Introduction to Stochastic Processes WS2018/19

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Reminder

- A conditional probability of an event A on the sample space Ω under the condition that an event B happened ($\Pr\{B\} > 0$), is defined by $\Pr\{A|B\} = \frac{\Pr\{A \cap B\}}{\Pr\{A\}}$.
- If there are n disjoint events $H_i,\ i = 1, \ldots, n$ such that $H_1 \cup H_2 \cup \ldots \cup H_n = \Omega$, then the law of the total probability says that

$$\Pr\{A\} = \sum_{i=1}^{n} \Pr\{A|H_i\}\Pr\{H_i\} = \sum_{i=1}^{n} \Pr\{A \cap H_i\},$$

  where the second equality is true because of the definition of conditional probability.
- *Example*: Imagine your way to the University. Suppose you have several possibilities: you can take a train, a bus or a car. Suppose you have a possibility to get on the train on 6 stations only. Let the event A be you've taken the train. Let the random variable X says on which station you have got on the train, it takes values in the set $\{1, 2, \ldots, 6\}$. Then events $\{X = i\},\ i = 1, \ldots 6$ are disjoint and give the whole sample space and we can write down the law of total probability in the form

$$\Pr\{A\} = \sum_{i=1}^{6} \Pr\{A|X = i\}\Pr\{X = i\}.$$

  Suppose that a random variable Y indicates the choice of the train, the bus or the car, having $\{t, b, c\}$ as a state space, then $A = \{Y = t\}$. The law of total probability takes the form:

$$\Pr\{A\} = \sum_{i=1}^{6} \Pr\{Y = t|X = i\}\Pr\{X = i\} = \sum_{i=1}^{6} \Pr\{Y = t, X = i\}.$$

Exercises

1. Explain in which order are two numbers $\Pr\{A \cap B\}$ and $\Pr\{A|B\}$ for any events A and B with $\Pr\{B\} > 0$.

2. **Baking pies.** Ms. K. has three sisters Elsa, Rosa and Polin and they like to bake time to time. Rosa bakes 60% of the pies, Polin bakes 10% of the pies and Elsa bakes 30% of the pies. Also we know that if Elsa bakes a pie, then she makes an apple pie in 20% of all cases, Rosa does that in 40% of the cases and Polin does that in 70% of the cases. What is the probability that Ms. K. finds an apple pie when she comes back home?

3. **Markov chain** Suppose you have a markov chain $(X_n)_{n\geq 1}$ with a state space $\sigma = \{0, 1, 2, 3\}$ and a transition probability matrix

$$\mathbb{P} = \begin{pmatrix} 1-p & p & 0 & 0 \\ 1-p & 0 & p & 0 \\ 1-p & 0 & 0 & p \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

   Find the following probabilities depending on $0 < p < 1$:

   (a) $\Pr\{X_1 = 2 | X_0 = 1\}$,

   (b) $\Pr\{X_1 = 0, X_2 = 0, X_3 = 0 | X_0 = 1\}$,

   (c) $\Pr\{X_1 = 0, X_3 = 0, X_5 = 0 | X_0 = 1\}$.

   (d) Find the probabilities $\Pr\{X_7 = i | X_5 = 0\}$, $i = 0, 1, 2, 3$. What is the sum of them?