + p(1-q)s^2 + pqs^3 = 0 \quad \Leftrightarrow \quad 1 - s + p(s^2 - 1) + pqs^2(s-1) = 0.$$

We will assume that $s \neq 1$ and divide by $s-1$. Further we will find the roots of the quadratic equation $pqs^2+p(s+1)-1=0$. We put the values of $p=2/3$ and $q=1/4$ and have the equation $1/6s^2+2/3s-1/3=0$, which has only one root in $(0,1)$ and it is $d=\sqrt{6}-2 \approx 0,4$.

(d) Since the reproduction is supercritical there is a nontrivial probability d of extinction for the population which starts with one cell. If the starting number of cells is fixed, say, k, then the extinction probability is d^k. Since the starting number is random, namely Γ, we have $d_\Gamma = E(d^\Gamma) = f_\Gamma(d)$, where f_Γ is a pgf for Γ.