

Initial Data for the Cauchy Problem in General Relativity Lecture 3

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Daniel Pollack

University of Washington

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Lecture 3: Solving the Einstein constraint equations

In 1952, Yvonne Choquet-Bruhat showed the existence of a local in time solution of the vacuum Einstein equations, $Ric(g) = 0$. In 1969 Choquet-Bruhat & Geroch established the existence of a maximal, globally hyperbolic evolution.

- The constraint equations, together with the second Bianchi identity, ensures that the wave coordinate gauge is evolved in time as one solves the reduced Einstein equations, yielding a solution of the full geometric equations.

Theorem (Choquet-Bruhat & Geroch, 1969)

Given an initial data set $(V; h, K)$ satisfying the vacuum constraint equations there exists a unique, globally hyperbolic, maximal, spacetime (M, g) satisfying the vacuum Einstein equations $Ric(g) = 0$ where $V \hookrightarrow M$ is a Cauchy surface with induced metric h and second fundamental form K . Moreover any other such solution is a subset of (M, g) .

In these results, a central role is played by the **existence of initial data sets solving the Einstein constraint equations**. We now turn our attention to this question.

Solving the Einstein constraint equations

The initial data for the Einstein field equations for (M^{n+1}, g) consist of specifying on an n -dimensional manifold Σ

- a Riemannian metric $\bar{\gamma}$
- a symmetric 2-tensor \bar{K}
- \mathcal{F} a collection of initial data for the non-gravitational fields.

The choices of initial data are constrained by the Gauss and Codazzi equations, which gives rise to the “Einstein constraint equations.

In terms of the data $(\bar{\gamma}, \bar{K}, \mathcal{F})$ above, the Einstein constraint equations are

$$\begin{aligned} \operatorname{div}_{\bar{\gamma}} \bar{K} - \nabla(\operatorname{tr} \bar{K}) &= J(\bar{\gamma}, \mathcal{F}) && \text{(Momentum constraint)} \\ R(\bar{\gamma}) - |\bar{K}|_{\bar{\gamma}}^2 + (\operatorname{tr} \bar{K})^2 &= 2\rho(\bar{\gamma}, \mathcal{F}) && \text{(Hamiltonian constraint)} \\ \mathcal{C}(\bar{\gamma}, \mathcal{F}) &= 0 && \text{(Non-gravitational constraints)} \end{aligned}$$

Concrete example: Einstein-Maxwell in 3+1 dimensions

The non-gravitational fields consist of an electric vector field \vec{E} and a magnetic vector field \vec{B} .

- current density: $J(\bar{\gamma}, \mathcal{F}) = (\vec{E} \times \vec{B})_{\bar{\gamma}}$
- energy density: $\rho(\bar{\gamma}, \mathcal{F}) = \frac{1}{2}(|\vec{E}|_{\bar{\gamma}}^2 + |\vec{B}|_{\bar{\gamma}}^2)$

The non-gravitational constraints are

$$\operatorname{div}_{\bar{\gamma}} \vec{E} = 0$$

$$\operatorname{div}_{\bar{\gamma}} \vec{B} = 0$$

Einstein-scalar fields

The vacuum Einstein equations come from a variational principle; the Einstein-Hilbert action

$$\mathcal{S}(g, \Psi) = \int_M R(g) dv_g.$$

Their Einstein-scalar field action is:

$$\mathcal{S}(g, \Psi) = \int_M [R(g) - \frac{1}{2} |\nabla \Psi|_g^2 - V(\Psi)] dv_g,$$

Where Ψ is a scalar valued function on M . From this we obtain the **Einstein-scalar field equations**:

$$G_{\alpha\beta} = T_{\alpha\beta} = \nabla_\alpha \Psi \nabla_\beta \Psi - \frac{1}{2} g_{\alpha\beta} \nabla_\mu \Psi \nabla^\mu \Psi - g_{\alpha\beta} V(\Psi)$$
$$\nabla_\mu \nabla^\mu \Psi = V'(\Psi).$$

(where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R(g) g_{\alpha\beta}$ is the Einstein curvature tensor, and we have deliberately neglected the factor of 8π for simplicity).

Einstein-scalar field Constraint Equations

Initial data on Σ^n consists of

- $\bar{\gamma}$ (the spatial metric)
- \bar{K} (the second fundamental form, or extrinsic curvature)
- $\bar{\psi}$ (the scalar field restricted to Σ)
- $\bar{\pi}$ (the normalized time derivative of Ψ restricted to Σ).

The constraint equations are then

$$\begin{aligned} \operatorname{div}_{\bar{\gamma}} \bar{K} - \nabla(\operatorname{tr} \bar{K}) &= -\bar{\pi} \nabla \bar{\psi} \\ R(\bar{\gamma}) - |\bar{K}|_{\bar{\gamma}}^2 + (\operatorname{tr} \bar{K})^2 &= \bar{\pi}^2 + |\nabla \bar{\psi}|_{\bar{\gamma}}^2 + 2V(\bar{\psi}). \end{aligned}$$

(Note: the scalar field does not introduce any new constraint equations.)

The conformal method (après Lichnerowicz, Choquet-Bruhat and York)

The constraint equations are highly underdetermined (in the $(3 + 1)$ -dimensional vacuum case, they consist locally of 4 equations for the 12 unknowns represented by the symmetric tensors $\bar{\gamma}$ and \bar{K}).

Split the initial data into two parts

- “conformal data”: regard as being freely chosen.
- “determined data”: found by solving the constraint equations, reformulated as a **determined** system of elliptic PDE.

General Criteria: For constant mean curvature (CMC) initial data, where $\tau = \text{tr}_{\bar{\gamma}} \bar{K}$ is constant, we want the equations to be “semi-decoupled”:

- First solve the nongravitational constraints.
- Then solve the conformally formulated momentum constraint.
- These solutions enter into the conformally formulated Hamiltonian constraint, which we solve for the remaining piece of determined data.

The conformal and determined data (vacuum case)

For the gravitational (vacuum) data, the free “conformal data” consists of

- γ , a Riemannian metric on Σ , representing a chosen *conformal class of metrics* $[\gamma] = \{\tilde{\gamma} = \theta^{\frac{4}{n-2}}\gamma : \theta > 0\}$.
- $\sigma = \sigma_{ab}$, a symmetric tensor which is divergence-free and trace-free w.r.t. γ (σ is a transverse-traceless or TT-tensor).
- τ , a scalar function representing the mean curvature of the Cauchy surface Σ in the resulting spacetime.

The “determined data” consists of

- ϕ , a positive function
- $W = W^a$, a vector field

Reconstructed data (vacuum case)

Use (ϕ, W) to reconstruct an initial data set $(\bar{\gamma}, \bar{K})$ from the conformal data set (γ, σ, τ) via:

$$\begin{aligned}\bar{\gamma} &= \phi^{\frac{4}{n-2}} \gamma \\ \bar{K} &= \phi^{-2}(\sigma + \mathcal{D}W) + \frac{\tau}{n} \phi^{\frac{4}{n-2}} \gamma\end{aligned}$$

here the operator \mathcal{D} is the conformal Killing operator relative to γ , $(\mathcal{D}W)_{ab} := \nabla_a W_b + \nabla_b W_a - \frac{2}{n} \gamma_{ab} \nabla_m W^m$, whose kernel consists of conformal Killing fields.

$(\bar{\gamma}, \bar{K})$ satisfy the vacuum constraint equations if and only if (ϕ, W) satisfy

$$\left\{ \begin{array}{l} \operatorname{div}(\mathcal{D}W) = \frac{n}{n-1} \phi^{\frac{2n}{n-2}} \nabla \tau \\ c_n^{-1} \Delta_\gamma \phi - R(\gamma)\phi + (|\sigma + \mathcal{D}W|_\gamma^2) \phi^{-\frac{3n-2}{n-2}} - \frac{n-1}{n} \tau^2 \phi^{\frac{n+2}{n-2}} = 0 \end{array} \right.$$

where $c_n = \frac{n-2}{4(n-1)}$.

Results (vacuum, CMC case with $\Lambda = 0$)

For cosmological (spatially compact) vacuum spacetimes and CMC initial data this approach yields a complete understanding of the question of the existence of solutions to the constraints equations.

Let $\mathcal{Y}([\gamma])$ denote the Yamabe invariant of the conformal class of metrics determined by γ .

Theorem (Choquet-Bruhat, York, Ó Murchadha, Isenberg)

Given a conformal initial data set (γ, σ, τ) the existence of a solution to the Lichnerowicz equation (and therefore, taking $W = 0$, to the vacuum constraints) is indicated as follows

	$\sigma \equiv 0, \tau = 0$	$\sigma \equiv 0, \tau \neq 0$	$\sigma \neq 0, \tau = 0$	$\sigma \neq 0, \tau \neq 0$
$\mathcal{Y}([\gamma]) < 0$	No	Yes	No	Yes
$\mathcal{Y}([\gamma]) = 0$	Yes	No	No	Yes
$\mathcal{Y}([\gamma]) > 0$	No	No	Yes	Yes

Concrete example II: Einstein-Maxwell

When we couple a Maxwell electromagnetic field in to the Einstein field equations the net effect, on the level of the constraint equations, is not very disruptive. Now the conformal initial data consists of $(\gamma, \sigma, \tau, E, B)$ where

- (γ, σ, τ) are the gravitational initial data
- E and B are **divergence free** vector fields providing the initial data for the electric and magnetic fields.

The Einstein-Maxwell Lichnerowicz equation is then

$$\Delta\phi - \frac{1}{8}R(\gamma)\phi + \frac{1}{8}(|\sigma + \mathcal{D}W|_{\gamma}^2)\phi^{-7} + \frac{1}{8}(|E|_{\gamma}^2 + |B|_{\gamma}^2)\phi^{-3} - \frac{1}{12}\tau^2\phi^5 = 0.$$

Note that the Maxwell field contributes a term with a positive coefficient and a negative power of ϕ . This allows it to be treated in the analysis in exactly the same way at the $|\sigma + \mathcal{D}W|_{\gamma}^2$ term.

Conformally formulated momentum constraint equation for the Einstein-scalar field system

Under the conformal rescaling $\bar{\gamma} = \phi^{\frac{4}{n-2}}\gamma$, we rescale the scalar field initial data as follows:

$$\bar{\psi} = \psi \quad \text{and} \quad \bar{\pi} = \phi^{\frac{2n}{n-2}}\pi.$$

The momentum constraint then becomes

$$\operatorname{div}_{\gamma}(\mathcal{D}W) = \frac{n-1}{n}\phi^{\frac{2n}{n-2}}\nabla\tau - \pi\nabla\psi.$$

- When τ is constant, the conformally formulated momentum constraint equation does not involve the conformal factor ϕ .
- $\operatorname{div}_{\gamma} \circ \mathcal{D}$ is a self-adjoint, second order, elliptic operator. On a compact manifold, $\ker(\operatorname{div}_{\gamma} \circ \mathcal{D}) = \{\text{conformal Killing vector fields}\}$. (If there are no conformal Killing vector fields, this equation has a unique solution for any choice of (ϕ, τ, ψ, π) .)

Conformally formulated Hamiltonian constraint equation for the Einstein-scalar field system

Let

$$\mathcal{R}_{\gamma,\psi} = c_n (R(\gamma) - |\nabla\psi|_{\gamma}^2), \quad \mathcal{A}_{\gamma,W,\pi} = c_n (|\sigma + \mathcal{D}W|_{\gamma}^2 + \pi^2)$$

and

$$\mathcal{B}_{\tau,\psi} = c_n \left(\frac{n-1}{n} \tau^2 - 4V(\psi) \right).$$

The Hamiltonian constraint equation for the Einstein-scalar conformal data $(\gamma, \sigma, \tau, \psi, \pi)$ (with a given vector field W satisfying the conformally formulated momentum constraint equation) is

$$\Delta_{\gamma}\phi - \mathcal{R}_{\gamma,\psi}\phi + \mathcal{A}_{\gamma,W,\pi}\phi^{-\frac{3n-2}{n-2}} - \mathcal{B}_{\tau,\psi}\phi^{\frac{n+2}{n-2}} = 0.$$

This is the Einstein-scalar field Lichnerowicz equation.

Analysis of the Einstein-scalar field Lichnerowicz equation

This equation differs from other matter/field Lichnerowicz equations (e.g. for vacuum, Maxwell, Yang-Mills, fluids) in **two very significant** ways:

- coefficient of linear term is $\mathcal{R}_{\gamma,\psi} = c_n (R(\gamma) - |\nabla\psi|_\gamma^2)$ vs. $R(\gamma)$.
- $\mathcal{B}_{\tau,\psi} = c_n \left(\frac{n-1}{n} \tau^2 - 4V(\psi) \right)$ may not, in general, have a fixed sign.

However

- The Lichnerowicz equation is **conformally covariant**: set

$$\begin{aligned}\tilde{\gamma} &= \theta^{\frac{4}{n-2}} \gamma & \tilde{\sigma} &= \theta^{-2} \sigma & \tilde{\tau} &= \tau \\ \tilde{\psi} &= \psi & \tilde{\pi} &= \theta^{\frac{2n}{n-2}} \pi\end{aligned}$$

ϕ solution w.r.t. $(\gamma, \sigma, \tau, \psi, \pi) \Leftrightarrow \phi$ solution w.r.t. $(\tilde{\gamma}, \tilde{\sigma}, \tilde{\tau}, \tilde{\psi}, \tilde{\pi})$.

- $\mathcal{A} \equiv 0 \Rightarrow$ a solution to the Lichnerowicz equation corresponds to a solution of the prescribed scalar curvature-scalar field equation

$$\mathcal{R}_{\tilde{\gamma},\psi} = -\mathcal{B}_{\tau,\psi}.$$

The Yamabe-scalar field conformal invariant

Definition

The Yamabe-scalar field conformal invariant is defined by

$$\mathcal{Y}_\psi([\gamma]) = \inf_{u \in H^1(\Sigma)} \frac{c_n^{-1} \int_{\Sigma} [|\nabla u|_{\gamma}^2 + c_n (R(\gamma) - |\nabla \psi|_{\gamma}^2) u^2] d\eta_{\gamma}}{\left(\int_{\Sigma} u^{\frac{2n}{n-2}} d\eta_{\gamma} \right)^{\frac{n-2}{n}}}.$$

- Hölder's inequality $\Rightarrow \mathcal{Y}_\psi([\gamma]) > -\infty$.
- $\mathcal{Y}_\psi([\gamma])$ is independent of the choice of background metric in the conformal class used to define it. It therefore defines an invariant of the conformal class and scalar field.

The conformal information from $\mathcal{Y}_\psi([\gamma])$

Define the *conformal scalar-field Laplace operator* $L_{\gamma,\psi}$ by

$$L_{\gamma,\psi} u = \Delta_\gamma u - c_n (R(\gamma) - |\nabla\psi|_\gamma^2) u.$$

(scalar-field analog of the conformal Laplace operator).

Proposition

The following conditions are equivalent:

- (i) $\mathcal{Y}_\psi([\gamma]) > 0$ (respectively $= 0, < 0$).
- (ii) *There exists a metric $\tilde{\gamma} \in [\gamma]$ which satisfies $(R(\tilde{\gamma}) - |\tilde{\nabla}\psi|_{\tilde{\gamma}}^2) > 0$ everywhere on Σ (respectively $= 0, < 0$).*
- (iii) *For any metric $\tilde{\gamma} \in [\gamma]$, the first eigenvalue, λ_1 , of the self-adjoint, elliptic operator $-L_{\tilde{\gamma},\psi}$ is positive (respectively zero, negative).*

Solving the Lichnerowicz equation I

On a compact manifold, in joint work with Y. Choquet-Bruhat and J. Isenberg (2007) we were able to establish the following

	$\mathcal{B}_{\tau,\psi} < 0$	$\mathcal{B}_{\tau,\psi} \leq 0$	$\mathcal{B}_{\tau,\psi} \equiv 0$	$\mathcal{B}_{\tau,\psi} \geq 0$	$\mathcal{B}_{\tau,\psi} > 0$
$\mathcal{Y}_\psi([\gamma]) < 0$	N	N	N	N&S Cond.	Y
$\mathcal{Y}_\psi([\gamma]) = 0$	N	N	Y	N	N
$\mathcal{Y}_\psi([\gamma]) > 0$	PR	PR	N	N	N

Table 1: Results for $\mathcal{A}_{\gamma,W,\pi} \equiv 0$ and $\mathcal{B}_{\tau,\psi}$ of determined sign.

Y The Lichnerowicz equation can be solved for that class of conformal data

N The Lichnerowicz equation has no positive solution

N&S There is a necessary and sufficient condition which needs to be checked

PR We have partial results

Solving the Lichnerowicz equation II

We also have

	$\mathcal{B}_{\tau,\psi} < 0$	$\mathcal{B}_{\tau,\psi} \leq 0$	$\mathcal{B}_{\tau,\psi} \equiv 0$	$\mathcal{B}_{\tau,\psi} \geq 0$	$\mathcal{B}_{\tau,\psi} > 0$
$\mathcal{Y}_\psi([\gamma]) < 0$	N	N	N	N&S Cond.	Y
$\mathcal{Y}_\psi([\gamma]) = 0$	N	N	N	Y	Y
$\mathcal{Y}_\psi([\gamma]) > 0$	PR	NR	Y	Y	Y

Table 2: Results for $\mathcal{A}_{\gamma,W,\pi} \neq 0$ and $\mathcal{B}_{\tau,\psi}$ of determined sign.

Y The Lichnerowicz equation can be solved for that class of conformal data.

N The Lichnerowicz equation has no positive solution.

N&S There is a necessary and sufficient condition which needs to be checked.

PR We have partial results.

NR We have no results indicating existence or non-existence.

Remarks on the Proofs I: Non-existence

We assume (via conformal invariance and Proposition 1) that $\text{sign}(\mathcal{R}_{\gamma,\psi}) = \text{sign}(\mathcal{Y}_{\psi}([\gamma]))$ and write the Lichnerowicz equation as

$$\Delta_{\gamma}\phi = \mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi)$$

where

$$\mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi) = \mathcal{R}_{\gamma,\psi}\phi - \mathcal{A}_{\gamma,W,\pi}\phi^{-\frac{3n-2}{n-2}} + \mathcal{B}_{\tau,\psi}\phi^{\frac{n+2}{n-2}}.$$

- All of the "N" entries in Tables 1 & 2 correspond to the situation where if ϕ were a positive solution on Σ then either $\mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi) \leq 0$ or $\mathcal{F}_{\gamma,\sigma,\tau,\psi,\pi}(\phi) \geq 0$ (but not identically zero). Integration then leads to an immediate contradiction.

Remarks on the Proofs II: Existence

We again assume that $\text{sign}(\mathcal{R}_{\gamma,\psi}) = \text{sign}(\mathcal{Y}_{\psi}([\gamma]))$. All the "Y" existence results are obtained by the method of sub- and super-solutions.

- Y's correspond to where one can directly find constant sub- and super-solutions.
- Y's correspond to where we first conformally transform the data via the positive solution to an well chosen linear equation, and then find constant sub- and super-solutions.
- The $\mathcal{A}_{\gamma,W,\pi} \not\equiv 0$ case (with $\mathcal{Y}_{\psi}([\gamma]) < 0$ and $\mathcal{B}_{\tau,\psi} \geq 0$) listed as "N&S Cond." may be reduced to the $\mathcal{A}_{\gamma,W,\pi} \equiv 0$ case, where this is the prescribed scalar curvature-scalar field problem

$$\mathcal{R}_{\tilde{\gamma},\psi} = -\mathcal{B}_{\tau,\psi}.$$

The necessary and sufficient condition for solving this problem in the pure scalar curvature case is due to A. Rauzy. cf. A very recent paper of D. Maxwell and J. Dilts: <http://arxiv.org/abs/1503.04172>

Remarks

- We obtain similar results when (Σ, γ) is asymptotically flat.
- Our constructions allow for rough (low regularity) initial data (cf. work of Y. Choquet-Bruhat and D. Maxwell).
- For existence cases we can also show the uniqueness of the solution within the conformal class.
- The most challenging and important area of current investigation for the constraint equations is the question of what happens when the mean curvature is not constant and the equations are fully coupled. Very important recent work in this area has been done by Holst, Nagy, and Tsogtgerel; Maxwell; and Dahl, Gicquaud and Humbert and others.

Applications of “Gluing” to the Constraint Equations

Gluing refers to a class of constructions in geometric analysis for combining known solutions of nonlinear partial differential equations to obtain new solutions.

Often a gluing construction is done with a topological modification of the underlying manifold on which the solution lives; the simplest such example is the “connected sum” operation.

The underlying connected sum can lead to two distinct constructions:

- The usual connected sum of two distinct disconnected summands (notation: $\Sigma_1 \# \Sigma_2$).
- The “Wormhole” construction. Here there is only one summand, the underlying topology is altered by adding a neck connecting two points (co-dimension n surgery).

Gluing is a standard technique in geometric analysis

Examples where it has played an important role include:

- Existence of anti-self-dual connections on 4-manifolds (Taubes)
- Donaldson & Seiberg-Witten invariants (Taubes, Kronheimer, Morgan, Mrowka)
- Pseudo-holomorphic curves and Gromov-Witten invariants (Gromov, Tian, Ruan, Taubes, Parker, Ionel)
- Manifolds with exceptional holonomy (Joyce)
- Metrics of constant scalar curvature (Schoen, Joyce, Mazzeo, Pacard, P., Mazzieri)
- Surfaces of constant mean curvature in \mathbb{R}^3 (Kapouleas, Mazzeo, Pacard, P.)
- Minimal surfaces (Kapouleas, Mazzeo, Pacard, Traizet)
- Special Lagrangian submanifolds (Joyce, Lee, Butscher, Haskins, Kapouleas)
- Kähler manifolds with constant scalar curvature & extremal Kähler metrics (Arzeto, Pacard, Singer)

General remarks regarding gluing constructions

- Gluing is a “singular perturbation” technique and as such it usually involves a hypothesis concerning the surjectivity of the linearization of the relevant equations about the known solutions (“nondegeneracy”).
- In all the examples listed, the relevant equations are *elliptic*.
- Often a gluing construction has a free parameter (e.g. “neck size”). In the limit, as the parameter tends to zero, the construction yields either the original known solutions or a singular version of these.
- Prior to applications in GR, all known gluing constructions involved a *global perturbation*. Away from the neck (where the connected sum takes place) one could prove that the new solution was a small deformation of the original ones.
 - ▶ The presence of this global perturbation is a reflection of the underlying equations satisfying a unique continuation property.
 - ▶ As we will see the constraint equations dramatically do not satisfy the unique continuation property. This is a reflection of their underdetermined nature.

Conformal gluing constructions

I will describe gluing constructions for the constraint equations in the context of the conformal method as described above. This allows one to perform either a connected sum or a wormhole construction in either of the following circumstances:

- For compact summands, we require that $\bar{K} \neq 0$ and that there do not exist conformal Killing fields which vanish at the points about which we wish to glue. (This is our “nondegeneracy” condition)
- For asymptotically flat or asymptotically hyperbolic summands we do not require any nondegeneracy conditions

Subsequently it was shown that one could relax the globally CMC requirement and only require the data to be CMC near the gluing points. Since in this setting the system does not semi-decouple this requires a nondegeneracy assumption on the surjectivity of the full linearized system obtained by the conformal method.

CMC Conformal gluing theorem

Theorem (Isenberg, Mazzeo, P. (2002))

Let $(\Sigma, \gamma, K; p_1, p_2)$ be a compact, marked, CMC solution of the Einstein constraint equations. We assume that (Σ, γ) has no conformal Killing fields which vanish at p_1 or p_2 and also that $K \not\equiv 0$. Then there is a one-parameter family of solutions $(\Sigma_\epsilon, \gamma_\epsilon, K_\epsilon)$ (for ϵ sufficiently small) of the Einstein constraint equations with the following properties. The manifold Σ_ϵ is constructed from Σ by adding a small neck connecting the two points p_1 and p_2 . Moreover, for small values of ϵ , the Cauchy data $(\gamma_\epsilon, K_\epsilon)$ is a small perturbation of the initial Cauchy data (γ, K) (with $(\gamma_\epsilon, K_\epsilon) \rightarrow (\gamma, K)$ as $\epsilon \rightarrow 0$) away from small balls about the points p_1, p_2 .

When (Σ, γ, K) is either asymptotically Euclidean or asymptotically hyperbolic, the same results hold even in the presence of conformal Killing fields. In the asymptotically Euclidean setting, time-symmetric ($K \equiv 0$) initial data is allowed.

Applications I

- There are no restrictions on the spatial topology of *asymptotically hyperbolic* solutions of the vacuum Einstein constraint equations.
- One may add black holes or wormholes to any spacetime with a CMC Cauchy surface (indicated by a marginally trapped surface)
 - ▶ Chruściel-Mazzeo verified the existence of spacetime developments whose event horizons have multiple connected components
- There are no restrictions on the spatial topology of *asymptotically flat* solutions of the vacuum Einstein constraint equations.
 - ▶ Requires the latter construction without the globally CMC hypothesis
- There exist globally hyperbolic, asymptotically flat solutions of the vacuum Einstein constraint equations with no *maximal* Cauchy surfaces.
- In subsequent work with Isenberg & Maxwell we extended the conformal CMC gluing construction to higher dimensions and non-vacuum data (e.g. Einstein-Maxwell, Yang-Mills, Vlasov, fluids)

Corvino-Schoen gluing

The earliest applications of gluing constructions to GR were given in Justin Corvino's 2000 PhD thesis. He demonstrated a different type of construction, initially working with time symmetric, asymptotically flat vacuum data (i.e. asymptotically flat, scalar flat metrics) he

- Performed a gluing construction which replaces a neighborhood of infinity with an exact slice of Schwarzschild
- Worked directly with the underdetermined constraint equation $R(\gamma) = 0$
- Was able to perform his perturbation with *compact support* within a large annulus. i.e. the original asymptotically flat data was left completely unchanged on an arbitrarily large compact set.

This led to the remarkable result

Theorem (J. Corvino (2000))

There exist a large class of globally hyperbolic vacuum spacetimes which are Schwarzschild at spatial infinity.

Local gluing constructions: initial data engineering

By combining the conformal (IMP) gluing construction with extensions of the Corvino technique due to Chruściel and Delay, we are able to establish a *local* gluing construction for the Einstein constraint equations.

Definition

Let (Σ, γ, K) be a set of initial data satisfying the Einstein vacuum constraint equations, and let $p \in \Sigma$ and let U be an open set containing p . The data has *No KIDs in U* if there do not exist non trivial solutions (N, Y) to the formal adjoint of the linearized constraint equations:

$$0 = \begin{pmatrix} 2(\nabla_{(i} Y_{j)} - \nabla^l Y_l \gamma_{ij} - K_{ij} N + \text{tr} K N \gamma_{ij}) \\ \nabla^l Y_l K_{ij} - 2K^l{}_{(i} \nabla_{j)} Y_l + K^q{}_l \nabla_q Y^l g_{ij} - \Delta N \gamma_{ij} + \nabla_i \nabla_j N \\ + (\nabla^p K_{lp} \gamma_{ij} - \nabla_l K_{ij}) Y^l - N \text{Ric}(\gamma)_{ij} \\ + 2NK^l{}_i K_{jl} - 2N(\text{tr} K) K_{ij} \end{pmatrix}$$

in U .

Local gluing constructions (continued)

- KIDs in U are in one-to-one correspondence with Killing fields within the domain of dependence of U in the spacetime development of the data (Montcrief)
- Under generic perturbations KIDs are absent in every open $U \subset \Sigma$ (Bieg-Chruściel-Schoen)
- The “no KIDs” condition will serve as our nondegeneracy assumption

Theorem (Chruściel, Isenberg, P. (2005))

Let $(\Sigma_1, \gamma_1, K_1)$ and $(\Sigma_2, \gamma_2, K_2)$ be a pair of smooth initial data sets which satisfy the vacuum ($\rho = 0$ and $J = 0$) constraint equations. Let $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$ be a pair of points, with open neighborhoods $U_1 \ni p_1$ and $U_2 \ni p_2$ in which the No KIDs condition is satisfied. There exists a smooth data set $(\Sigma_1 \# \Sigma_2, \hat{\gamma}, \hat{K})$ which satisfies the Einstein constraint equations everywhere, and which agrees with (γ_1, K_1) and (γ_2, K_2) away from $U_1 \cup U_2$. (the “neck” connecting Σ_1 and Σ_2).

Applications II (of the *local* gluing construction)

Remarks

- We have stated the connected sum version of the construction. One may also add (local) wormholes into given initial data sets.
- We can also allow for general non-vacuum data satisfying a strict dominant energy condition (which takes the place of the No KIDs assumption). The new glued solutions also satisfy the dominant energy condition but we do not control any additional equations which the non-gravitation fields may satisfy.
- The local gluing construction applies to generic initial data sets. Beyond the No KIDs assumption, no global conditions such as compactness, completeness, or asymptotic conditions are imposed.
- Results can likely be extended to non-vacuum models, e.g. Einstein-Maxwell, Yang-Mills, etc. The main issue is the assertion of a local Corvino type perturbation result for the relevant model.

Applications II: Spacetimes with no CMC Cauchy surfaces

The main application is the existence of spacetimes with no CMC slices:

Theorem (Chruściel, Isenberg, P. (2005))

There exist vacuum, maximally extended, spacetimes with compact Cauchy surfaces, which contain no compact, spacelike hypersurfaces with constant mean curvature.

- A CMC slice implies the existence of a CMC foliation. This gives a canonical time function (“CMC” or “York” time).

The prevalence of spacetimes with no CMC slices is largely open.

For a very recent gluing construction that even more dramatically demonstrates the huge flexibility among solutions to the constraint equations see “*Localizing solutions of the Einstein constraint equations*” by A. Carlotto and R. Schoen. <http://arxiv.org/abs/1407.4766>

Thank you very much for your attention!