Initial Data for the Cauchy Problem in General Relativity
Lecture 2

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Junior Scientist Andrejewski Days

100 years of General Relativity

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Introduction

This mini-course will be a brief tour through certain parts of mathematical relativity. Results will be presented mainly without proofs but we hope to present enough background to enable you appreciate some recent results in the area.

Here is a brief plan of the 4 lectures.

- **Lecture 1**: Introduction to Lorentzian geometry and causal theory.
- **Lecture 2**: The Einstein equations from the PDE perspective. The constraint equations and the local existence theorem of Choquet-Bruhat.
- **Lecture 3**: Solving the constraint equations via the conformal method
- **Lecture 4**: Topological censorship from the initial data point of view.

The final lecture is based on **Topological censorship from the initial data point of view**, (with Michael Eichmair and Gregory Galloway). (ArXiv:1204.0278) J. Differential Geometry **95** (2013), 389–405.

Lecture 2: The Einstein Field Equations

We wish to view the Einstein field equations as an (evolutionary) PDE system for an Lorentzian space-time M:

$$Ric(g) - \frac{1}{2}R(g)g + \Lambda g = 8\pi \frac{G}{c^4} T \tag{1}$$

- unknown: $g = g_{ab}$ is a Lorentz metric
- $Ric(g) = R_{ab}$ is the Ricci curvature
- $R(g)$ is the scalar curvature
- $T = T_{ab}$ is the stress-energy-momentum tensor. This encodes the non-gravitational physics (e.g. electromagnetic fields)
- We will work in units where $G = c = 1$. The factor of 8π comes from considering the Newtonian limit, so that these equations reduce to Newtonian gravity at speeds much slower than the speed of light.
- Λ is the "cosmological constant" which may be zero (but is currently thought to be positive in our Universe)

The Einstein Equations as a system of PDE

For simplicity, we will consider the vacuum Einstein equations, where $T \equiv 0$.
Exercise: by taking traces, one sees that here the Einstein equations reduce to

$$Ric(g) = 0. \tag{2}$$

Let's begin to understand this as a system of PDEs for the unknown metric g by trying to understand equation (2) as an equation consisting of derivatives for the metric components relative to a coordinate basis $\{\partial_\mu\}$ of T_pM. First recall that the Christoffel symbols are defined by

$$\nabla_{\partial_\alpha}\partial_\beta = \Gamma^\gamma_{\alpha\beta}\partial_\gamma,$$

Where ∇ is the Levi-Cevita connection (covariant derivative). (We employ the summation convention that we sum over repeated indices.)

Exercise: Show that, in terms of the metric

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}(\partial_\beta g_{\alpha\delta} + \partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta}).$$

The Einstein Equations as a system of PDE (cont.)

The Riemann curvature is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The components of the Riemann curvature tensor, relative to the coordinate basis, are defined by

$$R^{\alpha}_{\beta\gamma\delta} = \langle dx^{\alpha}, R(\partial_\gamma, \partial_\delta)\partial_\beta \rangle.$$

Exercise: this leads to the formula

$$R^{\alpha}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta} - \Gamma^{\alpha}_{\sigma\delta}\Gamma^{\sigma}_{\beta\gamma},$$

with the components of the Ricci tensor given by the sum

$$R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}.$$

This shows that the Ricci tensor is linear in the second derivatives of the metric, with coefficients which are rational in the components of the metric, and quadratic in the first derivatives of the metric, again with coefficients which are rational in g.

The Einstein Equations as a system of PDE (cont.)

Thus the vacuum Einstein equations are a second order system of quasi-linear (linear in the highest order derivatives) partial differential equations for the unknown metric g.

Systems of (nonlinear) PDE with *good algebraic properties* of the terms involving the highest order derivatives have been well studied and there are many methods to approach solving these types of equations. However (2) does not fall into any of these classes (e.g. elliptic, parabolic and hyperbolic systems of equations) in an arbitrary coordinate system.

The most significant difficulty with (2) from the PDE point of view is the high degree of non-uniqueness. This is due to the naturality of the equation. which leads to the coordinate or diffeomorphism invariance: if $\phi : M \to M$ is a diffeomorphism then

$$\phi^*(Ric(g)) = Ric(\phi^* g).$$

Thus, if g is a solution to (2) on M, so is $\phi^* g$.

The Einstein Equations as a system of PDE (cont.)

Another way to express this, locally, is as follows. Suppose that, with respect to coordinates $\{x^\mu\}$, we have a matrix of functions $g_{\mu\nu}(x)$ satisfying the quasi-linear system of PDE arising from (2) as previously described. If we perform a coordinate change $x^\mu \to y^\alpha(x^\mu)$, then the matrix of functions $\tilde{g}_{\alpha\beta}(y)$ defined by

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\alpha\beta}(y) = g_{\mu\nu}(x(y)) \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}$$

will also solve (2) (where the resulting x-derivatives are replaced by y-derivatives).

In the language of physics, one says that the diffeomorphism group expresses the gauge freedom of the Einstein field equations. Remarkably, in 1952, Yvonne Choquet-Bruhat proved that there is an underlying system of *hyperbolic* PDE governing the behavior of (2). This involves the introduction of a special set of coordinates (which in particular, breaks the diffeomorphism invariance) and the exploitation of the Bianchi identity together with the **Einstein constraint equations** to obtain a solution of the geometric equation.

The Geometry of Spacelike Hypersurfaces

Let (M, g) be a spacetime and let

$$i : V \hookrightarrow M$$

be an embedded *spacelike* hypersurface. This means that the induced metric $h = i^*(g)$ on V is Riemannian (i.e. has positive definite signature). Let η denote the timelike future-pointing unit normal vector field to V. If we let ∇ be the Levi-Cevita connection on (M, g) and ∇^V be the Levi-Cevita connection on (V, h) then recall that the second fundamental form, K on V, is defined by considering vector fields X and Y tangent to V and setting

$$\nabla_X Y = \nabla^V_X Y + K(X, Y)\eta,$$

so that, for each $p \in V$

$$K : T_pV \times T_pV \longrightarrow \mathbb{R}.$$

Exercise: using the fact that ∇ is torsion free and compatible with g, one can see that

$$K(X, Y) = g(\nabla_X \eta, Y) \implies K(X, Y) = K(Y, X),$$

so K is symmetric.

The Geometry of Spacelike Hypersurfaces (cont.)

A time function t on (M, g) is *adapted to* V if V is a level set of t. If $x = \{x^i\}$ are local coordinates on V then (x, t) form adapted local coordinates for M near V. With respect to such a coordinate system, the *lapse-shift* form for the vector field η is

$$\eta = N^{-1}\left(\frac{\partial}{\partial t} - X^i \frac{\partial}{\partial x^i}\right)$$

where N is call the lapse function and $X = X^i \frac{\partial}{\partial x^i}$ is called the shift vector field.

Exercise

In terms of h, N and X the ambient metric on M is expressed in these coordinates by

$$g = -N^2 dt^2 + h_{ij}(dx^i + X^i dt)(dx^j + X^j dt)$$

and the second fundamental form is given by

$$K_{ij} = K\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{1}{2}N^{-1}\left(\frac{\partial h_{ij}}{\partial t} - \mathcal{L}_X h_{ij}\right),$$

where $\mathcal{L}_X h_{ij}$ is the Lie derivative of the spatial metric h in the direction X.

The Geometry of Spacelike Hypersurfaces (cont.)

In particular, this gives a formula for the time derivative of the spatial metric

$$\frac{\partial}{\partial t} h_{ij} = 2NK_{ij} + \mathcal{L}_X h_{ij},$$

so in the special case when $N \equiv 1$ and $X \equiv 0$ (i.e. $\frac{\partial}{\partial t} = \eta$) we have

$$\frac{\partial}{\partial t} h_{ij} = 2K_{ij}.$$

Let's return to the general Einstein field equations (1) with $\Lambda = 0$, writing them with respect to an adapted local frame for $V \hookrightarrow M$. We define the Einstein tensor $G_{\mu\nu}$ via the left hand side of the equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{3}$$

Recall that the vacuum equations simplify to vanishing Ricci curvature:

$$G_{\mu\nu} = 0 \Longleftrightarrow R_{\mu\nu} = 0.$$

The Einstein Constraint Equations

The Gauss and Codazzi equations for $V \hookrightarrow M$ tell us that the ambient Einstein equations (3) on M impose relationship on the intrinsic and extrinsic curvatures of $(V, h) \hookrightarrow (M, g)$ and the components of the stress-energy-momentum tensor $T_{\mu\nu}$ in a local adapted frame.

Proposition (The Einstein Constraint Equations)

If (M, g) is a spacetime satisfying the Einstein field equations (3), and $V \hookrightarrow M$ is a spacelike hypersurface with induced Riemannian metric h and second fundamental form K then

$$R(h) - |K|_h^2 + (\mathrm{tr}_h K)^2 = 16\pi T_{00} = 2G_{00} = 2\rho \tag{4}$$

$$\mathrm{div} K - \nabla(\mathrm{tr}_h K) = 8\pi T_{0i} = G_{0i} = J \tag{5}$$

where $\mathrm{div} K = \nabla_j K^j{}_i$

The scalar function ρ is called the *local mass density* and the vector field J is called the *local current density* of the set (V, h, K). Equation (4) is called the *Hamiltonian constraint equation* and Equation (5) is call the *momentum constraint equation*.

Wave Equations

For a scalar function ϕ on a spacetime (M, g) we define the *wave operator* associated to the metric g to be the linear operator given by the trace of the Hessian:

$$\begin{aligned} \Box_g \phi &\equiv \nabla_\mu \nabla^\mu \phi \\ &= \frac{1}{\sqrt{-\det g_{\alpha\beta}}} \partial_\mu (\sqrt{-\det g_{\alpha\beta}}\, g^{\mu\nu} \partial_\nu \phi). \end{aligned}$$

We will make use of the following result

Theorem

Given an open set $U \in V$ and smooth functions f_1, f_2 on U, there exists a unique smooth solution ϕ defined on $D(U)$ for the problem

$$\Box g \phi = 0, \qquad \phi|_U = f_1, \qquad \frac{\partial \phi}{\partial t}|_U = f_2.$$

We will will actually make use of a non-trivial generalization of this result, namely that we have existence and uniqueness for solutions of a class of **second order quasi-linear hyperbolic equations**.

Wave Coordinates

The idea is that, despite the fact the the vacuum Einstein equations are not hyperbolic, we can identify a portion of the top order part of the operator which looks like the wave operator applied to the metric.

For the moment suppose that we already know the metric g in a spacetime neighborhood $\mathcal{O}(V)$ of a spacelike hypersurface V. We introduce "wave" or "harmonic" coordinates $\{x^\alpha\}$ by setting

$$H^\alpha \equiv \Box_g x^\alpha = 0 \qquad \text{in} \qquad \mathcal{O}(V)$$

$$x^0 = 0, \quad x^i = \bar{x}^i, \quad \text{and} \quad \frac{\partial x^\alpha}{\partial t} = 0 \qquad \text{on} \qquad V$$

(Greek letters vary among spacetime indices, Roman letters are used for spatial indices only, with 0 being the "time" coordinate.)

By specifying a coordinate system we have a good chance of "breaking" the Guage symmetry imposed by the diffeomorphism invariance of the geometric equations. However this is only useful if (1) the new equation takes a form that can be analyzed in these coordinates, and (2) we have a way of propagating a set of coordinates off of the initial spacelike hypersurface V so that they are indeed harmonic relative to the evolved metric (which is **not** known a priori!). The remarkable fact is that both of these conditions are indeed satisfied.

The Reduced Einstein Equations

The reason this is useful is that one can show that the Ricci curvature can be written as

$$R_{\alpha\beta} = R^H_{\alpha\beta} - H_{(\alpha,\beta)} \tag{6}$$

where

$$\begin{aligned} R^H_{\alpha\beta} &= -\frac{1}{2} g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + Q(g, \partial g) \\ &= -\frac{1}{2} \Box_g g_{\alpha\beta} + Q(g, \partial g) \end{aligned}$$

is the "harmonic" part and $H_{(\alpha,\beta)}$ vanishes in wave coordinates.

The **reduced vacuum Einstein Equations** are

$$R^H_{\alpha\beta} = 0.$$

This is a second order quasi-linear hyperbolic system for the metric g, so we can solve this provided we specify Cauchy data

$$g_{\alpha\beta} \quad \text{and} \quad \frac{\partial g_{\alpha\beta}}{\partial t} \qquad \text{on} \qquad V.$$

The Initial Data Set

Definition

An **initial data set** *for the* $(n+1)$*-dim'l vacuum Einstein Equations is a set* (V, h, K) *where* (V, h) *is an n-dim'l Riemannian manifold and* K *is a symmetric* $(0, 2)$ *tensor on* V*.*

We need to define the Cauchy data for the reduced Einstein equations from a given initial data set as above. First define

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & h_{ij} \end{pmatrix} \qquad \text{at} \qquad t = 0$$

This forces (recall the lapse and shift discussion)

$$\frac{\partial g_{ij}}{\partial t} = 2K_{ij} \qquad \text{at} \qquad t = 0.$$

We are still free to choose $\dfrac{\partial g_{0\beta}}{\partial t}$. We will do this so that

$$H_\alpha = 0 \qquad \text{initially on} \qquad V.$$

Propogating the wave coordinates

We have not yet made use of the vacuum constraint equations!

The contracted second Bianchi identity implies that the Einstein tensor is divergence free

$$\nabla^{\beta} G_{\alpha\beta} = 0.$$

This implies an evolution equation for H^{α}

$$\Box_g H_{\alpha} + \mathrm{l.o.t.} = 0 \tag{7}$$

where $\mathrm{l.o.t.}$ indicates a collection of lower order terms which are linear in H_{α}.

Since we have choosen $\frac{\partial g_{0\beta}}{\partial t}$ so that $H_{\alpha} = 0$ initially, if we can also insure that $\frac{\partial H_{\alpha}}{\partial t} = 0$, then by uniqueness for solutions to (7), we must have

$$H_{\alpha} \equiv 0 \qquad \text{on} \qquad \mathcal{O}(V).$$

This implies that the solution to the reduced Einstein equations is actually a solution to the full geometric vacuum Einstein equations

$$Ric(g) = 0.$$

Completing the argument: the role of the constraint equations

Proposition

The vacuum constraint equations for (V, h, K) imply that

$$\frac{\partial H_\alpha}{\partial t} = 0.$$

Sketch of Proof: The momentum constraint equation says

$$G_{0i} = \mathrm{div} K - \nabla(\mathrm{tr}_h K) = 0$$

This implies that

$$-\frac{1}{2} H_{0,i} - \frac{1}{2} \frac{\partial H_i}{\partial t} = 0.$$

However we have $H_{0,i} = 0$ on V so that

$$\frac{\partial H_i}{\partial t} = 0 \qquad \text{for} \qquad i = 1, \dots n.$$

Completing the argument: the role of the constraint equations (cont.)

The Hamiltonian constraint equation gives

$$G_{00} = R(h) - |K|_h^2 + (\mathrm{tr}_h K)^2 = 0$$

and this shows that

$$\begin{aligned} G_{00} &= -\frac{\partial H_0}{\partial t} - \frac{1}{2}\frac{\partial H_0}{\partial t} g_{00} \\ &= -\frac{\partial H_0}{\partial t} + \frac{1}{2}\frac{\partial H_0}{\partial t} \\ &= -\frac{\partial H_0}{\partial t} = 0 \end{aligned}$$

as desired. Therefore $\frac{\partial H_\alpha}{\partial t} = 0$ so that $H_\alpha \equiv 0$ on $\mathcal{O}(V)$. i.e. the coordinates we obtain are actually wave coordinates for the spacetime metric evolved from h on V by solving the reduced Einstein equations. This metric therefore satisfies the vacuum Einstein equations.

Initial data and local well posedness

In 1952, Yvonne Choquet-Bruhat established the existence of a local in time solution of the vacuum Einstein equations, $Ric(g) = 0$.

- The constraint equations, together with the second Bianchi identity, ensures that the wave coordinate gauge is evolved in time as one solves the reduced Einstein equations, yielding a solution of the full geometric equations.

Theorem (Choquet-Bruhat 1952)

Given an initial data set $(V; h, K)$ satisfying the vacuum constraint equations there exists a spacetime (M, g) satisfying the vacuum Einstein equations $Ric(g) = 0$ where $V \hookrightarrow M$ is a spacelike surface with induced metric h and second fundamental form K.

This is a local existence result. Nothing is claim about the “size” of (M, g) (one thinks of it simply as a “thickening of (V, h) into a small spacetime neighborhood). One would like a more global solution.

Maximal, globally hyperbolic developments

Theorem (Choquet-Bruhat & Geroch, 1969)

Given an initial data set $(V; h, K)$ satisfying the vacuum constraint equations there exists a unique, globally hyperbolic, maximal, spacetime (M, g) satisfying the vacuum Einstein equations $Ric(g) = 0$ where $V \hookrightarrow M$ is a Cauchy surface with induced metric h and second fundamental form K. Moreover any other such solution is a subset of (M, g).

This is a much more satisfactory result, but it still leaves open the most difficult questions concerning global existence.

As pointed already, in both of these results, a central role is played by the **existence of initial data sets solving the Einstein constraint equations**. We will turn our attention to this question in the next lecture.

Thank you again for your attention!