

# Initial Data for the Cauchy Problem in General Relativity Lecture I

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Junior Scientist Andrejewski Days  
100 years of General Relativity

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# Introduction

This mini-course will be a brief tour through certain parts of mathematical relativity. Results will be presented mainly without proofs but we hope to present enough background to enable you appreciate some recent results in the area.

Here is a brief plan of the 4 lectures.

- **Lecture 1:** Introduction to Lorentzian geometry and causal theory.
- **Lecture 2:** The Einstein equations from the PDE perspective. The constraint equations and the local existence theorem of Choquet-Bruhat.
- **Lecture 3:** Solving the constraint equations via the conformal method
- **Lecture 4:** Topological censorship from the initial data point of view.

The final lecture is based on **Topological censorship from the initial data point of view**, (with Michael Eichmair and Gregory Galloway).  
(ArXiv:1204.0278) J. Differential Geometry **95** (2013), 389–405.

## Lorentzian Manifolds

We start with an  $(n + 1)$ -dimensional Lorentzian manifold  $(M, g)$ .  $(M, g)$  is thus a pseudo-Riemannian manifold such that the metric

$$g : T_p M \times T_p M \longrightarrow \mathbb{R}$$

is a scalar product of signature  $(-1, +1, \dots, +1)$ . With respect to a Lorentzian orthonormal basis  $(e_0, e_1, \dots, e_n)$ , as a matrix,

$$[g_{ij}] = \text{diag}(-1, +1, \dots, +1).$$

**Example:** Minkowski space  $\mathbb{M}^{n+1}$  is the Lorentzian analogue of Euclidean space. For vectors  $X, Y \in T_p \mathbb{R}^{n+1}$  given in Cartesian coordinates on  $\mathbb{R}^{n+1}$  by

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^i \frac{\partial}{\partial x^i}$$

we define the Minkowski metric  $\eta$  by

$$\eta(X, Y) = -X^0 Y^0 + \sum_{i=1}^n X^i Y^i = \eta_{ij} X^i Y^j,$$

where  $\eta_{ij} = \varepsilon_i \delta_{ij}$  and  $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) = (-1, 1, \dots, 1)$ .

## Lorentzian Manifolds and basic causal theory

For any  $p \in M$ , a Lorentz manifold, we have a classification of vectors  $X \in T_p M$  into timelike, null or spacelike, as follows

$$X \text{ is } \begin{cases} \text{timelike} & \text{if } g(X, X) < 0 \\ \text{null} & \text{if } g(X, X) = 0 \\ \text{spacelike} & \text{if } g(X, X) > 0 \end{cases}$$

We extend this notion to smooth curves  $\gamma : (a, b) \rightarrow M$  as follows

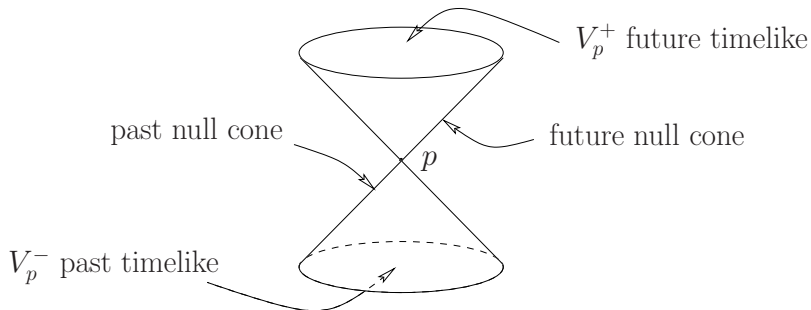
$$\gamma \text{ is } \begin{cases} \text{timelike} & \text{if } \gamma'(t) \text{ is timelike, } \forall t \in (a, b) \\ \text{null} & \text{if } \gamma'(t) \text{ is null, } \forall t \in (a, b) \\ \text{spacelike} & \text{if } \gamma'(t) \text{ is spacelike, } \forall t \in (a, b) \end{cases}$$

We say that  $\gamma$  is **causal** if  $\gamma'(t)$  is either timelike or null,  $\forall t \in (a, b)$ .

The world lines of particles follow causal curves, with light traveling on null curves (null geodesics) and massive particles traveling on timelike curves. At each point

$p \in M$  the set of timelike vectors form two disjoint open cones, which we'll denote as  $V_p^+$  and  $V_p^-$ , the interiors of the future and past light cones.

# The light cone at a point $p$



# Causal Structures and Conformal Structures

**Exercise:** Show that the assignment of a *causal structure* on a manifold  $M$  (i.e. the assignment of a smoothly varying light cone at each point  $p \in M$ ) is equivalent to the assignment a *conformal structure*, namely class of Lorentz metrics  $[g]$  where any two metrics  $g_1, g_2 \in [g]$  are conformally related:  $g_1 = \phi^2 g_2$  for some smooth, positive function  $\phi$  on  $M$ .

# Lorentzian Manifolds and basic causal theory

We'll say that the Lorentzian manifold  $(M^{n+1}, g)$  is **time orientable** if it admits a timelike vector field. This allows us to make a continuous choice of a **future** light cone  $V_p^+$  at each point of  $M$ .

## Definition

A **spacetime**  $(M^{n+1}, g)$  is a connected, time-oriented Lorentzian manifold.

Let  $T$  denote a timelike vector field defining the time orientation on  $M$ . For any nonzero causal vector  $v \in T_p M$ ,  $g(v, T)$  is either positive or negative. If  $g(v, T)$  is negative we say that  $v$  is **future pointing** (since  $v$  then lies in  $V_p^+$ ) and if  $g(v, T)$  is positive we say that  $v$  is **past pointing** (since  $v$  then lies in  $V_p^-$ ).

A causal (timelike, null) curve  $\gamma$  is said to be **future pointing** if  $\gamma'$  is future pointing at each point along  $\gamma$ .

## Future and Past sets

We say  $p \ll q$  if there is a future pointing timelike curve in  $M$  from  $p$  to  $q$ , and  $p < q$  if there is a future pointing causal curve in  $M$  from  $p$  to  $q$ .  $p \leq q$  means that either  $p = q$  or  $p < q$ .

### Definition

Let  $A$  be a subset of  $M$

$$I^+(A) = \{p \in M : q \ll p \text{ for some } q \in A\}$$

$$J^+(A) = \{p \in M : q \leq p \text{ for some } q \in A\}$$

$I^+(A)$  is called the **chronological future** of  $A$  and  $J^+(A)$  is called the **causal future** of  $A$ . The past sets  $I^-(A)$  and  $J^-(A)$  are similarly defined.

The sets  $I^+(A)$  and  $I^-(A)$  are always open (**exercise**) but for  $J^+(A)$  and  $J^-(A)$  no general statement holds without further assumption (remove a point from Minkowski spacetime to see that these need not be closed). However, we do have (**exercise**)

$$I^+(A) = I^+(I^+(A)) = I^+(J^+(A)) = J^+(I^+(A)) \subseteq J^+(J^+(A)) = J^+(A).$$



## Strong Causality

We need to impose a reasonable causality condition on our spacetimes in order to prohibit pathologies (such as closed timelike curves) and make them amenable to analysis.

The **strong causality condition** holds at  $p \in M$  if, given any neighborhood  $U$  of  $p$ , there is a neighborhood  $V \subseteq U$  of  $p$  such that every causal curve segment with endpoints in  $V$  lies entirely in  $U$ . A spacetime  $M$  is said to be **strongly causal** if strong causality holds at each point  $p \in M$ .

Strong causality prohibits the existence of closed causal curves, but is much stronger:

### Lemma

*Suppose that strong causality holds in a compact subset  $K \subset M$ . If  $\gamma : [0, b) \rightarrow M$  is a future inextendible causal curve that starts in  $K$ , then it eventually leaves  $K$  and does not return, i.e.,  $\exists t_0 \in [0, b)$  such that  $\gamma(t) \notin K \forall t \in [t_0, b)$ .*

So a future inextendible causal curve can not be contained forever within a compact set on which strong causality holds. (**Exercise:** Prove the Lemma.)

# Global hyperbolicity

## Definition

$(M, g)$  is **globally hyperbolic** if it is **strongly causal** and for every pair  $p < q$ , the set

$$J(p, q) = J^+(p) \cap J^-(q)$$

is compact (“internal compactness”).

Mathematically, global hyperbolicity often plays a role analogous to geodesic completeness in Riemannian geometry, but as the name suggests (and as we will see), it is also related to the solvability of hyperbolic PDE. Global hyperbolicity is also connected to the (strong) cosmic censorship conjecture introduced by Roger Penrose, which says that, generically (globally hyperbolic) solutions to the Einstein equations do not admit *naked singularities* (singularities visible to some observer).

# Consequences of Global hyperbolicity

The following are some consequences of global hyperbolicity:

## Theorem

Let  $(M, g)$  be a globally hyperbolic spacetime. Then

1. The sets  $J^\pm(A)$  are closed, for all compact subsets  $A \subset M$ .
2. The sets  $J^+(A) \cap J^-(B)$  are compact, for all compact subsets  $A, B \subset M$ .
3. If  $p < q$ , then there is a maximal future directed causal geodesic from  $p$  to  $q$  (no causal curve from  $p$  to  $q$  can have greater length).
4. If we have convergent sequences on  $M$ ;  $p_n \rightarrow p$  and  $q_n \rightarrow q$  and  $p_n \leq q_n$ , then  $p \leq q$  (i.e. the causality relation  $\leq$  is closed on  $M$ ).

In the way that for Riemannian manifolds **completeness** insures the existence of minimizing geodesics between points (recall the Hopf-Rinow theorem), global hyperbolicity is the condition which insures the existence of maximal causal geodesic segments (cf. 3. above).

# Domains of Dependence

$A \subset M$  is called **achronal** if there is no pair of points  $p, q \in A$  that can be connected by a timelike curve. Let  $A \subset M$  be achronal, we define the **future and past domains of dependence** (also called *Cauchy developments*) of  $A$  as follows

$$D^+(A) = \{p \in M : \text{every past inextendible causal curve from } p \text{ meets } A\},$$
$$D^-(A) = \{p \in M : \text{every future inextendible causal curve from } p \text{ meets } A\}.$$

( $p$  is a *future endpoint* of a causal curve  $\gamma$  if for any Lipschitz parametrization  $\gamma : [0, \infty) \rightarrow M$ , we have that for any neighborhood  $U$  of  $p$ ,  $\exists T = T(U)$  such that  $\gamma(t) \in U, \forall t \geq T$ .  $\gamma$  is future inextendible if it does not have a future endpoint.)

The **domain of dependence** of  $A$  is

$$D(A) = D^+(A) \cup D^-(A)$$

Since information travels along causal curves,  $D(A)$  consist of the set of points in spacetime which are (potentially) influenced by *every point* in the set  $A$ , to either the past or the future. If physics is deterministic then initial data on  $A$  should completely determine the state of the theory on all of  $D(A)$ .

## Domains of dependence and global hyperbolicity

Domains of dependence are tied to global hyperbolicity because the interior of the domain of dependence (viewed itself as a spacetime) is globally hyperbolic:

### Proposition

Let  $A \subset M$  be achronal.

- (1) Strong causality holds on  $\text{int } D(A)$ .
- (2) Internal compactness holds on  $\text{int } D(A)$ , i.e., for all  $p, q \in D(A)$ ,  $J^+(p) \cap J^-(p)$  is compact.

We wish to find a condition on an achronal subset  $A$  that will insure that the domain of dependence of  $A$  is all of  $M$ .

$$D(A) = M.$$

This will insure that the entire spacetime is deterministic relative to  $A$ , so that we can try to approach an analytical theory (namely the Einstein field equations) via an evolutionary perspective by prescribing initial data on  $A$ , determining the spacetime metric by solving a system of PDE in  $D(A)$  (2nd lecture).

# Cauchy surfaces

## Definition

A **Cauchy surface**  $S$  is an achronal subset of  $M$  which is met by every inextendible causal curve in  $M$ .

If  $S$  is a Cauchy surface for  $M$  then  $S = \partial I^+(S) = \partial I^-(S)$ , from this one can show that  $S$  is a closed  $C^0$  hypersurface. The existence of Cauchy surfaces and global hyperbolicity for the **entire spacetime** are closely connected.

## Theorem (Geroch)

Let  $M$  be a spacetime.

1. If  $M$  is globally hyperbolic then it admits a Cauchy surface.
2. If  $S$  is a Cauchy surface for  $M$  then  $M$  is homeomorphic to  $\mathbb{R} \times S$ .

Thus we see that for globally hyperbolic spacetimes, the topology of a Cauchy surface  $S$  determines the topology of the entire spacetime. At the end of this lecture we will make some remarks regarding the strengthening of this result to the smooth category.

## Cauchy surfaces (cont.)

Sketch of Proof: For 1. let  $\mu$  be a probability measure on  $M$  so  $\mu$  is a positive measure with  $\mu(M) = 1$ . Let  $f^-(p) = \mu[J^-(p)]$  and  $f^+(p) = \mu[J^+(p)]$  and using these, define a positive function  $f : M \rightarrow \mathbb{R}$  by

$$f(p) = \frac{f^-(p)}{f^+(p)} = \frac{\mu[J^-(p)]}{\mu[J^+(p)]}.$$

One can show that  $f$  is continuous, and strictly increasing along future directed causal curves. The claim is that the level sets of  $f$  are each Cauchy surfaces. This is demonstrated by showing that both (1)  $f^-(p) \rightarrow 0$  along every *past* inextendible causal curve, and (2)  $f^+(p) \rightarrow 1$  along every *future* inextendible causal curve. This shows that  $f$  attains all values of  $(0, \infty)$  along every inextendible causal curve, and therefore each such curve intersects each level set precisely once.

To prove 2. one introduces a future directed timelike vector field  $X$  scaled so that the time parameter of each integral curve of  $X$  extends from  $-\infty$  to  $\infty$  with  $t = 0$  corresponding to  $S$ . The flow of  $X$  provides the desired homeomorphism.

# The topology of globally hyperbolic spacetimes

## Proposition

If a spacetime has a Cauchy surface  $S$  then

$$D(S) = M$$

Sketch of Proof: Let  $p \in M$  and let  $\gamma$  be an inextendible timelike geodesic through  $p$ . The  $\gamma$  intersects  $S$  in exactly one point. So  $p$  is in one of the sets  $S$ ,  $I^+(S)$  and  $I^-(S)$ . Since  $S$  is a Cauchy surface these sets are disjoint. Also  $J^\pm(S)$  and  $I^\mp(S)$  are disjoint. This shows

$$J^\pm(S) = M \setminus I^\mp(S),$$

so  $J^\pm(S)$  are closed sets. Since  $p \in I^-(S)$  implies  $p \notin D^+(S)$ , we have  $D^+(S) \subset J^+(S)$ . On the other hand one can see that  $J^+(S) = S \cup I^+(S) \subset D^+(S)$  so we see that  $J^+(S) = D^+(S)$ . Reversing time orientation above we see that  $J^-(S) = D^-(S)$ . These together show that  $D(S) = M$ .



## Additional Remarks

In summary we have seen that a spacetime  $M$  is globally hyperbolic if and only if it admits a Cauchy surface  $S$ , it's global topology is  $\mathbb{R} \times S$  and  $D(S) = M$ .

A **time function** on a Lorentzian manifold  $(M, g)$  is a function that is strictly increasing along any future directed causal curve. In the sketch of the proof of Geroch's Theorem we introduced the time function  $f$ .

The existence of a *smooth* Cauchy surface and a *smooth* time function leading to a splitting  $M = \mathbb{R} \times S$  (as a diffeomorphism) was only rigorously established in a series of papers from 2003 – 2006 by Bernal and Sánchez.

# Boundary Conditions

In the next lecture we will introduce the Einstein field equations upon which general relativity is based. Historically there are two cases of restrictions on the topology/geometry which have received the most attention. From the perspective of analysis we can view these as choices of (spatial) boundary conditions.

**(1) Cosmological spacetimes:** here we assume that the Cauchy surface  $S$  is compact without boundary (so this is the empty boundary condition).

**(2) Asymptotically flat spacetimes – Isolated gravitational systems:** There are a number of different ways to introduce this notion. We take the initial data approach that will be discussed further in the next talk.

A Cauchy surface  $S$  in a spacetime  $(M, g)$  inherits a natural geometry as a submanifold. The relevant geometric data on  $M$  is the induced metric  $h$  (Riemannian if  $S$  is spacelike) and second fundamental form  $K$ . We will explore more about the relationship between  $(M, g)$  and  $(S, h, K)$  in the next talk. Here we simply express the notion that  $(M, g)$  is asymptotically flat in terms of restrictions on  $(S, h, K)$ .

## Boundary Conditions: Asymptotic flatness

We want to assume that the geometry of  $(S, h, K)$  is such that 'near infinity' it is asymptotic to a  $t = \text{constant}$  slice of Minkowski space  $(\mathbb{M}^{3+1}, \eta)$ . In other words it should be asymptotic to  $(\mathbb{R}^3, \delta, 0)$  where  $\delta$  is the Euclidean metric and 0 represents the trivial second fundamental form.

To be more precise, suppose we have a compact subset  $C \subset S$  for which  $S \setminus C = \bigcup_{m=1}^k E_m$ , where the  $E_m$  are pairwise disjoint, and each diffeomorphic to the exterior of a ball in Euclidean space, i.e.  $\mathbb{R}^3 \setminus \{|x| \leq 1\}$ .

Then we say that  $(S, h, K)$  is *asymptotically flat* with decay rate  $q$  provided each  $E_m$  admits coordinates for which we have

$$|\partial_x^\alpha (h_{ij} - \delta_{ij})(x)| = O(|x|^{-|\alpha|-q})$$

and

$$|\partial_x^\beta K_{ij}(x)| = O(|x|^{-|\beta|-1-q})$$

for  $|\alpha| \leq \ell + 1$  and  $|\beta| \leq \ell$ , for some  $\ell \in \mathbb{Z}_+$  (chosen depending on the need). (For simplicity, we can take  $q = 1$ , though  $q > \frac{1}{2}$  generally gives sufficient decay.)

## Asymptotically flat spacetimes

The Einstein field equations in the context of asymptotically flat spacetimes leads to a number of the major themes. These include

- Notions of Mass and the Penrose inequality
- Black holes, censorship and gravitational radiation

Each of these topics have their origins in the *Schwarzschild* spacetimes, a family of rotationally symmetry, explicit solutions of the vacuum Einstein equations which has been very important not only for physics but also for geometry.

The Schwarzschild metric takes the form, in coordinates  $(t, x) \in \mathbb{R} \times \mathbb{R}^3$ ,

$$\bar{g}_S(x) = - \left( \frac{1 - \frac{m}{2|x|}}{1 + \frac{m}{2|x|}} \right)^2 dt^2 + \left( 1 + \frac{m}{2|x|} \right)^4 \delta$$

where  $\delta$  is the Euclidean metric. The parameter  $m$  is called the *mass* of the spacetime. The spacelike slice  $t = 0$  is asymptotically flat and conformally flat with vanishing scalar curvature, and the metric  $g_S(x) = \left( 1 + \frac{m}{2|x|} \right)^4 \delta$  extends to a complete metric on the set  $\mathbb{R}^3 \setminus \{0\}$ , with two asymptotically flat ends.

# Schwarzschild Geometry

We will refer to this Riemannian metric as the “Schwarzschild metric”. The two-sphere  $|x| = \frac{m}{2}$  inside this slice is totally geodesic, and the three-manifold has a reflection symmetry across it. This minimal sphere is called the *horizon* of the time symmetric slice. In the Schwarzschild black-hole space-time itself, this horizon is the central leaf of the three-dimensional null hypersurface comprising the actual *event horizon*.

The “asymptotic simplicity” model for isolated gravitational systems proposed by Penrose has been very influential. This model assumes existence of smooth conformal completions to study global properties of asymptotically flat space-times.

We will come back to the geometry of the Schwarzschild metric, and its horizon, in the fourth lecture in this series.

Thank you very much for your attention!